Built-Up Members in Plastic Design

STRENGTH OF LONGITUDINALLY STIFFENED PLATE PANELS WITH LARGE b/t

by

Tsuneo Tsuiji

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STRENGTH OF LONGITUDDINALLY STIFFENED
PLATE PANELS WITH LARGE b/t

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LONGITUDINALLY STIFFENED PLATE PANELS
WITH LARGE $b/c$"

By TSUNEKO TSUNI
June, 1965

Note: "SHOULD READ" will be annotated as "S.R."
19  change $\frac{a}{h_p} = 0.16$ to read $\frac{a}{h_p} = 0.187$
change $\gamma_s = 10.0$ ksi to read $\gamma_s = 11.0$
change $\gamma_p = 36.0$ ksi to read $\gamma_p = 33.7$ ksi

23  eqn 4.2  S.R.
$\phi = \phi_i - \frac{\phi_i - \phi_0}{d_4}$
where $\phi_i$, $\phi_0$ = and the $0^{th}$

$\Delta l_{i+2}$ = the length of the segment from $i$ to $i+2$
$A =$

The slope is
$\theta = \phi_i + \phi_4 \frac{1}{2} \frac{\phi_i - \phi_4}{A_{i+2}} \Delta l_{i+2}$

24  eqn 4.5  S.R.
$\theta_{i+1} = \theta_i + \phi_4 \Delta l_i + \frac{1}{2} (\phi_{i+1} - \phi_4) \Delta l_i$

25  eqn 4.6  S.R.
$\phi \frac{dV}{dd} = (-1) \phi \cos \theta \phi \sin \theta$

Note: Capital "S" denotes a non-dimensional segment length
Lower case "s" (i.e. $d_4$) denotes a dimensional segment length

eqn 4.7  S.R.
$\frac{dW}{dd} = (1 - \phi \cos \theta) \phi \sin \theta$

26  eqn 4.8  S.R.
$\frac{dV}{dd} + V \sin \theta + W \cos \theta = 0$

27  eqn 4.11  S.R.
$M = M_0 - V_0 (y - y_0) - \frac{E}{2} (V - V_0)^2 - g \phi \cos \theta (y - y_0) \sin \theta$
$- \frac{E}{2} (V - V_0)^2 + g \phi \cos \theta (y - y_0) \sin \theta$

28  eqn 4.15  S.R.
$N_{i+1} = N_i + g \phi \cos \theta - \phi \cdot \delta \phi \cos (\phi \sin \theta - \sin \theta)$
$M_{i+1} = M_i - V_0 \Delta x - \frac{g \phi \delta \phi \cos \theta}{2} \left[ \frac{1}{2} (\Delta x)^2 \right] + \phi \cdot \delta \phi \cos \theta$
$- \phi \delta \phi \cos (\phi \sin \theta - \sin \theta)$
\[ \Delta x = \cos \Theta \cdot (Ax) - \left( \frac{\Phi_1}{3} + \frac{\Phi_3}{6} \right) \sin \Theta \cdot (Ax)^2 \]
\[ \Delta y = \sin \Theta \cdot (Ax) + \left( \frac{\Phi_1}{3} + \frac{\Phi_3}{6} \right) \cos \Theta \cdot (Ax)^2 \]
\[ L = \frac{\Delta x}{\Delta y} \]

\[ \sigma = \sigma_x + \left( \frac{\sigma_x}{\partial x} \right) \left( \frac{\sigma_x}{\partial x} \right) \Delta x \]
\[ + \frac{1}{2} \left[ \left( \frac{\sigma_y}{\partial y} \right) - \left( \frac{\sigma_y}{\partial y} \right) \right] \left( \frac{\Delta y}{\partial x} \right)^2 \]

\[ 
\[ \sigma_{xx} = \sigma_x + \frac{1}{2} \left[ \left( \frac{\sigma_{xx}}{\partial x} \right) + \left( \frac{\sigma_{xx}}{\partial x} \right) \right] \left( \frac{\Delta x}{\partial x} \right)^2 \]

\[ \frac{\Delta x}{x} = -\frac{\Delta x}{x} \cos \Theta - \left( \frac{\sigma_y}{\partial y} \right) \left( \frac{\Delta y}{\partial y} \right)^2 \]
\[ \frac{\Delta y}{y} = \frac{\Delta y}{y} \sin \Theta + \left( \frac{\sigma_y}{\partial y} \right) \left( \frac{\Delta y}{\partial y} \right)^2 \]

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Table 1 - a Moment-Curvature-Thrust Relationships

**Negative Bending**

**Case 1:** For \( \frac{M}{Ecr} \) S.R.
\[ \frac{E_c}{Ecr} - \Phi_1 \]

**Case 2:** For \( \frac{M}{Ecr} \) S.R.
\[ \left( \frac{E_c}{Ecr} \right) \left( \frac{E_c}{Ecr} - \left( \frac{E_c}{Ecr} \right) \right) + \frac{f}{E} \left( 1 - \frac{E_c}{Ecr} \right) \left( \frac{E_c}{Ecr} - \left( \frac{E_c}{Ecr} \right) \right) \left( \frac{A}{E} \right) \]
\[ + \left( 1 - \frac{E_c}{Ecr} \right) \left( \frac{E_c}{Ecr} \right) \left( \frac{A}{E} \right) - \Phi_2 \left( \frac{E_c}{Ecr} - \left( \frac{E_c}{Ecr} \right) \right) \left( \frac{A}{E} \right) \]

**4th Case:** \( \frac{M}{Ecr} \) S.R.
\[ \frac{E_c}{Ecr} - \left( \frac{E_c}{Ecr} \right) \left( \frac{A}{E} \right) - \left( \frac{E_c}{Ecr} \right) \left( \frac{A}{E} \right) - \frac{E_c}{Ecr} \left( \frac{A}{E} \right) \left( \frac{A}{E} \right) \]
\[ + \frac{f}{E} \left( \frac{E_c}{Ecr} - \left( \frac{E_c}{Ecr} \right) \right) \left( \frac{A}{E} \right) \left( \frac{A}{E} \right) \]
Last case on p. 39, \( \frac{m}{n} \) S.R.

\[
\left( \frac{x}{y} \right)^{1/2} \left[ \frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right) \right] - \alpha \left( \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

1st case on p. 40, \( \frac{m}{n} \) S.R.

\[
\frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right) + \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

Last case on p. 40, \( \frac{m}{n} \) S.R.

\[
\frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right) - \alpha \left( \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

Case 1 (2nd from top), \( \frac{m}{n} \) S.R.

\[
\frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right) + \alpha \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

2nd case from bottom, \( \frac{m}{n} \) S.R.

\[
\frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} - \frac{E_{0r}}{E_{cr}} \right) + \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

Bottom case, \( \frac{m}{n} \) S.R.

\[
\frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{0r}}{E_{cr}} + \frac{E_{0r}}{E_{cr}} \right) \left( \frac{A_{cr}}{A} \right)
\]

Change \( \alpha \) to read \( \alpha \).

Change \( y \) to \( x \).

Change \( x \) to \( y \).
This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

(Date) Professor Alexis Ostapenko
Professor in Charge

Professor Ferdinand P. Beer
Head of the Department
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9. VITA
AN ANALYTICAL INVESTIGATION OF THE ULTIMATE STRENGTH OF LONGITUDDINALLY STIFFENED PLATE PANELS HAVING LARGE PLATE WIDTH TO THICKNESS RATIOS (PLATE BUCKLES BEFORE THE ULTIMATE AXIAL STRENGTH OF PANEL IS REACHED), AND SUBJECTED TO COMBINED AXIAL AND UNIFORMLY DISTRIBUTED LATERAL LOADS IS PRESENTED. OF INTEREST IN THIS STUDY ARE STIFFENED PANELS AS USED IN SHIP BOTTOM PLATING.

THE EMPHASIS IS PLACED ON THE EFFECT OF LATERAL LOADS, RESIDUAL STRESSES AND B/T RATIO ON THE ULTIMATE AXIAL STRENGTH OF THE PANELS. THE RESULTS WERE OBTAINED NUMERICALLY ON A DIGITAL COMPUTER.

THE FOLLOWING EFFECTS ON THE ULTIMATE AXIAL LOAD WERE ESTABLISHED:

1. LATERAL LOADS CAUSE A SIGNIFICANT REDUCTION.
2. COMPRESSIVE RESIDUAL STRESSES IN THE PLATE HAVE REDUCING INFLUENCE.
3. A LARGE POSTBUCKLING STRENGTH CAN BE EXPECTED FOR PANELS WITH LARGE PLATE WIDTH TO THICKNESS RATIOS.

A COMPARISON BETWEEN THE ANALYTICAL AND TEST RESULTS SHOWS THAT THE METHOD OF ANALYSIS DEVELOPED IN THIS STUDY CAN PREDICT THE ULTIMATE STRENGTH OF STIFFENED PANELS WITH RELATIVELY GOOD ACCURACY.
1. INTRODUCTION

It is well known that a flat plate under axial thrust can carry additional loads after the elastic buckling of the plate by reason of the constraint of the unloaded edges against the movement in the plane and in the lateral direction \(1,2,3\).

Since in the panels used as ship bottom plating the plate represents the predominant portion of the cross-sectional area, the same phenomenon, the so called postbuckling strength, can be expected for the longitudinally stiffened plate panels which are subjected to the axial and uniformly distributed lateral loads.

Test results show that stiffened plate panels with a relatively high plate width to thickness ratio, \(\frac{b}{t} = 60\), collapsed under a higher axial load than the plate buckling load \(4,5,6,7\).

In spite of the importance of predicting the ultimate strength for the design of stiffened plate panels, no analytical work has been done because of the complexity of the stress analysis in the plate after buckling.

Analytical studies on the postbuckling behavior of flat plates under the axial load have been performed by several investigators \(3\). Their results were presented as the relationship between the average stress in the plate and the unloaded edge strain.

By assuming that the stress-strain relationship in the plate after its buckling can be represented by the average stress vs. edge
strain curve of a flat plate, the ultimate strength of stiffened plate panels can be obtained without complex analysis of the stress in the plate. The problem is then reduced to the ultimate strength analysis of the panels which consist of a plate and a stiffener having different material properties. Although many formulas for the stress-strain relationship in the postbuckling range of the flat plate are available, Koiter's formula is the most suitable for this problem.

The set and fulfilled objectives of this study are as follows:

(1) To develop a procedure for the ultimate strength analysis of stiffened plate panels simultaneously subjected to axial and lateral loads.

(2) To evaluate the effect of the lateral load on the ultimate strength.

(3) To evaluate the effect of residual stresses in the plate on the ultimate strength.

(4) To evaluate the effect of the b/t ratio of the plate on the ultimate strength.
2. ASSUMPTIONS

The following assumptions are the basis of this study:

(1) The stress-strain relationship of the stiffener is idealized as shown in Fig. 1-a.

The stress-strain relationship for the plate is shown in Fig. 1-b. The elastic-plastic curve is assumed in tension, and the elastic-non-linear-plastic curve in compression. The non-linear part in this curve is given by Koiter's equation.

(2) The residual stress distribution does not vary along the length.

(3) There are no initial imperfections.

(4) The axial load is applied at the centroid of the cross section, and the lateral loading is applied uniformly on the plate side of the panel.

(5) No strain reversal is assumed to have taken place before reaching the ultimate load of the panel.

(6) Stresses in the plate and the flange of the stiffener are constant through the thickness.

(7) The effect of shear deformation is neglected.

(8) The geometry of the cross-section does not change due to bending and buckling of the plate.

(9) Plane cross-section remains plane after deformation.
3. MOMENT-CURVATURE-AXIAL THRUST RELATIONSHIPS

In the determination of the ultimate load capacity of a beam-column, it is necessary to know the moment versus curvature relationships under an axial load, namely M-Ø-P curves. If the M-Ø-P curves for the given cross-section are available the ultimate axial load for the column of a given length can be obtained from the column curves. These column curves are obtained by numerically integrating the differential equations derived in Art. 4.

The moment-curvature-axial thrust relationship depends on the magnitude of the axial load, the moment, the dimensions of the cross-section, the material properties of the members, namely the yield stresses of the plate and the stiffeners, and distribution and magnitude of residual stresses.

The significant dimensions of a cross-section are shown in Fig. 2. It should be noted that the cross-sections shown represent a typical portion of a stiffened panel as indicated in the inserted sketch.

3.1 Residual Stresses

The idealized residual stress distribution in the plate is assumed as shown in Fig. 3. The tensile residual stress, which is of the same magnitude as the yield stress of the plate, is near the connection of the stiffener. The compressive residual stress is distributed
uniformly in the remaining portion of the plate.

Since these residual stresses should initially be in equilibrium, the width of the tensile stress zone $c$ can be obtained from

$$c = \frac{\sigma_r}{\sigma_r + \sigma_{yp}} b$$

(3.1)

where $\sigma$ is the stress and the subscripts $r$ and $yp$ designate the compressive residual stress and the yield stress of the plate, and $b$ is the width of the plate, that is, the spacing of the stiffeners.

For the convenience of later computations, the equation for $c$ is nondimensionalized, namely

$$\frac{c}{b} = \frac{\sigma_r}{\sigma_{cr}} = \frac{\sigma_{cr}}{\sigma_{cr} + \sigma_{yp}}$$

(3.2)

where $\sigma_{cr}$ is the elastic buckling stress of the plate.

3.2 **Buckling Stress of Plate**

Under the assumption that the plate in one subpanel has no effect on the deformations of the plate in the adjoining subpanel, the plate can be considered to be simply supported along the longitudinal edges, that is, at the connection of the stiffener. This assumption is justified by the fact that plates in neighboring subpanels are expected to buckle in an antisymmetrical mode.

The critical stress of a simply supported plate is given by
where \( E \) is the modulus of elasticity, \( t \) is the thickness of the plate, \( \mu \) is Poisson's ratio, and \( K \) is the plate buckling coefficient which is constant for a given ratio \( L/b \), \( L \) being the length of the plate.

Since the ratio \( L/b \) for longitudinally stiffened panels is usually larger than 3, coefficient \( K \) is approximately equal to 4. Then the critical stress is a function of the ratio \( t/b \) only, namely

\[
\sigma_{cr} = K \frac{\pi^2 E}{12(1-\mu^2)} \left( \frac{t}{b} \right)^2
\]

(3.3)

This equation is applicable only to plates subjected to uniform compression. When residual stresses exist, the stress in the plate is no longer uniform. But since the width \( c/2 \) of the tensile residual stress zones is narrow in comparison with the width of the plate and these zones are at the edges where deflection of the plate is small, the effect of the tensile residual stresses on the buckling stress of the plate can be assumed to be negligible. It is assumed that the externally applied stress at which the plate buckles is reduced by an amount equal to the compressive residual stress.

3.3 Maximum Stress in Plate

The stresses in the plate after buckling are distributed as shown in Fig. 4(10).
Koiter's formula gives in the elastic range the average stress in the plate for a given strain at the connection of the stiffeners. Experiments have shown that it is reasonable to assume that the maximum average stress in the plate is the average stress corresponding to the yield strain at the connection of the stiffener. However, if residual stresses are as shown in Fig. 3, yielding will start at point A rather than at the stiffener. The maximum average stress in the plate under this limitation will be greater than for yielding at the stiffener, and it can be computed from the strain being greater than yield strain at the stiffener.

The stress distribution in the plate is unknown, and it is assumed to be parabolic.

Under this assumption the maximum average stress in the plate can be obtained by solving simultaneously the following two equations: Koiter's equation giving the average stress $\sigma_p/\sigma_{cr}$ as a function of the strain at the connection of the stiffener, and the relationship between the strain at the stiffener and the maximum average stress in the plate as controlled by yielding at point A of Fig. 3.

\[
\left(\frac{\sigma_p}{\sigma_{cr}}\right)_{\text{max}} = 1.2 \left(\frac{\varepsilon_e}{\varepsilon_{cr}}\right)^{0.6} - 0.65 \left(\frac{\varepsilon_e}{\varepsilon_{cr}}\right)^{0.2} + 0.45 \left(\frac{\varepsilon_e}{\varepsilon_{cr}}\right)^{-0.2} \quad (3.5)
\]

\[
\frac{\varepsilon_e}{\varepsilon_{cr}} = \frac{2(\varepsilon_{yp}) + 3 \left(1 - \frac{c}{b}\right)^2 - 1 \left(\frac{\sigma_p}{\sigma_{cr}}\right)_{\text{max}}}{3 \left(1 - \frac{c}{b}\right)^2 - 1} \quad (3.6)
\]
If the plate can not carry any additional load after buckling, the maximum stress in the plate is the buckling stress of the plate.

3.4 Relationship between Moment, Curvature and Thrust

3.4.1 Bending in Negative Direction

The stress distribution in the cross-section for negative bending under axial load and with residual stresses is shown in Fig. 5.

The axial force equilibrium for this distribution of stresses can be written in the following form:

\[
P = AE_{s} - \frac{1}{2} E_{s} (E_{s} - E_{e}) A_{w} - (E_{s} - \sigma_{ys}) A_{f} - (E_{s} - \sigma_{p} + \sigma_{r}) A_{p} \]

\[
- \frac{1}{2} f (E_{s} - \sigma_{ys}) A_{w} - \frac{1}{2} g (\sigma_{ys} + E_{e}) A_{w}
\]

where \( P \) is the axial load,
\( A_{w}, A_{f}, A_{p} \) are the areas of the stiffener web, the stiffener flange, and the plate,
\( E_{s} \) and \( E_{e} \) are the strains in the flange and the plate at the connection of the stiffener,
\( \sigma_{ys} \) is the yield stress of the stiffener,
\( \sigma_{p} \) is the average stress in the plate,
\( \sigma_r \) is the compressive residual stress in the plate, 
f and g are the nondimensionalized yield penetration depths measured from the flange of the stiffener and from the plate, respectively.

The moment about the z-axis can be written,

\[
M = \frac{1}{2} \left( \frac{1}{3} d - \alpha d \right) E \left( \varepsilon_s - \varepsilon_e \right) A_w + \frac{1}{2} \left( d - \alpha d - \frac{1}{3} + d \right) (E \varepsilon_s - \sigma_s) A_w \\
+ \left( d - \alpha d \right) (E \varepsilon_s - \sigma_s) A_t - \alpha d \left( E \varepsilon_s - \sigma_p + \sigma_r \right) A_p \\
- \frac{1}{2} g \left( \alpha d - \frac{1}{3} g d \right) \left( \sigma_s + E \varepsilon_e \right) A_w 
\]

(3.8)

where 
\( d \) = the height of the stiffener, 
\( \alpha \) = the nondimensionalized distance from the plate to the centroid of the cross section. It can be expressed by

\[
\alpha = \frac{A_t}{A} + \frac{1}{2} \left( \frac{A_w}{A} \right) 
\]

(3.9)

where \( A \) is the total area of the cross-section.

The relationship between the strains in the plate and in the flange of the stiffener, and the yield penetration in the stiffener can be given from the cross-sectional geometry by

\[
\varepsilon_s = \varepsilon_e - \phi d 
\]

(3.10)

\[
f d = 1 + \frac{\varepsilon_e - \varepsilon_s}{\varepsilon_s - \varepsilon_e} d 
\]

(3.11)
where $\phi$ is the curvature.

Equations 3.3, 3.4 and 3.5 can be nondimensionalized in the following way,

\[
\frac{\varepsilon_s}{\varepsilon_{cr}} = \frac{\varepsilon_e}{\varepsilon_{cr}} - \frac{1}{\alpha} \left( \frac{\phi}{\phi_{cr}} \right)
\]  
(3.13)

where

$\varepsilon_{cr} =$ the strain corresponding to the plate buckling stress,

$\varepsilon_{cr} = \sigma_{cr}/E$,

$\phi_{cr} =$ the curvature corresponding to the moment

$M_{cr} = \sigma_{cr} s$,

$s =$ the section modulus with respect to the plate.

\[
f = 1 + \frac{\varepsilon_e - \varepsilon_s}{\varepsilon_s - \varepsilon_e}
\]  
(3.14)

\[
g = \frac{\varepsilon_e + \varepsilon_{ys}}{\varepsilon_s - \varepsilon_e}
\]  
(3.15)

Substitution of Eq. 3.7 into Eqs. 3.8 and 3.9 gives

\[
f = 1 - \alpha \left( \frac{\varepsilon_e}{\varepsilon_{cr}} - \frac{\varepsilon_{ys}}{\varepsilon_{cr}} \right) \frac{1}{\phi_{cr}}
\]  
(3.16)

and

\[
g = \alpha \left( \frac{\varepsilon_e}{\varepsilon_{cr}} + \frac{\varepsilon_{ys}}{\varepsilon_{cr}} \right) \frac{1}{\phi_{cr}}
\]  
(3.17)
Similarly, the nondimensionalized equations for the axial force and the moment are

\[
P = \frac{P}{P_{cr}} = \frac{E_s}{E_{cr}} \left( \frac{E_s - E_e}{E_e} \right) \frac{A_r}{A} - \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} \right) \frac{A_f}{A} - \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} + \frac{S_r}{S_{cr}} \right) \frac{A_p}{A}
\]

\[
- \frac{1}{Z} f \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} \right) \frac{A_w}{A} - \frac{1}{Z} g \left( \frac{S_{ys}}{S_{cr}} + \frac{E_s}{E_{cr}} \right) \frac{A_w}{A}
\]

(3.18)

and

\[
M = \frac{M}{M_{cr}} = \frac{1}{(\frac{S}{Ad})} \left[ \frac{1}{Z} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_s}{E_{cr}} - \frac{E_e}{E_e} \right) \frac{A_r}{A} + \frac{1}{Z} f \left( 1 - \alpha - \frac{1}{3} \right) \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} \right) \frac{A_w}{A} \right.
\]

\[
+ \left( 1 - \alpha \right) \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} \right) \frac{A_f}{A} - \alpha \left( \frac{E_s}{E_{cr}} - \frac{S_{ys}}{S_{cr}} + \frac{S_r}{S_{cr}} \right) \frac{A_p}{A}
\]

\[
- \frac{1}{Z} g \left( \alpha - \frac{1}{3} \right) \left( \frac{S_{ys}}{S_{cr}} + \frac{E_s}{E_{cr}} \right) \frac{A_w}{A} \right]
\]

(3.19)

where

\[
P_{cr} = \text{the critical axial load given by } S_{cr} A,
\]

\[
M_{cr} = \text{the critical moment given by } S_{cr} S,
\]

\[
\frac{S}{Ad} = \text{the nondimensionalized section modulus given by}
\]

the following expression:

\[
\frac{S}{Ad} = \frac{1}{3\alpha} \left( \frac{A_f}{A} + 2\alpha - 3\alpha^2 \right)
\]

(3.20)

From Eqs. 3.18 and 3.19 with Eqs. 3.16 and 3.17 the axial force and the moment for each strain state can be obtained in accordance with the following procedure:
(1) When both $E_s$ and $E_e + \frac{\sigma_c}{E}$ are smaller than the yield strain of the stiffener $E_{ys}$, $f$ and $g$ are equal to zero.

(2) When $E_s$ is smaller than $E_{ys}$, the terms containing $\frac{E_s}{E_{cr}}$ and $\frac{E_{ys}}{E_{cr}}$ are equal to zero.

(3) When the stress in the plate $\sigma_p$ is between $\sigma_{cr}$ and the tensile yield stress of the plate $\sigma_{yp}$, the term $\left(\frac{\sigma_p - \sigma_f}{\sigma_{cr}}\right)$ is identical to $\frac{E_e}{E_{cr}}$.

(4) When $E_e$ is larger than $E_{cr}$, $\sigma_p$ is computed from Koiter's formula, Eq. 3.5, using $E_e$.

(5) When $E_e$ is larger than $E_{p_{max}}$, $\sigma_p$ is constant and equal to $\sigma_{p_{max}}$.

An example of the application of this procedure is given below for the strain state, $E_{cr} > E_e + E_r > E_{yp}$, $E_{ys} > E_s > E_{ys}$.

The strain diagram for this case is shown in the first row in Table 1-a.

The equilibrium equation for the axial force is

$$\frac{P_{cr}}{P} = \frac{E_s}{E_{cr}} - \frac{1}{2} \left(\frac{E_s - E_e}{E_c} A_w - \left(\frac{E_s - E_e}{E_{cr}}\right) A_p\right)$$

Substitution of Eq. 3.13 into the above equation gives a simple form for $P/P_{cr}$:

$$\frac{P}{P_{cr}} = \frac{E_e}{E_{cr}} - \frac{q}{\sigma_{cr}}$$

The moment is then
All other strain states for the bending in the negative direction, and the equations for the corresponding axial thrust and moment are listed in Table 1-a.

The ultimate bending moment is computed as the limit of the load carrying capacity of the cross-section under a constant axial load. The following two cases develop depending on the position of the neutral axis and thus, on the dimensions of the cross-section, the properties of the materials, and the magnitude of the axial thrust.

(1) The neutral axis is in the web of the stiffener.

$$\frac{M}{M_{cr}} = \frac{1}{Z} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_t}{E_{cr}} - \frac{E_p}{E_{cr}} \right) \frac{A_w}{A} - \alpha \left( \frac{E_t}{E_{cr}} - \frac{E_p}{E_{cr}} \right) \frac{A_p}{A}$$  \hspace{1cm} (3.23)

where \( n \) is the nondimensionalized distance of the neutral axis from the plate and is defined by

$$n = -\frac{1}{2 (\frac{S_{cr}}{A}) \left( \frac{A_w}{A} \right)} \left[ \frac{P}{P_{cr}} - \left( \frac{S_{cr}}{A} \right) \left( \frac{A_e}{A} \right) + \left( \frac{S_{cr}}{S_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$$  \hspace{1cm} (3.24)

(2) The neutral axis is in the plate.

$$\left( \frac{M}{M_{cr}} \right)_{max} = \frac{\alpha}{S \left( \frac{A}{A} \right)} \left( \frac{S_{cr}}{S_{cr}} \frac{S_{cr}}{S_{cr}} \right) \frac{A_p}{A}$$  \hspace{1cm} (3.25)

where \( \frac{S_{cr}}{S_{cr}} \) is the stress in the plate and is computed from
3.4.2 Bending in Positive Direction

Figure 6 shows the general strain state for the cross-section under a positive moment. By the same procedure as in the previous section, the moment-curvature-axial thrust relationship can be obtained.

The axial force in the nondimensionalized form is

\[
\frac{P}{P_{cr}} = \frac{E_s}{E_{cr}} + \frac{1}{Z} \left( \frac{E_s}{E_{cr}} - \frac{E_s}{E_{cr}} \right) \frac{A_P}{A} - \frac{1}{Z} f \left( \frac{E_s}{E_{cr}} - \frac{E_s}{E_{cr}} \right) \frac{A_{fr}}{A}
\]

\[
+ \left( \frac{\sigma_p}{\sigma_{cr}} - \frac{\sigma_s}{\sigma_{cr}} - \frac{\varepsilon_s}{\varepsilon_{cr}} \right) \frac{A_P}{A} - \frac{1}{Z} g \left( \frac{\varepsilon_s}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\varepsilon_{cr}} \right) \frac{A_{fr}}{A}
\]

The nondimensionalized moment is

\[
\frac{M}{M_{cr}} = \frac{1}{(s/A)} \left[ \frac{1}{Z} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_s}{E_{cr}} - \frac{E_s}{E_{cr}} \right) \frac{A_{fr}}{A} + \frac{1}{Z} f \left( 1 - \frac{1}{3} f \right) \left( \frac{E_s}{E_{cr}} - \frac{E_s}{E_{cr}} \right) \frac{A_{fr}}{A} \right]
\]

\[
+ \alpha \left( \frac{\sigma_p}{\sigma_{cr}} - \frac{\sigma_s}{\sigma_{cr}} - \frac{\varepsilon_s}{\varepsilon_{cr}} \right) A_P + \frac{1}{Z} g \left( \frac{1}{3} - \frac{1}{Z} g \right) \left( \frac{\varepsilon_s}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\varepsilon_{cr}} \right) \frac{A_{fr}}{A}
\]

\[
\cdot \left( 1 - \alpha \right) \left( \frac{E_s}{E_{cr}} - \frac{E_s}{E_{cr}} \right) \frac{A_{fr}}{A} \right]
\]

(3.23)
The yield penetration distances \( f \) and \( g \) are given by

\[
f = 1 - \alpha \left( \frac{\varepsilon_e}{\varepsilon_{cr}} + \frac{\sigma_{ys}}{\sigma_{cr}} \right) \frac{1}{\left( \frac{\sigma}{\sigma_{cr}} \right)}
\]

and

\[
g = \alpha \left( \frac{\varepsilon_e}{\varepsilon_{cr}} - \frac{\sigma_{ys}}{\sigma_{cr}} \right) \frac{1}{\left( \frac{\sigma}{\sigma_{cr}} \right)}
\]

Recalling the procedure described in the previous section, the equations of the equilibrium of the axial force and the moment for each state of strain can be obtained from Eqs. 3.28 and 3.29. These results are listed in Table 1-b.

The maximum bending moment for the positive bending can be considered for the following three cases in accordance with the position of the neutral axis.

1. The neutral axis is in the web of the stiffener. The maximum moment is

\[
\left( \frac{M}{M_{cr}} \right)_{\text{max}} = \frac{1}{Z} \left( \frac{\sigma_{ys}}{\sigma_{cr}} \right) \left[ \frac{P}{P_{cr}} + \frac{A_e}{A} \left( \frac{\sigma_{ys}}{\sigma_{cr}} - \frac{\sigma_{cr} + \sigma_{cr}}{\sigma_{cr}} \right) \right]
\]

where \( \eta \) is the nondimensionalized distance from the plate to the neutral axis and is given by

\[
\eta = \frac{1}{Z} \left( \frac{\sigma_{ys}}{\sigma_{cr}} \right) \left( \frac{P}{P_{cr}} + \frac{A_e}{A} \left( \frac{\sigma_{ys}}{\sigma_{cr}} - \frac{\sigma_{cr} + \sigma_{cr}}{\sigma_{cr}} \right) \right)
\]

2. The neutral axis is in the flange of the stiffener
where \( \frac{\sigma_s}{\sigma_{cr}} \) is the stress in the flange of the stiffener and can be determined from

\[
\frac{\sigma_s}{\sigma_{cr}} = \left( -\frac{\sigma_{yx} A_w}{A} - \left( \frac{\sigma_{pmax}}{\sigma_{cr}} - \frac{\sigma_{r}}{\sigma_{cr}} \right) \frac{A_p}{A} + \frac{P}{P_{cr}} \right) \frac{1}{\left( \frac{A_f}{A} \right)} \tag{3.35}
\]

(3) The neutral axis is in the plate

\[
\left( \frac{M}{M_{cr}} \right)_{max} = \frac{\alpha}{(A_d)} \left( \frac{\sigma_{yx}}{\sigma_{cr}} + \frac{\sigma_{r}}{\sigma_{cr}} \right) \frac{A_p}{A} \tag{3.36}
\]

where

\[
\frac{\sigma_{p}}{\sigma_{cr}} = \left( \left( \frac{\sigma_{yx}}{\sigma_{cr}} \right) \frac{A_w}{A} + \frac{P}{P_{cr}} \right) \frac{1}{\left( \frac{A_f}{A} \right)} \tag{3.37}
\]

3.5 Numerical Examples

3.5.1 Procedure of Computation

The moment versus curvature curves under the given axial load and with residual stresses in the plate can be computed from the equilibrium equations of the moment and the axial force. Since Koiter's equation is used for the stress-strain relation of the plate in post-buckling range, it may be an easy way for the computation of \( M-\psi-P \) curves to start in assuming a strain in the plate.

The steps of the computational procedure are as follows:
(1) Assume a strain in the plate, $\varepsilon_e$.

(2) Compute the average plate stress corresponding to the assumed strain, $\sigma_p$.

(3) Compute the curvature from the equilibrium equation for the axial load,

(4) Compute the moment corresponding to the curvature computed in step (3).

(5) Change the strain in the plate and repeat steps (2) to (4).

(6) After the computed moment becomes larger than 99% of the maximum moment, the moment is assumed to change linearly.

By this procedure 200 points of the $M-\theta$ curve were computed for each of a series of axial loads. Numerical computation was carried out on a GE225 digital computer. The program was written in Fortran language. The computer program is summarized in the form of the block diagram in Fig. 7.

3.5.2. Numerical Values for Example Problem

The dimensions and the material properties of the example panel are chosen the same as in the panels which were tested at Fritz Engineering Laboratory of Lehigh University in 1963 (6).

Dimensions of Cross-Section:

Aspect Ratio of Plate = $\frac{L}{b} = 3.8$
Area of Stiffener \[ \frac{A_s}{A_p} = 0.16 \]

Flange Area of Stiffener \[ \frac{A_t}{A_s} = 0.474 \]

Plate Width to Thickness Ratio \[ \frac{b}{t} = 60 \]

Material Properties:

Yield Stress of Plate \[ \sigma_{yp} = 36.0 \text{ ksi} \]

Yield Stress of Stiffener \[ \sigma_{ys} = 40.0 \text{ ksi} \]

Modulus of Elasticity \[ E = 29.9 \times 10^3 \text{ ksi} \]

Poisson's Ratio \[ \mu = 0.3 \]

Residual Stresses:

Compressive Residual Stress \[ \sigma_r = 4.0 \text{ ksi} \]

Tensile Residual Stress \[ 36.0 \text{ ksi} \]

The following three cases of the stress condition in the plate were considered in this study.

(1) No residual stress in the plate. The maximum stress in the plate is then
   (a) The stress computed from Eqs. 3.5 and 3.6 (with the postbuckling strength),
   (b) The buckling stress of the plate (without the postbuckling strength).
(2) The residual stresses shown in Fig. 4 are in the plate and the maximum stress in the plate is the same as in the case of (1-a).

3.5.3 **Numerical Results**

a. **Moment Capacity of Cross-Section**

The maximum moments for the cross-section under various axial loads are plotted in Fig. 8. In this figure, the solid line represents the case with the postbuckling strength and without residual stresses. The dash-dot line shows the reduction of the moment capacity due to residual stresses. The broken line represents the case without both the postbuckling strength and residual stresses.

Since the cross-section is unsymmetric and the axial load is applied at the centroid of the cross-section, the positive moment capacity of the cross-section increases with the axial load until the stress in the plate becomes a maximum.

For the same reason and owing to the difference in the material properties between the plate and the stiffener, the moment at the maximum axial load is not zero, but has some negative value.

b. **Moment-Curvature-Thrust Curves**

The M-\(\Theta\)-P curves for three cases, which are defined in Art. 3.5.2, are shown in Fig. 9.
The solid line shows the case with the postbuckling strength and without residual stresses. The broken line represents the cross-section without the postbuckling strength and without residual stresses. The dash-dot line shows the case with both the postbuckling strength and residual stresses.

The following characteristics of the M-0-P curves are observed:

(1) The positive moment is maximum under a certain axial load, for this example $P/P_{cr} = 0.8$.

(2) Moment capacities for positive and negative bending are not the same.

(3) The moment at the zero curvature under a large axial load is not zero, but has some value.

The computation of 200 points for the M-0-P curve required about two minutes of computing time.
4. ULTIMATE STRENGTH OF STIFFENED PLATE PANELS

In order to obtain the ultimate strength of stiffened plate panels under combined axial load and uniformly distributed lateral loads, a stepwise integration procedure can be used with given M-ф-P curves.

At the start of this procedure, relationships between forces, moments, deflections and geometry must be established for a small segment cut out from a panel.

Jun Kondo developed these relationships and presented a numerical method for the computation of the ultimate strength of stiffened plate panels in Reference 11. In his method, the maximum length of panels was computed for a given axial and lateral load by a stepwise procedure in which the curvature and the moment at the mid-height of the panels were chosen as initial values in the integration.

The following assumptions are necessary for the derivation of the equilibrium equations and geometrical relationships:

(1) The axial thrust is constant along the length of the stiffener.
(2) The curvature changes linearly in a small segment.
(3) Forces and moments are applied at the centroid of the cross section.

The numerical results are presented in the form of column curves which give the slenderness ratio versus the axial load in Art. 4.2.3. The
effect of the plate width to thickness ratio on the ultimate strength is presented in Art. 4.2.3.

4.1 Relationship between Forces and Deformations

The small segment cut out by two adjacent cross-sections is shown in Fig. 10. Forces, moments, and uniformly distributed lateral load are also shown in this figure.

The curvature can be written as the ratio of the slope change to small segment length.

\[
\phi = \frac{d\theta}{ds} \quad (4.1)
\]

where \( \theta \) = the slope.

If the curvature changes linearly in the small segment, the curvature at any point in this segment can be expressed by

\[
\phi = \phi_i + \frac{\phi_{i+1} - \phi_i}{\Delta s} s \quad (4.2)
\]

where

\( \phi_i, \phi_{i+1} \) = the curvatures at the \( i^{th} \) and the \( i+1^{th} \) cross-sections,

\( \Delta s \) = the length of the segment,

\( s \) = the distance of any point in the segment from \( i^{th} \) cross-section.

The slope is obtained by integrating Eq. 4.1 with respect to \( s \),

\[
\theta = \theta_i + \phi_i s + \frac{1}{2} \frac{\phi_{i+1} - \phi_i}{\Delta s} s^2 \quad (4.3)
\]
The slope at the end of the segment is
\[ \theta_{i+1} = \theta_i + \phi_i \Delta s + \frac{1}{2} (\phi_{i+1} - \phi_i) \Delta s \] (4.5)

The equilibrium conditions can be expressed by

\[ \frac{dV}{ds} = (1 - \phi \alpha d) g b \sin \theta \] (4.6)

\[ \frac{dH}{ds} = (1 - \phi \alpha d) g b \cos \theta \] (4.7)

\[ \frac{dM}{ds} + V \sin \theta + H \cos \theta = 0 \] (4.8)

The cross-sectional forces, V and H, and the moment, M, at any point in the segment can be obtained by integrating Eqs. 4.6, 4.7 and 4.8.

\[ V = V_i + g b (y - y_i) + g b \cdot \alpha d (\cos \theta - \cos \theta_i) \] (4.9)

\[ H = H_i + g b (x - x_i) - g b \cdot \alpha d (\sin \theta - \sin \theta_i) \] (4.10)

\[ M = M_i - V_i (y - y_i) - \frac{gb}{2} (y - y_i)^2 + g b \cdot \alpha d (y - y_i) \cos \theta_i \\
- H_i (x - x_i) - \frac{gb}{2} (x - x_i)^2 + g b \cdot \alpha d (x - x_i) \sin \theta_i \] (4.11)

In the above integrations, the following relationships are used

\[ \frac{dy}{ds} = \sin \theta \] (4.12)

\[ \frac{dx}{ds} = \cos \theta \] (4.13)
At the end of the segment (the \(i+1^{th}\) cross-section) these forces and the moment can be written as

\[
V_{i+1} = V_i + gb \cdot \Delta y + gb \cdot \alpha d \left( \cos \theta_{i+1} - \cos \theta_i \right) \tag{4.14}
\]

\[
H_{i+1} = H_i + gb \cdot \Delta x + gb \cdot \alpha d \left( \sin \theta_{i+1} - \sin \theta_i \right) \tag{4.15}
\]

\[
M_{i+1} = M_i - V_i \Delta y - H_i \Delta x - gb \left( \frac{1}{2} (\Delta y)^2 + \frac{1}{2} (\Delta x)^2 \right) - \alpha d \cdot (\Delta y) \cos \theta_i + \alpha d \cdot (\Delta x) \sin \theta_i \tag{4.16}
\]

where

\(\Delta x, \Delta y = \) linear components of the length of the segment

\(\Delta s\) in x and y directions,

\(q = \) the uniformly distributed lateral load,

\(b = \) the width of the plate.

Considering large deflections \(\Delta x\) and \(\Delta y\) can be written from the geometry of the deflection curve in the following form.

\[
\Delta x = \cos \theta_i \cdot (\Delta S) - \left( \frac{\phi_i}{3} + \frac{\phi_{ii}}{6} \right) \sin \theta_i \cdot (\Delta S)^2 \tag{4.17}
\]

\[
\Delta y = \sin \theta_i \cdot (\Delta S) - \left( \frac{\phi_{ii}}{3} + \frac{\phi_{iii}}{6} \right) \cos \theta_i \cdot (\Delta S)^2 \tag{4.18}
\]

The length along the centroidal line is simply a summation of the lengths of the segments and can be written

\[
L = \sum_j \Delta S \tag{4.19}
\]

where

\( j = \) the number of segments.
For the convenience of computation and discussion of the results, Eqs. 4.3, 4.5 and 4.14 to 4.19 can be nondimensionalized in the following way:

The slope at any point in the segment is

$$
\theta = \theta_i + \left( \frac{\phi_i}{\phi_{cr}} \right) \left( \frac{S}{r} \right) \left( \frac{r}{\alpha d} \right) E_{cr}
$$

$$
+ \frac{1}{2} \left[ \left( \frac{\phi_{i+1}}{\phi_{cr}} \right) - \left( \frac{\phi_i}{\phi_{cr}} \right) \right] \left( \frac{S}{r} \right) \left( \frac{\alpha d}{r} \right) E_{cr} \tag{4.20}
$$

The slope at the \( i+1 \)th cross-section is

$$
\theta_{i+1} = \theta_i + \frac{1}{2} \left[ \left( \frac{\phi_{i+1}}{\phi_{cr}} \right) + \left( \frac{\phi_i}{\phi_{cr}} \right) \right] \left( \frac{S}{r} \right) \left( \frac{r}{\alpha d} \right) E_{cr} \tag{4.21}
$$

The vertical and horizontal forces at the \( i+1 \)th cross-section are:

$$
\frac{V_{i+1}}{G_{cr} A r} = \frac{V_i}{G_{cr} A r} + \left( \frac{gb \alpha d}{G_{cr} A} \right) \left( \frac{4 \chi}{r} \right)
$$

$$
+ \left( \frac{gb \alpha d}{G_{cr} A} \right) \left( \frac{\alpha d}{r} \right) \left( \cos \theta_{i+1} - \cos \theta_i \right) \tag{4.22}
$$

and

$$
\frac{H_{i+1}}{G_{cr} A r} = \frac{H_i}{G_{cr} A r} + \left( \frac{gb \alpha d}{G_{cr} A} \right) \left( \frac{4 \chi}{r} \right)
$$

$$
- \left( \frac{gb \alpha d}{G_{cr} A} \right) \left( \frac{\alpha d}{r} \right) \left( \sin \theta_{i+1} - \sin \theta_i \right) \tag{4.23}
$$
The moment at the \(i+1\)th cross-section is
\[
\frac{M_{i+1}}{M_{cr}} = \frac{M_i}{M_{cr}} - \left( \frac{V_i \alpha_d}{\sigma_{cr} A r} \right) \left( \frac{4y}{r} \right) - \left( \frac{H_i \alpha_d}{\sigma_{cr} A r} \right) \left( \frac{4x}{r} \right) \\
- \left( \frac{g \alpha_d}{\sigma_{cr} A} \right) \left\{ \frac{v}{2} \left( \frac{4y}{r} \right)^2 + \frac{v}{2} \left( \frac{4x}{r} \right)^2 \right\} \\
- \left( \frac{\alpha_d}{r} \right) \left[ \left( \frac{4y}{r} \right) \cos \theta_i - \left( \frac{4x}{r} \right) \sin \theta_i \right] \right\} \tag{4.24}
\]

The \(x, y\) components of \(\Delta s\) are
\[
\frac{\Delta x}{r} = \frac{4s}{r} \cos \theta_i - \left[ \frac{1}{3} \left( \frac{\phi_i}{\alpha_d} \right) + \frac{1}{6} \left( \frac{\phi_{ii}}{\alpha_d} \right) \right] \left( \frac{4s}{r} \right)^2 \left( \frac{r}{\alpha_d} \right) \delta \tag{4.25}
\]
\[
\frac{\Delta y}{r} = \frac{4s}{r} \sin \theta_i + \left[ \frac{1}{3} \left( \frac{\phi_i}{\alpha_d} \right) + \frac{1}{6} \left( \frac{\phi_{ii}}{\alpha_d} \right) \right] \left( \frac{4s}{r} \right)^2 \left( \frac{r}{\alpha_d} \right) \delta \tag{4.26}
\]

where \(r\) is the radius of gyration about \(z\) axis shown in Fig. 2.

The nondimensionalized radius of gyration \(r/\alpha_d\) can be established from the geometry of the cross section by
\[
\frac{r}{\alpha_d} = \frac{1}{\alpha} \sqrt{\left( 1 - 2\alpha \right) \frac{A_f}{A} + \left( \frac{1}{3} - \alpha \right) \frac{A_r}{A} + \alpha^2} \tag{4.27}
\]

The length of the plate panel is given by
\[
\frac{L}{r} = \sum_j \frac{\Delta s}{r} \tag{4.28}
\]

The deflection at the mid-height and the chord length of the panel are
\[
\frac{\bar{y}}{r} = \sum_j \frac{\Delta y}{r} \tag{4.29}
\]
4.2 Numerical Examples

4.2.1 Procedure of Computation

The stepwise procedure is used to compute the ultimate length of the panels.

Since the lateral load is applied uniformly and the deflection of the panel can be assumed to be symmetric as shown in Fig. 11, the shear force and the slope at the mid-height of the panel are equal to zero. Therefore, if the starting point for the computation is chosen at the mid-height of the panel, the curvature and the moment at this point can be taken as the initial values in the computation.

The steps of the computational procedure for a given axial thrust and lateral loading are as follows:

(1) Assume the curvature at the mid-height of the panel and read the corresponding moment from the M-Ø-P curve which was computed in Art. 3.

(2) Select a segment length.

(3) Assume the curvature at the next point.

(4) Compute the slope at the next point, x and y components of the segment length from Eqs. 4.21, 4.26, and 4.25. Compute the deflection at the preceding point of the panel and the chord length from Eqs. 4.25 and 4.26.
(5) Compute the horizontal and vertical forces and the moment at this point from Eqs. 4.22, 4.23 and 4.24 with \( \Delta x/r \) and \( \Delta y/r \) computed in step (3).

(6) Read the curvature corresponding to the moment in step (4) from the M-\( \theta \)-P curve.

(7) If the difference between the computed curvature and the curvature assumed in step (2) is larger than a certain value, reassume the curvature and repeat steps (3) to (7). When this difference becomes small, repeat steps (2) to (7) to obtain the next point.

(8) Compute the segment length so that the zero moment is at the end of this segment. Compute the slope and forces at this point as the boundary values for the pinned-ends shown in Fig. 12. Compute the deflection at mid-height, the chord length and the total length of the panel.

(9) Continue the computation of steps (2) to (6) until the slope computed in step (3) changes its sign.

(10) Compute the segment length so that the slope at the end of this segment is zero. This corresponds to the fixed-end condition shown in Fig. 11. Compute the moment and forces at the fixed-end. Compute the deflection at mid-height, the chord length and the total length corresponding to the fixed condition.

(11) Change the curvature at the mid-height of the panel and
repeat steps (2) to (9) until the value of the length computed in step (9) begins to decrease.

(12) Compute the maximum length for both pinned-end and fixed-end conditions. The maximum length is obtained by maximizing the quadratic function which satisfies three sets of points \( \left( \frac{L}{r}, \left( \frac{\theta}{\theta_{cr}} \right) \right) \) near the ultimate point, where \( j = 1, 2, \) and 3 as shown in Fig. 12.

(13) Plot the curves of \( P/P_{cr} \) versus \( L/r \).

Numerical computation was carried out on the digital computer GE225. The program was written in Fortran language. This program is summarized in the form of a block diagram in Fig. 13. About one minute was required for the computation of the maximum slenderness ratio for a given set of axial and lateral loads. The slenderness ratios for the pinned and the fixed conditions were computed at the same time.

4.2.2 Numerical Values for Example Problem

The dimensions of the panel, material properties, the residual stress pattern are the same as in the examples of Art. 3.5.2.

In order to evaluate the effect of \( b/t \) ratio, three additional values of \( b/t \), 50, 75 and 90, were used beside the value of 60 of the example. Three intensities of lateral load were considered in this study, namely, 6.5 psi, 13.0 psi, and 20.0 psi.
4.2.3 Numerical Results of Computation

The maximum slenderness ratios for given axial loads were computed. The results are tabulated in Table 2-a for the panels simply supported at the loaded edges and in Table 2-b for the fixed-end panels. These results are plotted in Figs. 14 and 15, where the curves are for the ultimate load versus the slenderness ratio of the panels.

It is interesting to observe in Fig. 14 that the effect of the axial load on the ultimate slenderness ratio of the pinned-end panels is quite insignificant for axial loads smaller than about $P/P_{cr} = 0.75$. For the fixed-end panels (see Fig. 15), however, this effect is cancelled by the negative moment near the fixed ends. Therefore, the slenderness ratio decreases quite noticeably with an increasing axial load.

The reduction of the ultimate load due to residual stresses is shown in Fig. 14 and 15 by broken lines. This reduction is slightly larger in panels with a small slenderness ratio.

Figures 14 and 15 show also the effect of the lateral load on the ultimate load of the panels. There are three sets of curves for the three intensities of lateral load, 6.5 psi, 13.0 psi and 20.0 psi.

The maximum slenderness ratios were computed for four values of $b/t$ under a constant lateral load $q = 6.5$ psi. In this computation residual stresses were neglected. The results are given in Table 3 and plotted in Fig. 16 and 17 as curves of ultimate axial load versus $b/t$ ratio. The solid line shows the ultimate load for panels including the postbuckling strength and the broken line represents the ultimate load.
without the postbuckling strength.

4.3 **Comparison with Test Results**

The test results, which were obtained at Fritz Engineering Laboratory of Lehigh University in 1963, are plotted in Fig. 15. Point numbers correspond to the test specimen numbers of Reference 6. The slenderness ratios of these specimens were computed for the cross-section which consists of a plate and a stiffener as shown in Fig. 2, not for the total cross-section. Specimen T-13 was tested under 6.5 psi lateral load and T14 under 13.0 psi.

Since T13 had some initial eccentricity so that the initial deflection due to lateral load was almost zero, the specimen failed under an axial load which was too high. This is reflected in the plot by the position of the T13 point relative to the broken line for q = 6.5 psi.

On the other hand, the test result of T14 shows a good correlation between the analytical result and the test.
5. CONCLUSIONS

5.1 Conclusions

The following conclusions can be drawn from the results of this study:

(1) The comparison of the test results with the computed curves shows a relatively good correlation. This indicates that the procedure is sufficient for design purposes.

(2) Figures 14 and 15 show a large reduction of the ultimate strength of the longitudinally stiffened plate panels due to uniformly distributed lateral load.

(3) A large reduction of the ultimate strength takes place due to residual stresses, and this reduction is slightly larger for panels with small slenderness ratios.

(4) Figures 16 and 17 show that the panel with a large b/t ratio has a large postbuckling strength.

5.2 Recommendations for Future Research

As an extension of this study, the numerical analysis should be done for other panels with dimensions in the range of practical ship bottom plating. Results of such an analysis should be summarized in the form of design nomographs analogous to the nomographs developed in Ref.11.
To improve the analytical method it is important to study the effect of the following factors on the ultimate strength of longitudinally stiffened plate panels:

(1) Residual stresses in the stiffener.

(2) Change of the cross-sectional shape due to buckling of the plate.

(3) Initial imperfections.

Experimental studies may be desirable to verify analytical results.
6. NOMENCLATURE

A \quad \text{total area of the cross-section}

A_f \quad \text{flange area of the stiffener}

A_p \quad \text{plate area, } A_p = bt

A_s \quad \text{area of the stiffener}

A_w \quad \text{web area of the stiffener}

b \quad \text{width of the plate}

c \quad \text{width of the zone of the tensile residual stress}

d \quad \text{depth of the stiffener}

E \quad \text{modulus of elasticity}

f \quad \text{nondimensionalized yield penetrating distance from the flange of the stiffener}

g \quad \text{nondimensionalized yield penetrating distance from the plate}

H \quad \text{horizontal force}

H_i \quad \text{horizontal force at the } i^{\text{th}} \text{ section}

H_{i+1} \quad \text{horizontal force at the } i+1^{\text{th}} \text{ section}

K \quad \text{plate buckling coefficient}

L \quad \text{length of the panel}

M \quad \text{moment at any point}

M_{cr} \quad \text{moment expressed as } \sigma_{cr} S

M_i \quad \text{moment at the } i^{\text{th}} \text{ section}

M_{i+1} \quad \text{moment at the } i+1^{\text{th}} \text{ section}

n \quad \text{nondimensionalized distance from the plate to the neutral axis}

P \quad \text{axial thrust}
$P_{cr}$ axial thrust given as $A \sigma_{cr}$
$q$ uniformly distributed lateral load
$r$ radius of gyration about z axis
$s$ length along the centroidal line
$\Delta s$ length of the segment
$S$ section modulus with reference to the plate
$t$ thickness of the plate
$V$ vertical force
$V_i$ vertical force at the $i^{th}$ section
$V_{i+1}$ vertical force at the $i+1^{th}$ section
$x$ coordinate $x$ at any point
$x_i$ coordinate $x$ at the $i^{th}$ section
$y$ coordinate $y$ at any point
$y_i$ coordinate $y$ at the $i^{th}$ section
$\alpha$ nondimensionalized distance between the plate and the centroid of the cross-section.
$\varepsilon_{cr}$ strain in the plate at the buckling load
$\varepsilon_e$ strain in the plate at the connection to the stiffener
$\varepsilon_s$ strain in the flange of the stiffener
$\theta$ slope at any point
$\theta_i$ slope at the $i^{th}$ section
$\theta_{i+1}$ slope at the $i+1^{th}$ section
$\mu$ Poisson's ratio
$\sigma_{cr}$ buckling stress of the plate
$\sigma_p$ average stress in the plate
\( \sigma_{p\text{max}} \) maximum average stress in the plate
\( \sigma_{yp} \) yield stress of the plate
\( \sigma_{ys} \) yield stress of the stiffener
\( \sigma_r \) compressive residual stress in the plate
\( \phi \) curvature at any point
\( \phi_{cr} \) curvature given as \( \frac{\varepsilon_r}{\alpha d} \)
\( \phi_i \) curvature at the \( i^{th} \) section
\( \phi_{i+1} \) curvature at the \( i+1^{th} \) section
7. TABLES AND FIGURES
## Table 1-a: Moment-Curvature-Thrust Relationships (Negative Bending)

<table>
<thead>
<tr>
<th>Strain Diagram</th>
<th>Limit</th>
<th>Axial Force $P/P_{cr}$</th>
<th>Moment $M/M_{cr}$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_{cr} &gt; \varepsilon_{e} + \varepsilon_{t} &gt; \varepsilon_{p}$</td>
<td>$\frac{P}{P_{cr}} = \frac{\phi}{\phi_{cr}}$</td>
<td>$\frac{1}{(S_{Ad})} \left[ \frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{cr}}{A} \right) - \alpha \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{p}}{A} \right) \right]$</td>
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<tr>
<td></td>
<td>$\varepsilon_{ys} &gt; \varepsilon_{s} &gt; 0$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_{cr} &gt; \varepsilon_{e} + \varepsilon_{t} &gt; \varepsilon_{p}$</td>
<td>$\frac{E_{e}}{E_{cr}} - \frac{\phi}{\phi_{cr}} - \left( \frac{E_{e} - \sigma_{ys} - \sigma_{t}}{\sigma_{cr} + \sigma_{cr}} \right) \left( \frac{A_{p}}{A} \right)$</td>
<td>$\frac{1}{(S_{Ad})} \left[ \frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{cr}}{A} \right) + \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{p}}{A} \right) \right]$</td>
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<tr>
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<td>$\varepsilon_{ys} &gt; \varepsilon_{s} &gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_{max} &gt; \varepsilon_{e} + \varepsilon_{t} &gt; \varepsilon_{cr}$</td>
<td>$\frac{E_{e}}{E_{cr}} - \frac{\phi}{\phi_{cr}} - \left( \frac{E_{e} - \sigma_{ys} - \sigma_{t}}{\sigma_{cr} + \sigma_{cr}} \right) \left( \frac{A_{p}}{A} \right)$</td>
<td>$\frac{1}{(S_{Ad})} \left[ \frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{cr}}{A} \right) - \alpha \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{p}}{A} + \frac{\sigma_{ys} + \sigma_{t}}{\sigma_{cr} + \sigma_{cr}} \right) \left( \frac{A_{p}}{A} \right) \right]$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_{ys} &gt; \varepsilon_{s} &gt; 0$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_{max} &gt; \varepsilon_{e} + \varepsilon_{t} &gt; \varepsilon_{cr}$</td>
<td>$\frac{E_{e}}{E_{cr}} - \frac{\phi}{\phi_{cr}} - \left( \frac{E_{e} - \sigma_{ys} - \sigma_{t}}{\sigma_{cr} + \sigma_{cr}} \right) \left( \frac{A_{p}}{A} \right)$</td>
<td>$\frac{1}{(S_{Ad})} \left[ \frac{1}{2} \left( \frac{1}{3} - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{cr}}{A} \right) + \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_{cr}}{E_{cr} - E_{e}} \right) \left( \frac{A_{p}}{A} \right) \right]$</td>
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<td></td>
<td>$\varepsilon_{ys} &gt; \varepsilon_{s} &gt; 0$</td>
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TABLE 1-a  MOMENT-CURVATURE-THRUST RELATIONSHIPS (NEGATIVE BENDING) - Cont.

<table>
<thead>
<tr>
<th>Strain Diagram</th>
<th>Limit</th>
<th>Axial Force P/P&lt;sub&gt;cr&lt;/sub&gt;</th>
<th>Moment M/M&lt;sub&gt;cr&lt;/sub&gt;</th>
</tr>
</thead>
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<tr>
<td><img src="image1" alt="Strain Diagram 1" /></td>
<td>( \varepsilon_g &gt; \varepsilon_y + \varepsilon_r &gt; -\varepsilon_g )</td>
<td>( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{(\frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}})(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
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<tr>
<td></td>
<td>( \varepsilon_s &gt; \varepsilon_y )</td>
<td>( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{(\frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}})(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
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<td><img src="image2" alt="Strain Diagram 2" /></td>
<td>( \varepsilon_g &gt; \varepsilon_y + \varepsilon_r &gt; -\varepsilon_g )</td>
<td>( \frac{-\alpha \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right)(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
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<td>( \varepsilon_s &gt; \varepsilon_y )</td>
<td>( \frac{-\alpha \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right)(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
</tr>
<tr>
<td><img src="image3" alt="Strain Diagram 3" /></td>
<td>( \varepsilon_g &gt; \varepsilon_y + \varepsilon_r &gt; -\varepsilon_g )</td>
<td>( \frac{\varepsilon_r}{\varepsilon_{cr}} - \frac{(\frac{\varepsilon_r}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}})(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
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<td>( \varepsilon_s &gt; \varepsilon_y )</td>
<td>( \frac{\varepsilon_r}{\varepsilon_{cr}} - \frac{(\frac{\varepsilon_r}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}})(A_r + A_s)}{A} )</td>
<td>( \frac{1}{2} \left[ 1 - \frac{1}{3} - \alpha \right] \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) + \frac{1}{2} \left( 1 - \alpha - \frac{1}{3} - \alpha \right) \left( \frac{\varepsilon_y}{\varepsilon_{cr}} - \frac{\varepsilon_s}{\sigma_{cr}} \right) \left( \frac{A_r}{A} \right) )</td>
</tr>
<tr>
<td>Strain Diagram</td>
<td>Limit</td>
<td>Axial Thrust $P/P_{cr}$</td>
<td>Moment $M/M_{cr}$</td>
</tr>
<tr>
<td>----------------</td>
<td>-------</td>
<td>------------------------</td>
<td>------------------</td>
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<tr>
<td>$E_e &gt; E_s + E_r &gt; 0$</td>
<td>$\frac{E_e}{E_{cr}} - \frac{\phi}{\phi_{cr}}$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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<tr>
<td>$-E_{gs} &gt; E_s$</td>
<td>$\frac{E_e}{E_{cr}} - \left( \frac{E_e + E_{gs}}{E_{cr}} \right) \left( \frac{A_r}{A} + \frac{A_p}{A} \right)$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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<tr>
<td>$E_s &gt; E_e + E_r &gt; 0$</td>
<td>$\frac{E_e}{E_{cr}} - \left( \frac{E_e + E_{gs}}{E_{cr}} \right) \left( \frac{A_r}{A} + \frac{A_p}{A} \right) + \frac{\alpha}{2} \left( \frac{E_e}{E_{cr}} + \frac{E_{gs}}{E_{cr}} \right)^2 \left( \frac{A_r}{A} \right) \frac{1}{\phi_{cr}}$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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<td>$-E_{gs} &gt; E_s$</td>
<td>$\frac{E_e}{E_{cr}} - \left( \frac{E_e + E_{gs}}{E_{cr}} \right) \left( \frac{A_r}{A} + \frac{A_p}{A} \right) + \frac{\alpha}{2} \left( \frac{E_e}{E_{cr}} + \frac{E_{gs}}{E_{cr}} \right)^2 \left( \frac{A_r}{A} \right) \frac{1}{\phi_{cr}}$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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<tr>
<td>$E_e &gt; E_s + E_r &gt; 0$</td>
<td>$\frac{E_e}{E_{cr}} - \left( \frac{E_e + E_{gs}}{E_{cr}} \right) \left( \frac{A_r}{A} + \frac{A_p}{A} \right) + \frac{\alpha}{2} \left( \frac{E_e}{E_{cr}} + \frac{E_{gs}}{E_{cr}} \right)^2 \left( \frac{A_r}{A} \right) \frac{1}{\phi_{cr}}$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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<td>$-E_{gs} &gt; E_s$</td>
<td>$\frac{E_e}{E_{cr}} - \left( \frac{E_e + E_{gs}}{E_{cr}} \right) \left( \frac{A_r}{A} + \frac{A_p}{A} \right) + \frac{\alpha}{2} \left( \frac{E_e}{E_{cr}} + \frac{E_{gs}}{E_{cr}} \right)^2 \left( \frac{A_r}{A} \right) \frac{1}{\phi_{cr}}$</td>
<td>$\frac{1}{(\frac{E}{E_{cr}})} \left[ \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_r}{A} \right) + \alpha \left( \frac{E_e}{E_{cr}} - \frac{E_r}{E_{cr}} \right) \left( \frac{A_p}{A} \right) \right]$</td>
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### TABLE 1-b  MOMENT-CURVATURE-THRUST RELATIONSHIPS (POSITIVE BENDING) - Cont.

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<th>Strain Diagram</th>
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<th>Axial Thrust ( P/P_{cr} )</th>
<th>Moment ( M/M_{cr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_{e} + \varepsilon_{r} &gt; \varepsilon_{y} )</td>
<td>( \frac{\varepsilon_{e}}{\varepsilon_{cr}} - \left( \frac{\varepsilon_{e} + \varepsilon_{y}}{\varepsilon_{cr}} \right) \left( \frac{A_{f} + A_{t}}{A} \right) )</td>
<td>( \frac{1}{2} \left[ -\frac{1}{2} \left( 1 - \alpha \right) \left( \varepsilon_{e} - \varepsilon_{s} \right) \left( \frac{A_{f}}{A} \right) + \frac{1}{2} \left( 1 - \alpha \right) \left( \varepsilon_{s} + \varepsilon_{cr} \right) \left( \frac{A_{f}}{A} \right) \right] )</td>
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</tr>
<tr>
<td>( - \varepsilon_{y} &gt; \varepsilon_{s} )</td>
<td>( \frac{\sigma_{e} - \varepsilon_{e} - \varepsilon_{s}}{\sigma_{cr}} \left( \frac{A_{f} + A_{t}}{A} \right) + 2\alpha \left( \frac{\sigma_{s}}{\sigma_{cr}} \right) \left( \frac{A_{f}}{A} \right) \left( \frac{\varepsilon_{e}}{\varepsilon_{cr}} \right) )</td>
<td>( \left[ 1 - \alpha \right] \left( \frac{\varepsilon_{e} + \varepsilon_{s}}{\varepsilon_{cr}} \right) \left( \frac{A_{f}}{A} \right) + \alpha \left( \frac{\sigma_{e} - \varepsilon_{e} - \varepsilon_{s}}{\sigma_{cr}} \right) \left( \frac{A_{f}}{A} \right) - \frac{1}{2} \alpha \left( \frac{\sigma_{e} - \varepsilon_{e} - \varepsilon_{s}}{\sigma_{cr}} \right) \left( \frac{A_{f}}{A} \right) )</td>
<td></td>
</tr>
</tbody>
</table>

(1) \( \frac{\varepsilon_{e}}{\varepsilon_{cr}} = \frac{\varepsilon_{e}}{\varepsilon_{cr}} - \frac{1}{\alpha} \left( \frac{\phi}{\phi_{cr}} \right) \)

(2) For negative bending \( f = 1 - \alpha \left( \frac{\varepsilon_{e}}{\varepsilon_{cr}} - \frac{\varepsilon_{s}}{\varepsilon_{cr}} \right) \left( \frac{1}{\phi_{cr}} \right) \)

(3) For negative bending \( g = \alpha \left( \frac{\varepsilon_{e}}{\varepsilon_{cr}} + \frac{\varepsilon_{s}}{\varepsilon_{cr}} \right) \left( \frac{1}{\phi_{cr}} \right) \)

(4) For positive bending \( f = 1 - \alpha \left( \frac{\varepsilon_{e}}{\varepsilon_{cr}} + \frac{\varepsilon_{s}}{\varepsilon_{cr}} \right) \left( \frac{1}{\phi_{cr}} \right) \)

(5) For positive bending \( g = \alpha \left( \frac{\varepsilon_{e}}{\varepsilon_{cr}} + \frac{\varepsilon_{s}}{\varepsilon_{cr}} \right) \left( \frac{1}{\phi_{cr}} \right) \)
TABLE 2. EFFECT OF LATERAL LOADING AND RESIDUAL STRESS ($\frac{b}{L} = 60$)

a. PINNED-ENDS AT LOADED EDGES

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<th>20.0</th>
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<tr>
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<td>0.133</td>
<td>0.133</td>
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<td>P/\sigma_{cr}</td>
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<td>29.67</td>
<td>22.80</td>
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<td>39.28</td>
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b. FIXED ENDS AT LOADED EDGES

<table>
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<tr>
<th></th>
<th>6.5</th>
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<td>Q</td>
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<td>P/\sigma_{cr}</td>
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<td>65.59</td>
<td>49.71</td>
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TABLE 3. EFFECT OF $\frac{b}{t}$ RATIO.  $q = 6.5$ psi

### a. PINNED ENDS AT LOADED EDGES

<table>
<thead>
<tr>
<th>$\frac{b}{t}$</th>
<th>P.B.S.</th>
<th>50</th>
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<th>75</th>
<th>90</th>
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</thead>
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<td>$\frac{P}{P_{cr}}$</td>
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<td>W.O.</td>
<td>W.</td>
<td>W.O.</td>
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<tr>
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<td>-</td>
<td>-</td>
<td>-</td>
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<td>1.60</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>43.39</td>
</tr>
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<td>1.40</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>22.86</td>
<td>-</td>
</tr>
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<td>-</td>
<td>-</td>
<td>54.39</td>
<td>-</td>
</tr>
<tr>
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<td>20.86</td>
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<td>25.50</td>
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<td>0.80</td>
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### b. FIXED ENDS AT LOADED EDGES

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<th>P.B.S.</th>
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<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
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<td>W.</td>
<td>W.O.</td>
</tr>
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</tr>
<tr>
<td>1.60</td>
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<tr>
<td>1.40</td>
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<td>-</td>
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<td>80.29</td>
<td>85.35</td>
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</table>

P.B.S. = Post-Buckling Strength  
W = With  
WO = Without
Fig. 1 STRESS STRAIN RELATIONSHIPS
Fig. 2 TYPICAL CROSS SECTION RELATIONSHIP

Fig. 3 IDEALIZED RESIDUAL STRESS PATTERN
Fig. 4 STRESS DISTRIBUTION IN PLATE AFTER BUCKLING

Fig. 5 STRESS DIAGRAM FOR NEGATIVE BENDING

Fig. 6 STRESS DIAGRAM FOR POSITIVE BENDING
Fig. 7 BLOCK DIAGRAM FOR M-Φ-P
Fig. 8  MAXIMUM MOMENT VS. AXIAL THRUST CURVE FOR EXAMPLE CROSS-SECTION (b/t = 60)
Fig. 9 MOMENT CURVATURE THRUST CURVE
Fig. 10  FORCES AND MOMENT ON SEGMENT
Fig. 11 DEFLECTION SHAPE OF PANEL

Fig. 12 MIDHEIGHT CURVATURE VS. SLENDERNESS RATIO CURVE
Fig. 13 BLOCK DIAGRAM FOR COLUMN CURVE
Fig. 13-a NOTATION FOR BLOCK DIAGRAM FOR COLUMN CURVE COMPUTATION

$L_{tn}, L_{tn-1}$: Length of panels for fixed-ends
$L_{sn}, L_{sn-1}$: Length of panels for pinned-ends
$M_{Mr}$: Moment
$\left(\frac{M}{Mr}\right)_o$: Moment at mid-height of panels
$\left(\frac{M}{Mr}\right)_{f,n}$: Fixed-end moment
$\left(\frac{M}{Mr}\right)_{max}$: Maximum fixed-end moment
$n$: Integer
$p_{Mr}$: Axial load
$Q$: Lateral load
$\Delta Q$: Lateral load increment
$\Delta S$: Segment length increment
$X_{fn}$: Chord length for fixed-end panels
$X_{f max}$: Maximum chord length for fixed-end panels
$X_{sn}$: Chord length for pinned-end panels
$X_{s max}$: Maximum chord length for pinned-end panels
$\Delta X$: x component of $\Delta S$
$\gamma_{fn}$: Deflection at mid-height for fixed-end panels
$\gamma_{f max}$: Maximum deflection at mid-height for fixed-end panels
$\gamma_{sn}$: Deflection at mid-height for pinned-end panels
$\gamma_{s max}$: Maximum deflection at mid-height for pinned-end panels
$\Delta Y$: y component of $\Delta S$
$\theta$: Rotation
\( \theta_{en} \) : Rotation at simply supported edge

\( \theta_{s_{max}} \) : Maximum rotation at simply supported edge

\( \frac{\phi}{\alpha_{cr}} \) : Curvature

\( (\frac{\phi}{\alpha_{cr}})_{p} \) : Curvature at mid-height of panels

\( (\frac{\phi}{\alpha_{cr}})_{sn} \) : Curvature at simply supported edge

\( (\frac{\phi}{\alpha_{cr}})_{s_{max}} \) : Curvature at simply supported edge corresponding to \( \theta_{s_{max}} \)

\( (\frac{\phi}{\alpha_{cr}})_{f_{n}} \) : Curvature at fixed edge

\( (\frac{\phi}{\alpha_{cr}})_{f_{max}} \) : Curvature at fixed edge corresponding to \( (\frac{M}{M_{cr}})_{f_{max}} \)
Fig. 14 ULTIMATE STRENGTH CURVE FOR EXAMPLE PANELS
(PINNED ENDS AT LOADED EDGES b/t = 60)
Fig. 15 ULTIMATE STRENGTH CURVE FOR EXAMPLE PANELS
(FIXED ENDS AT LOADED EDGES b/t = 60)
Fig. 16 EFFECT OF $b/t$ RATIO ON ULTIMATE STRENGTH
(PINNED ENDS AT LOADED EDGES $q = 6.5$ psi)
Fig. 17  EFFECT OF b/t RATIO ON ULTIMATE STRENGTH
(FIXED ENDS AT LOADED EDGES q = 6.5 psi)
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9. VITA

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