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CONTENTS

INFLUENCE OF SHEAR ON THE FULL PLASTIC MOMENT OF BEAMS

I. ABSTRACT

It is well known that most recent developement in steel structures is "Plastic Design" which will be used in stead of elastic design. In plastic design, the full plastic moment of beams would be the most important concept on which every calculation bases. So far as shear influence is concerned, its reducing effect on the full plastic Mt is not so large in the case of usual structural members. However, in the case of short beams or under high shear, its influence would be of conciderable high order and its characteristics of M-ϕ and P-δ diagrams have not been studied enough. In this sense, 5 beam tests (12WF27) were planned and some theoretical solutions which were deduced from Mises' yielding condition and differencial equation of deflection was obtained.

Consequently, so-called "Yang's Effect" could have one new explanation and the characteristics of shear influence have been made clearer than before.

II. INTRODUCTION

(a) GENERAL

As to the developement of plastic design, most of structures will be designed by this theory. In plastic design of steel structures, the full plastic moment (M_p) is the most important concept. In short, the elastic- design has the theoretical inelastic phenomena as the limiting criterion and in the case of plastic design, the max. load carrying capacity is considered as the limiting criterion, unless we think of the factor of safety which is not subject of this report. However, in both cases the working load or stress will stay in elastic range.

It is well known that M_p will decrease its value under existence of shear forces and the reduction had calculated by various methods which show that the reduction is only a few percent when $l/d > 4$-5 but $l/d < 4$ it can be conciderable amount especially in the case of I-beams. However, theoretically or experimentally what kind of influence will occur in M-ϕ curve and how much is the deflection due to shear, have not shown yet. It will be shown that shear has a conciderable deduction at the first stage of M-ϕ curve theoretically and that will be certified by experimental works as described below.

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(b) HISTORICAL BACKGROUND

1. Theoretical Concideration

Though a lot of tests had done before, the remarkable theoretical developements have done recently. Horne, Onat and Leth were admirable attempts to investigate theoretically the effect of shear stresses with reference to the reduction to be expected in full plastic moment(M_p).

Horne treated the problem of a cantilever beam with a concentrated load at the free end as one of plane stress problem and used Tresca's yielding condition as the limiting criterion. After yielding occurs, He proved, the remaining elastic part will take all the shear loads, because plastic zone can not carry any shear stress. From this fact he shew the reduction in moment from the full plastic moment.

More beautiful theoretical treatments had been done by Onat and Leth at Brown University. Extending Horen's work, they used Velocity Field (Kinematically Admissible, Upper Bound Theorem) and Stress Field (Statically Admissible, Lower Bound Theorem), the truevalue will exist between these two bounds.

As a matter of fact, this elastic-plastic problem is so complicated that at present no exact solutions are available. In other words, all theories including this one are the stress solutions which do not satisfy the compatibility equation.

2. Experimental Works

Many beam tests were conducted before in order to certify the application of the plastic theory to the structural members, but most of them have no attention to the influence of shear. Of course the effect usually is not so large and this will be the reason.

At Lehigh University, large steel structural model tests with regards to the plastic design has been performed. That tests shew thet behavior of beams clearly in plastic range as well as in elastic range. However, there is little experimental information with reference to the shear effects. We can find in Progress Report No.8, Johnston, Yang and Beedle point out some of the implication of shear yielding of the web, present several pictures of beam section in which the web yielded in shear and also in Progress Report No.5, Yang, Beedle and Johnston, some figures which shows the M-ϕ curves something like the following figure (see Fig. 1). We shall call that phenomena "Yang's Effect". They explained it by thinking of stress concentration and residual stresses, that is, " Although no positive proof is given, the fact that as-delivered specimens consistently show a lower strength than that predicted by the simple plastic theory may thus be explained partly on the bases of stress concentration and residual stress in the member. There are an increased number of local plastic yield zones in the beam section will result in a higher ϕ-value even though the moment is kept unchanged since portions of the beam yield which are assumed to be elastic in the theory."

This idea would be correct partly, but this phenomena can also be explained by shear influence on full plastic moment of the beams.

In England the influence of shear forces on the deformation characteristics of I-beams has been studied by Baker and Roderick experimentally.

Most recently at University of Illinois, W. J. Hall tested two continuous beams to study the influence of shear.

He said, " On the basis of these tests it is concluded that no measurable reduction in the moment capacity was indicated."

(c) THEORETICAL CALCULATION

There is no theoretical calculation of M-ϕ curves including shear influence, so in this report some theoretical calculations which were deduced from Mises' yielding condition and differential equation of deflection of beams, will be shown.

(d) TESTS CONDUCTED

In order to certify the phenomena of shear influence 5 Beam Tests which consists of three third-point loading specimens and two concentrated loading specimens, were conducted.

In the former case, the curvature due to pure bending moment was measured at the center portion of beam and at the same time the curvature which has shear influence was measured at the vicinity of supports.

Effects of concentrated load were also tested in the latter specimens.

III. TEST

(a) COUPON TESTS

Before main tests, coupon specimens were tested by using 60 kips capacity Hydraulic-Type Testing Machine and the Templin Auto-graphic recorder. At first, 12 specimens (8 from flange, 4 from web) which were cut from specimens S-1, S-2, S-4 and S-5 (S-3 having no margin length, that was concidered to do coupon tests after the bending test) were prepared, but according to the good agreements of tests, 8 specimens were only tested actually. (see Table 1)

(b) SPECIMENS

5 specimens were taken from 5-12WF27 beams which were only available at that time, that is , the choice of this section has no special meaning.

All specimens are shown in Fig. 3 . To prevent local web buckling, two or one stiffeners were attached as can be seen in Fig.3 just beneath the loading points.

The characteristics of this section were checked before, under consideration of local buckling and lateral buckling.

b/t	-	16
d/w	-	48
ld/bt	-	166

(c) TEST SET-UP

The schewmatic sketch of Set-up is shown in Fig. 6 Photogragh of the actual test (S-) set-ups are shown.

The center span of 3 ft long is taken for pure bending test in third points loading. The influence of shear can be measured in the portions of loading. In ~~this~~ these tests the main portion is the measurements of curvature. In order to measure ~~these~~ them, three methods were used:

1. Vertical Type
2. Horizontal Type
3. SR4-Gage Type

Each type are shown in Fig. .

Three types are shown in Fig. 7 . Each type has some advantages and at the same time disadvantages, too. Vertical type has good sensitivity and easy reading as compared with Horizontal type, but it will be easily affected by the local imperfection (i.e. local twisting, local buckling, lateral buckling etc.) As regards SR4 type , it is very sensitive but affected easily by local yield line etc., (so its' application should be taken under carefull attention)

In short, the superiority of measuring type does not depend on its' own excellent quality, but the kind of test on which gages will be used.

A 14WF136 section is used as the base beam, but furthermore to prevent the error due to bending of base beam, the second base beam for dial gages (can be seen in Fig. 6) was used. Under some assumptions, the bending of this base beam was calculated.

To obtain the rotation of loading section by which the deflection of a cantilever beam should be corrected and the application of the following theoretical calculation will be available, and to certify the occurence of lateral buckling , two level bars were attached. (See Fig. 6)

(d) TEST PROCEDURE

The sequence of load increments were taken about 5-kips interval in elastic range. The loads were applied slowly and during maintaining constant load readings of dial gages,level bars and SR4-gages were taken.

When yielding occurs under constant load in the plastic range, a ~~special~~ period of time must elapse to permit the penetration of yield zones into the specimen. 15 minutes were usually used. After yielding occurs, the criterion of loading of course was changed to the centeral deflection, that is to say, at the first stage about 0.1 in. - at the last stage about 0.3 in.

Loads were kept constant while readings were taken. Curve of th centeral deflection was plotted against load during each test as a check on the proper functioning of apparatus.

IV. THEORETICAL ANALYSIS

(a) NOMENCLATURE

usually at front of book

σ_y = Yielding stress level
σ_x = Normal stress in x-direction
y_o = Distance between the neutral axis of beam and the edge of yielding zone
η = y/y_o
d = Depth of beam
M_{p_o} = Full plastic moment due to pure bending
M = Moment at x
P = Load
b = Breadth of beam
τ = Shearing stress
J_2 = The second invariant of stress deviation

K = Yielding limit in simple shear
σ_γ = Normal stress in γ-direction
ξ = Pl/M_{po} Reduction Factor of shear influence

n = l/d Slender ratio for shear influence
ξ_{ult} = ξ at ultimate stage
y_s = Deflection due to shear

G = Modulus of shear
K = Shape factor for shear(3/2 for rectangular section)
V = Shearing forces
A = Sectional area which can carry shear stress
ϕ_s = Curvature due to shear
ϕ_{s1} = Curvature due to shear at x=1

E = Young's modulus
ϕ_m = Curvature due to moment
ϕ_{m1} = Curvature due to moment at x=1

M_y = Full plastic moment of beam

φ = x/l

(b) ASSUMPTIONS AND LIMITATIONS

A canti-lever beam with rectangular cross-section is treated in this section and we have to allow the following assumptions and limitations:

1. The problem will be analyzed as a problem of plane stress
2. Only elastic portion can carry shearing forces
3. Mises' yielding condition is satisfied at three points
4. This solution is only a stress solution
5. Fixed end can have warping.

(c) REDUCTION FACTOR OF THE FULL PLASTIC MOMENT DUE TO SHEAR

Assuming $\sigma_z=\tau_{zx}=\tau_{zy}=0$, in elastic zone

$-y_0 \leq y \leq y_0$, (See Fig. 2)

$$\sigma_x = \sigma_y \frac{y}{y_0} = \sigma_y \eta \tag{1}$$

Equations of equilibrium must be satisfied,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \qquad \text{and,} \tag{2}$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 . \tag{3}$$

By differentiating eq.(1), We can get from eq.(2),

$$\frac{\partial \tau}{\partial y} = \sigma_y \frac{y_0' y}{y_0^2}$$

$$\therefore \ \tau_{xy} = \int_{-y_0}^{y} \sigma_y \frac{y_0'}{y_0^2}\, y\, dy = \frac{\sigma_y y_0'}{2}(\eta^2 - 1) . \tag{4}$$

From eq.(3),

$$\frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau}{\partial x} = \frac{\sigma_y}{2}\frac{y_0'^2}{y_0}\left(3\frac{y^2}{y_0^2} - 1\right).$$

$$\therefore \ \sigma_y = \int_{-y_0}^{y} \frac{\sigma_y}{2}\frac{y_0'^2}{y_0}\left(\frac{3y^2}{y_0^2} - 1\right)dy = \frac{\sigma_y}{2} y_0'^2 \eta(\eta^2 - 1), \tag{5}$$

According to plastic analysis, (see Fig. 2)

$$M(x) = Px = M_{po} - \sigma_y \frac{b y_0^2}{3} \tag{6}$$

$$M_{po} = \sigma_y Z = \sigma_y \frac{bd^2}{4} . \tag{7}$$

From eq.(1) we can get by differentiating,

$$y_0^2 = \frac{3}{\sigma_y b}(M_{po} - Px) \tag{8}$$

$$2 y_0 y_0' = -\frac{3P}{\sigma_y b} . \tag{9}$$

Mises' yielding condition ;

$$J_2 = K^2 \quad \text{OR} \quad \sigma_v^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x\sigma_y + 3\tau^2 \tag{10}$$

where J_2 = the 2nd invariant of stress deviation and K = yielding limit in simple shear.

and σ_v = (σ_{yield}) in tension

From eq. (1),(4),(5) and (10), we have,

$$\left(\frac{\sigma_r}{\sigma_y}\right)^2 = \eta^2 + \frac{\eta_0'^{4}}{4}\eta^2(\eta^2-1)^2 - \frac{\eta_0'}{2}\eta^2(\eta^2-1) + \frac{3y_0'^{2}}{4}(\eta^2-1)^2. \quad (11)$$

We assumed

$$\left(\frac{\sigma_r}{\sigma_y}\right)^2 = 1 \qquad at \quad \eta = \pm 1 \quad and \quad \eta = 0. \quad (12)$$

From this,

$$\eta_0'^{2} = \frac{4}{3} \fallingdotseq 1.33 \quad (13)$$

Equating eq.(8) to square of y_0 which is deduced from eq. (9), we get,

$$y_0^2 = \frac{3}{\sigma_y b}(M_{po} - Px) = \frac{9P^2}{4\sigma_y^2 b^2 y_0'^2}. \quad (14)$$

Substituting (13) and (7) into (14), the following eq. can be seen easily,

$$\xi^2 + \frac{64}{9}n^2\xi - \frac{64}{9}n^2 = 0 \qquad at \quad x = l, \quad (15)$$

where $\xi = Pl/M_{po}$ Reduction Factor of shear influence,

$l/d = n$ - Slender ratio for shear influence.

$$\xi_{ult} = \frac{32}{9}n^2\left[\sqrt{1+\frac{9}{16n^2}} - 1\right]. \quad (16)$$

By using Maclaurin's expansion formula, numerical calculation is done easily and the results are shown in Fig. 4 .

(d) CURVATURE AND DEFLECTION DUE TO SHEAR

(1) Curvature due to shear

In the case of cati-lever beam, in elastic range shearing force diagram is clearly uniform, but in plastic stage distribution of shearing force is not constant under existence of yielding zones. Distribution of shear stress is not uniform over cross-section, therefore, a plane cross-section before will not remain as a plane but a curved surface. As centroid of each cross-section, however, will slide each other vertically, slope of deflection due to shear should be equal to shear strain at centroid of each cross-section. Hence, we have the eq. of slope as follows :

$$\frac{dy_s}{dx} = \frac{(\tau_{xy})_{y=0}}{G} = \frac{K}{G} \cdot \frac{V}{A} \; . \tag{17}$$

Of course, at the fixed end the fixed end condition is not satisfied completely by reason of warping of cross-section, but now this approximation may be taken, for specimens, S-1, S-2 and S-3, can warp under third-point loading position which is thought as a fixed end.

In elastic range, shearing force diagram is uniform, then from eq. (17) it is easy to see

$$\frac{d^2 y_s}{dx^2} = 0 \; . \tag{18}$$

This means that we can find some additional deflection as well known, but can not find any change of curvature. In ~~plastic~~ stage, sectional area A which carry shear stress is not constant but variable due to yielding zones. That is,

$$\frac{dy_s}{dx} = \frac{K}{G} \cdot \frac{V}{A} = \frac{K}{G} \frac{P}{y_0 b} \, , \tag{19}$$

$$\therefore \quad \phi_s = \frac{d^2 y_s}{dx^2} = \frac{KP}{bG} \frac{d}{dx}\left(\frac{1}{y_0}\right) . \tag{20}$$

Making use of eq. (9) and (8),

$$\phi_s = \frac{3KP^2}{2b^2 G \sigma_y \left(\frac{3}{4}d^2 - \frac{3Px}{\sigma_y b}\right)^{3/2}} \quad . \tag{21}$$

After some calculations,

$$\phi_s\Big|_{x=\ell} = \phi_{s\ell} = \frac{K\sigma_y d\,\xi^2}{4\sqrt{3}\,G\ell^2(1-\xi)^{3/2}} \quad . \tag{22}$$

Making no-dimentional expression,

$$\frac{\phi_{s\ell}}{\phi_y} = \frac{K}{8\sqrt{3}}\left(\frac{E}{G}\right)\frac{\xi^2}{n^2(1-\xi^2)^{3/2}} \tag{23}$$

$$\frac{\phi_{s\ell}}{\phi_y}\Bigg|_{\text{rect.}} = \frac{\sqrt{3}}{16}\,\frac{E}{G}\,\frac{\xi^2}{n^2(1-\xi)^{3/2}} \tag{24}$$

for rectalgular shape.

(2) Curvature due to moment

Next, let us consider curvature due to moment ϕ_m. According to plastic theory,

$$M = Px = \sigma_y\left(Z - \frac{b\sigma_y^2}{3E^2\phi^2}\right) = \sigma_y\left(\frac{bd^2}{4} - \frac{b\sigma_y^2}{3E^2\phi^2}\right) \tag{25}$$

$$\therefore \quad \phi_m^2 = \frac{4b\sigma_y^3}{3E^2(bd^2\sigma_y - 4Px)} \tag{26}$$

Changing form,

$$\phi_m\Big|_{x=l} = \phi_{ml} = \frac{2\sigma_y}{\sqrt{3}\,dE(1-\xi)^{1/2}} \tag{27}$$

$$\frac{\phi_{ml}}{\phi_y} = \frac{1}{\sqrt{3}(1-\xi)^{1/2}} \tag{28}$$

Then total ϕ at x = 1, is,

$$\frac{\phi_l}{\phi_y} = \frac{\phi_{ml}}{\phi_y} + \frac{\phi_{sl}}{\phi_y} = \frac{1}{\sqrt{3}(1-\xi)^{1/2}} + \frac{\sqrt{3}}{16}\frac{E}{G}\frac{\xi^2}{n^2(1-\xi)^{3/2}} \tag{29}$$

where $\xi = Pl/M_{po} = \frac{M}{M_{po}}$ and $\xi_y = \frac{M_y}{M_{po}} = \frac{2}{3} \le \xi \le \xi_{ult}$.

Strain hardening range will be discussed later.

(e) M- ϕ CURVES (THE CHARACTERISTICS OF SOLUTION)

The problem is where the discontinuity of M- ϕ curve comes from ? (see Fig. 5) In order to ascertain this, we are going to investigate the characteristics of the solution.

The equation on which the solution bases is,

$$\frac{dy_s}{dx} = \frac{KP}{Gb}\frac{1}{y_0} = \frac{2KP}{Gbd}\frac{1}{\zeta} \quad , \text{ where } \zeta = \frac{y_0}{d/2} \qquad (19)$$

Since

$$y_0^2 = \frac{3}{4}d^2\left(1 - \frac{Px}{M_{PO}}\right), \quad \zeta^2 = \frac{y_0^2}{d^2/4} = 3\left\{1 - \frac{P\ell}{M_{PO}}\left(\frac{x}{\ell}\right)\right\},$$

$$\therefore \zeta^2 = 3\left[1 - \xi \cdot \varphi\right] \quad , \text{ where } \quad \varphi = \frac{x}{\ell} \qquad (30)$$

Three dimentional diagram of eq. (30) is shown in Fig. We can see easily that in the plane at which x = 1 or $\varphi = 1$, ζ has the discontinuity of its first derivative at $\xi = 2/3$. This means $\frac{d}{dx}\left(\frac{1}{\zeta}\right)$ must have the discontinuity at $\xi = 2/3$.

On the other hand, we can see obviously this phenomena from the following condition, that is,

$$\lim_{\substack{y_0 \to d/2 \\ x = \ell}} \frac{d}{dx}\left(\frac{1}{y_0}\right) \neq 0 \quad .$$

To be able to find this phenomena in many previous works is very interesting, for example, Fig. 25 in reference (23) shows clearly this phenomena.

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References

No.	Authors	Title
1	Amirikian A.	Future Developement in Welded Steel Buildings, Weld. J. Vol. 27 No.8 Aug. 1948
2	Baker J. F.	A Review of Recent Investigation in Behavior of Steel Frames in the Plastic Range, J. of the Institute of C.E. Jan. 1949
3	Donnel L.H.	Plastic Flow as an Unstable Process, J. of Appl. Mech. Vol.9 June 1942
4	Green A.P.	The Plastic Yielding of NotchedBar due to Bending, Q. J. of Mechanics and Appl. Math. 6. 1953
5	Greenberg H.J. & Prager W.	On Limit Design of Beams and Frames, Tech. Rep. No.5 Brown Univ. Oct. 1949
6	Hall W.J.	Shear Deflection of Wide Flange Steel Beams in the Plastic Range, Univ. Ill. Nov. 1954
7	Hill R.	The Mathematical Theory of Plasticity, Chap.6-9 Oxford 1950
8	Horne M.R.	The Plastic Theory of Bending of Steel Beams with Particular References to the Effect of Shear Forces, Proc. Roy. Soc. London Vol207 '51
9	Johnston, Yang & Beedle	Prog. Rep. No.8 An Evaluation of Plastic Analysis as Applied to Structural Design, Weld. J. Vol. 32 1953
10	Leth C. A.	The Effect of Shear Stress on the Carrying Capacity of I-Beam, Brown Univ. March 1954
11	Luxion W.W. & Johnston B. G.	Prog. Rep.No.1 Plastic Behavior of Wide Flange Beams, Weld. J. Vol. 27 Nov. 1948
12	Nadai A.	Theory of Flow and Fracture of Solids, McGraw-Hill, New York 1950
13	Nadai A.	Plasticity, McGraw-Hill 1931
14	Onat E. T. & Shield R. T.	The Influence of Shearing Forces on the Plastic Bending of Wide Beams, Brown Univ.'54
15	Onat E. T. & Prager W.	Limit Analysis of Arches, J. of Mech. and Physics of Solid 1 1953
16	Prager W. & Hodge P. G.	Theory of Perfectly Plastic Solids, John Wiley & Sons Inc. New York 1951
17	Prager W.	The Stress Strain Laws of the Mathematical Theory of Plasticity--A Survey of Recent Progress, J. Appl. Mech. Vol. 15 No.3 1948
18	Roderick J.W. & Phillips I.H.	Carrying Capacity of Simply Supported Mild Steel Beams, Eng. Structures 1949
19	Roderick J. W.	Further Tests on Beams and Portals, Trans. Inst. Welding 3 1940 No. 2
20	Roderick J.W.	Theory of Plasticity-Elements of Simple Theory, Phil. Mag. Ser.7 Vol 34
21	Topractsoglou A. A.	Connections for Welded Rigid Portal Frames, Dissertation for Ph. D. Lehigh Univ. 1950
22	Van Den Broek J. F.	Theory of Limit Design, John wiley and Sons Inc. New York 1948
23	Yang, Beedle & Johnston	Prog. Rep. No.5 Residual Stresses and the Yield Strength of Steel Beams, Weld. J.31 '52