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Lateral Buckling of I-Section Column With Eccentric End Loads in Plane of the Web

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When a relatively slender column of the symmetrical I or WF shape is loaded eccentrically in the plane of the web by longitudinal end loads it may fail by elastic instability involving a combination of lateral bending and twisting. Timoshenko has shown that in the case of the solid rectangle a considerable end eccentricity causes the critical buckling load to be only slightly less than the Euler critical load for the same strip loaded axially. The corresponding solution for the WF or I-column, presented herein, shows that relatively small eccentricities of the longitudinal end load applied in the plane of the web cause a large falling off from the Euler critical load.

For the ideally straight column, with a material perfectly elastic up to the yield point, there would be a critical *l/r* ratio for any given eccentricity, column length, and material. Below this critical ratio the column would fail by yielding, according to the secant or eccentricity formula. Above the critical ratio the failure would be by elastic instability with respect to lateral bending and twisting.

Application of the equations is made to two structural wide-flange sections, of light and medium weight, and the results are presented in the form of diagrams.

The behavior of the ideally straight and uniform I-section column, under the various positions of loading shown in Fig. 1, will be reviewed briefly and particular attention given to loading position 3. Application of the results is made to some structural-steel sections, and the limits are indicated within which the usual secant formula for eccentrically loaded columns may be expected to apply.

In all of the cases to be discussed it is assumed that the cross-sectional dimensions of the column are sufficiently thick to maintain the shape and prevent local buckling in the elastic range. It is noted that the torsional properties of the section may affect the behavior in load positions 1, 3, and 4.

I-Section Under Axial Load: Position 1

A brief discussion of the axial-load condition is introduced for completeness. The problem is one of pure stability, and the critical buckling stress for integral failure is given by the modified Euler formula developed by Considère, Engesser, von Kármán, and others

$$\sigma_{cr} = \frac{\pi^2 \bar{E}}{\left(\frac{l_0}{r}\right)^2} \qquad [1]$$

where

σ_{cr} = average compressive stress at buckling load
$\bar{E}$ = effective modulus, equivalent to E below proportional limit, but modified by shape of cross section and stress-strain relations above proportional limit (1)[2]
l_0 = equivalent pin-ended length
r = radius of gyration

The axially loaded I-section column also has a tendency to buckle by twisting on its own axis. In the case of thin-wall sections of certain cross section and length, twisting buckling may occur at lower loads than the bending or Euler buckling. This type of buckling is important in aircraft construction and has been studied in general by Wagner (2) and Kappus (3) and for the I-section in particular by Lundquist (4). The critical twisting-buckling load for the centrally loaded I-section column may be written in terms of properties given in some handbooks on structural sections (8).

$$\sigma_{cr} = \frac{\bar{G}J + \frac{I_y h^2 \pi^2 \bar{E}}{4L_0^2}}{I_y + I_x} \qquad [2]$$

where

J = torsion constant (polar moment of inertia for circular sections)
$\bar{G}$ = effective shearing modulus
I_y = moment of inertia about principal axis in plane of web
I_x = moment of inertia about principal axis normal to plane of web
h = distance between flange centroids

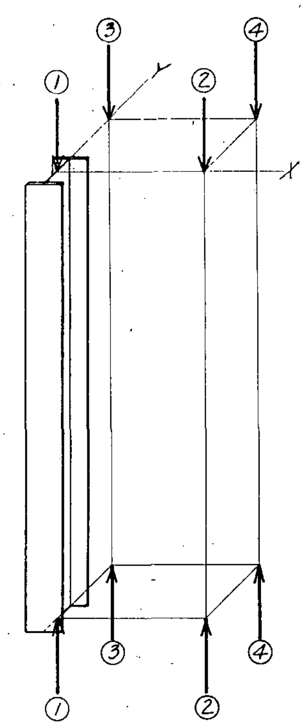

Fig. 1 Various Load Positions Applied to I-Section Columns

The critical buckling stresses by Equation [2] for structural-steel wide-flange sections are usually considerably above the yield point but in some of the lightest-weight sections the critical stress is only slightly above the yield point for structural steel. Such sections should be investigated for torsional buckling if made of high-yield-point alloy steels.

The Torsion Constant

The torsion constant J appears in Equation [2] and in the solu-

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[2] Numbers in parentheses refer to the References at the end of the paper.

tion to be discussed for load positions 3. The torsion constant J of a narrow rectangular strip of thickness t and breadth b is given by (5)

$$J = \frac{bt^3}{3} - 0.210t^4 \qquad [3]$$

The approximate torsion constant of the open structural section may be built up as the summation of the J values as given by Equation [3] for the component rectangular elements. The correct J values will be somewhat larger, and the author has evaluated corrections by means of soap-film tests which permit accurate calculation of the torsion constant of the structural I-beam and wide-flange sections (6, 7), including the effect of variable flange thickness and fillets. The results have been verified experimentally (7), and were used to evaluate the handbook (8) values.

The I-Section Eccentrically Loaded in Position 2

The behavior of the column eccentrically loaded in position 2 is not one of pure buckling. The limiting Euler load can never be reached, but the column will bend more and more in its weakest direction until it is no longer useful, either because of local yielding or excessive deflection. For materials with a linear stress-strain relationship, the secant formula gives the relationship between average axial stress and the maximum stress due to combined direct load and bending, based on the usual assumptions for bending with small deflections

$$\sigma_a = \frac{\sigma_{\max}}{1 + \frac{e}{s} \sec \frac{l}{2r} \sqrt{\frac{\sigma_a}{E}}} \qquad [4]$$

where

σ_a = average stress = $\frac{P}{A}$

e = eccentricity

s = $\frac{S}{A}$ where S = section modulus

$\frac{e}{s}$ = "eccentricity ratio"

l = length of column

r = radius of gyration

Salmon (9) credits the first derivation of the "eccentricity" or "secant" formula to H. Scheffler in the year 1858. For materials with a linear or nearly linear stress-strain relationship up to the yield point, the formula may be used to determine the average stress at which the yield-point stress is reached, and a safe proportion of this average stress may be used as the working load.

I-Section Eccentrically Loaded in Position 3

When an ideally straight and uniform column of the symmetrical I- or WF-shape is loaded eccentrically in the plane of the web by longitudinal end loads as in position 3, it may fail by elastic instability, involving a combination of lateral bending and twisting. In such a case, the secant formula may not be used as a design criterion.

S. Timoshenko (10) has presented a method for evaluating the lateral buckling of I-section beams under various types of loading; in particular, the I-beam subjected to pure bending in the plane of the web. Timoshenko (10) also presents the solution for the lateral buckling of the rectangular cross section loaded by eccentric end loads, applied in the plane of the center line of the long axis. In the case of the rectangle it is shown that considerable end eccentricity causes the critical buckling load to fall but little below the Euler critical load for the same strip loaded axially.

The corresponding solution for the I-column, herein presented, shows that relatively small eccentricities of longitudinal end load applied in the plane of the web cause a large falling off from the Euler critical load. The relative difference in the behavior of the rectangular cross section and the I-beam cross section is due to the respective differences in the ratios between the torsional and bending stiffnesses of these cross sections. In the case of the narrow rectangle, the torsional stiffness is numerically about 1.5 times as great as the lateral bending stiffness, whereas, conversely, in the case of the structural shape, the lateral bending stiffness is relatively much greater than the torsional stiffness, ranging from 15 times as great for deep narrow I-beams to as much as 120 times as great for broad wide-flange (WF) sections.

Nomenclature

The following nomenclature is in addition to that previously cited:

x, y, z = coordinate axes

u, v, w = corresponding displacements

β = angle of rotation

P = load

M = moment

e = eccentricity of load at end

$B_1 = EI_y$ = lateral bending stiffness factor

$C = JG$ = torsional stiffness factor

$a = \frac{h}{2}\sqrt{\frac{B_1}{C}}$ = torsion-bending constant for I-beams with free warping of flanges restricted[3]

$$R_1 = \frac{C}{2e^2}, \qquad R_2 = \frac{\pi^2 a^2}{l^2}, \qquad R_3 = \frac{2\pi^2 B_1}{l^2}$$

Before discussing the I-section loaded in position 3, Timoshenko's solution for the rectangle will be briefly reviewed and the results presented in a somewhat different form.

Timoshenko (10) shows for this loading that the differential equations of the buckled rectangular strip, involving lateral bending and twisting, respectively, are

Eqn. f. p. 243 Ref. 10

$$\left.\begin{aligned} B_1 \frac{d^2u}{dz^2} &= P(\beta e - u) \\ C \frac{d\beta}{dz} &= -\frac{du}{dz} Pe \end{aligned}\right\} \qquad [5]$$

The strip is assumed as free to rotate in the xz and yz planes at each end, but is held against twisting at both ends.

By eliminating β between these equations the following is obtained

Eqn. g, p. 243 Ref. 10

$$\frac{d^3u}{dz^3} + \left[\frac{P^2e^2}{B_1C} + \frac{P}{B_1}\right]\frac{du}{dz} = 0 \qquad [6]$$

Integrating Equation [6], it is found that the smallest value of P which will hold the strip in its buckled position is given by

Eqn. h, p. 243 Ref. 10

$$\frac{P^2e^2}{B_1C} + \frac{P}{B_1} = \frac{\pi^2}{l^2} \qquad [7]$$

If e becomes very large while Pe remains constant, the solution for pure bending is obtained

$$M_{cr} = Pe = \frac{\pi}{l}\sqrt{B_1C} \qquad [8]$$

[3] Reference (8), "Structural Shapes."

If e becomes zero in Equation [7], the Euler load is the result

$$P_{cr} = \frac{B_1\pi^2}{l^2} \quad \text{[9]}$$

The solution of Equation [7] as a quadratic, using only the positive result, may be written conveniently as follows

$$P_{cr} = R_1\left(\sqrt{1+\frac{R_3}{R_1}}-1\right) \quad \text{[10]}$$

where

$$R_1 = \frac{C}{2e^2} \quad \text{and} \quad R_3 = \frac{2\pi^2 B_1}{l^2}$$

Equation [10] may be used for any value of eccentricity e. The application of Equation [10] to the I-beam or WF-section will give results that are too low because of the relative increase in the ratio of $\frac{B_1}{C}$ for such sections.

Timoshenko shows that for relatively large eccentricities the critical load for the rectangular strip falls but little below the Euler critical load. For example, when $e = 0.1l$ the critical load is about 6 per cent smaller than that given by Euler's formula. Such is not the case for the I-section, however, and relatively small eccentricities cause a large falling off from the Euler load.

The twist of the I-beam in "pure torsion" with flanges free to warp longitudinally is expressed by the following

$$C\frac{d\beta}{dz} = M_z \quad \text{[11]}$$

where M_z = twisting moment.

Timoshenko (10) shows that, when longitudinal warping of the flanges is restricted, lateral shears due to bending of the

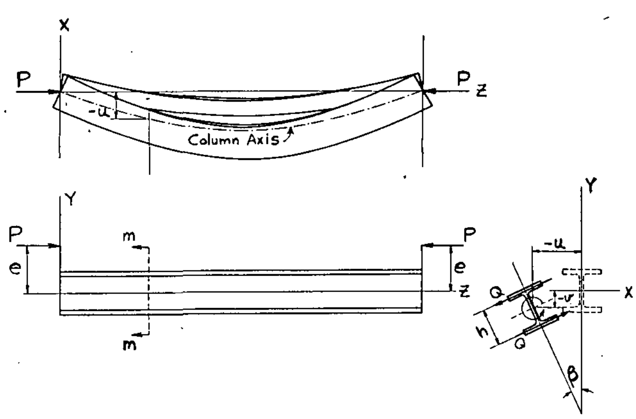

Fig. 2 Laterally Buckled I-Beam With Eccentric End Loads in Plane of Web

flanges are set up. The torsional moment is then expressed satisfactorily by the following

$$C\frac{d\beta}{dz} - \frac{B_1h^2}{4}\frac{d^3\beta}{dz^3} = M_z \quad \text{[12]}$$

The bending stiffness of one flange is taken as very nearly equal to $B_1/2$.

The buckled condition of the I-beam or WF-section for load position 3 is shown in Fig. 2. The differential equations for this equilibrium condition (similar to Equation [5] for the rectangle) must now include the flange-bending effect as in Equation [12]

$$\left.\begin{aligned} B_1\frac{d^2u}{dz^2} &= P(\beta e - u) \\ C\frac{d\beta}{dz} - \frac{B_1h^2}{4}\frac{d^3\beta}{dz^3} &= -\frac{du}{dz}Pe \end{aligned}\right\} \quad \text{[13]}$$

In Equations [13] it is assumed that v displacements are inappreciable in comparison with end eccentricity e. Equations [13] may be combined into the following by eliminating β as a variable

$$\frac{d^5u}{dz^5} - \left(\frac{1}{a^2}-\frac{P}{B_1}\right)\frac{d^3u}{dz^3} - \left(\frac{P}{B_1a^2}+\frac{P^2e^2}{B_1Ca^2}\right)\frac{du}{dz} = 0 \quad \text{[14]}$$

let

$$A_1 = \left(\frac{1}{a^2}-\frac{P}{B_1}\right), \quad A_2 = \left(\frac{P}{B_1a^2}+\frac{P^2e^2}{B_1Ca^2}\right)$$

and

$$N_1 = \sqrt{+\frac{A_1}{2}+\frac{1}{2}\sqrt{A_1^2+4A_2}}$$

$$N_2 = \sqrt{\frac{A_1}{2}-\frac{1}{2}\sqrt{A_1^2+4A_2}}$$

In terms of the constants N_1 and N_2, the integration of Equation [14] may be written

$$u = D_1 + D_2\cosh N_1z + D_3\sinh N_1z + D_4\cosh N_2z + D_5\sinh N_2z \quad \text{[15]}$$

D_1, D_2, D_3, D_4, and D_5 are constants of integration to be evaluated by end conditions.

Case A. Assume that the column is free to rotate in the xz and yz planes at each end but that twisting of the ends about the z axis is prevented. The deflection u and the lateral bending moment $\left(B_1\frac{d^2u}{dz^2}\right)$ will be zero at each end. The flanges are free to warp at the ends so the local bending in each flange $\left(\frac{B_1h}{4}\frac{d^2\beta}{dz^2}\right)$ also equals zero at each end. The following end conditions may therefore be used:

(1) $u = 0$ for $z = 0$

(2) $\frac{d^2u}{dz^2} = 0$ for $z = 0$

(3) $\frac{d^2\beta}{dz^2} = \frac{B_1}{Pe}\frac{d^4u}{dz^4} + \frac{1}{e}\frac{d^2u}{dz^2} = 0$, for $z = 0$

(4) $u = 0$ for $z = l$

(5) $\frac{d^2u}{dz^2} = 0$ for $z = l$

From conditions 1, 2, and 3

(1) $0 = D_1 + D_2 + D_4$

(2) $0 = D_2N_1^2 + D_4N_2^2$

(3) $0 = D_2\left(\frac{B_1}{Pe}N_1^4 + \frac{N_1^2}{e}\right) + D_4\left(\frac{B_1}{Pe}N_2^4 + \frac{N_2^2}{e}\right)$

From 2 and 3, since $N_1^2 \neq N_2^2$, $D_2 = D_4 = 0$ therefore $D_1 = 0$

Equation [15] is now reduced to

$$u = D_3\sinh N_1z + D_5\sinh N_2z \quad \text{[16]}$$

Substituting end conditions 4 and 5 into Equation [16]

$$0 = D_3\sinh N_1l + D_5\sinh N_2l$$

$$0 = D_3N_1^2\sinh N_1l + D_5N_2^2\sinh N_2l$$

Again, since $N_1^2 \neq N_2^2$

$$\sinh N_1 l \sinh N_2 l = 0$$

or substituting N_1 and N_2 in terms of A_1 and A_2

$$\sinh\left(l\sqrt{\frac{A_1}{2} + \frac{1}{2}\sqrt{A_1^2 + 4A_2}}\right)\sin\left(l\sqrt{-\frac{A_1}{2} + \frac{1}{2}\sqrt{A_1^2 + 4A_2}}\right) = 0$$

since the sinh term is always greater than zero, the sine term must equal zero. The lowest value of the sine term to give zero is π, resulting in the following equation which determines the critical buckling load

$$\sqrt{-\frac{A_1}{2} + \frac{1}{2}\sqrt{A_1^2 + 4A_2}} = \frac{\pi}{l} \quad \text{[17]}$$

or, squaring both sides, and substituting for A_1 and A_2

$$\frac{P}{2B_1} - \frac{1}{2a^2} + \frac{1}{2}\sqrt{\frac{P^2}{B_1^2} + \frac{2P}{B_1 a^2} + \frac{1}{a^4} + \frac{4P^2e^2}{B_1 C a^2}} = \frac{\pi^2}{l^2} \quad \text{[18]}$$

When $e = 0$ Equation [18] reduces to the Euler load for the pin-ended column

$$P_{cr} = \frac{B_1 \pi^2}{l^2}$$

When e approaches ∞ as Pe remains constant, Equation [18] reduces to the case of pure bending

$$M_{cr} = Pe = \frac{\pi}{l}\sqrt{B_1 C}\sqrt{\frac{\pi^2 a^2}{l^2} + 1} \quad \text{[19]}$$

Equation [19] is identical to that derived by Timoshenko for pure bending.[4]

Equation [18] may be solved directly as a quadratic for the positive value of P_{cr} for any eccentricity of load

$$P_{cr} = R_1\left[\sqrt{(1 + R_2)\left(1 + R_2 + \frac{R_3}{R_1}\right)} - (1 + R_2)\right] \quad \text{[20]}$$

where

$$R_1 = \frac{C}{2e^2}, \qquad R_2 = \frac{\pi^2 a^2}{l^2}, \qquad R_3 = \frac{2\pi^2 B_1}{l^2}$$

When R_2 is placed equal to zero in Equation [20], the solution reduces to that for the rectangle, as given by Equation [10].

Case B. Assume that the column is fixed against lateral rotation in the xz plane at each end, free to rotate in the yz plane, and held against twisting at each end as before.

The end conditions in this case are as follows:

(1) $u = 0$ for $z = 0$

(2) $\frac{du}{dz} = 0$ for $z = 0$

(3) $\frac{d\beta}{dz} = 0$ for $z = 0 = \frac{B_1}{Pe}\frac{d^3u}{dz^3} + \frac{1}{e}\frac{du}{dz}$

(4) $u = 0$ for $z = l$

(5) $\frac{du}{dz} = 0$ for $z = l$

[4] Reference (10), Equation 159, p. 261.

The solution follows along the same lines as for case A and the result is finally found to be

$$P_{cr} = R_1\left[\sqrt{(1 + 4R_2)\left(1 + 4R_2 + \frac{4R_3}{R_1}\right)} - (1 + 4R_2)\right] \quad \text{[21]}$$

When $e = 0$ the critical load for fixed ends is obtained

$$P_{cr} = \frac{4B_1\pi^2}{l^2} \quad \text{[22]}$$

When e approaches ∞ as Pe is held constant, the equation for pure bending is obtained

$$M_{cr} = Pe = \frac{2\pi\sqrt{B_1 C}}{l}\sqrt{\frac{4\pi^2 a^2}{l^2} + 1} \quad \text{[23]}$$

It should be noted that Equations [21] and [23] may be obtained directly from the corresponding Equations [20] and [19] for pin ends, by the substitution of a reduced length $l/2$ for l.

After studying the foregoing problem the author distributed an abstract, including Equation [20], to several engineers. Mr. H. N. Hill, research structural engineer of the Aluminum Company of America, pointed out that both Equation [7] and Equation [18] agree with the following relationship

$$\frac{M^2}{M_{cr}^2} + \frac{P}{P_{cr}} = 1 \quad \text{[24]}$$

In Equation [24], M_{cr} is the critical moment for lateral buckling of either the rectangle or I-section (Equation [8] or [19]) for pure moment, and P_{cr} is the Euler critical load for buckling in the weak direction. Equation [24] may be solved for M

$$M = M_{cr}\sqrt{1 - \frac{P}{P_{cr}}} \quad \text{[25]}$$

Equation [25] gives the end bending moment which will cause instability when added to a strut loaded axially with load P. Equation [25] is the basis of design criteria for this problem.[5]

Figs. 3 and 4 show the application of Equation [20] to light- and medium-weight 14-in. wide-flange structural sections, some of the properties[6] of which are given in Table 1.

TABLE 1 PROPERTIES OF WIDE-FLANGE STRUCTURAL SECTIONS

General dimensions, in.	Area, sq in.	Weight, lb per ft	Torsion constants: J (in.^{4})	Torsion constants: a (in.)	Radii of gyration: Maximum	Radii of gyration: Minimum
14 × 6$^3/_4$	8.81	30	0.41	70.93	5.73	1.41
14 × 12	24.71	84	4.48	76.66	6.13	3.02

The calculations in Figs. 3 and 4 were developed for various eccentricity ratios e/s and various slenderness ratios l/r. The eccentricity ratio e/s is the ratio between end eccentricity of load and the kern distance.

In Fig. 3 the critical stress $\left(\sigma_{cr} = \frac{P_{cr}}{A}\right)$ by Equation [20], for ends free to rotate, is plotted against the slenderness ratio in the weak direction, and comparison is made with the Euler curve. In Fig. 4 the same critical stresses are plotted against the slenderness ratio in the strong plane. Also plotted in Fig. 4 are average stresses (σ_a), at which the maximum stress would reach the minimum specification yield-point stress value of 33 kips per sq in. for structural steel, as calculated by the secant formula

$$\sigma_a = \frac{33}{1 + \frac{e}{s}\sec\frac{l}{2r}\sqrt{\frac{\sigma_a}{E}}}$$

[5] Reference (8), "Structural Aluminum Handbook," p. 58.

[6] Reference (8), "Structural Shapes."

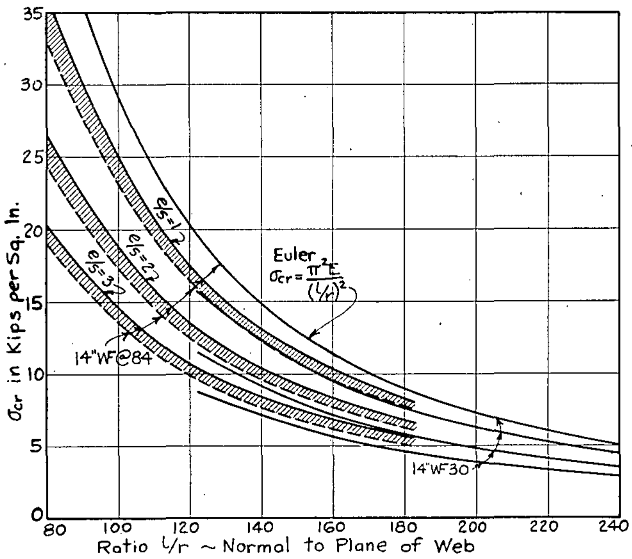

FIG. 3 CRITICAL STRESSES FOR LATERAL BUCKLING FOR TWO WIDE-FLANGE SECTIONS

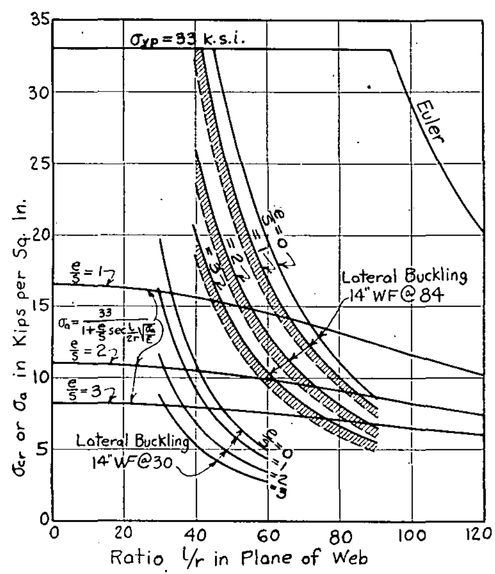

FIG. 4 AVERAGE STRESS AT YIELDING OR LATERAL BUCKLING FOR TWO WIDE-FLANGE SECTIONS

The curves in Fig. 4 indicate for various eccentricity ratios the relative ranges of column slenderness in which inelastic failure or elastic instability would occur for the two steel shapes which were considered, assuming a linear stress-strain relationship.

It should be noted that the foregoing solution neglects the additional moment due to v displacements in the yz plane. An approximate correction which will be on the safe side may be made as follows:

1 Calculate the critical load by Equation [20].

2 Calculate approximate additional deflection[7] at the center as follows

$$\delta = e\left(\sec \frac{l}{2r}\sqrt{\frac{\sigma_a}{E}} - 1\right)$$

r in this case is the maximum radius of gyration, about the x-axis.

[7] Reference (10), p. 13.

3 Recalculate critical load for lateral buckling using $e + \delta$ in place of e.

The foregoing would result in an "overcorrection" and the actual critical load would be between the uncorrected and the corrected values. In the case of the 14-in. WF at 84, with a maximum moment of inertia only about 4 times the minimum, the correction is appreciable and is shown by the dashed lines in Figs. 3 and 4. The actual buckling load would lie in the shaded area between the dashed and solid lines. In the case of the 14-in. WF at 30, with a maximum moment of inertia 16 times the minimum, the correction is negligible.

THE I-SECTION COLUMN ECCENTRICALLY LOADED IN POSITION 4

Load position 4 in Fig. 1 represents the most general case with the thrust line in neither principal plane of the I-section. The maximum stress in such a case is sometimes calculated by resolving the end moment into the components in the principal planes and superposing the results given by the secant formula for the stress due to each effect. Such a procedure neglects direct stresses due to torsion which in the case of torsionally weak I-sections may amount to more than that added by the direct deflections considered by the secant formula.

The author has made an approximate solution of this problem to verify the foregoing conclusions; however, the results are not considered to be necessary in this paper.

CONCLUSION

In conclusion, it is well to ask: "Just what is the structural significance of the types of behavior which have been discussed?" Real columns are not ideally straight and the end conditions are rarely as simple as those tested herein. The actual column is usually framed as an integral part of a structure, in which case the eccentricities introduced through frame action will not remain constant, but usually will reduce for loads below the pin-ended Euler load. Nevertheless the study of certain fundamental types of structural behavior may lead to a better understanding of what may happen in a structure, and thereby result in improved procedures of structural design.

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