Plastic Design of Multi-Story Frames

ULTIMATE STRENGTH OF LATERALLY LOADED COLUMNS

by
Le-Wu Lu
Hassan Kamalvand

November, 1966

Fritz Engineering Laboratory Report No. 273.52
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This work has been carried out as part of an investigation sponsored jointly by the Welding Research Council and the Department of the Navy with funds furnished by the following:

American Iron and Steel Institute
American Institute of Steel Construction
Naval Ships Systems Command
Naval Facilities Engineering Command

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Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

November, 1966

Fritz Engineering Laboratory Report No. 273.52
SYNOPSIS

Analytical procedures are developed for computing the maximum carrying capacity of steel columns subject to combined axial thrust and lateral load. It is assumed that the columns are permitted to deflect only in the plane of the applied load and that failure is always caused by excessive bending in the same plane. Numerical results are obtained for four types of columns with different loading and support conditions and are presented in the form of interaction curves relating axial thrust, lateral load and slenderness ratio. The analytically obtained results are compared with the predictions based on an empirical interaction formula and good agreement is observed.
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1. **INTRODUCTION**

Compression members subjected to lateral (or transverse) loads occur frequently in building frames, bridge trusses and other important engineering structures. They are usually proportioned to satisfy some limiting stress criteria set by specifications or codes. The stresses developed at any cross section in such a member consist of (1) the axial stress caused by the compressive force, (2) the primary bending stress due to the lateral load, and (3) the secondary bending stress produced by the so-called secondary moment, which is the product of the resulting deflection of the section and the compressive force. The last stress introduces instability effect into the members and becomes particularly important for columns with high slenderness ratios and carrying large compressive forces. The procedures for computing the secondary moments and stresses in elastic columns are described in books on stability theory.\(^1,2,3\)

Although elastic analysis has been used extensively in design computations, it does not give accurate indications of the true load-carrying capacity. Laterally loaded columns generally fail by excessive bending after the stresses in some portions of the member exceed the elastic limit. To determine the ultimate strength of such a column, it is necessary to perform a stability analysis that considers the elastic-plastic behavior of the various sections. Unfortunately, the required analysis is often too complex
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for practical applications, and recourse is sometimes made to empirical formulas,\textsuperscript{3, 4} (formulas providing approximate estimates of the column strength).

The purpose of this paper is to develop efficient methods, particularly adaptable to computer programming,\textsuperscript{1} for performing elastic-plastic stability analysis for a variety of columns and to present numerical results in a form suitable for design use.

1.1 PREVIOUS WORK

In contrast to the extensive work done on columns subjected to combined axial force and end moments,\textsuperscript{2, 5, 6, 7, 8} only a few attempts have been made to study the strength of laterally loaded columns. Wright developed an approximate formula for the case of a column loaded by a concentrated load at the midspan [Case (a) in this paper].\textsuperscript{9} The same case was also studied by Ketter who developed interaction curves for a variety of columns.\textsuperscript{6} An empirical method for estimating the ultimate strength of laterally loaded columns was proposed by Horne and Merchant using the modified Rankine Formula.\textsuperscript{3} The validity of the method has not been verified by comparing the estimated strength with the strength determined either from exact solutions or from laboratory experiments.

1.2 SCOPE OF INVESTIGATION AND ASSUMPTIONS

The loading and support conditions of the four cases of
laterally loaded columns investigated in the paper are shown in Fig. 1. The columns are assumed to be prismatic and made of "as-rolled" wide-flange shapes. In all the cases, the lateral load, \( R \) or \( w \), causes bending moment about the major axis of the cross section.

For each case, a method of solution is first developed and numerical results are then given for columns with slenderness ratios ranging from 20 to 100. The results are presented as interaction curves. All computations were performed on A36 (yield stress = 36 ksi) steel columns, but the results can also be used for other columns with different yield stress levels by the proper adjusting of the slenderness ratio (see Summary and Conclusions). The computed ultimate strength is compared with the ultimate strength predicted by the empirical interaction formula contained in the AISC Specification.

The following assumptions are made in the solutions:

1. The stress-strain properties of the column material are elastic and perfectly plastic and the effect of strain hardening is neglected.

2. For a given combination of axial force and bending moment acting at a section, there exists a unique value of curvature. This means that the deformation of a section depends only on the final values of the axial force and bending moment and that the actual history of loading does not affect the resulting curvature.
3. The effect of shear is small and can be neglected.

4. Weak-axis buckling and lateral-torsional buckling are effectively prevented so that failure is always caused by excessive bending in the plane of the applied lateral load.

In performing numerical calculations, it is further assumed that the axial force $P$ is applied first and maintained at a constant value as the lateral load increases or decreases.
2. GENERAL PROCEDURE OF NUMERICAL INTEGRATION

The methods of solution to be developed subsequently for the individual cases utilize a common numerical integration procedure. It was first used by von Kármán in his studies on eccentrically loaded columns\textsuperscript{10} and was later modified by Chwalla, Ojalvo, and others for use in the analysis of restrained columns.\textsuperscript{11,12} It is further modified in this paper to take into account the effect of lateral load and variation in end conditions.

2.1 INTEGRATION PROCEDURE

Figure 2 shows a portion of a laterally loaded column whose deformed configuration in the elastic and elastic-plastic range is to be determined. The applied forces consist of the axial force, $P$, lateral load, $q$, end moment, $M_0$, and support reaction, $V_0$. They are considered as the known quantities in this discussion. It is required to determine the deflection curve of the member when the initial slope, $\theta_0$, at the left end is assigned a specific value. This can be accomplished by applying a segment-by-segment integration process, starting from the left end. For the first segment, whose length is chosen to be $\rho_1$, the deflection at its mid-point (shown as a dot in Fig. 2) is approximately equal to

$$\delta_{al} = \theta_0 \frac{\rho_1}{2} \quad (1)$$
and the corresponding bending moment is

\[ M_{al} = P \delta_{al} + V_0 \rho_{1/2} - M_0 - \left[ \text{Moment Due to } q \text{ Applied Over } \rho_{1/2} \right] \]  

(2)

This moment is assumed to be the average moment of the entire segment. The average curvature, \( \phi_{al} \), can be determined from the moment-curvature-thrust (M-\( \phi \)-P) relationship which includes both the elastic and inelastic range of cross sectional response. (The properties of the M-\( \phi \)-P relationships used in this paper will be briefly described later.) When \( \phi_{al} \) is known, the slope and deflection at the end of the first segment can be computed from the expressions

\[ \theta_1 = \theta_0 - \phi_{al} \rho_{1} \]  

(3)

\[ \delta_1 = \theta_0 \rho_{1} - \frac{1}{2} \phi_{al} \rho_{1}^{2} \]  

(4)

and the corresponding bending moment is given by

\[ M_1 = P \delta_{1} + V_0 \rho_{1} - M_0 - \left[ \text{Moment Due to } q \text{ Applied Over } \rho_{1} \right] \]  

(5)

The values of \( \theta_1 \) and \( \delta_1 \) determined from Eqs. 3 and 4 will be used as the initial values to start the integration for the second segment. Again, the deflection and bending moment at the mid-point of the segment are first computed.
\[ \delta_{a2} = \delta_1 + \theta_1 \frac{\rho_2}{2} \] (6)

and

\[ M_{a2} = P \delta_{a2} + V_0 (\rho_1 + \frac{\rho_2}{2}) - M_0 - \left[ \text{Moment Due to } q \text{ Applied Over } \rho_1 + \rho_2/2 \right] \] (7)

The average curvature, \( \phi_{a2} \), of the segment is then found from the \( M-\phi-P \) relationship. The slope, deflection, and bending moment at the end of the segment can then be determined from the equations

\[ \theta_2 = \theta_1 - \phi_{a2} \rho_2 \] (8)

\[ \delta_2 = \delta_1 + \theta_1 \rho_2 - \frac{1}{2} \phi_{a2} \rho_2^2 \] (9)

and

\[ M_2 = P \delta_2 + V_0 (\rho_1 + \rho_2) - M_0 - \left[ \text{Moment Due to } q \text{ Applied Over } \rho_1 + \rho_2 \right] \] (10)

Repeated calculations can be carried out for as many segments as necessary. The calculations may be terminated when certain specified conditions are satisfied. For instance, in analyzing columns with a given length, L, the integration may be terminated when the total accumulated length is equal to L/2. It
is also possible to terminate the calculations when the slope of the deflection curve becomes zero or negative. If the integration process is terminated after completing n segments (Fig. 2), the last set of the numerical results gives the values of the slope, deflection, and bending moment of the point which is located at a distance $\rho_1 + \rho_2 + \ldots + \rho_n$ from the left end.

The procedure described above can be effectively programmed on a digital computer, and extensive computations can be made for many columns with different loading and support conditions. The initial conditions required to start the integration and the criteria used to terminate it are, of course, somewhat different for the different cases. They will be discussed in some detail when the method of solution for each case is presented.

2.2 MOMENT-CURVATURE-THRUST RELATIONSHIPS

The M-\(\phi\)-P relationships used in the computations were determined for the 8W31 section by a separate computer program. In this program a moment vs. curvature curve was developed for a constant axial thrust by dividing the cross section into a large number of finite elements. The strains of the elements are related to the curvature of the section and the stresses to the applied bending moment. The relationship between the applied moment and the resulting curvature can therefore be found through equilibrium and compatibility conditions of these elements. The details of the
method and the computer program are described elsewhere.\textsuperscript{14}

The basic program can be easily modified to take into account the effect of residual stresses. In this study, only residual stresses resulting from differential cooling rate are considered. The distribution and magnitude of the residual stresses adopted in the calculations are the same as those used in the previous studies on beam-columns.\textsuperscript{5,6,13} When the resulting $M-\phi-P$ relationships are used in the numerical integration process, the final results (deformations and ultimate strength) will automatically include the influence of residual stresses.

Although the $M-\phi-P$ curves and other cross sectional properties used in the analysis were based on the $8W31$ section, the numerical results obtained, after proper nondimensionalization, are valid for other column sections also. It has been found in another study that the $M-\phi-P$ curves of the $8W31$ section are close to the average $M-\phi-P$ curves of most column sections.\textsuperscript{15}
3. **METHOD OF SOLUTION AND RESULTS FOR CASE (a)**

The problem to be solved for this case is as follows: for a given column (with known length and cross sectional properties) subjected to a specified axial force, determine the maximum lateral load, $R_{\text{max}}$, that can be safely carried by the member. The first step in the solution is to find a systematic approach for determining the response of the column to the varying lateral load. The desired response is usually represented by a load versus center deflection ($R - \delta_c$) curve or a load versus end slope ($R - \theta_0$) curve. Once the complete response curve is obtained, the maximum load can be easily determined from the peak of the curve.

Referring to Fig. 2, the special conditions applicable to Case (a) are: $M_0 = 0$, $q = 0$, and $V_0 = \frac{R}{2}$. With these conditions, the integration process may proceed according to the scheme described above after $R$ and $\theta_0$ are assigned specific values. In carrying out the actual numerical computations, it is convenient to first specify a value of $R$ and to perform repeated computations for a number of selected $\theta_0$ values. The procedure is then repeated for other $R$ values. In all the computations, the integration is terminated at a point where the slope of the deflection curve becomes zero. This is done because the actual deflection curve of the column has a zero slope at the midspan. The distance from the left end of the column to the point of zero slope obviously will
vary with the assumed initial slope, $\theta_0$.

Since the purpose of the computations is to obtain numerical results for several columns with specified slenderness ratios, it is more convenient to select the segment lengths as multiples or fractions of the radius of gyration of the cross section. Both the slenderness ratio and the radius of gyration used in the subsequent discussions are computed for the major axis. In order to improve the accuracy of the final results, the length selected for a given segment is varied according to the bending moment found at the end of the previous segment. The following criteria are used in the selection of segment length

\[ \rho = 2r \text{ if } 0 \leq |M| < 0.8 \ M_{pc} \]  

\[ \rho = r \text{ if } 0.8 \ M_{pc} \leq |M| < 0.9 \ M_{pc} \]  

\[ \rho = 0.1r \text{ if } 0.9 \ M_{pc} \leq |M| < M_{pc} \]

in which $M_{pc}$ is the reduced plastic moment corresponding to the specified axial force. As an example, if at the end of the tenth segment the bending moment $M_{10}$ is found to be 0.85 $M_{pc}$, then the length for the eleventh segment should be $\rho_{11} = r$. Thus, starting from the left end, the segment length decreases from $2r$ to $r$ and finally to $0.1r$, if the bending moment equals or exceeds 0.9 $M_{pc}$. For certain cases, in which the bending moment is always less than...
0.9 \( M_{pc} \), a segment length equal to 0.1\( L_r \) is also used for the last few increments before terminating the computations. This adjustment permits a more precise determination of the location at which the slope of the deflection curve becomes zero. When the integration process carried out for a given case is completed, the total distance included in the computations is usually given as a multiple of \( r \), say \( \lambda r \).

To provide a systematic way of specifying the value of the lateral load, a reference load is used. This load, denoted by \( R_{pc20} \), is defined as the plastic load of the shortest column (\( L/r = 20 \)) considered in the computations and is given by

\[
R_{pc20} = \frac{4 M_{pc}}{L} = \frac{M_{pc}}{5r}
\]  (12)

In all the numerical computations, the lateral loads are always specified as fractions of the reference load. In a similar manner, the axial force, \( P \), is specified as a fraction of the axial yield load, \( P_y \).

The results of computations made for the case with \( P = 0.4 P_y \) are shown in Fig. 3. Each curve gives the relationship between the initial slope, \( \theta_0 \), and the resulting zero-slope distance \( \lambda r \) (or \( \lambda \)) for a specified value of \( R \). Initially, an increase in \( \theta_0 \) results in a corresponding increase in \( \lambda \). The rate of increase in \( \lambda \) is gradually reduced as \( \theta_0 \) increases; and, corresponding
to some assumed $\theta_0$, a maximum $\lambda$ is eventually reached. At this point a further increase in $\theta_0$ causes a decrease in $\lambda$. Although the relationship between $\lambda$ and $\theta_0$ tends to reverse for larger $\theta_0$ values, the bending moment $M_\lambda$ at the point of zero slope is found to increase continuously with $\theta_0$. For some specified initial slope, say $\theta_0^p$, the bending moment in some segment may reach the reduced plastic moment, $M_{pc}$, of the section before the slope of the deflection curve becomes zero. When this occurs, the integration process is discontinued and no further computations will be performed for larger assumed initial slopes. Each $\theta_0-$ curve therefore terminates at a maximum initial slope equal to $\theta_0^p$. A total of twenty-three $\theta_0-$ curves was prepared (sixteen are shown in Fig. 3) from which the lateral load versus end slope relationships given in Fig. 4 were obtained.

The procedure for determining the load-end slope curves is illustrated in Fig. 3 for a column with $L/r = 60$. A vertical line is first drawn from the point where $\lambda$ equals 30. The points of intersection of this line with the various $\theta_0-$ curves give the end slopes of the column when it is loaded by the specified lateral loads. The results obtained are plotted as small circles in Fig. 4, and, through these circles, the desired load-end slope curve (labelled 60) can be constructed. The peak of the curve gives the ultimate load of the column. The lateral load is nondimensionalized with respect to the individual plastic load, $R_{pc}$, not with respect.
to the reference load $R_{pc20}$, as was done in Fig. 3. This permits a closer examination of the effect of instability on the strength and behavior of the five columns.

The procedures described above for obtaining the $\theta_0 - \lambda$ charts and the $\frac{R}{R_{pc}} - \theta_0$ curves have been repeated for the following specified axial forces: $P = 0.2P_y$, $0.6P_y$, $0.8P_y$, and $0.9P_y$. The maximum lateral loads obtained for these cases together with those determined from Fig. 4 are summarized in the form of ultimate strength interaction curves in Fig. 5. Each curve is for a particular column and gives the combinations of axial force and lateral load that can be safely supported by the column. The lateral load is now nondimensionalized with respect to the simple plastic load, $R_p$, of the individual columns (assuming that no axial force is applied to the columns). The interaction curves can be directly used in analysis and design computations and also in checking the validity of the existing design approximations. ${}^{3,4,9}$
4. METHOD OF SOLUTION AND RESULTS FOR CASE (b)

The numerical integration procedure illustrated in Fig. 2 is again used to develop the load-deformation curves of Case (b) columns by utilizing the appropriate boundary and loading conditions. An elementary analysis will show that the bending moment is generally higher at the midspan than at the two ends. The center portion of the column always yields first. Although some yielding would also occur near the ends, no plastic hinges have been found to form at these locations when the column is loaded by the maximum lateral load. Hence, the appropriate conditions to be used in the integration process are \( \theta_0 = 0, q = 0, \) and \( V_0 = \frac{R}{2} \). The trying variable that must be assumed before starting the integration is the end moment \( M_0 \).

For a specified value of \( R \), a number of \( M_0 \) values are first assumed and computations are carried out for each assumed \( M_0 \) to determine the zero-slope distance \( \lambda r \). When the integration process reaches the point of zero slope, the deflection at that point, \( \delta_\lambda \), is also determined. A chart, similar to that given in Fig. 3, can then be prepared to give the relationships between \( \delta_\lambda \) and \( \lambda \) for a series of specified \( R \) values. From this chart the lateral load versus center deflection \( (R - \delta_c) \) curves of the columns can be determined. The maximum lateral loads that can be resisted by the columns are again given by the peaks of the curves.
The maximum loads determined for the five selected columns subjected to various specified axial loads are presented in Fig. 6. The lateral load is again nondimensionalized with respect to the simple plastic load of the individual member,

\[ R_p = \frac{8M_p}{L} \]  

(13)

A comparison of the interaction curves of Fig. 6 with those of Fig. 5 indicates that the effect of instability is less pronounced in fixed-end columns than in simply-supported columns. The fixed-end columns are usually stiffer and have less deflections at the maximum load. Consequently, the secondary moment which causes the instability effect is also less.
5. METHOD OF SOLUTION AND RESULTS FOR CASE (c)

Because of the difference in loading condition, the method developed previously for Case (a) columns is not directly applicable to this case, although the basic numerical integration procedure can still be used. In the present case, the reaction \( V_0 \) at the left end depends not only on the lateral load, \( q \), but also on the span length, \( L \). It is not possible to fully specify \( V_0 \) and to start the integration without actually knowing the zero-slope distance, \( \lambda r \) (which should be equal to \( L/2 \) if the assumed initial slope \( \theta_0 \) is correct). For this reason, each column has to be analyzed individually for a number of selected lateral loads in order to determine its complete load-deformation relationship.

Figure 7 shows the procedure used to obtain the end slopes of a given column when it is loaded by specified axial and lateral loads. The slenderness ratio of the column is equal to 60, and the specified loads are \( P = 0.4 \, P_y \) and \( w = 0.66 \, w_{pc} \), in which \( w_{pc} \) is the plastic load of the member and is given by

\[
W_{pc} = \frac{8M_{pc}}{L^2} \tag{14}
\]

To determine the end slopes, a number of trial \( \theta_0 \) values are first
selected and the numerical integration process is carried out for each value. The appropriate conditions for integration are: $M_0 = 0$, $q = w$, and $V_0 = wL/2$. In each case, when the accumulated length covered by the computations reaches $L/2$ or $30\pi$, the integration is terminated and the slope $\theta_{30}$ recorded. Unless the $\theta_0$ happens to be correct, $\theta_{30}$ is usually different from zero. When the assumed $\theta_0$'s are plotted against the recorded $\theta_{30}$'s, the curve given in Fig. 7 is obtained. The correct end slopes for the column are found to be 0.0144 rad. and 0.0224 rad. These results determine two points on the load versus end slope curve (Fig. 8). Additional points on the curve can be obtained by repeating the procedure for other specified lateral loads. When all the five columns are analyzed by this method, the $\frac{w}{w_{pc}} - \theta_0$ curves shown in Fig. 8 are obtained, from which the maximum lateral loads can be determined.

Figure 9 summarizes the results of ultimate strength computations made for the five columns. The maximum lateral load is nondimensionalized with respect to the simple plastic load, $w_p$, which is determined from Eq. 14 by substituting $M_p$ for $M_{pc}$. A close examination of the interaction curves given in Figs. 5 and 9 indicates that on a nondimensional basis the reduction in load-carrying capacity due to instability is greater in Case (c) columns than in Case (a) columns. This can again be attributed to the difference in the resulting deflection produced by the two types of loading.
6. _METHOD OF SOLUTION AND RESULTS FOR CASE (d)_

The determination of the ultimate strength of Case (d) columns is complicated by the fact that the columns may fail in two different modes. For columns in which the bending moment is caused primarily by the lateral load, plastic hinges tend to form at the ends prior to the attainment of the maximum load. This situation is shown as Case I in Fig. 10. The available conditions to be used in the integration process are: \( M_0 = M_{pc}, q = w, \) and \( V_0 = \frac{wL}{2}, \) and the unknown quantity that must be determined by trial is the end slope \( \theta_0. \) The problem is thus seen to be the same as that for Case (c) columns and the method of solution described previously for that case can be directly applied.

In columns having high slenderness ratio and subjected to heavy axial load, the bending moment at the center may be amplified significantly by the secondary moment. As a result, the center moment may become larger than the moment at the two ends, and initial yielding is likely to occur near the midspan. Although in some cases, limited yielding can also take place at the ends, no plastic hinges are found to develop at these locations. The ends of the column can therefore be assumed to remain fixed for the purpose of analysis. This case is shown as Case II in Fig. 10, and the known conditions are: \( \theta_0 = 0, q = w, \) and \( V_0 = \frac{wL}{2}. \) It can be easily recognized that apart from the difference in loading condition the
problem to be solved is essentially the same as that involved in the analysis of Case (b) columns. For a given column subjected to specified axial and lateral loads, the quantity to be determined by the trial-and-error procedure is the end moment $M_0$. A series of trial $M_0$ values is first selected and numerical integration is carried out for each selected value to determine the slope $\theta_c$ at the midspan. A plot, similar to the $\theta_0$ versus $\theta_c$ plot given in Fig. 7, can then be obtained between the trial $M_0$ value and the resulting slope $\theta_c$. From this plot the correct $M_0$ value which produces zero slope at the midspan can be determined.

For certain combinations of slenderness ratio and axial force, it may be difficult to predict which case is the governing case. In such situations, trial calculations must be performed for both cases.

Figure 11 summarizes the numerical results obtained for the five selected columns. The lateral load is again nondimensionalized with respect to the simple plastic load, $w_p$, which is given by

$$w_p = \frac{16M_p}{L^2}$$  \hspace{1cm} (15)

The interaction curves presented in Fig. 11 can be compared with those given in Fig. 6 to assess the effect of variation in loading
condition. The same curves can also be compared with the interaction curves in Fig. 9 to evaluate the influence of end restraint.
7. COMPARISON OF RESULTS WITH INTERACTION FORMULAS

In designing members carrying combined axial compression and bending stresses, use is frequently made of the so-called interaction formulas. Three such formulas are given in Part 1 of the AISC Specification for use in allowable-stress design. Although these formulas were originally developed for predicting the stresses in the elastic range, studies have shown that they also provided good estimates of the ultimate strength. These studies were made primarily for columns which are loaded by combined axial force and end moments. In the following, a similar study will be made for the four cases of laterally loaded columns analyzed in the paper.

The formula which will be used in the comparison is Formula (7a) given in the Specification. It can be written in terms of ultimate loads as

\[
\frac{P}{P_{cr}} + \frac{C^{R}}{(1 - \frac{P}{P_{e}})R_{P}} = 1.0
\]  

(16a)

for Case (a) and Case (b) columns, or as

\[
\frac{P}{P_{cr}} + \frac{C^{W}}{(1 - \frac{P}{P_{e}})w_{P}} = 1.0
\]  

(16b)
for Case (c) and Case (d) columns. There are three new terms, \( P_{cr} \), \( P_e \), and \( C_m \), in these equations; they are defined as follows:

\[
P_{cr} = \text{critical load of the member loaded concentrically by axial force only. For the columns under consideration, it is the buckling load about the strong axis.}
\]

\[
P_e = \text{elastic critical load for buckling in the plane of the lateral load.}
\]

\[
C_m = \text{a coefficient depending on loading and support conditions. It assumes the following values for the cases of columns considered herein}
\]

\[
\begin{align*}
\text{Case (a)} & \quad C_m = 1 - 0.2 \frac{P}{P_e} \\
\text{Case (b)} & \quad C_m = 1 - 0.6 \frac{P}{P_e} \\
\text{Case (c)} & \quad C_m = 1.0 \\
\text{Case (d)} & \quad C_m = 1 - 0.4 \frac{P}{P_e}
\end{align*}
\]

The maximum loads determined by the analytical procedures for the four types of columns are compared in Fig. 12 with the interaction formula in a manner suggested by Mason, Fisher and Winter.\textsuperscript{17}
In general, the interaction formulas are seen to give good predictions for all the cases investigated. For Cases (a) and (c), the formula tends to underestimate the carrying capacity in the low axial force range, but the difference is usually small. The same trend can also be observed for Case (b) and Case (d) columns, except that in these cases the difference (on conservative side) seems to be somewhat larger for low slenderness ratio columns. It may be concluded from this study that the AISC interaction formula is valid not only in predicting the elastic range stresses but also in estimating the ultimate strength.
8. SUMMARY AND CONCLUSIONS

A general method for determining the load-deformation relationships of laterally loaded columns in the elastic and elastic-plastic range has been developed. It employs a segment-by-segment integration process, using the available moment-curvature-thrust data as the basic impact. By properly utilizing the boundary and support conditions, the method can be effectively used to obtain solutions to a variety of column problems. Specific applications have been made to the four cases of columns shown in Fig. 1. For each case the general method was applied to obtain ultimate strength solutions for five selected columns with slenderness ratios ranging from 20 to 100. The results are given in Figs. 5, 6, 9, and 11 in the form of interaction curves relating the axial thrust, maximum lateral load and slenderness ratio.

All the results obtained in this study are for wide-flange columns subjected to lateral loads producing bending moment about the major axis of the cross section. It has been assumed that failure is always due to excessive bending in the plane of the applied load and that lateral-torsional buckling or local buckling does not occur throughout the loading history. In all the calculations the influence of cooling residual stresses were taken into account through the use of the special moment-curvature-thrust relationships that include the effect of these stresses. Although the interaction curves were prepared for A36 steel with a nominal
yield stress of 36 ksi, they can be applied to steels of other yield stress levels by substituting an equivalent slenderness ratio:

\[
\left( \frac{L}{r} \right)_{\text{equ}} = \left( \frac{L}{r} \right) \sqrt{\frac{36}{\sigma_y}}
\]

(18)

This substitution will yield exact results if the residual stress has the same distribution pattern over the cross section and the same proportion of the yield stress for the different steels.²

The maximum lateral loads determined by the analytical procedures have been compared in Fig. 12 with the AISC interaction formula and good agreement has been observed.
9. ACKNOWLEDGMENTS

The work described in this paper was first carried out by the junior author, when he was a Byllesby Research Fellow in the Fritz Engineering Laboratory, Department of Civil Engineering, Lehigh University. The research was performed under the supervision of the senior author who formulated the analytical procedures and prepared portions of the computer program.

The senior author has been closely associated with a general investigation on "Plastic Design of Multi-Story Frames", from which the subject matter studied in the paper was developed. The investigation is sponsored by the Welding Research Council and the U. S. Navy Department. Funds are supplied by the American Iron and Steel Institute, American Institute of Steel Construction, Naval Ship Systems Command and Naval Facilities Engineering Command. Technical guidance for the project is provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council. Dr. T. R. Higgins is Chairman of the Lehigh Project Subcommittee.

Dr. B. P. Parikh was responsible for the development of the moment-curvature relationships used in the computations. His assistance is gratefully acknowledged.
10. NOTATION

\( C_m \) = a coefficient contained in the AISC interaction formula

\( L \) = length of column

\( M \) = bending moment

\( M_a \) = average bending moment of segment

\( P \) = axial force

\( P_{cr} \) = critical load of column if axial force alone existed

\( P_e \) = elastic critical load about strong axis

\( P_y \) = axial yield load

\( q \) = general distributed lateral load

\( R \) = concentrated lateral load

\( R_{pc} \) = maximum value of \( R \) according to plastic theory. The reduction in moment-carrying capacity due to axial force is included, but the effect of column instability is not.

\( R_p \) = maximum value of \( R \) according to simple plastic theory

\( r \) = radius of gyration about major axis

\( V \) = shear force

\( V_0 \) = shear force at beginning of first segment = support reaction

\( w \) = uniformly distributed lateral load

\( w_{pc} \) = maximum value of \( w \) according to plastic theory (See \( R_{pc} \))

\( w_p \) = maximum value of \( w \) according to simple plastic theory

\( \delta \) = deflection

\( \delta_a \) = average deflection of segment
\( \theta \) = slope

\( \theta_0 \) = initial end slope

\( \lambda \) = a parameter used to define zero-slope distance

\( \rho \) = length of segment

\( \sigma_y \) = yield stress of material

\( \phi \) = curvature
Fig. 1 Cases of Laterally Loaded Columns Studied in the Paper

Fig. 2 Numerical Integration Scheme
Fig. 3 A Sample $\theta_0 - \lambda$ Chart
Fig. 4  Lateral Load Vs. End Slope Curves for Five Columns Subjected to Concentrated Load
Fig. 5  Ultimate Strength Interaction Curves for Simply Supported Columns Subjected to Concentrated Load
Fig. 6  Ultimate Strength Interaction Curves for Fixed-End Columns Subjected to Concentrated Load
Fig. 7 Determination of End Slopes of Simply Supported Column Subjected to Uniformly Distributed Load

\[
\frac{P}{P_y} = 0.4 \\
\frac{L}{r} = 60 \\
\frac{w}{w_{pc}} = 0.66
\]

Fig. 8 Lateral Load Vs. End Slope Curves for Simply Supported Columns Subjected to Uniformly Distributed Load
Fig. 9 Ultimate Strength Interaction Curves for Simply Supported Columns Subjected to Uniformly Distributed Load
Fig. 10  Deformation Modes of Case (d) Columns
Fig. 11 Ultimate Strength Interaction Curves for Fixed-End Columns Subjected to Uniformly Distributed Load
Fig. 12 Comparison of Analytical Results With AISC Formula
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