

Torsional Rigidity of Structural Sections

A Method of Determining This Property by the Soap Film Analogy

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WHEN a shaft of circular cross section is twisted, a section that was plane before twisting remains very nearly plane. Internal resistance to the twist of the shaft is maintained by shearing stresses which are proportional to the distance from the center line of the shaft. The total resisting moment may be expressed as:

$$T = JG\theta \dots \dots \dots [1]$$

in which T = torsional moment in inch-pounds

J = polar moment of inertia

G = shearing modulus

θ = angle of twist in radians per inch

At any distance, c , from the center line of the shaft, the shearing stress is:

$$\tau = \frac{Tc}{J} \dots \dots \dots [2]$$

These simple relationships between torsional moment, twisting deformation, and shearing stress only hold for circular sections. In a shaft of any other cross section, a plane section warps during twisting, and the polar moment of inertia of the section has, in general, no direct relation to the torsional problem. It is possible, however, to determine either mathematically or experimentally a torsion constant which may be substituted for J in Eq. 1. Denoting the torsion constant by K , then:

$$T = KG\theta \dots \dots \dots [3]$$

and the torsional shearing stress at any point in a non-circular section is proportional to $\frac{T}{K}$ multiplied by some function of the thickness of the material at the point in question. It is of primary importance, in designing any non-circular section for torsion, to be able to calculate the torsion constant.

Treatises on the torsion problem show that a general analysis involves the solution of a partial differential equation, which may be written in the following form:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \dots \dots \dots [4]$$

In this equation, ϕ is a torsional stress function in terms of the coordinate axes of the section under consideration. Saint Venant¹ was the first to develop correctly the general solution of the torsion problem, finding the torsion constant and stress distribution in such shapes as the ellipse, rectangle, square, and triangle.

EXPLANATION OF MEMBRANE ANALOGY

In 1903, Prandtl² showed that the general equations applicable to the torsion problem could be satisfied me-

*I*N the April 1935 issue of "Proceedings" there appeared the results of an interesting study of the torsional properties of standard structural sections, made by Inge Lyse, M. Am. Soc. C.E., and Mr. Johnston. That investigation had been supplemented by soap film experiments, as well as by actual torsion tests, but the resulting paper did not discuss in detail the soap film work. This Mr. Johnston has ably presented in the following article, which describes the apparatus and procedure used in the study above-mentioned, to develop by means of Prandtl's membrane analogy, general formulas for the torsional constant of structural shapes. This work was conducted as a regular research project of the Fritz engineering laboratory at Lehigh University.

chanically by stretching a thin membrane across an opening having the same shape as the section to be investigated and distending the membrane by a slight variation in pressure. The membrane analogy was first used successfully by Griffith and Taylor³ and later by Trayer and March,⁴ both investigations dealing with sections common in aircraft construction.

In developing the theory of the membrane analogy it is assumed that the tension in the soap bubble film, Fig. 1, is the same in all directions and that it is independent of location or shape. It is further assumed that the uniform pressure under the film acts in a direction parallel with the z axis rather than normal to the surface of the film.

This assumption is relatively true if the bubble is distended only a slight amount. It can then be shown, by considering the equilibrium of a small part of the film, that the equation of the surface of the film is:^{2,5}

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{F} \dots \dots \dots [5]$$

In this case, z , x , and y = rectangular coordinates
 p = unit pressure under the film
 F = unit tensile force in the film

The similarity between Equations 5 and 4 makes evident

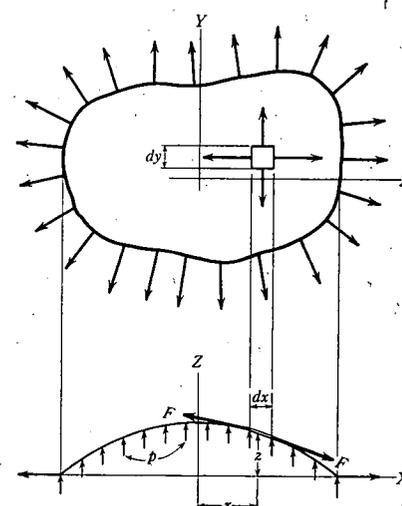


FIG. 1. FORCES IN A DISTENDED SOAP FILM

Based on the Assumption That Tension Is the Same in All Directions and That Pressure Under the Film Acts Parallel with the Z -Axis

the analogy between the distended membrane and the torsion problem. Prandtl showed that the torsional rigidity, K , is proportional to the total volume of the displaced bubble; that the torsional shearing stress at any point on the cross section is proportional to the maximum slope of the bubble at that point; and that contour lines on the bubble represent the direction of maximum shearing stress.

The possibilities of error in using this analogy arise

both from the assumptions made in the derivation and from difficulties encountered in the actual application. The derivation of the soap film analogy is dependent on the fact that for small angles the sine is nearly equal to the tangent. The deviation of sine from tangent increases with increased inclination of film surface, and the results are affected in two different ways: First, by the assumption that the sine is equal to the tangent, and second, by the assumption that the pressure on the film was parallel with the Z-axis. While these two errors partially offset each other, the net error for a given non-circular section is not readily calculable as a function of average film slope. In actual application, the error is minimized by using a circular section beside the section under study as an index of comparison, making it possible to obtain directly the unknown torsional properties from the slope and volume ratios of the two bubbles.

In the study of structural sections at Lehigh University, elsewhere referred to, fifty-seven differently proportioned H-beam and I-beam shapes were tested by means of the membrane analogy. Only volume measurements were made, as it was the purpose of the series to establish formulas for the torsion constant. Four standard circular-hole sections were made with varying diameters, and these were calibrated against each other and against rectangular and square sections of known torsional properties. A calibration curve, made from the results of thirty such tests, provided both for correcting theoretical errors and for volume and pressure changes which occurred during direct volume measurements.

Several different techniques or methods have been used by investigators in applying the membrane analogy to torsion. An air-tight box is generally used as a base, over which is clamped an aluminum or brass plate in which a hole has been cut to reproduce the section of the member to be studied, with a circular hole to serve as the comparison index. A film of soap solution is then spread over the holes and enough air is introduced into the compartment below to give the bubbles a convexity of the desired degree. Measurements are then taken of volumes and slopes, as required by the nature of the problem.

CHARACTERISTICS OF A SUITABLE SOAP SOLUTION

A soap film for use in membrane analogy experiments must be tough and durable and of such consistency that it will maintain a uniform thickness. The composition of soap films has been the subject of considerable investigation, and reference may be made to the work of Sir

James Dewar,⁶ a pioneer in this field. Trayer and March⁴ tried out several compositions and finally obtained best results with a Dewar solution made of "a very small quantity of triethylamine oleate added to a fifty per cent solution of glycerin in distilled water." In the present work the following solution was used:

Triethynolamin oleate.....	10 per cent
Glycerin.....	30 per cent
Water.....	60 per cent

The proportions are by weight. The triethynolamin oleate is less expensive than the oleate used by Trayer



FIG. 3. RECEPTACLE FOR HOLDING SOAP FILMS

and March but gave very satisfactory results in the work here described. Some of the films lasted for three or four days after having been subjected to test.

The soap solution must be kept in an air-tight container and should be used only under conditions of constant temperature. Loss of water by evaporation will change the consistency of the film and the corresponding tensile force which it develops. Precautions should be taken to see that no grease or acid contaminates the film, as these will alter its properties and shorten its life.

METHOD OF MEASURING VOLUMES

The volume of the circular index bubble may be calculated on the basis of its being a spherical segment from direct observations of its altitude made with a micrometer screw suspended over the center of the bubble and supported on a suitable frame.

In previous investigations, the volume of the test bubble was usually established indirectly by means of contour-line plotting. A pointed micrometer, held in an adjustable sliding frame over the bubble, was lowered until the point just touched the surface at any desired height. A pencil attached to the upper end of the micrometer traced off the contour lines on a drawing board as the point was moved around the bubble. The volume was then calculated by planimetering the contour-line areas at equal height increments.

This method has the advantage of reproducing directly the stress lines of the section and also affords a means of plotting vertical sections from which the slopes of the bubbles may be approximately determined and shearing stress values calculated. The disadvantages are that the accuracy of the results is dependent on a double measurement involving the repeated use of micrometer and planimeter, and a large amount of time is required both in observation and calculation.

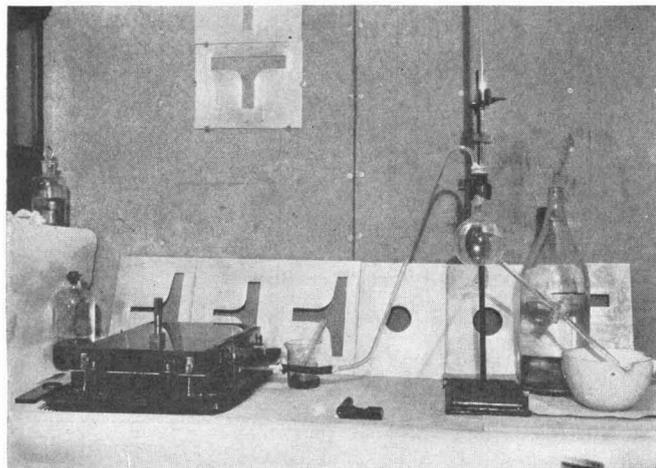


FIG. 2. GENERAL VIEW OF TESTING APPARATUS

In the present study, the volume was determined by measuring directly the total displacement of the two bubbles and subtracting from the total increase in volume the calculated volume of the circular bubble. Water from a burette, shown in Fig. 2, was introduced into an air-tight flask which served as a displacement chamber. The air from this chamber entered a compartment under the soap films to "blow" the bubbles to the desired height. The total volume of water introduced was not

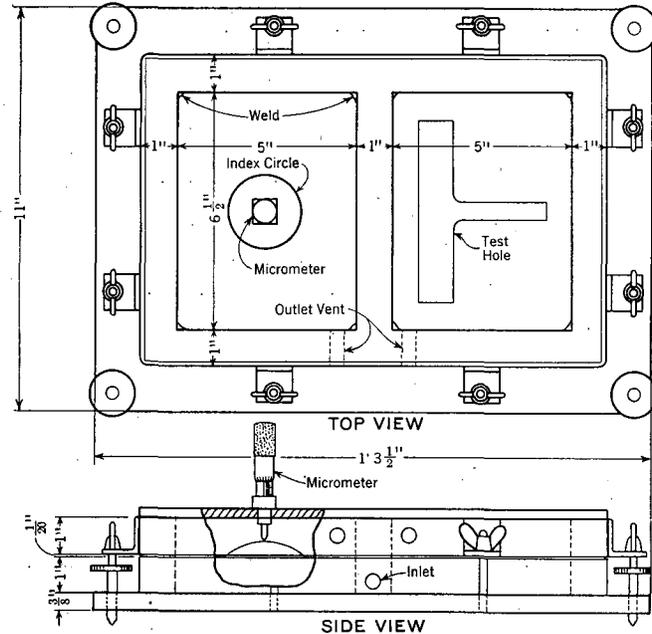


FIG. 4. DETAIL OF RECEPTACLE FOR HOLDING SOAP FILMS

exactly equal to the total volume of the two bubbles because of the slight increase in pressure in the compartment. The discrepancy was taken care of by an experimentally determined correction or calibration curve made between sections of known torsional properties.

The direct method here described has the advantages of speed and accuracy, but gives no information as to the magnitude or direction of stress. It is believed, however, that the most important application of the analogy is in determining the torsion constant of any section and that the importance of the theoretical stresses depends on the degree to which they are localized and the ductility of the material. These factors can only be determined by actual torsion tests.

DESCRIPTION OF THE APPARATUS USED

A general view of the apparatus set up for operation is shown in Fig. 2. The displacement chamber was made with a 250-cc flask, to the bottom of which was welded a drainage tube with a stop-cock. A two-hole rubber cork in the top of the flask received a 50-cc burette, graduated to 0.1 cc, and a glass tube for transmitting air to the box.

The receptacle for holding the soap films is shown in Fig. 3 and detailed in Fig. 4. The lower part consists of a flat plate to which is soldered the double box frame made of bars 1 in. square, welded together. Thumb screws in each of the four corners provide a means of supporting and leveling the apparatus. Eight stud bolts are fitted in the plate to receive and tighten down the upper box with wing nuts and washers. The entry tube for the air is of brass, screw-threaded at the end to enter a tapped hole in the lower frame. The air had free access to both lower chambers by means of a recess in the end

of the lower-frame center-piece. The top of the lower frame is machined to receive the two aluminum test pieces. The detachable upper frame is made the same shape as the lower part and is machined on both faces. All joints in the apparatus were sealed. Over the upper chamber rested a removable section of plate glass 1/4 in. thick, fitted with a pointed micrometer for measuring the height of the circular bubble.

Stock aluminum sheets 0.05 in. thick were used for the plates. The holes were laid out with a scribe and roughed in with a power jig-saw. Small steel files were used to cut close to the line, and the final shaping was done with fine emery cloth. The under side of the cuts was beveled in order to prevent the edge of the soap film from dropping below the upper surface of the plate. The bevel was made at an angle of 45 deg. and was brought nearly up to meet the upper surface.

PROCEDURE IN MAKING TESTS

In making the tests, the following order of procedure was followed:

1. A continuous coating of clean, hard grease was placed on the upper surface of the lower frame. The aluminum test pieces were clamped into position with even pressure.
2. The level and alignment of the plates were checked with a small spirit level and with a micrometer.
3. All grease was wiped from the surfaces of the aluminum plates.
4. An even layer of soap film was stretched across the holes by means of a smooth piece of celluloid or bakelite which had received an even distribution of the soap solution along its lower part.
5. Surplus drops of solution were drained from the films.
6. A drop of soap solution was applied to the point of the micrometer measuring screw, and the glass cover was set in place.

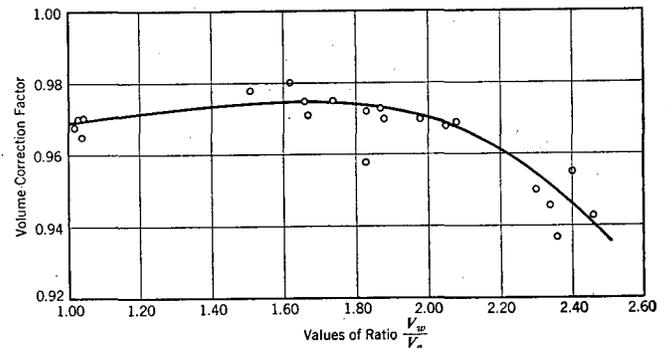


FIG. 5. CALIBRATION CURVE FOR OBTAINING VOLUME CORRECTION FACTOR

Corrections for Theoretical Errors and Volume and Pressure Changes Were Made by the Use of This Curve

7. The micrometer was set at the elevation desired for the initial reading, which should give the bubble a slight upward curvature.
8. Water was slowly introduced from the burette until the soap film just touched and ran up on the point of the micrometer screw. This procedure gave results accurate to ± 0.001 in.
9. The burette reading was recorded, estimating to the nearest 0.01 cc.
10. The micrometer screw was set in position for the final reading and operation No. 8 was repeated.
11. The final reading of the burette was recorded.

The following precautions were observed throughout the progress of the work:

1. The room and all parts of the apparatus were kept at constant temperature. As aids to this end, the glass parts of the apparatus were not contacted by the hands more frequently than essential to the test. Further, the water in the burette and flask were mixed before use, in order to equalize their temperature.
2. The tightness of the apparatus and the constancy of the temperature were checked by blowing a bubble to a definite height and observing its behavior over a period of time.
3. All parts of the apparatus were kept free from grease and acid. Whenever the bubbles burst quickly, all apparatus was cleaned a second time. Acidity may be caused by an excess of carbon dioxide in the atmosphere.
4. To insure uniformity of pressure change, each test in any series was started with the water level in the displacement flask at about the same height.

TEST RECORDS AND CALCULATIONS

The calibration curve, based on tests between sections of known torsional properties, is shown in Fig. 5 for one of the soap solutions used in the tests. The ordinates represent factors by which the total volume of water introduced from the burette is multiplied to obtain the actual total volume of the two bubbles. The abscissas represent ratios of the total volume of water to the volume of the circular test bubble. It seems desirable to use a circular section of such diameter as to keep $\frac{V_w}{V_c}$ below 2.00, as this will give a correction factor which is very nearly constant. For a solution of any given consistency, the pressure correction curve will vary—hence the importance of maintaining the composition of the soap film solution and the temperature of the room constant throughout any series of tests.

A sample order of computation for a typical test follows:

Diameter of test circle	2.100 in.
J = polar moment of inertia of circle	1.909 in.
V_w = volume of water drawn from burette (average of three tests which check closely)	10.82 cc
V_c = volume of circular bubble calculated from altitudes	5.90 cc
$\frac{V_w}{V_c}$ =	1.84
Correction factor (Fig. 5) =	0.973
V_b = total volume of bubbles = 0.973×10.82 =	10.53 cc
V_t = volume of test section bubble = $10.53 - 5.90$ =	4.63 cc
$\frac{V_t}{V_c}$ =	0.785
K = torsion constant of test section = 0.785×1.909 =	1.499 in. ⁴

The manner in which the results of the tests at Lehigh University were used to develop general formulas for the torsion constant of structural sections has been described in "Structural Beams in Torsion," which appeared in PROCEEDINGS for April 1935. Acknowledgment is made to Professors Inge Lyse and Joseph B. Reynolds, of Lehigh University, whose assistance and guidance made this work possible.

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