A. INTRODUCTION

The use of influence lines for the design of bridges subjected to live loads has become a standard practice, even to the extent that no other method is accepted.

The influence lines allow the determination of the maximum moment, shearing force, axial load, etc. for a given section in a bridge member under live loads. A logical extension of this method to the design of bridge slabs is the development of influence surfaces (two-dimensional influence lines). They allow to determine the maximum moment (and shearing force, twisting moments etc., if desired) at a given point of the slab subjected to concentrated wheel loads. The proper detailing of the slab can be readily handled, once the extreme moment values are known. It is quite evident that use of influence surfaces to determine the maximum bending moments in the slab will lead to more economical designs than the present semi-empirical rules.

B. ENGINEERING CONCEPT OF INFLUENCE SURFACES

(1) Mathematical Theory of Influence Lines
(Theory of Green's Functions of Beams)

(a) Simply Supported Beam    (Figure 1)

(i) Deflection $G(u,x)$

$$
G(u,x) = \begin{cases} 
\frac{P(L-x)u}{5EI} \left[ 2L(L-u) - (L-x)^2 - (L-u)^2 \right] & (0 \leq u \leq x) \\
\frac{P(L-x)}{5EL} \left[ 2L(L-x) - (L-u)^2 - (L-x)^2 \right] & (x \leq u \leq L) 
\end{cases} \quad (1)
$$
(ii) Bending Moment $M(u,x) = EI \frac{\partial^2 G(u,x)}{\partial u^2}$

$M(u,x) = \begin{cases} \frac{P(L-x)}{L} & (0 \leq u \leq x) \\ \frac{Px(L-u)}{L} & (x \leq u \leq L) \end{cases}$ ............ (2)

From $M(u,x)$, influence line $m(u)$ and bending moment diagram $M(x)$ can be derived.

Influence Line $m(u)$ for influence point $u$:

(linear)

Moment Line $M(x)$ for loading point $x$:

(linear)

(b) One Edge Built-in Beam (Fig. 3)

(i) Deflection $G(u,x)$

Figure 3

$G(x,u) = \begin{cases} \frac{1}{6EI} \left[ R_1(u^3-3Lu^2u) + 3P(L-u)^2u \right] & (0 \leq u \leq x) \\ \frac{1}{6EI} \left[ R_1(u^3-3Lu^2u) + P(3(L-u)^2u-(u-x)^2) \right] & (x \leq u \leq L) \end{cases}$ ........ .... (3)

where $R_1 = \frac{P}{2L^3} (3(L-x)^2L-(L-x)^3)$

(ii) Bending Moments $M(u,x) = EI \frac{\partial^2 G(u,x)}{\partial u^2}$

$M(u,x) = \begin{cases} \frac{Pu}{2L^3} (L-x)^2(2L+x) & (0 \leq u \leq x) \\ \frac{P}{2L^3} \left\{ u^3 \frac{L-x}{2L+x} -(u-x) \right\} & (x \leq u \leq L) \end{cases}$ ............ (4)

Influence Line $m(u)$

(3rd order parabola with respect to $x$)
C. ENGINEERING CONCEPT OF INFLUENCE SURFACES OF PLATES

(a) Basic Differential Equation

of Plates (isotropic)

\[ D \Delta \Delta W = q(x,y) \] .................. (5)

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \] Laplacian

\( w(u,v,x,y) \): Deflection surface of a given plate

\( q(x,y) \): Load acting on the plate

with prescribed boundary conditions.

(b) Definition of Green's Functions for the Deflection of a Plate

If the load \( q(x,y) \) is a concentrated load whose magnitude \( P=1 \) acting at the point \( (x,y) \), the deflection surface \( W(u,v,x,y) \) due to this load is called Green's function for the deflection of a given plate.

Green's function \( W(u,v,x,y) \) must satisfy the eq(5) with the prescribed boundary conditions. —— logical extension of influence lines to the plate problems (two dimensional field).

(c) Influence Functions for Bending Moments of Plate

\( M_x(u,v,x,y), M_y(u,v,x,y) \) (rectangular coordinates) are

\[ \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^2 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = q(x,y) \]

* if a given plate is orthotropic, the basic equation is given by
given by:

\[ M_x(u,v;x,y) = -D \left( \frac{\partial^2 w}{\partial u^2} + \nu \frac{\partial^2 w}{\partial v^2} \right) \] ..........................(6)

\[ M_y(u,v;x,y) = -D \left( \frac{\partial^2 w}{\partial u^2} + \nu \frac{\partial^2 w}{\partial v^2} \right) \]

(d) An Example of Influence Functions.

\[ \text{Infinite plate strip with simply supported parallel edges (isotropic)} \]

Figure 6

\[ W(u,v;x,y) = \frac{\alpha}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( 1 - \frac{n\pi}{a} (v-y) \right) \frac{n\pi}{a} (v-y) \sin \frac{n\pi}{a} \sin \frac{n\pi}{a} \]

(upper sign for \( V < y \))

(lower sign for \( V > y \)) ..............................(7)

\[ M_x(u,v;x,y) = \frac{P}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ (1+\nu) + (1-\nu) \frac{n\pi}{a} (v-y) \right] \cosh \frac{n\pi}{a} (v-y) \sin \frac{n\pi}{a} \sin \frac{n\pi}{a} \]

(upper sign for \( M_x \))

(lower sign for \( M_y \)) ..............................(8)

or closed form expressions:

\[ M_x(u,v;x,y) = \frac{P}{2\pi^2} \left[ (1+\nu) \log \frac{\cosh \frac{n\pi}{a} (v-y) - \cos \frac{n\pi}{a} (u+x)}{\cosh \frac{n\pi}{a} (v-y) - \cos \frac{n\pi}{a} (u-x)} \right] \]

\[ M_y(u,v;x,y) = \frac{P}{2\pi^2} \left\{ \frac{\sin \frac{n\pi}{a} (v-y) - \sin \frac{n\pi}{a} (v-y)}{\cosh \frac{n\pi}{a} (v-y) - \cos \frac{n\pi}{a} (u+x)} \right\} \]

...........................(9)

From moment influence functions \( M_x(u,v;x,y) \), \( M_y(u,v;x,y) \) influence surfaces for \( m_x(u,v) \), \( m_y(u,v) \) and moment surfaces for \( M_x(x,y) \), \( M_y(x,y) \) can be derived, and those surfaces are presented by contour line diagrams. (Fig. 7)

* for \( y < v \) the sign preceding \((v-y)\) must be changed.
Influence Surfaces \( m_x(u,v) \)

3-dimensional appearance of influence surface \( m_x(u,v) \)

\[ \text{Figure 7} \]

\[ \text{Figure 8} \]

D. Historical Background

1. H.M. Westergaard -- First attempt based on Müller-Breslau Principle
   "Computation of Stresses in Bridge Slabs Due to Wheel Loads"
   Public Roads (March 1930)

2. Newmark, N.M.
   "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams"
   University of Illinois, Bulletin No. 304, (1938)

3. Baron, F.M.
   "Influence Surfaces for Stresses in Slabs"
   ASME Transactions, (1941)

4. Pucher, A. -- developed General Theory of Influence Surfaces
   "Momentenein fluss flachen rechteckigen Platten"
   Deutscher Ausschuss Für Eisenbetonbau Heft. 90, (1938)
   "Einflussfelder elastischer Platten"
   Springer Verlag, Vienna (1951)

E. Scope of Study *

*Pucher and other investigators did a considerable amount of work on this subject. (Especially Pucher published a book of influence surface diagram.*)

*locit. (E.4)
Unfortunately the known surfaces cover only cases of single slabs of rectangular shape. The main objective of this research program is the development of new theoretical solutions for:

- a. continuous slabs
- b. skewed slabs
- c. orthotropic slabs
- d. slabs on elastic foundations
- e. flat slabs
- f. study of possibility to determine the influence surfaces by experiments for unusual cases.

**F. APPLICATION OF INFLUENCE SURFACES**

(i) for a distributed load $p(x,y)$

$$M(u,v) = \int \int p(x,y) \ m(u,v;x,y) \ dx\ dy$$

(ii) for a line load $p(s)$

$$M(u,v) = \int p(s) \ m(u,v;x,y) \ ds$$

(iii) for several concentrated loads $P_i(x_i,y_i)$

$$M(u,v) = \sum P_i(x_i,y_i) \ m(u,v;x_i,y_i)$$

(Similar to the influence lines!)

Volume of influence surface above value $7/8 \pi$

$$V = 1.338 \times 10^{-5} a$$

such that it can be ordinarily neglected in computing the Mx-moment. (similar for My) In actual computation, Simpson's Rule is employed. Careful computation will yield very accurate results (maximum error $< 5\%$).

**G. EXPERIMENTAL VERIFICATION OF INFLUENCE SURFACES**

Since the influence surfaces are based on the ordinary plate theory, the accuracy of the obtained results is correct.
within limitations of the theory of elasticity, and hence far superior to semi-empirical rules.

The theory developed by Pucher has been experimentally verified by several investigators.

(1) R.G. Sturm and R.L. Moore

The Behavior of Rectangular Plates Under Concentrated Load. Jour. of Applied Mech. (June 1937)

(2) Ir. H.J. Kist, Ir. A.L. Bouma


Especially, the second reference assured very successfully the consistency between the theory and experiments. However, the moments at the influence point inside of slabs become infinitely large so that the influence surfaces contain singular points. This is due to the application of a concentrated load.

Actually, the portion of the plate just under the load must be subjected to rather high compression because of highly localized load. Therefore it is impossible to apply the ordinary plate theory. Instead, the theory of thick plates (three dimensional theory of plates) must be applied (Nadai's Elastische Platten" 1923). Nevertheless, such a disturbance is so localized (St. Venant's Principle) that the accuracy of the theory is practically not affected because the value of influence surface above the value \( \frac{7}{8\pi} \) is usually negligible as stated before.

H. PRACTICAL APPLICATION OF INFLUENCE SURFACES

It should be pointed out that the use of influence surfaces is by no means restricted to concrete slabs but has its applications
in case of steel decks such as open grid floors, battleship decks, corrugated sheets and plywood plates, etc. Such plates can be regarded as orthotropic plates.

The extension of the theory of influence surfaces to orthotropic plates is the first phase of this research program. Besides the bending problem of plates, the theory of influence functions has other extensive applications, i.e., all classes of eigen value problems of plates (vibration as well as instability problems of plates), transient phenomena in plates due to dynamic loading, thermal effects, etc.

I. POSSIBLE MATHEMATICAL METHODS FOR THE STUDY OF INFLUENCE SURFACES

General Approach (Singularity Method)

\[ W(u,v;x,y) = W_0(u,v;x,y) + W_1(u,v;x,y) \]

where

\[ W_0(u,v;x,y) \]: particular (singular) solution of \( D\Delta W = q(x,y) \)

\( (W_0 \) contains the singularity of influence surface ; \( r^2 \log r \)

\[ W_1(u,v;x,y) \]: homogeneous solution of \( D\Delta W = 0 \) (Biharmonic functions)

\( W_1 \) must be determined such that sum \( W_0 + W_1 \) will fulfill the prescribed boundary conditions.

(1) Differential equations

Infinite, semi-infinite plate strips, rectangular plates with simply supported edges.

\[ W_0(u,v;x,y) \): solution for infinite plate strips given in (7).

\[ W_1(u,v;x,y) = \sum_{n=1}^{\infty} \left( A_n e^{i \pi n \nu a} + B_n e^{i \pi n \nu a} + C_n \left( \frac{n \pi \nu}{a} \right) e^{i \pi n \nu a} + A_n \left( \frac{n \pi \nu}{a} \right) e^{i \pi n \nu a} \right) \sin \frac{n \pi \nu u}{a} \]

...homogeneous solution for \( D\Delta W = 0 \)
(2) Fourier Integrals

quicker and simpler way to find the solution for infinite, semi-finite plate strip starting from Navier's solution for a rectangular plate, especially useful for slabs on elastic foundation.

(3) Application of Theory of Complex Variables

(a) Conformal Mapping plates with polygonal shapes and simply supported edges (isotropic)

Introduction of Moment Invariant \( M \)
\[
M = M_x + M_y = -D(1+\nu)\Delta W
\]
\[
D\Delta W = q(x, y) \quad \therefore \quad -(1+\nu)\Delta M = q(x, y) \quad \ldots \text{Poisson's equation.}
\]

Boundary Condition \( M = 0 \)

\( M \) can be obtained by conformal mapping of Green's function for a unit circle to required domain.

\( M_x, M_y \) can be derived without finding deflection \( W \).

(b) Muschel's Method

Similarity between bending of plates and plane stress problems

\[
\begin{align*}
\Delta \Delta P &= 0 \quad \text{E: Airy's stress function} \\
\Delta \Delta W &= 0
\end{align*}
\]

\[
\Delta \Delta W \rightarrow \frac{\partial^4 W}{\partial z \partial \bar{z}^2} = 0 \quad (z = x + iy, \bar{z} = x - iy)
\]

Singular solution:

\[
W_0 = \frac{P}{8\pi} r^2 \log r = \frac{P}{8\pi} \text{Re} \left[ z \bar{z} \log z \right]
\]

\[
W_1 = \text{Re} \left[ z \varphi(z) + \psi(z) \right] \quad \ldots \text{Goursat's stress function}
\]

\( \varphi(z), \psi(z) \) are analytic functions.

With the aid of theory of function, \( W_1 \) will be determined such that \( W = W_0 + W_1 \) may satisfy boundary conditions.
(4) Slabs Continuous Over Multiple Cross Beams

Application of integral equation.

\[ W(u,v;x,y) = G(u,v;x,y) - \sum_{i=1}^{K} \int_{0}^{z_i} \frac{\partial^4 W(z_i,c_i,x,y)}{\partial z_i^4} G(u,v;z_i,c_i) dz_i \]

where \( G(u,v;x,y) \) is the Green's function for a particular plate.

Solution can be easily obtained by making use of orthogonality of eigen functions belonging to \( G(u,v;x,y) \).

APPENDIX

1. Green's Function for the deflection of a simply supported rectangular plate (isotropic) (Fig. 10).

(a) Navier's solution (double Fourier series)

(Timoshenko "Plates & Shells", p. 122-125)

\[ G(u,v;x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{P}_{mn}(x,y) \mathcal{P}_{mn}(u,v) \frac{\lambda_{mn}}{\lambda_{mn}^2} \]

\[ \mathcal{P}_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \text{Normalized eigen function belonging to Green's Function } G(u,v;x,y) \]

\[ \lambda_{mn} = \sqrt{\left( \frac{mn}{a} \right)^2 + \left( \frac{mn}{b} \right)^2} \quad \text{eigen values of } G(u,v;x,y) \quad \ldots \ldots (10) \]

(b) Levy's solution (single Fourier series)

Navier's solution (10) can be transformed into Levy's solution (12) with the aid of following mathematical formula (A. Knesner, Die Integralgleichungen und ihre Anwendungen in der math. Physik S.157u167 Braunschweig 1922)

\[ \sum_{n=1}^{\infty} \frac{\cos n\pi x}{(n^2+k^2)^2} = -\frac{1}{2k^4} + \frac{\pi^2}{4k^2} \cdot \cosh kx \cdot \sinh^2 k\pi + \frac{\pi}{4k^3} \cdot \cosh k(\pi-x) \]

\[ + \frac{\pi x}{4k^3} \cdot \frac{\sinh k(\pi-x)}{\sinh k\pi} \quad (0 \leq x \leq 2\pi) \quad \ldots \ldots (11) \]
The following are the results of transformation.

(i) \( v \geq y \)

\[
\frac{P a^2}{W^3D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \sinh \frac{n\pi}{a} (b-v) \right] \left\{ \sinh \frac{n\pi}{a} - \frac{n\pi}{a} \cosh \frac{n\pi}{a} \right\} + \frac{\sinh \frac{n\pi}{a}}{\sinh \frac{n\pi b}{a}}
\]

\[
\frac{n\pi}{a} \cosh \frac{n\pi}{a} (b-v) - \frac{n\pi b}{a} \frac{\sinh \frac{n\pi v}{a}}{\sinh \frac{n\pi b}{a}} \right] \left\{ \sinh \frac{n\pi}{a} \sin \frac{n\pi x}{a} \right\}
\]

(ii) \( v \leq y \)

\[
\frac{P a^2}{W^3D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \sinh \frac{n\pi}{a} (b-y) \right] \left\{ \sinh \frac{n\pi}{a} - \frac{n\pi}{a} \cosh \frac{n\pi}{a} (b-y) \right\}
\]

\[
+ \frac{\sinh \frac{n\pi}{a} (b-y)}{\sinh \frac{n\pi b}{a}} \left\{ \frac{n\pi}{a} \cos \frac{n\pi v}{a} - \frac{n\pi b}{a} \frac{\sinh \frac{n\pi v}{a}}{\sinh \frac{n\pi b}{a}} \right\}
\]

\[
\cdots \cdots \cdots (12)
\]

2. **Green's Function for the deflection of semi-infinite plate strip (Fig. 11).**

The solution can be easily derived by making \( b \to \infty \) in the Levy's solution (12) derived above.

\( b \gg 0 \) \( \Rightarrow \) \( \sinh \frac{n\pi b}{a} \sim \frac{1}{2} e \frac{n\pi b}{a}, \cosh \frac{n\pi b}{a} \sim 1 \frac{n\pi b}{a} \)

\[
W(u, v; x, y) = \frac{P a^2}{2W^3D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \left\{ 1 + \frac{n\pi}{a} (v-y) \right\} \right] \left\{ \sinh \frac{n\pi}{a} \sin \frac{n\pi x}{a} \right\}
\]

\[
\begin{align*}
\text{upper sign for } v \leq y \\
\text{lower sign for } v \geq y
\end{align*}
\]

\[
\cdots \cdots \cdots (13)
\]

3. **Green's Function for the deflection of infinite plate strip.**

Since the second series of eq. (13) represents the image of the first series with respect to \( x \)-axis, it is obvious that the first series is the required solution.
\[ W(u,v;x,y) = \frac{Pa^2}{2\pi D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \frac{\sin \frac{\pi}{a} (v-y)}{\sin \frac{\pi}{a} (u-y)} \right] e^{\frac{n\pi}{a}(v-y)} \sin \frac{\pi m u}{a} \sin \frac{\pi m x}{a} \]

This is the solution given in (7)

4. Derivation of closed form expressions (9) for \( M_x(u,v;x,y) \) \( M_y(u,v;x,y) \) of an infinite plate strip.

In order to sum up the series solution for \( M_x, M_y \) given (d), the following mathematical formula can be applied.

\[ \sum_{n=1}^{\infty} \frac{r^n}{n} \cos nx = -\frac{1}{2} \log (1-2r \cos x+r^2) \]

(14)

\[ \sum_{n=1}^{\infty} r^n \cos nx \frac{1-r \cos x}{1-2r \cos x+r^2} -1 = \frac{1}{2} (\frac{1-r^2}{1-2r \cos x+r^2} -1) \]

(15)

for values of \(|r| < 1\). (Whittaker & Watson's Modern Analysis p.190, ex.1).

\[ \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{n\pi}{a} (v-y)} \frac{\sin \frac{\pi m u}{a} \sin \frac{\pi m x}{a}}{\sin \frac{\pi}{a} (u-y)} \]

\[ = \frac{1}{4} \log \frac{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u+x)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u-x)} \]

and

\[ \sum_{n=1}^{\infty} e^{\frac{n\pi}{a} (v-y)} \frac{\sin \frac{\pi m u}{a} \sin \frac{\pi m x}{a}}{\sin \frac{\pi}{a} (u-y)} \]

\[ = \frac{1}{4} \left( \frac{\sinh \frac{\pi}{a} (v-y)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u+x)} - \frac{\sinh \frac{\pi}{a} (v-y)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u-x)} \right) \]

Therefore,

\[ M_x(u,v;x,y) = \frac{P}{\beta \pi} \left[ (1+\nu) \log \frac{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u+x)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u-x)} + (1-\nu) \frac{\pi}{a} (v-y) \right] \]

\[ M_y(u,v;x,y) = \frac{P}{\beta \pi} \left[ \frac{\sinh \frac{\pi}{a} (v-y)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u+x)} - \frac{\sinh \frac{\pi}{a} (v-y)}{\cosh \frac{\pi}{a} (v-y) - \cos \frac{\pi}{a} (u-x)} \right] \]