The moment will therefore be \( \frac{\pi r^2}{2} \cdot R \cdot \frac{3}{8} r = \frac{3}{8} \pi r^3 R \).

The moment of the weight, acting with a leverage \( 2r \), is \( 2wR \). The equation of equilibrium is \( \frac{3}{8} \pi r^3 R = 2wR \), or \( \frac{3}{8} \pi r^3 \)

\[
\frac{R}{2r} x \cdot 2y = \text{area} = \frac{R}{r} xy. \quad \frac{R}{r} xy \, d \, x = \text{elementary solid}.
\]

\[
\frac{R}{r} y x^2 \, d \, z = \text{moment of elementary rectangle}.
\]

But from the equation of the circle we have \( y = \sqrt{2r^2 - x^2} \)

\[
\int \frac{R}{r} x y \, d \, x = \frac{R}{r} \int (2r^2 - x^2)^{\frac{1}{2}} x \, d \, x \quad \text{make} \quad r - x = z, \quad d \, x = -dz, \quad 2r \, x = x^2 = r^2 - z^2. \]

Substitute these values we obtain \( \frac{R}{r} \int (2r^2 - x^2)^{\frac{1}{2}} x \, d \, x = \frac{R}{r} f(r^2 - z^2)^{\frac{1}{2}} (r - z) (-dz) = \frac{R}{r} \left[ f(r^2 - z^2)^{\frac{1}{2}} r \, dz + f'(r^2 - z^2)^{\frac{1}{2}} z \, dz \right]. \)

The first of these integrals taken between the limits +\( r \) and −\( r \) is the area of a semi-circle and is consequently equal to \( \frac{\pi r^2}{2} \); hence the value of the first term becomes \( \frac{\pi r^3}{2} \).

The second term becomes \( \frac{(r^2 - z^2)^{\frac{3}{2}}}{3} \) (see demonstration of ungula, Prob. 14), and is equal to 0 when \( z = +r \) or \( z = -r \); it therefore disappears, and the volume of the solid becomes \( -\frac{R}{r} \cdot \frac{\pi r^3}{2} = -\frac{1}{2} \pi r^2 R \); a result which is evidently correct, since the volume is half that of the cylinder \( \pi r^2 R \).

To find the distance to the centre of gravity we must divide the integral of \( \frac{R}{r} y x^2 \, d \, x \) by the volume. Making similar substitutions to those used in finding the volume, we obtain \( \frac{R}{r} \int y x^2 \, d \, x = \frac{R}{r} f(r^2 - z^2)^{\frac{1}{2}} (r - z) (-dz) = \frac{R}{r} f(r^2 - z^2)^{\frac{1}{2}} (r^2 - 2rz + z^2) (-dz) = \frac{R}{r} [r^2 f(r^2 - z^2)^{\frac{1}{2}} r \, dz - 2r f(r^2 - z^2)^{\frac{1}{2}} z \, dz + f'(r^2 - z^2)^{\frac{1}{2}} z^2 \, dz]. \)

The first of these integrals is a semi-circle, hence \( r^2 f(r^2 - z^2)^{\frac{1}{2}} r \, dz = r^3 \)

\[
\frac{\pi r^4}{2} = \frac{\pi r^4}{2}.
\]

The second, \( f(r^2 - z^2)^{\frac{1}{2}} z \, dz \), as we have seen, becomes = 0.

The third, \( f'(r^2 - z^2)^{\frac{1}{2}} z^2 \, dz \), is proved, in the problem of the ungula, to be \( \frac{1}{8} r^2 f'(r^2 - z^2)^{\frac{1}{2}} z \, dz \), which, between the limits +\( r \) and −\( r \), becomes \( \frac{\pi r^4}{8} \). (See note to Prob. 14.)