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INELASTIC BUCKLING OF ECCENTRICALLY LOADED COLUMNS

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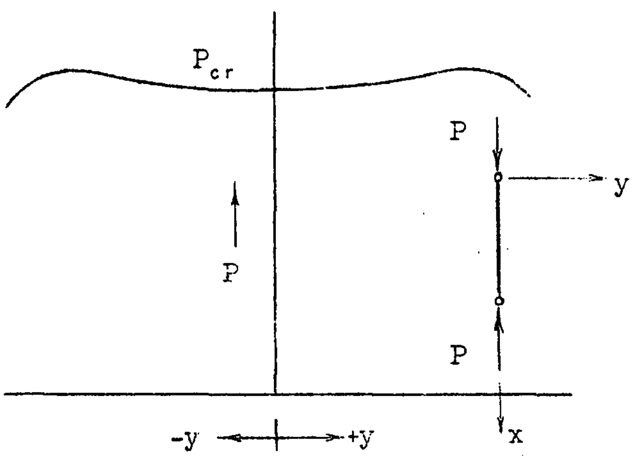
INTRODUCTION

Column type members have been classified in many different ways, however, the distinction usually made is between the following two conditions of loading,

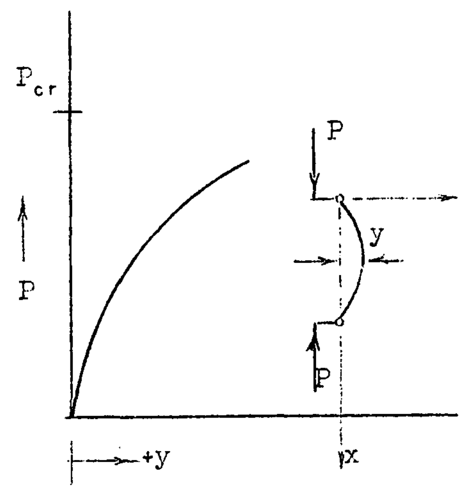
1. concentric compression, and
2. eccentric compression

Strictly speaking the concentrically loaded column is a limiting case of the eccentrically loaded one in which the eccentricity is zero. Nevertheless, there is a fundamental difference between these two conditions. Consider for illustration figures (1) and (2).



Centerline Defl.

Fig. 1



Centerline Defl.

Fig. 2

As shown in Fig. 1 (the load-deflection curve for the centrally compressed member), no deflection is produced until a certain critical load, P_{cr}, is reached. At this load the members undergoes a sudden lateral deflection, but the direction of the deflection is not known in advance. This sudden movement constitutes the instability condition discussed in several of the previous talks in this series, however, that value of axial thrust, P_{cr}, as was shown in seminars (2) and (6), is not necessarily the maximum load the member will sustain.

If on the other hand, a member is eccentrically compressed as shown in Fig. 2, a deflection of unambiguous sign is produced from the start. This condition of predictable -vs.- non-predictable deflections constitutes the major difference between these seemingly interrelated conditions of loading. The second condition (that of eccentric loading) will not lend itself readily to the method of inelastic solution as used in the former problem since for that case it was only necessary to determine the existance of one equilibrium position other than the straight form. The latter case is somewhat more involved since it requires the determination of two possible adjacent deflected positions satisfying equilibrium of the same loading.

Before going to the inelastic case it is well that we determine the elastic behavior of members loaded eccentrically.

ELASTIC BEHAVIOR

For the column loaded as shown in Fig. (3), the differential equation (for small deflections) is

$$\frac{d^2y}{dx^2} = \frac{-M}{EI} = \frac{-P(e + y)}{EI} \quad \text{.........................(1)}$$

or

$$y'' + k^2y = -k^2e \quad \text{.............................(2)}$$

where $k^2 = \frac{P}{EI}$

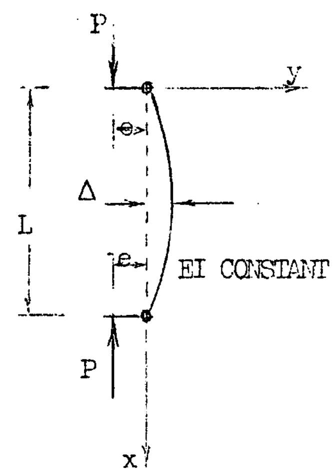


Fig. 3

This equation has the solution

$$y = A \sin kx + B \cos kx - e$$

which upon substitution of the boundary conditions at $x = 0$ and $x = L$ gives

$$y = \frac{e(1-\cos kL)}{\sin kL} \sin kx + e(\cos kx - 1) \quad \ldots\ldots(3)$$

Noting that at $x = \frac{L}{2}$, $y = \Delta$,

$$\Delta = \frac{e(1 - \cos kL)}{\sin kL} \sin \frac{kL}{2} + e\left[\cos \frac{kL}{2} - 1\right]$$

or

$$\frac{\Delta}{e} = \frac{1-\cos u}{\cos u} \quad \ldots\ldots(4)$$

where $u = \frac{kL}{2}$

A plot of equation (4) is shown as Fig. (4) where load has been expressed non-dimensionally as the ratio P/P_e where P_e is the Euler buckling load for the concentrically loaded column.

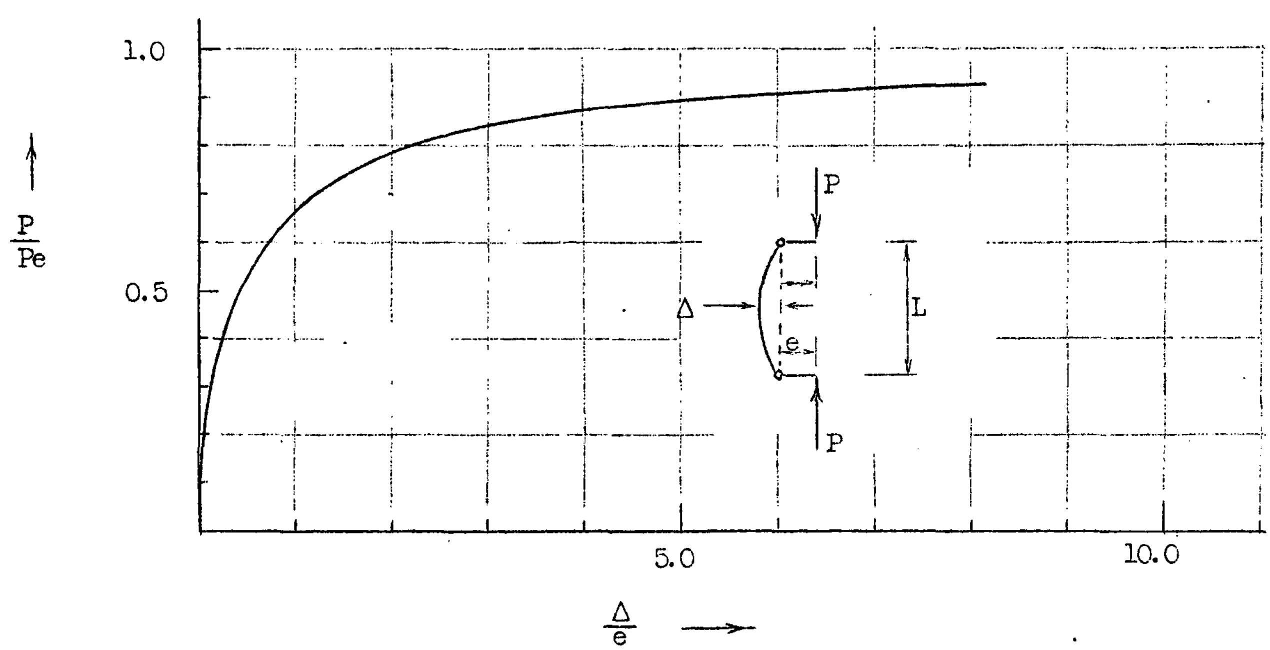


Fig. 4

As indicated in this plot or by solution of Eq. (4), instability* occurs at $P/P_e = 1$, providing this point can be reached and still have the member react as assumed by the small deflection theory express by equation (4). Realizing that an infinite deflection is slightly hard to reach and that the stress situation at the center section is a combination of the effects of an axial thrust and a moment $P(e + \Delta)$; it is safe to assume that somewhere along the line, certain of the most highly stressed fibers exceed their elastic limit and yield. As the compressive load, P, is increased beyond this initial yield point, plastic flow spreads from the edge toward the inside of the cross-section and in the direction of the bar. Therefore, the flexural rigidity of the member gradually decreases. Consequently, the deflection y, and hence the moment of the external forces will increase at a higher rate than for the elastic case. Finally there is reached a point at which a critical load, P_{cr}, is reached which cannot be exceeded. Each deflection that is greater than the deflection Δ_{cr} produced by P_{cr} corresponds to a value of P less than P_{cr}. Fig. (5) shown a typical load deflection curve for such a condition.

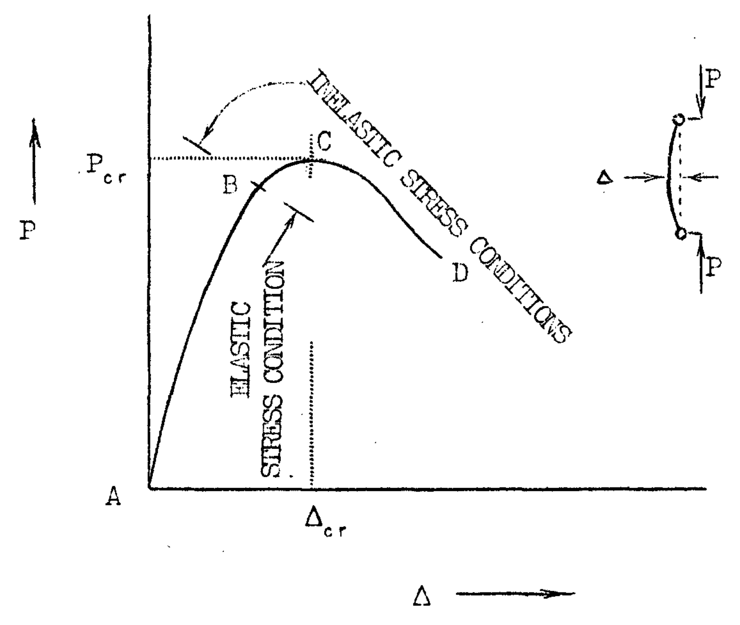


Fig. 5

*(that is, bifurcation of the equilibrium position)

Part AB of the plot corresponds to the elastic deflection solution shown in Fig. (4). BC defines the stable inelastic positions of equilibrium with C corresponding to the ultimate carrying capacity of the member (moreover, the buckling load). That portion of the curve CD represents unstable positions of equilibrium.

It is with the determination of the load P_{cr} that the remainder of this discussion will be concerned.

METHOD OF SOLUTION

In general the reasoning to be used is as follows:

a. Since what we are looking for is the maximum value of P (indicated as P_{cr} in Fig. 5.), solution can be achieved by solving the equation.

$$\frac{dp}{d\Delta} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

But this assumed that P is an express function of Δ (i.e. $P = f(\Delta)$).

b. However, to determine the $P - \Delta$ equation, it is first necessary to determine the expression to be used in place of Equation (1) of the elastic solution, which defines the basic load-curvature relation.

c. Then with this expression, $P = f(\Delta)$ can be obtained by assuming some deflection curve (e.g. y = g (Sin qx) to substitute in the equation.

$$\phi = \frac{d^2y}{dx^2} \; *$$

(Note: ϕ = curvature = reciprocal of the radius of curvature)

Even though the exact deflection curve is that which results in the lowest potential energy of the system, stability problems of this type (as was illustrated in the Ritz solution of lecture No. 2) are relatively insensitive to changes in the assumed deflection configuration providing these deflection equations conform to the geometric boundary conditions of the member.

The first thing to review then is how equation (1) was derived so that an analogous system can be employed

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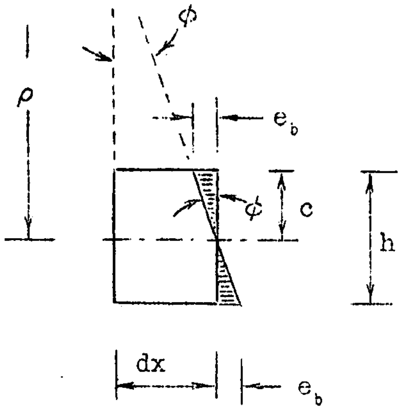
* Here as for the elastic case a small deflection solution will be assumed such that

$$\phi = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \quad \text{can be approximately by} \quad \phi = \frac{d^2y}{dx^2}.$$

for the inelastic case. Consider for illustration a differential section of a beam loaded such that the beam bends as well as compresses. Since elastic conditions are implied, superposition of the independent effects of the thrust and bending moment acting on the section allows us to consider just the bending part since it alone contributed to the variation in strain only the section (and thereby as is shown in Fig. 6 determines the curvature ϕ).



From similar triangles (see Fig. 6)

$$\frac{\rho}{dx} = \frac{c}{e_b}$$

or

$$\frac{e_b}{dx} = \frac{c}{\rho}$$

Fig. 6

but $\frac{e_b}{dx} = \epsilon_b$, the unit strain in the extreme fiber due to bending.

This then reduces to

$$\rho = \frac{c}{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

But c and ϵ can be related to the external forces, the stiffness of the material, and the dimension of the beam, according to the following equations

$$M = \frac{I}{c}\,\sigma_b \qquad \text{and} \qquad E = \frac{\sigma_b}{\epsilon_b}$$

Equation (6) can be rewritten as

$$\rho = \frac{EI}{M} \quad \text{or} \quad \phi = \frac{1}{\rho} = \frac{M}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

INELASTIC BEHAVIOR

Two basic assumption of material properties must be made. There are:

a. the Bernoulli - Navier hypothesis that bending strains are proportional to the distance from the neutral axis can be extended to include inelastic as well as elastic deformations, and

b. for all strains greater than the initial yield point strain, ϵ_y, the stress is consistent with Fig. 7., the yield point stress, σ_y.

Solution will be considered only for the rectangular member, however, the method can be extended to other types of cross-section.

Since we are concrened with solution to the inelastic problem, there are two different distributions of stress possible, as indicated in Figures 8a and 9a.

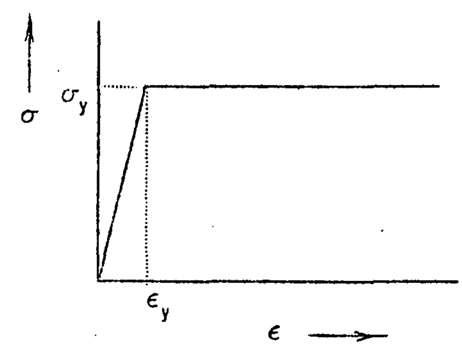


Fig. 7

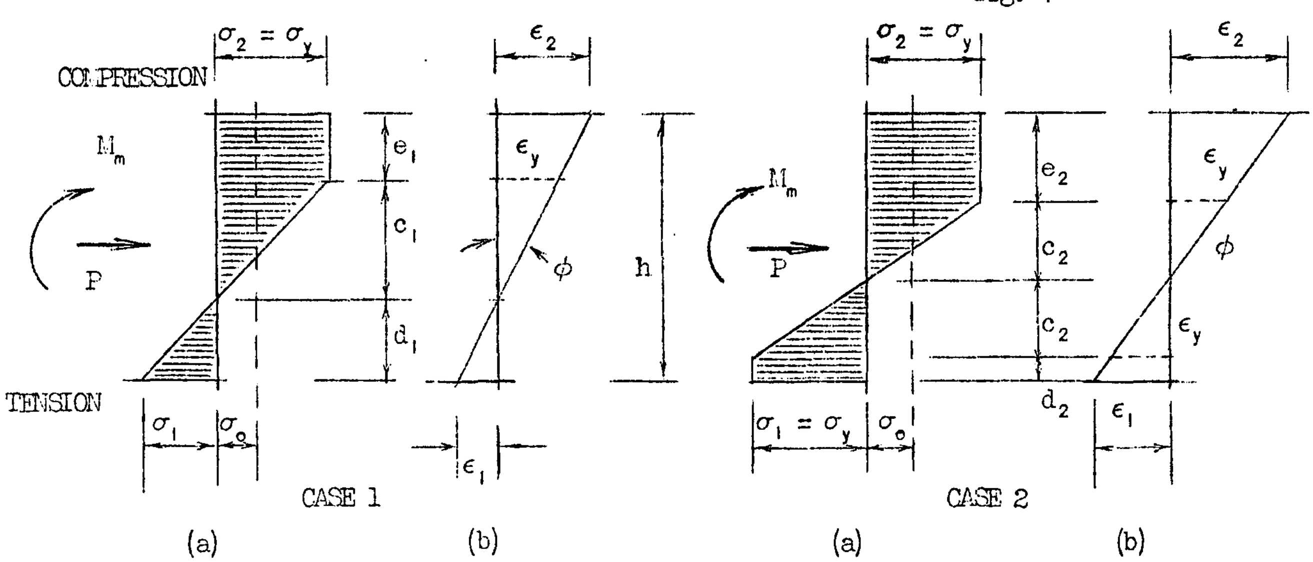


Fig. 8

Fig. 9

The first case, (shown in Fig. 8) will be the only one considered herein, however, the second can be solved in a like manner.

CASE 1 SOLUTION:

From the stress distribution pattern, the external loads M_n and P can be determined from the equilibrium conditions

$$P = b\int_h \sigma dz \quad \ldots\ldots (8)$$

and

$$M_n = b\int_h \sigma z dz \quad \ldots\ldots (9)$$

Note: b = width of section.

z denotes the fiber distance from the σ_o - axis. Since the stress function is defined by the straight line relations shown in Fig. (8a), the intergals (8) and (9) are readily computed. We then have two equations which enable us to obtain expression for the variables e_1 and d_1. Then by noting that σ_o = P/A (the average stress) and that $e_1 + c_1 + d_1 = h$

$$\frac{e_1}{h} = \frac{3M_n}{(\sigma_y - \sigma_o)bh^2} - \frac{1}{2} \quad \ldots\ldots (10)$$

and

$$\frac{c_1}{h} = \frac{9\left[\sigma_y - \sigma_o - \frac{2M_n}{bh^2}\right]^2}{8(\sigma_y - \sigma_o)^3} \sigma_y \quad \ldots\ldots (11)$$

From the strain distribution pattern (Fig. 8b)

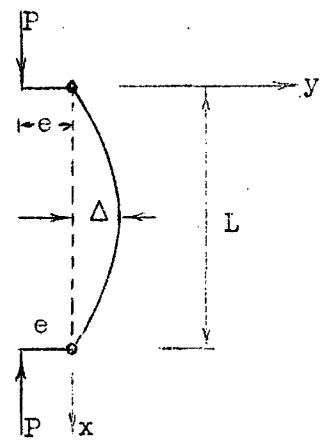
$$\phi = \frac{\epsilon_2 - \epsilon_1}{h} = \frac{\epsilon_y}{c_1}, \text{ but since } E = \frac{\sigma_y}{\epsilon_y}, \quad \phi = \frac{\sigma_y}{Ec_1}$$

Introducing the value of c_1 as determined from eq. (11), the stress condition, gives

$$\phi = \frac{8\sigma_o\left[\frac{\sigma_y}{\sigma_o} - 1\right]^3 h}{9E\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_o} - 1\right) - \frac{M_n}{P}\right]^2} \quad \ldots\ldots (12)$$

This then is the basic load-curvature relation corresponding to Eq. (1) of the elastic solution. Proceeding as outlined in the Section on METHOD OF SOLUTION, assume a deflection curve of the form

$$y = \Delta \operatorname{Sin} \frac{\pi X}{L}$$



The curvature at mid-height of the column is then expressed by the equation

$$\phi = \frac{d^2y}{dX^2} = \frac{\pi^2}{L^2}\Delta \qquad \text{.........................(13)}$$

Combing equations (12) and (13) leads to a relation between Δ and σ_o of the form

$$9E\Delta\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_o}-1\right)-\Delta-e\right]^2 - 2\sigma_o h\left[\frac{\sigma_y}{\sigma_o}-1\right]^3\frac{L^2}{\pi^2} = 0 \qquad \text{.....................(14)}$$

where M_m has been replaced by $P(\Delta + e)$, the moment at the mid-height.

Equation (14) is the expression for $P-\Delta$ referred to in part (b) of METHOD OF SOLUTION where P is expressed as the average stress $\sigma_o = P/A$. To obtain the maximum value of P, that is P_{cr}, solve

$$\frac{d\sigma_o}{d\Delta} = 0 .$$

This results in a second degree equation in the variable Δ,

$$3\Delta^2 - 4\Delta\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_o}-1\right)-e\right] + \left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_o}-1\right)-e\right]^2 = 0$$

for which

$$\Delta = \frac{1}{3}\left[\frac{h}{2}\left(\frac{\sigma_y}{\sigma_o}-1\right)-e\right] \qquad \text{..(15)}$$

Substituting eq. (15) in eq. (14) and rearranging terms gives

$$\sigma_c = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}\left[\frac{\frac{\sigma_y}{\sigma_c} - 1 - \frac{2e}{h}}{\frac{\sigma_y}{\sigma_c} - 1}\right]^3 \quad \text{.......................................(16)}$$

where $\sigma_c = \sigma_o$ the critical average compressive stress.

A plot of this equation as well as that which would have resulted from solution to the second loading condition, illustrated in Fig. (9), is shown in Fig. (10).

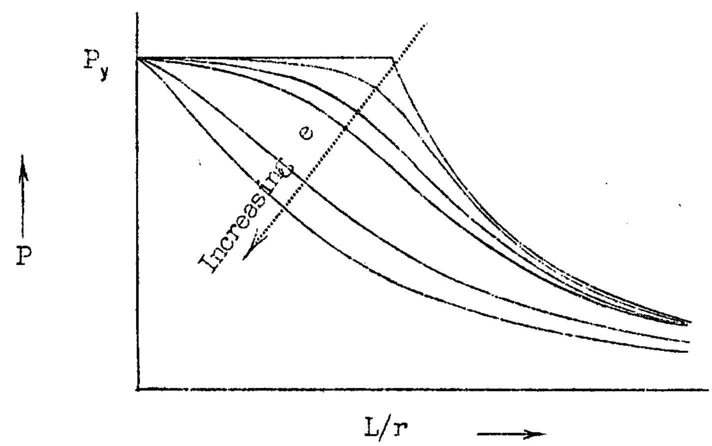


Fig. 10

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A P P E N D I X

INFLUENCE OF COOLING RESIDUAL STRESSES ON BUCKLING OF ECCENTRICALLY LOADED WF COLUMNS (WEAK AXIS SOLUTION)

Columns, as they are used in structures, contain certain initial stresses which are neglected in conventional analysis. These "locked-in" stress can be due to several conditions; however, the two more important ones are: a. differential cooling of the member during the rolling process, and b. cold bending of the specimen when straightening. In this discussion only the influence of the cooling type will be considered. Moreover, it will be assumed that the initial stress pattern is that shown in Fig. 11.

Since the weak axis buckling behavior is desired, the problem is comparable to the rectangular section solution presented herein before. Rather than go thru the algebra of a solution as in the former case, only the method will be described and typical resulting curves presented.

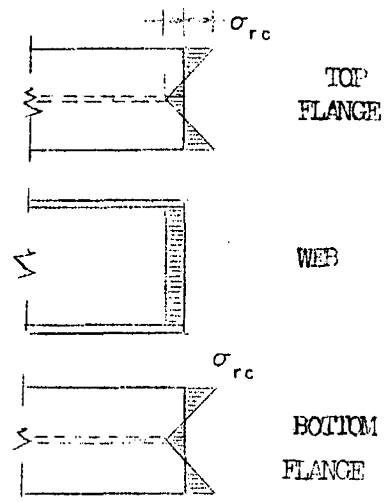


Fig. 11

Proceeding as previously, applied stress and strain patterns similar to (8) and (9) must be considered in order to evaluate the basic load-curvature relation of the section. This can be carried out as before by using equations (8) and (9), however, it is possibly easier to illustrate this inter-relation between the variables M, P and ϕ by use of graphs. One such graph is shown as Fig. 12 where P has been held constant at $0.6P_y$. The solution neglecting residual stress is shown solid, whereas that including residual stress (assuming $\sigma_{rc} = 0.3\sigma_y$) is dashed.

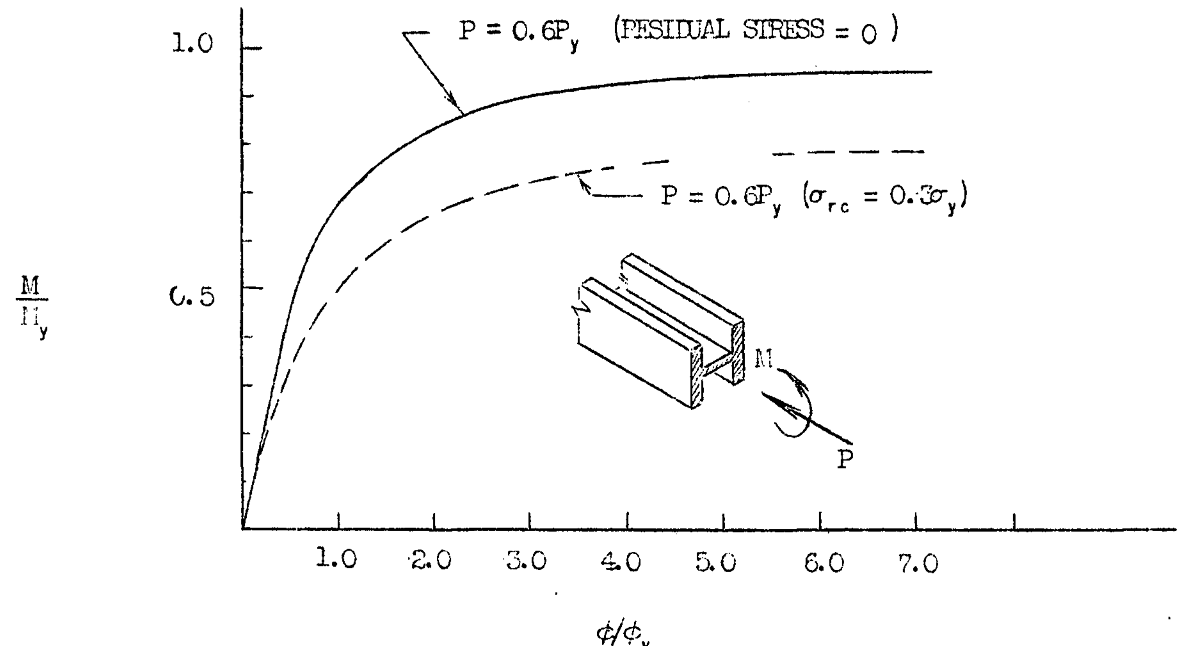


Fig. 12

To illustrate the reduction in buckling strength due to this change in $M-\phi$ behavior consider the column shown in Fig. 13. Here as in Fig. 12. it is assumed that P is held constant at $0.6P_y$. Assuming that the deflection curve is of the sin form, curvature at the center section can be expressed by the equation.

$$\phi = \frac{\pi^2}{L^2}\Delta \text{ ....}(17)$$

Moment at the center section is due to the combined effects of M_o and P, i.e.

$$M = M_o + P\Delta \text{ , ...}(18)$$

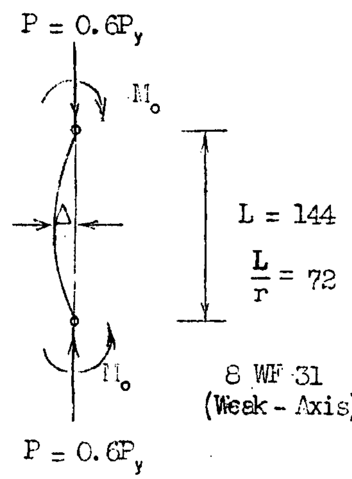
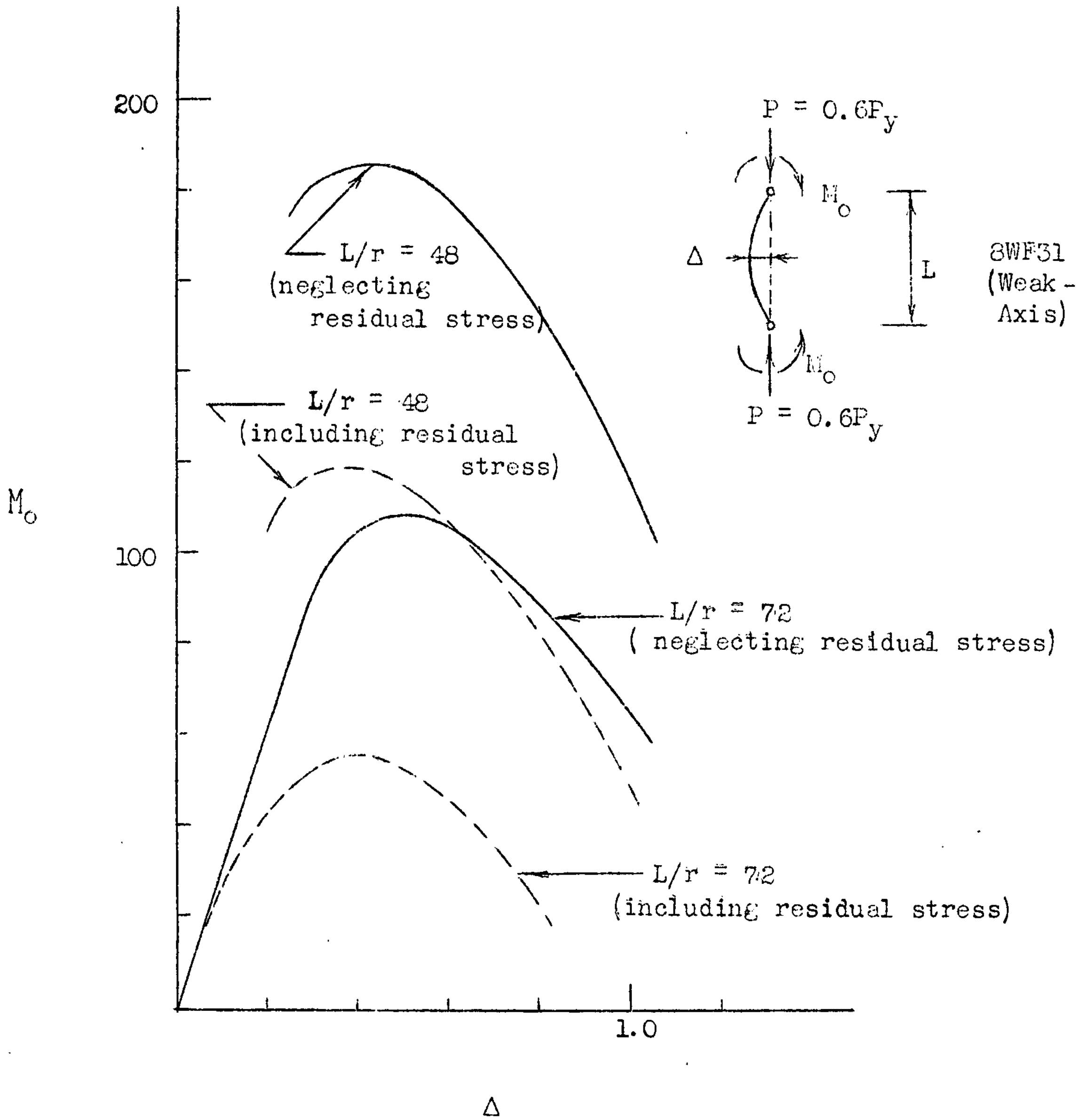


Fig. 13

It is possible then by selecting various values of Δ to compute M_o from equations (17), (18) and the graph of Fig. 12. Such was done for the problem column of Fig. (13) as well as for one where L = 96 inches, (L/r = 48). These plots are shown in Fig. (14). As you will note there is appreciable reduction in carrying capacity.



200
100
M_o
1.0
Δ
L/r = 48
(neglecting
residual stress)
L/r = 48
(including residual
stress)
L/r = 72
(neglecting residual stress)
L/r = 72
(including residual stress)
P = 0.6P_y
M_o
Δ
L
8WF31
(Weak -
Axis)
M_o
P = 0.6P_y

Fig. 14