DISCUSSION

by

Robert L. Ketter and Lynn S. Beedle

of

"Strength of Columns Elastically Restrained and Eccentrically Loaded"
by

P.P. Bijlaard, G.P. Fisher, and George Winter

The work of the project at Cornell University sponsored by the Column Research Council and the Bureau of Public Roads, is a real contribution to engineering knowlege. Nearly every structural compression member is a restrained beam-column and yet most current specifications are based on the strength of the pin-ended member although an estimate is made of the effective length due to end restraint. The authors have developed a solution to the elastically restrained column problem by limiting the study to members that do not fail by local or lateral-torsional instability. This solution will be basic to future work in this field.

It is the purpose of this discussion to consider some aspects of the experimental results, to present a precise method for determining the plastic reduction factor, η , to examine the "shape factor" mentioned by the authors, and to suggest a tentative form for design curves.

With regard to the nomenclature it is understood that the term "η" is the plastic reduction factor. However, on page 14, γ' is called the "plastic reduction factor". This should be cleared up.

The term "c" used by the authors is apparently a factor less than the Euler length factor, K[17]. (See Fig. 5b.) However, in the earlier Cornell work[8] which is used in the paper, there is no question but that "c" is equal to the Euler length factor. Comment on this point seems in order.

The use of a square shape is justified for checking theory. The testing of a limited number of I-shape bent about the weak axis is also justified to confirm the findings, thus eliminating the variable of lateral-torsional instability.

It is considered that attention should next be directed, however, to the wide-flange shape with flexure forced about the strong axis, since it is expected that most structural columns will be loaded in this manner. Lateral-torsional buckling will occur when columns without complete lateral bracing are bent in this manner. A lower carrying capacity results[18], and theoretical studies are currently under way at Lehigh in an effort to predict this behavior.

Yielding of Steel

The use of whitewash as a strain-indicating device is certainly valuable, its use extending back for many years. Lyse and Johnston[19] used this technique. R. S. Johnston[20] used a

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(17) "The Basic Column Formula", Column Research Council Technical Memorandum No. 1, May, 1952.
(18) "Plastic Deformation of Wide-Flange Beam-Columns", by R. L. Ketter, E. L. Kaminsky and L. S. Beedle, ASCE Proceedings Separate No. 330 (Vol. 79) October, 1953.
(19) "Structural Beams in Torsion", by Inge Lyse and B. G. Johnston, Transactions, ASCE, V. (1935) p.
(20) "Compressive Strength of Column Web Plates and Wide Web Columns", by R. S. Johnston, U.S. Bur. Standards Technologic Paper 327, V. 20 (1926) pp. 33-82.

mixture of portland cement and water for the same purpose. The objective, of course, is to more clearly reveal the flaking of mill scale.

The authors have used a factor of 15% to account for the fact that indication of first yield by whitewash was consistently above the computed value. This was done both for as-delivered and annealed members. This delayed indication of yielding by whitewash might also be attributed to an irregular mill scale or else the flaking might not have been observed when it first developed. A considerable body of test data at Lehigh University indicates that for typical wide-flange shapes yielding occurs at less than (not greater than) the nominal computed value. The cause is attributed to residual stresses. Considering such initial stresses, fair agreement has been obtained between the theoretical yield load and the load at which flaking of whitewash was first observed. (For example see Figs. 27-31 of Ref. 18. See also Ref. 21.)

No issue is taken with the fact that an upper yield point effect exists. But it is doubtful if it can account for any such increase as shown, for example, in Fig. 15(b).

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(21) "Residual Stress and the Compressive Strength of Steel", by A. W. Huber and L. S. Beedle, Fritz Lab. Report No. 220A.9, Lehigh University, Dec. 1, 1953, (Scheduled for publication in Welding Journal).

Influence of Residual Stress

The authors mention "a minor effect of residual stress" as indicated by the rounded knee of the curve in Fig. 13 for the as-delivered I-section. Recent studies have shown that similar deviations from linearity have a significant effect(21). From Fig. 22 of Ref. 14, the maximum strength of the as-delivered I-shape column tested by the authors at zero eccentricity is 94 kips or an average stress of 34 ksi (area = 2.76 sq. in.). The yield stress level for this as-delivered 4I9.5 material is 38.0 ksi. Thus the reduction in strength is 10% in a region in which a compression member should carry full yield point stress in the absence of residuals and eccentricities. (cL/r = 51.8)

It is to be expected that the residuals would be less in the more compact I-shape than in most wide-flange shapes. But it must also be remembered that the stress-strain diagram on which the authors base their theory (Fig. 1) has a shape not unlike the "cross-section" test of the as-delivered I-shape, as presented in Fig. 13. (Chwalla's work was based on a steel whose proportional limit, σ_p, was 0.80 times the yield stress level, σ_y. In Fig. 13, $\sigma_p = 0.75\ \sigma_y$.) Thus the influence of residuals had already been taken into account, empirically, in the theory.

Of course as the eccentricity ratio ec/r^2 increases the influence of residuals diminishes rapidly (as shown in Ref. 18), until, for a member subjected to bending alone, there is no influence upon the load-carrying capacity.

Plastic Reduction Factor

The authors' solution is based on the premise that for the given full-length column which is eccentrically loaded and elastically restrained there corresponds a shorter, hypothetical, pin-ended column subjected to an end eccentricity of load such that it behaves in the same manner as the original member. The solution, then, is basically that of determining the equivalent, hypothetical member. Knowing this length and the adjusted eccentricity, strengths could be predicted based on pin-ended column collapse solutions.

Aside from the use of the moment-distribution equation,

$$\epsilon = \frac{2\eta EK}{2\eta EK + \beta_0} e \qquad (8)$$

the most important aspect of the problem is the determination of η , the plastic reduction factor. The authors suggest a straight-line approximation, plotted in Fig. 8, which is given as Eq. 25. A method for determining a more precise value of η will now be examined - a method that may be applied equally well to any material whose moment-curvature relationship may be obtained.

The plastic reduction factor, η , first appears in Eq. 1

$$-\eta EI \frac{d^2y}{dx^2} = P \cdot y \qquad (1)$$

Thus, the secant to the moment-curvature curve in the inelastic range is ηEI. If the M-Ø curve is non-dimensionalized as shown in Fig. 18, then η is the secant to this curve. As pointed out by the authors, η decreases in value as load increases until the critical load (or moment) is reached. Thus the correct value of η is that which exists at the critical load.

A method for obtaining the critical load for an eccentric column was presented in Ref. 18 and it was based on knowledge of the moment-curvature (M-Ø) relationship. Methods for obtaining the M-Ø curve including the influence of axial thrust and other variables were also presented there. From the calculations of moment vs centerline deflection leading up to Fig. 38 of Ref. 18 the centerline moment values corresponding to critical load could be obtained. Hence, precise η -values may be obtained from Fig. 26 of Ref. 18, for example, for a WF shape bent about the weak axis. The determination of η from an M-Ø curve is illustrated in Fig. 18.

These η -values are plotted in Fig. 19 as influenced by axial load and slenderness ratio. The curves in Fig. 20 are for a WF shape (8WF31) bent about the strong axis.

Throughout the writers' calculations, it is assumed that the deflected shape is that of a partial cosine curve. This is in contradiction to fact, but, if the boundary conditions are satisfied, an error in shape has but little influence on the critical load.[18] As a matter of fact it appears that the authors make the same simplifying assumption on p. 9.

The advantage of the more precise determination of η is that the theory may be applied to any material and, hence, to steel members containing residual stresses. A disadvantage is in the necessity for computing M-Ø curves. These, however, may be non-dimensionalized and once computed for a given shape, further calculation is unnecessary. Further, if design curves are eventually to be used, then the more accurate work seems justified. An additional advantage is that an arbitrary shape factor need not be used. This matter will be considered next.

Shape Factor

The authors suggest in Table 3 a modification of the shape factors as presented by Bleich on p. 45 of Ref. 9. (It is noted that Bleich's shape factor for "Case 2" is different by 50% from the value suggested by the authors.) In Bleich's method, the "equivalent eccentricity", K, ($K = \epsilon c/r^2$) is to be multiplied by the shape factor. The authors multiply the effective eccentricity (ϵ) by the shape factor, μ. Unless c/r^2 of the shape under question is the same as c/r^2 of the rectangle (this ratio is also equal to A/S) then the two methods are not equivalent. Bleich notes that his values of μ "must be considered as crude approximations only, since the effect of cross-sectional form upon the buckling strength of eccentrically-loaded columns is by no means cleared up".

One influence of shape may be seen by comparing Fig. 19 and Fig. 20. Another may be seen by comparison of Fig. 37 and Fig. 38 of Ref. 18 and these two curves are re-plotted in part in Fig. 21 herein. An alternate to the more precise determination of μ, however, is not to use it at all. The procedure suggested above for determining the plastic reduction factor, η, and the determination of critical loads in eccentric columns outlined in Ref. 18 makes a separate consideration of shape factor unnecessary. It is already inherent in the calculations.

Fig. 22 shows the variation in shape factor for an 8WF31 shape bent about the strong axis. It was derived from Fig. 21.

It shows that μ may not be considered as constant for a given L/r or for a given axial load.

Comparison of Methods

Referring to the seven steps suggested by the authors as the "Simplified Method" (pp. 24-25), the only significant change of the above procedure is in Step 2 (determination of η) and Steps 6 and 7. The steps, in outline, are as follows:

(1) (no change) Calculate β_o.

(2a) Assume effective lengths, cL, for a range of P/Py values.

(2b) Determine η from Fig. 19 for the assumed values of P/Py and cL/r.

(3) (no change) Check "c" using Ref. 8.

(4) (no change) Repeat Steps 1, 2, and 3 as necessary.

(5) (no change) Using correct η -values for each P/Py determine the relation between ϵ and e from Eq. (8).

(6) Not necessary

(7a) From Fig. 21 obtain ϵ/r for the various P/Py-values and cL/r values obtained above.

(7b) Multiply the results of Steps (5) and (7a) to obtain values of P/Py vs e/r.

The above alternate procedure has been used for the annealed square bars and the results are plotted in Fig. 23 together with the data from the authors' Fig. 16a. This alternate procedure correlates well with experiment and with the authors' procedure.

The authors have suggested that design curves must be developed. The following paragraphs outline one method by which they may be obtained and present some typical curves.

Design Curves

In the design of a building frame it is expected that a structural unit would be developed based on a preliminary choice of cross-section for each member. The columns in this framework would require checking; that is, with a given resultant eccentricity delivered to the column it is desired to know whether or not the member is stable and, if so, can there be an additional saving in material by decreasing the size of the member.

A sample set of possible design curves is shown in Fig. 24. The following steps would be used in a design problem involving an eccentrically loaded column.

1. Given: A member of known L/r with a known axial load and eccentricity. To Find: Allowable load on the column.
2. Compute β_o, the spring constant of the elastic restraints. This would be obtained from the analysis of the frame -- already completed.
3. For two β_o/EK values obtain the allowable load, P, at the given end moment (or eccentricity).
4. Interpolate (or extrapolate) between the values obtained from the two β_o/EK curves. Compare the result with the given P.
5. Improve the design by increasing or decreasing the section modulus of the member based on the results of Step 4. Steps 2, 3, and 4 would be followed through, again, as a check.

The striking thing about the use of Fig. 24 is that it is unnecessary to consider the plastic reduction factor, η, or the

Euler length factor "c". These factors appear only in the calculations leading up to Fig. 24. The curves are developed as follows:

1. An effective slenderness ratio, cL/r, is selected.

   (Knowns: EI, cL/r)

2. Select η from curves like Figs. 19 or 20.

   (Knowns: EI, cL/r, η, P/P_y)

3. From a suitable $M/M_y - \phi/\phi_y$ curve (for example, Fig. 19), determine M/M_y and ϕ/ϕ_y noting that

$$\eta = \frac{M_{cr}/M_y}{\phi/\phi_y} = \frac{M_{cr}}{\phi}\left(\frac{1}{EI}\right) \tag{26}$$

4. From cL/r of Step 1 and P/P_y of Step 2, obtain the end moment for the pin-ended eccentrically loaded column, $M = P \cdot \epsilon$. This may be done, either from curves like Fig. 21 or by assuming a cosine curve and making use of Step 3 above.

   (Knowns: EI, cL/r, η, P/P_y, $M_\epsilon = P \cdot \epsilon$)

5. Assume β_o/EK and compute $\beta_o/\eta EI$.

6. Obtain "c" from Graph II of Ref. 8 and then compute L.

   (Knowns: EI, cL/r, η, P/P_y, $M_\epsilon = P \cdot \epsilon$, β_o)

7. Compute e/ϵ from equation 8, as

$$\frac{e}{\epsilon} = 1 + \frac{\beta_o}{2\,\eta\,EK} = \frac{M_e}{M_\epsilon} \tag{8a}$$

   where M_e is the moment at the end of the elastically restrained column of length L.

8. From Step 4 and Step 7 obtain M_e by direct multiplication. This may be expressed as M_e/M_y and there is now known the relationship between the stiffness (β_o/EK), the full length of member (L), the critical axial load (P) and the end eccentricity (e) or end moment (M_e).

One set of curves is needed for each basic shape. The "WF Section" curve shown in Fig. 24 is for the 8WF31 shape. Although there is considerable variation in WF shapes, it is probable that a few typical M/Ø curves covering groups of WF shapes will prove to be sufficiently accurate.

For any given member, of course, the allowable load at a given eccentricity may quickly be determined from Fig. 24. This has been done for the authors' 1-1/2 inch square annealed bars and the results are shown in Fig. 23. A suitable length correction is made due to difference in material properties since Fig. 24 (as well as those preceeding it) was developed for σ_y = 40 ksi. (The adjusted L/r is found from the expression,

$$(L/r)_a = (L/r)\sqrt{\sigma_y/40.0}\quad)$$

The agreement between the results obtained and the tests and the authors' analysis is most satisfactory.

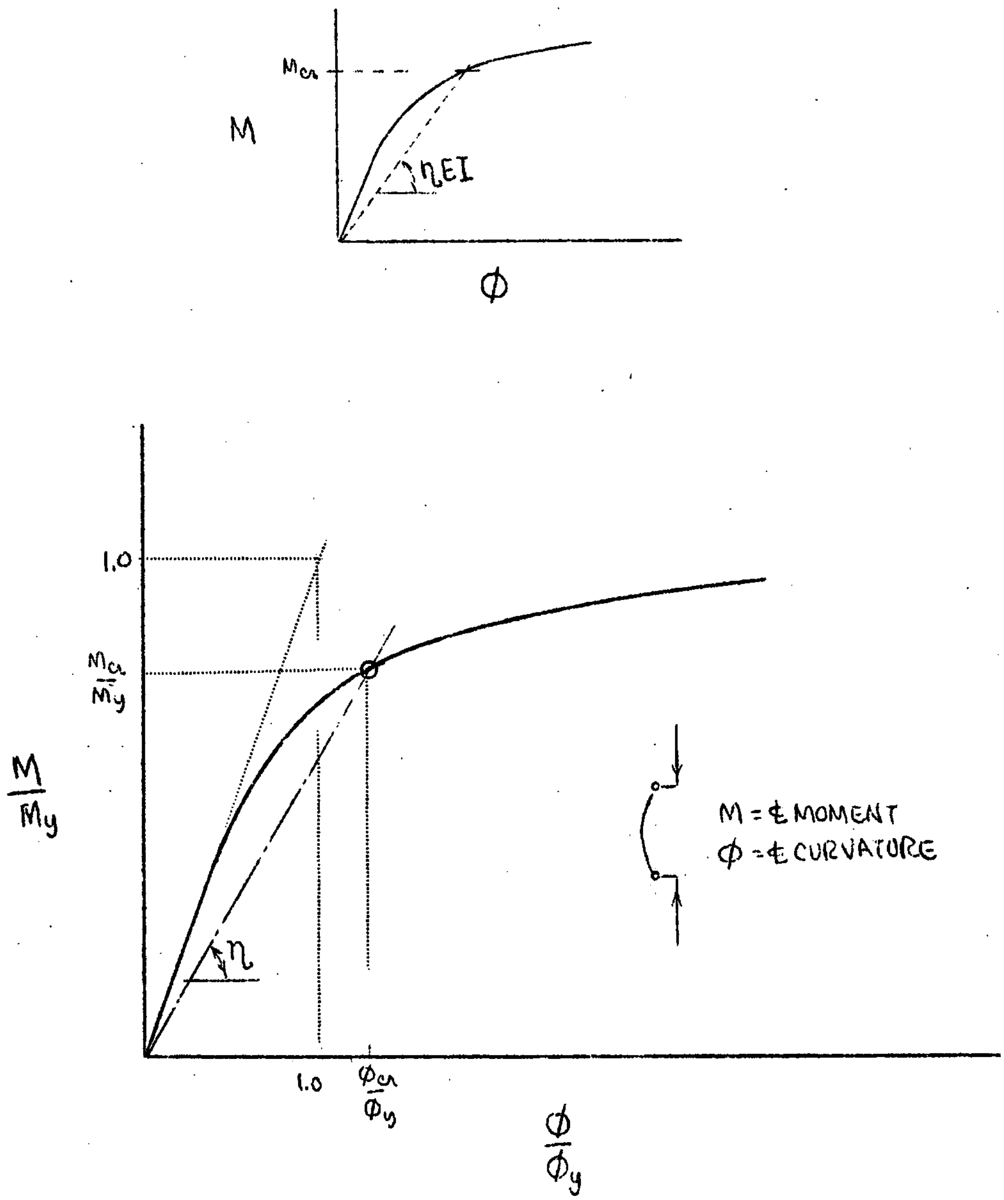


FIG. 18 TYPICAL MOMENT-CURVATURE RELATIONSHIP

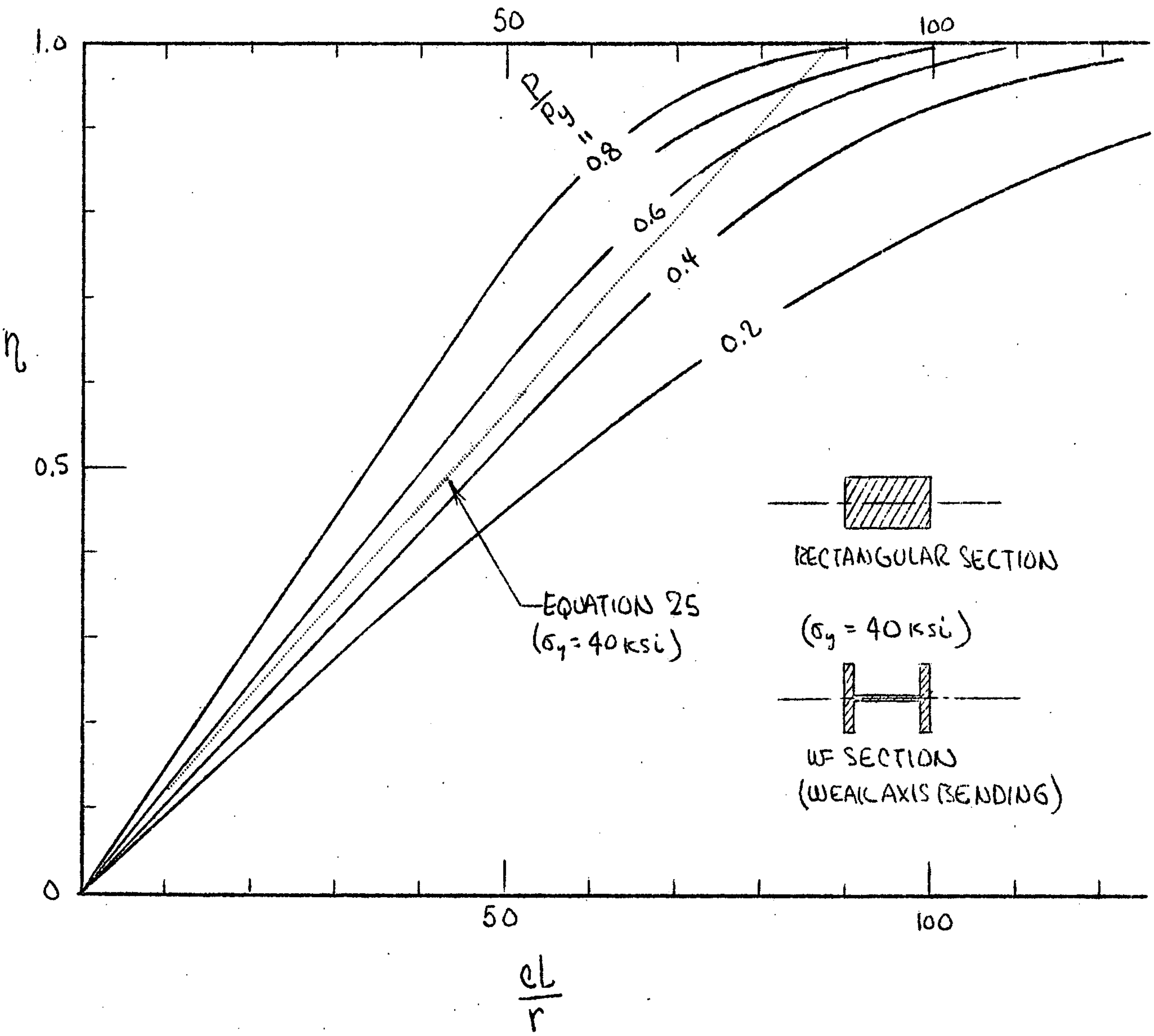


FIG. 19 PLASTIC REDUCTION FACTOR, η, FOR RECTANGLE

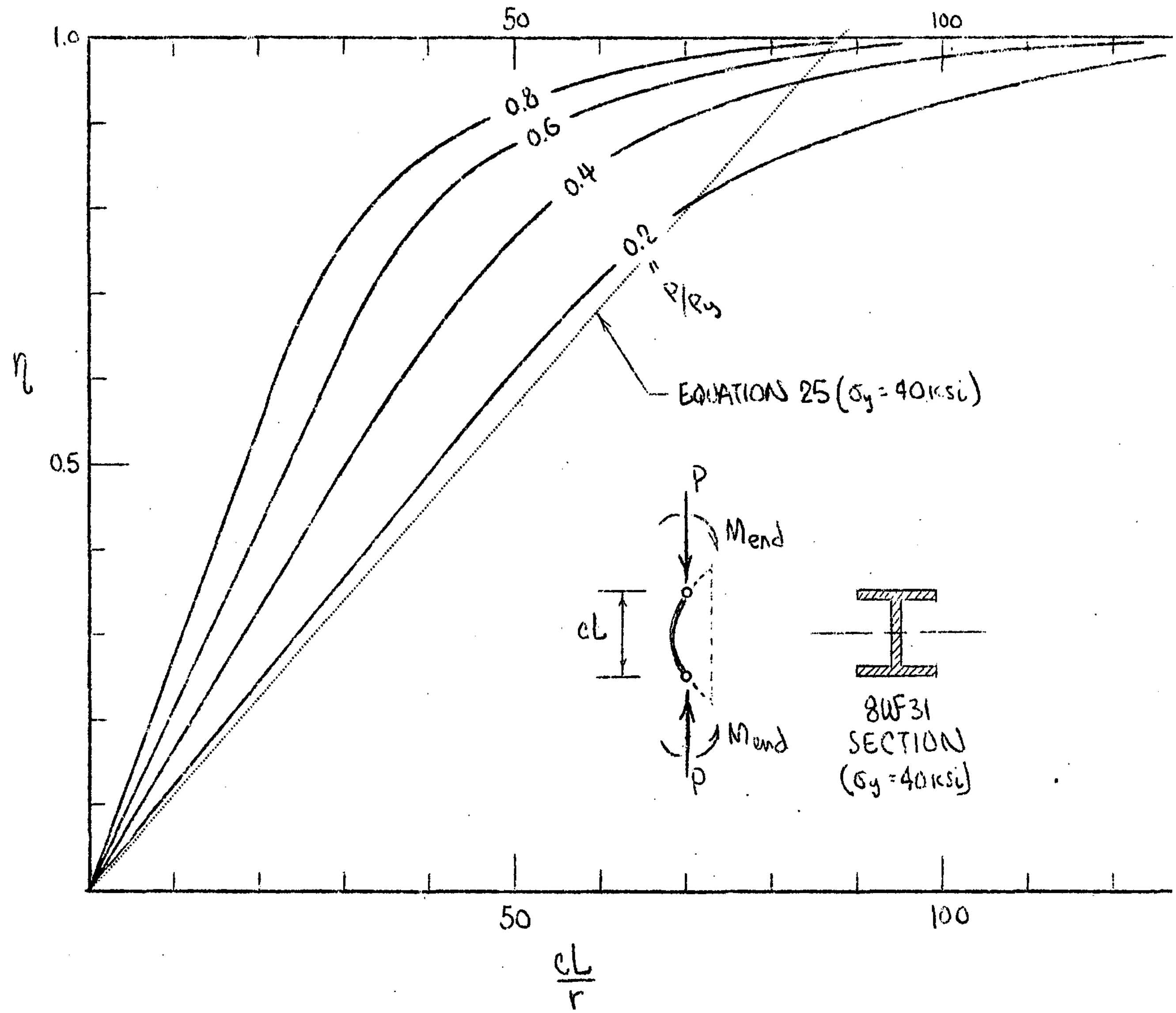


FIG. 20 PLASTIC REDUCTION FACTOR, η, FOR 8WF31 SECTION (STRONG AXIS BENDING)

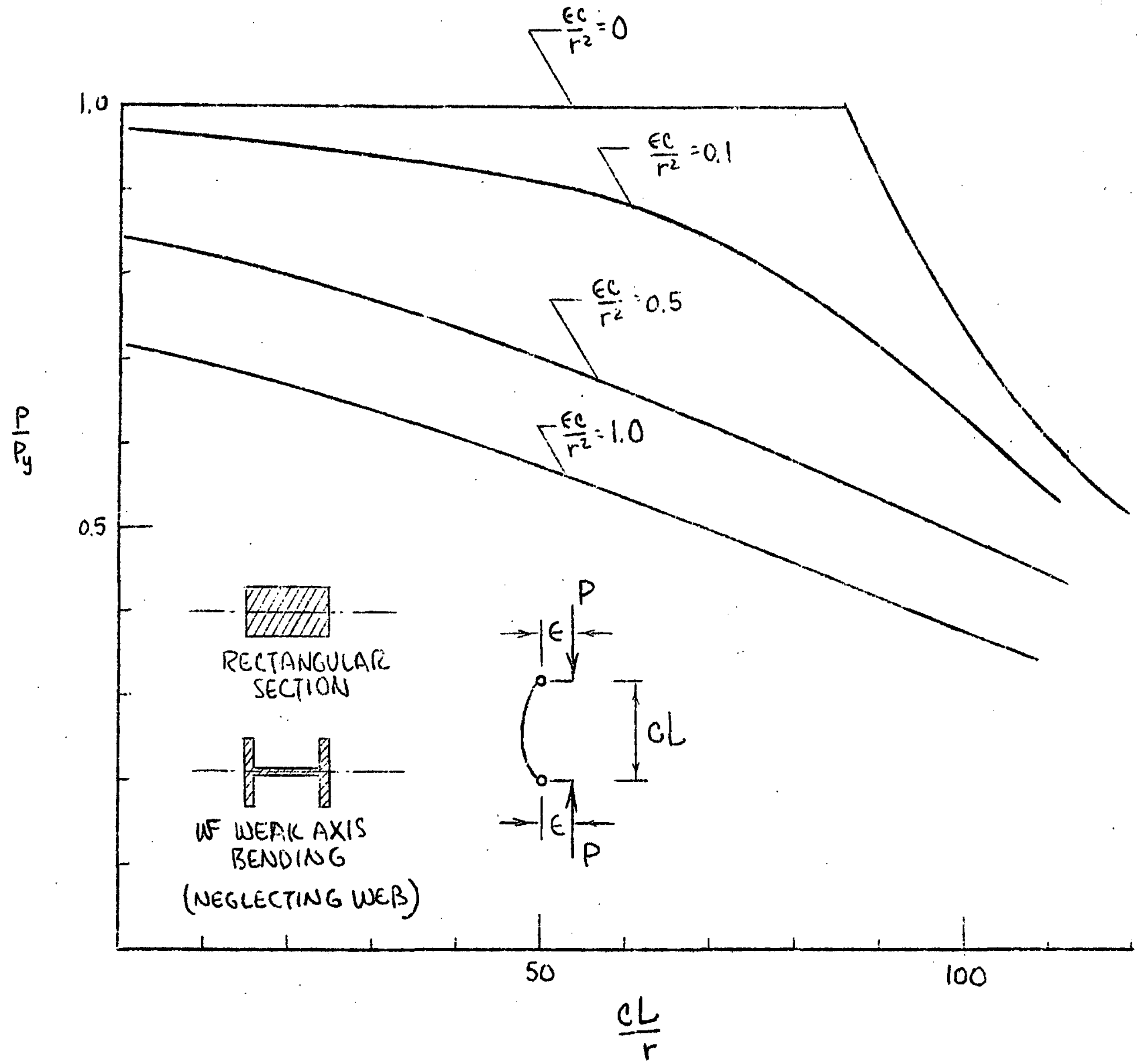


FIG. 21a. MAXIMUM STRENGTH OF PIN-ENDED, ECCENTRICALLY LOADED COLUMNS (18)

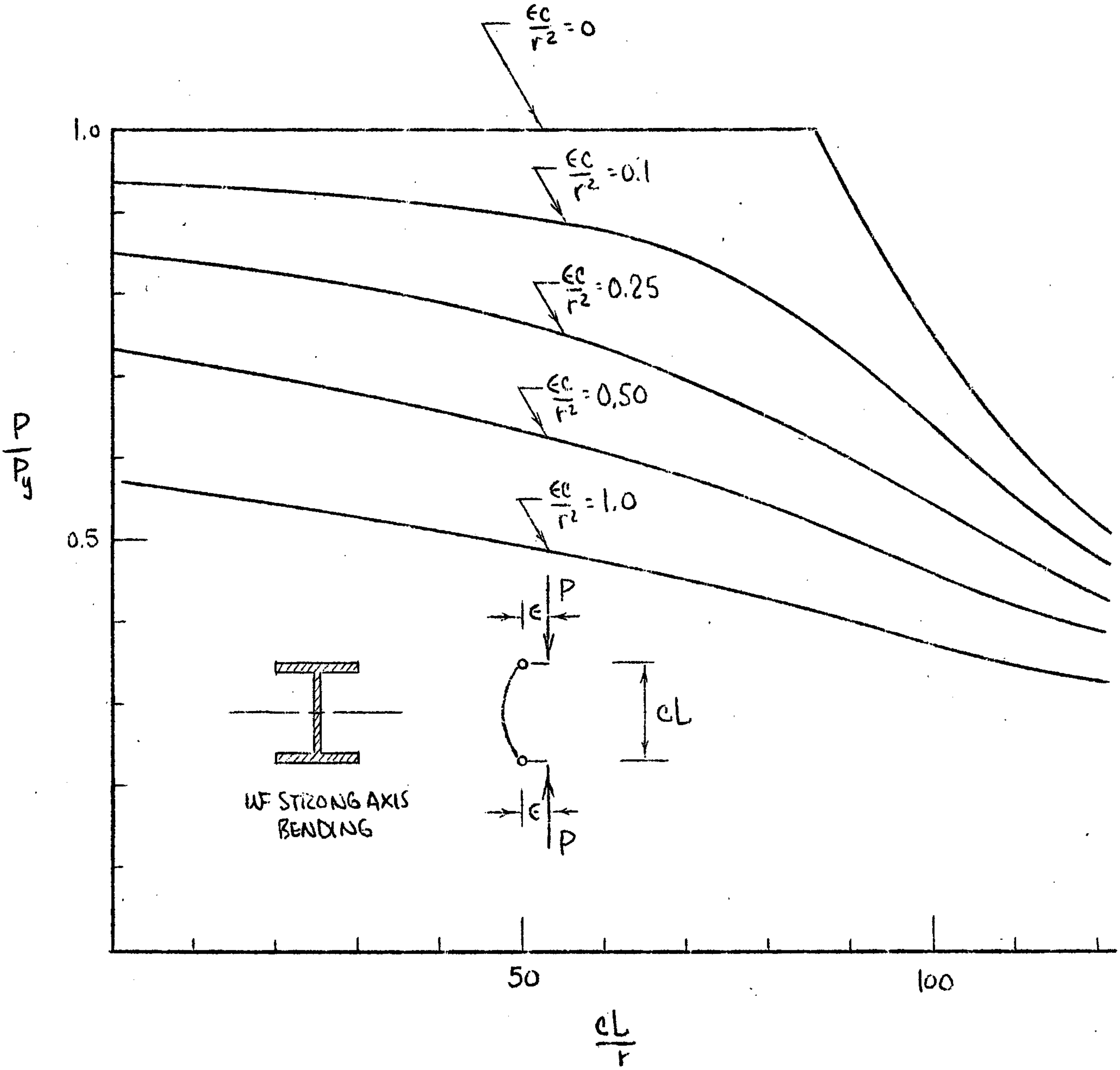
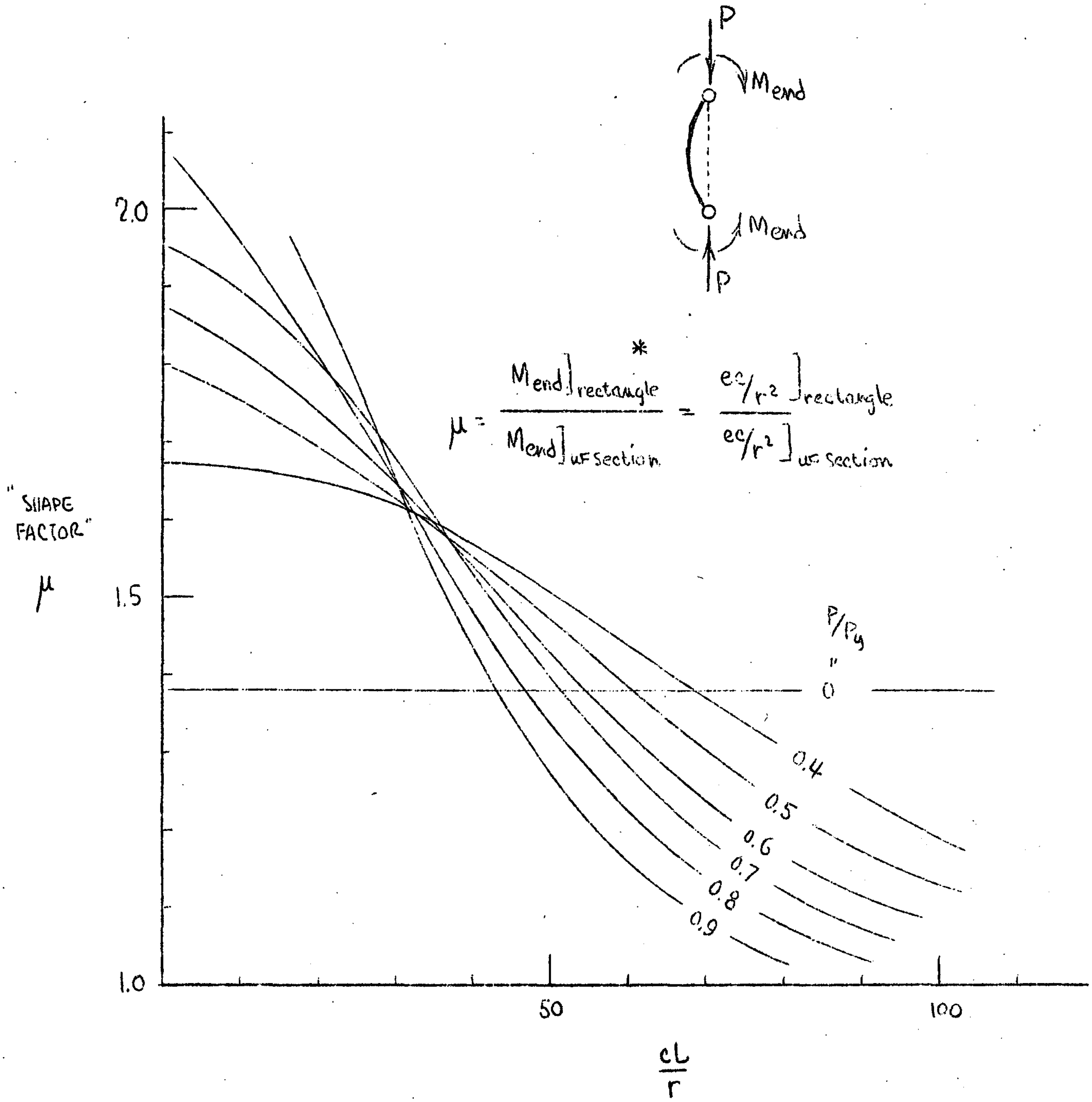


FIG. 21b. MAXIMUM STRENGTH OF PIN-ENDED, ECCENTRICALLY LOADED COLUMNS (18)



$$\mu = \frac{M_{end}]_{rectangle}}{M_{end}]_{WF\,section}} = \frac{ec/r^2]_{rectangle}}{ec/r^2]_{WF\,section}}$$ *

* HERE IT IS ASSUMED THAT $M_y]_{rectangle} = M_y]_{WF}$ and $P_y]_{rectangle} = P_y]_{WF}$. ($\sigma_y = 40$ ksi)

FIG. 22 "SHAPE FACTOR" FOR 8WF31 SECTION

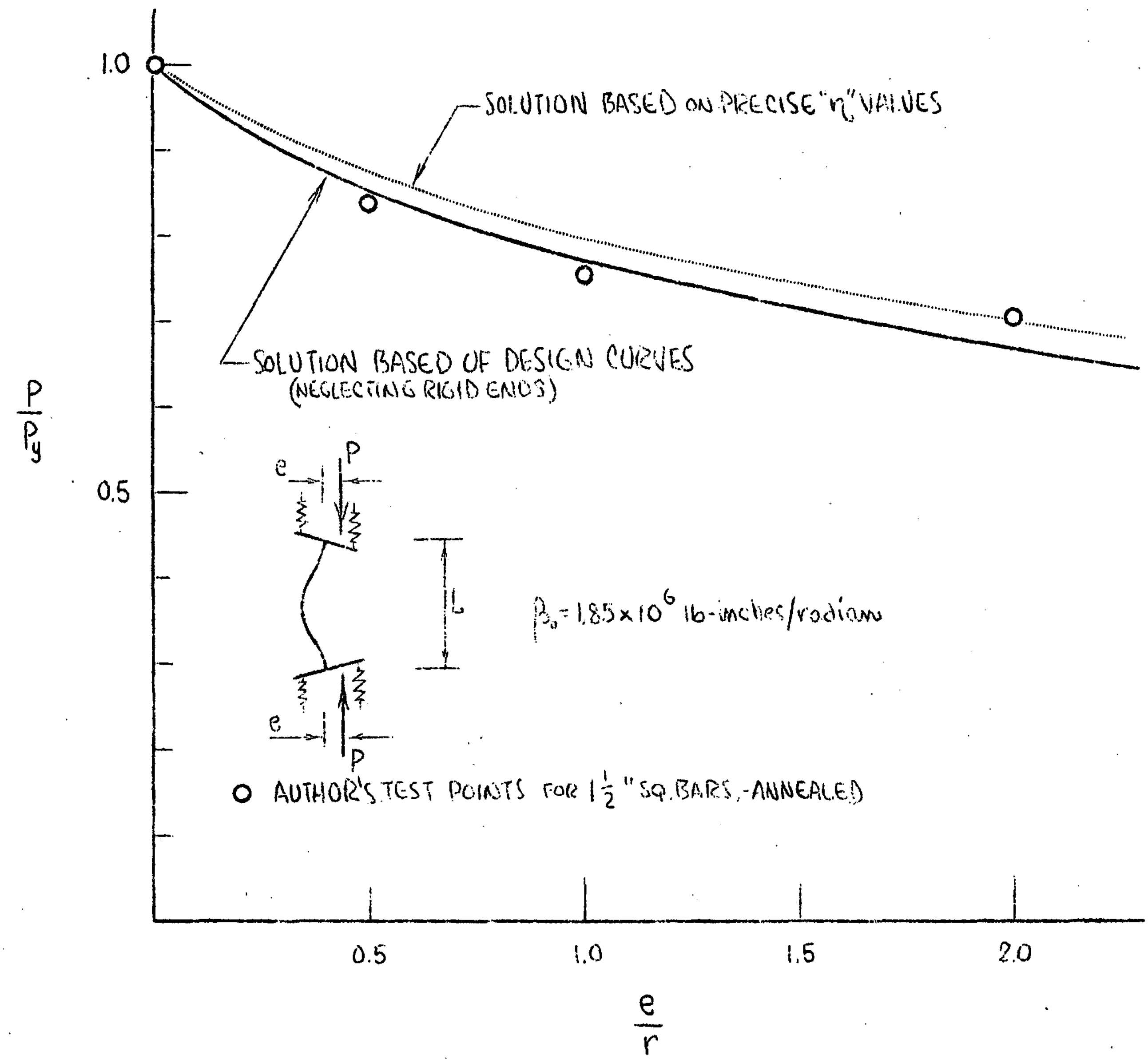


FIG. 23 MAXIMUM STRENGTH OF SQUARE BARS
($L/r = 80.6$)

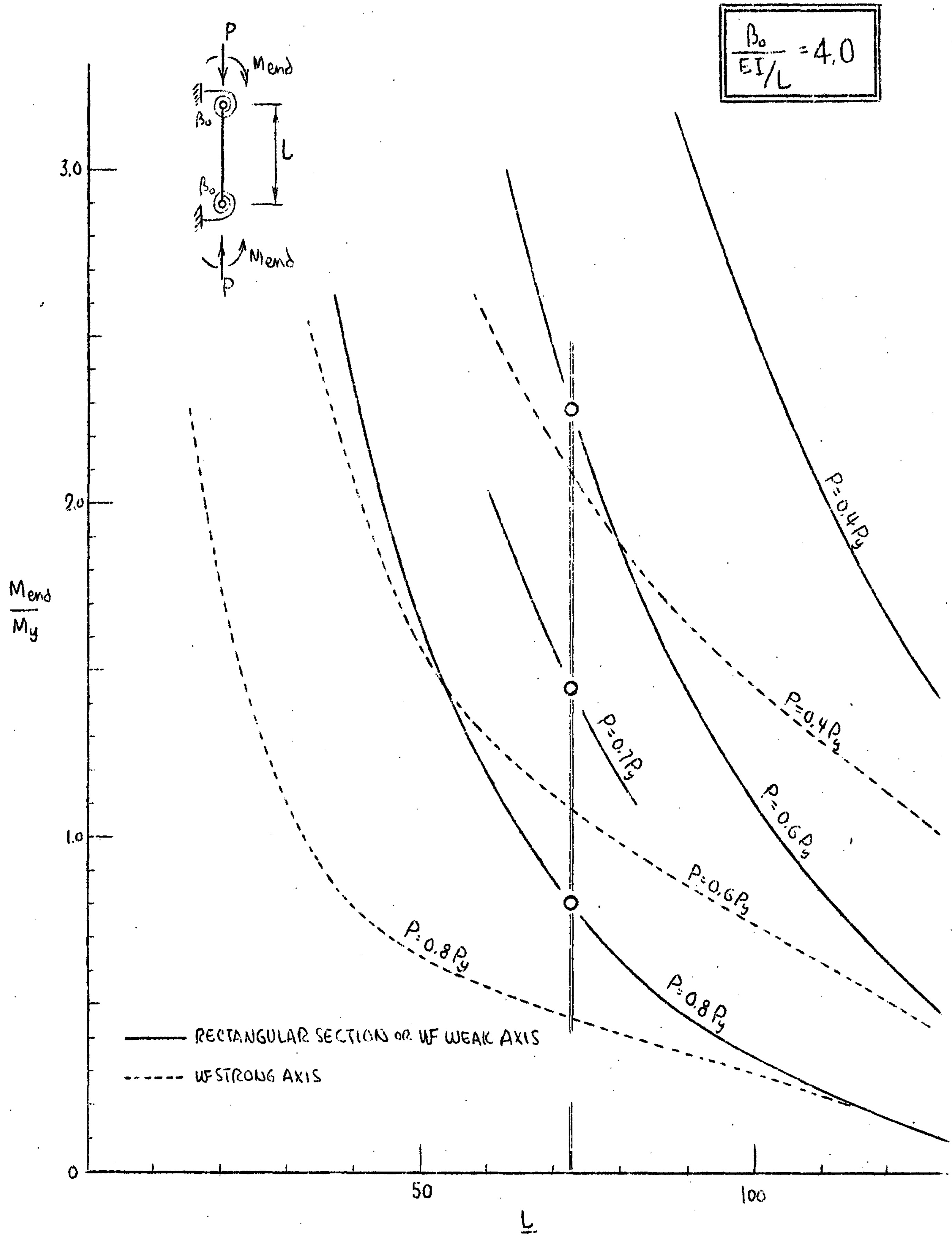


FIG. 24a DESIGN CURVES FOR ELASTICALLY RESTRAINED COLUMNS (ULTIMATE STRENGTH)

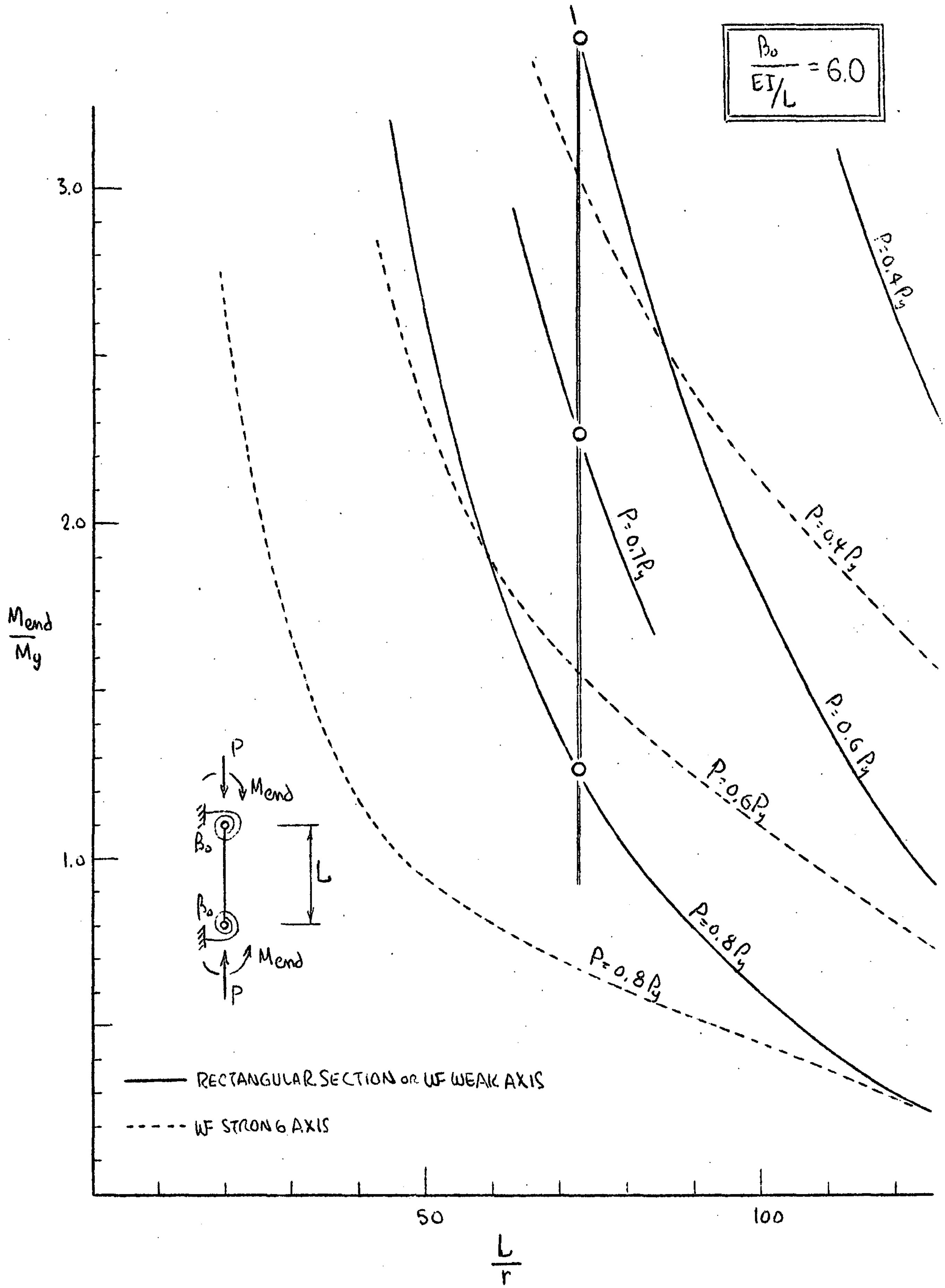


FIG. 24b DESIGN CURVES FOR ELASTICALLY RESTRAINED COLUMNS (ULTIMATE STRENGTH)