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IN PRESTRESSED CONCRETE BOX-BEAM BRIDGES

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THEORETICAL ANALYSIS OF LOAD DISTRIBUTION
IN PRESTRESSED CONCRETE BOX-BEAM BRIDGES

by
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ABSTRACT

This investigation is a theoretical study of the vehicular load distribution in spread box-beam bridges. The box-beam bridge superstructures are composed of a number of precast, prestressed concrete box-beams, equally spaced and spread apart, and a cast-in-place composite slab. A method of analysis is developed for beam-slab bridges, and is particularly applied to spread box-beam bridges. In this method, the bridge superstructure is reduced to an articulated structure by introducing a series of beam and plate elements.

The validity of the theoretical analysis is verified by comparison with the results of field tests on four different spread box-beam bridges. An extensive study of all of the parameters involved in the analysis has been made. Over three hundred different spread box-beam bridges are analyzed under different types of loading in order to observe the pattern of live-load distribution. Based upon the results obtained, design procedures for the determination of lateral live-load distribution are developed and recommended.
1. INTRODUCTION

1.1 General

In recent years several new types of bridge superstructures have been constructed which utilize precast concrete beams. Initially, multi-beam bridges utilized precast, prestressed concrete beams which were placed on the supports side by side, and laterally connected through continuous longitudinal shear keys. Lateral post-tensioning or bolting was used to hold the beams together, and a wearing surface served as the deck. The next development in the design and construction of multi-beam bridges was the use of the cast-in-place reinforced concrete slab and the elimination of lateral post-tensioning or bolting.

In the state of Pennsylvania, precast, prestressed concrete box-beams have been used in multi-beam bridges. The latest development in this state is the spread box-beam bridge, in which the beams are equally spaced and spread apart, with a cast-in-place composite slab. This is the type of bridge for which the load distribution is to be investigated.

The Pennsylvania Department of Highways has adopted some of the provisions listed in the ASSHO specifications for I-beam bridges,¹ to be used in the design of spread box-beam bridges. According to these provisions, set forth by PDH,² the interior beams should be designed using a live-load distribution factor of $\frac{S}{5.5}$, where $S$ is the center-to-center beam spacing.
For exterior beams, the live-load distribution factor is based on the reaction of the wheel load obtained by assuming the flooring to act as a simple span between beams.

The problem of load distribution in spread box-beam bridges has been under investigation at Lehigh University since 1964. The investigation was initiated by a pilot field study of the Drehersville Bridge, and continued with field study of the Berwick, Brookville, White Haven, and Philadelphia bridges. The theoretical investigation reported here was begun by the author in 1967.

1.2 Object and Scope of Study

The theoretical analysis reported here concerns the load distribution in spread box-beam bridges. Spread box-beam bridges are composed of a number of precast, prestressed box-beams and a cast-in-place slab. Fig. 1 shows a typical transverse cross-section of this type of bridge.

The purpose of this study is to develop a method of analysis to describe the behavior of spread box-beam bridges under the application of live-loads. Based upon this theoretical analysis, design recommendations are suggested for the fraction of live-loads to be carried by the box-beams.

The scope of this study can be outlined as follows:

1. Only simple span right bridges are studied.

2. Load distribution is investigated for the service load (working load) range.
3. A theoretical analysis is developed to describe the behavior of spread box-beam bridges under the application of any type of vertical loading.

4. The effect of sidewalk or safety curb is considered.

5. Maximum moments produced in any beam due to standard design vehicles are determined.

6. The results of this theoretical analysis are compared with those of the experimental study on the type of bridge under consideration.

7. The influence of significant factors in the analysis is studied.

8. A simplified procedure for the determination of lateral load distribution is recommended.

The method of analysis described here is applicable to any beam-slab type bridge superstructure, with only slight modifications. In the formulation of the method only box-beams are considered, but modifications which should be made for other types of beams are also specified.

1.3 Previous Studies

The problem of load distribution in bridges has been under investigation by many researchers in this country and abroad for several decades. The results of their work have provided some insight into the behavior of bridges under the applied loads. The variety of types of bridges, and the complex structure of each
type, have made it impossible to develop a unique and exact solution to this problem. Recent advancements in science and technology, on the other hand, make it possible to re-evaluate methods and techniques of analysis and to obtain a more accurate solution to complex problems. Research within the past decade on the problem of load distribution has contributed more to the understanding of the behavior of bridges than all other research previous to that time. This achievement is due to the availability of the new generation of fast computers, and to the development of more sophisticated and rigorous methods of analysis.

Research in the problem of load distribution began in the 1930's, following the construction of several composite structures. One of the earlier studies was by Timoshenko, who used a method of equating deflections at beam intersections to analyze a hinged grid system. Hetenyi analyzed a system composed of a hinged grillage of beams by assuming no rotation at the intersection of longitudinal and transverse members, and by using Fourier series to express the concentrated loads. Pippard and deWaele developed a method in which it was assumed that the longitudinal members did not rotate and the transverse members were replaced by a continuous medium.

An extensive study of slab beam bridges was begun in 1936 at the University of Illinois. Newmark developed a distribution procedure for the analysis of slabs continuous over flexible beams. In this method, the moments and shears were calculated.
for each term of the Fourier series expanded for the loads. The procedure is analogous to Cross's moment distribution for the analysis of continuous beams and frames. Based on this procedure, Newmark and Siess\textsuperscript{13} analyzed fifty-two I-beam bridges of various spans and beam stiffness. In addition, several scale-model I-beam bridges were tested in order to verify the accuracy of the results predicted by their theoretical analysis.

As noted by Viest, Fountain and Siess,\textsuperscript{14} the first specifications for the design of composite highway bridges were published in 1944, as a part of the "Standard Specifications" of the American Association of State Highway Officials. Since the publication of the 1944 edition of the AASHO specifications, numerous field tests and theoretical analyses have been carried out, resulting in revision of, and addition to, these specifications. Some of the significant methods of analysis are discussed below.

One method, originated by Guyon and Massonet, involves the use of the orthotropic plate theory. In this method an equivalent orthotropic plate replaces the actual bridge superstructure. Guyon\textsuperscript{16} first used this method for the analysis of grillages without torsional stiffness and Massonet\textsuperscript{16} then expanded it to include the effect of torsion. This method was used by Morrice and Little\textsuperscript{17} for the analysis of load distribution in prestressed concrete bridges by Roseli and Walther,\textsuperscript{18} and by many others for multi-beam bridges.

For the analysis of load distribution in multi-beam
bridges Arya,\textsuperscript{19} Pool,\textsuperscript{20} and Khachaturian\textsuperscript{21} considered a series of beams hinged at the top corners, and developed a method of solution using simple beam theory and total potential energy. Abdel-Samad\textsuperscript{22} studied the behavior of elastic thin-walled multi-cell box girders by using the variational method of generalized coordinates developed by Vlasov.\textsuperscript{23} In this study the effects of transverse and longitudinal stiffeners were included whereas the effects of torsional moment and transverse axial elongation were neglected.

Another approach used to analyze bridges for load distribution is the folded plate theory. Two different methods are commonly used for the analysis of folded plates. One method is the so-called ordinary method\textsuperscript{24}, \textsuperscript{25}, \textsuperscript{26} in which the longitudinal response of the plate is governed by the beam theory, and the transverse response by the theory for a continuous one-way slab. This method neglects the effect of torsional moments, in-plane shearing deformations, transverse axial elongation, and longitudinal bending moments. The second method is a stiffness method of analysis developed by Goldberg and Leve\textsuperscript{27} in which plane-stress and two-way slab theories are combined. This method takes into account the effect of all of the above-mentioned neglected quantities in the first method. Scordelis\textsuperscript{28} investigated the problem of load distribution of a simply supported multi-cell box-beam bridge, using the folded plate theory developed by Goldberg and Leve.
2. METHOD OF ANALYSIS

2.1 Basic Assumptions

The bridge considered here is the beam-slab type, supported by prestressed concrete spread box-beams, with a simple span, and without intermediate diaphragms. The design standards of the Pennsylvania Department of Highways specify that intermediate diaphragms must be used when the span length exceeds 45 feet. On the other hand, the results of field tests on the Philadelphia Bridge\(^8\) showed that the intermediate diaphragm had little effect on the load distribution factors. Therefore, in the theoretical analysis developed here, the effect of the intermediate diaphragm is not being considered.

The standard precast prestressed box-beam is shown in Fig. 2. The idealized beams considered in the analysis are prismatic, thereby assuming the same geometrical and mechanical properties throughout their lengths.

Other basic assumptions made in this analysis can be itemized as follows:

1. The full composite action between the beams and the slab is ensured.
2. The slab thickness is uniform throughout the bridge.
3. The beams are equally spaced and spread apart.
4. The beams and the slab are made of a linearly elastic, homogeneous and isotropic material.
5. The beam cross-section is rigid, so that the distortion of the cross-section is negligible in comparison with the deformation of the whole section as a unit.

6. The shear deformations are negligible.

7. The beams have negligible warping rigidity in comparison with pure torsional rigidity.

8. The deformations are assumed to be small.

9. The end diaphragms are free to rotate out of their own planes but rigid against in-plane bending and twisting.

2.2 Basic Elements and Coordinate System

Considering each beam with the portion of slab directly on top of it as a unit, a series of joint-lines are introduced between the slab and the beams. Fig. 3 shows the location of these joint-lines. The bridge is composed of a finite number of beam and plate elements which are attached along the joint-lines. The right-handed coordinate system shown in Fig. 3 is a typical reference axis system, which was used in analyzing the beam and the plate elements. A typical element i is connected to element (i-1) through the joint-line j, and to element (i+1) through the joint-line (j+1).

2.3 Method of Solution

Considering the defined beam and plate elements, the
bridge superstructure is reduced to an articulated system. To obtain a solution to this highly indeterminate structure, a flexibility type of analysis is employed. Based upon assumption 9, all of the beam and plate elements will behave as if they were simply-supported at both ends. To proceed with the solution, first a series of cuts are introduced along the joint-lines. The unknown stress resultants existing along each joint-line consist of three forces $S$, $H$ and $R$ acting along the $x$, $y$, and $z$ axes respectively, and a moment $M$ about the $x$ axis. These stress resultants are shown for a typical point along the joint-lines in Fig. 4. The reference axes for the beam and plate elements are also shown in this figure.

The beam and plate elements under the edge forces, as well as the applied external loads, are next analyzed. The analysis of the plate involves two different approaches. First, a small-deflection bending analysis for the applied external loads, the edge force $R$, and the edge moment $M$ (see Fig. 4), and second, a stress analysis for the in-plane edge forces $H$ and $S$. For the analysis of beams under the applied external load and the edge forces, a combined action of torsion, axial force, and bi-axial moment is considered. The applied external load is considered to be distributed over a rectangular area with dimensions $2c$ and $2d$. For the analysis of the beam elements, the external load is considered to be distributed over a line segment with dimension $2c$.

The deformation of the beam and plate elements along the
joint-lines consists of three linear displacements in the x, y and z directions, while the fourth is the rotation about the x axis. The continuity conditions along the joint-lines are satisfied by equating the deformations of the adjacent beam and plate elements. The continuity conditions along each joint-line provide four equations in terms of the unknown stress resultants. For a bridge with \( N_B \) number of beams, the number of joint-lines is \( 2 (N_B - 1) \), and the number of unknown stress resultants is \( 8 (N_B - 1) \). There are also exactly \( 8 (N_B - 1) \) equations of continuity which can be solved simultaneously to find the unknowns. After the unknown stress resultants along the joint-lines are found, the forces and moments in the beam elements can be evaluated.

2.4 External Loads

The live-load used in the design of bridge superstructures consists of either standard truck loads or the alternative uniformly distributed lane loads as specified in the AASHO specifications.\(^1\) The standard HS20-44 truck will be considered here for the live-load, since the uniformly distributed lane loads will govern the design only when the span length of the bridge is very large (larger than 120 feet for reactions and 140 feet for bending moments). The HS20-44 truck consists of a 20-ton standard truck with a 16-ton semi-trailer. The spacing between the front and the drive axles is 14 feet, and the distance between the drive and rear axles varies from 14 feet to 30 feet. The variable
axle spacing is provided to approximate more closely the trucks now in use, and to provide a more satisfactory design loading for continuous spans.

The drive and rear wheels consist generally of two tires each. The contact area between the wheels and the surface of the bridge varies depending on the dimensions and the inflation pressure of the tires, and the wheel loads. As shown in Fig. 5, the contact area has an oval shape that can be assumed as a rectangular area with dimensions $2c_1$ and $2d_1$. The dimension $2c_1$ is assumed to be 10 inches for all wheels. The dimension $2d_1$ is assumed to be 24 inches for the drive and rear wheels, and 12 inches for the front wheels. The distribution of load through the wearing surface will increase the dimensions of the loaded area on the top of the slab to $2c$ and $2d$, as shown in Fig. 5. Since for the type of bridge under consideration a 1/2 inch monolithic wearing surface is included in the slab thickness, the dimensions $2c$ and $2d$ are taken to be the same as $2c_1$ and $2d_1$ respectively.

2.5 Curbs and Parapets

The bridge superstructures contain the cast-in-place reinforced concrete curbs and parapets. Basically, the curbs and parapets are not installed as load-carrying members, and in the present design procedures no account is taken of the curbs and parapets in the design of exterior beams for the live-load. On the other hand, the results of the field tests$^{3,6,7,8}$
consistently showed the full effectiveness of the curbs in composite action with the exterior beams. The composite action of parapets with curbs, however, was not fully effective in the Philadelphia Bridge test.\textsuperscript{8}

In the theoretical analysis developed here, the curbs (and the parapets if so desired) can be considered to be effective in composite action with the exterior beams. The interior beam elements are symmetrical with respect to the $z$-axis, whereas with the consideration of the curbs (or curbs and parapets both), the exterior beam elements are no longer symmetrical. Therefore, the bi-axial bending of the exterior beams would become coupled.
3. ANALYSIS OF PLATE ELEMENTS

3.1 General

A typical plate element isolated from the rest of the bridge superstructure is shown in Fig. 6. The plate element is simply-supported along the two ends, and is free along the two lateral sides. The length of the plate element, which is the span length of the bridge, is designated by $L$. The width of the plate element, which is the clear spacing of the beams, is designated by "$a$". The wheel-load $P$ is distributed over an area of the plate element with dimensions of $2c$ and $2d$. The intensity of this distributed load, $p$, is:

$$p = \frac{P}{4cd} \quad (3.1)$$

The $x$ and $y$ coordinates of the centroid of the loaded area are designated by $\xi$ and $\eta$, respectively.

For the analysis of the beam elements, as explained before, the wheel loads are considered to be distributed over a line segment with the dimension of $2c$. This line loading was considered since there would be no difference in the results of the analysis of the beam elements whether the load is distributed over a rectangular area or a line segment. The intensity of this distributed line load $p_b$ on the beam elements is:

$$p_b = 2d \cdot p \quad (3.2)$$

As noted in Fig. 4, the $x$ and $y$ coordinates of the centroid of
the line loading on the beam elements, are also designated by $\xi$ and $\eta$. When the results of the analysis of all beam and plate elements are to be combined, the quantities associated with the beam and plate elements will take the superscript $i$, $(i+1)$, etc., and those associated with the joint-lines will take the superscript $j$, $(j+1)$, etc.

3.2 Fourier Series Expansion of Wheel Loads

To proceed with the analysis of the beam and plate elements, the wheel loads are expanded in terms of the Fourier series. Since both the plate and the beam elements are simply-supported at the ends, a Fourier sine-series expansion is employed in the longitudinal direction along the $x$-axis, with the period of $2L$.

The Fourier sine-series expansion for the distributed line loading $p_b$, on the beam elements yields:

$$\bar{P}_b = p_b \sum_{n=1}^{\infty} Q_n \sin \alpha_n x$$  \hspace{1cm} (3.3)

where

$$Q_n = \frac{u}{n \pi} \sin \alpha_n \xi \sin \alpha_n \eta$$  \hspace{1cm} (3.4)

$$\alpha_n = \frac{n \pi}{L}$$  \hspace{1cm} (3.5)

and $n$ is an integer-number defining different terms of the series, that is,

$$n = 1, 2, 3, 4, \ldots \ldots$$

For the plate elements a strip of the loaded area, with
an infinitesimal width of $dy_1$ at a distance $y_1$ from the $x$-axis, is expanded by the Fourier sine-series. Fig. 7 shows the location of the defined strip loading and the intensity of this strip loading $p_1$, is:

$$p_1 = p \, dy_1$$

(3.6)

where $y_1$ can vary between the following limits:

$$(T_1-d) \leq y_1 \leq (T_1+d)$$

(3.7)

Once the results of the analysis of the plate elements under the distributed strip loading $p_1$ are obtained, it would take only a simple integration to obtain the results for the complete loaded area. The significance of this approach will become clear later.

The Fourier sine-series expansion for the distributed strip loading $p_1$, yields:

$$\tilde{p}_1 = p_1 \sum_{n=1}^{\infty} Q_n \sin \alpha_n x$$

(3.8)

where $Q_n$ and $\alpha_n$ are given in Eqs. (3.4) and (3.5) respectively.

The deflected surface of the plate elements, due to the strip loading, and assuming a small-deflection bending analysis, can also be expressed by sine-series:

$$w_1 = \sum_{n=1}^{\infty} Y_n(y) \sin \alpha_n x$$

(3.9)

where $Y_n(y)$ is a function of $y$ and the integer-number $n$, and is independent of the variable $x$. From Eq. (3.9) it can easily be verified that all of the force and geometric boundary conditions at
the simply-supported ends, that is, at x=0 and x=L, are automatically satisfied. This explains the reason behind the selection of the sine-series for the expansion of the wheel loads.

Since the material is assumed linearly elastic, and the deformations small, the principle of superposition is valid. The validity of this principle for the bridge under consideration has been verified by the field test. Therefore, in the following analysis, only one general term of the Fourier sine-series of the loading will be considered. The final results will be obtained by superimposing the results of the analysis for all the terms in the series. Considering a particular value of in Eq. (3.3), the loading for the beam elements reduces to:

\[ \bar{p}_{bn} = p_n \sin \alpha_n x \]  \hspace{1cm} (3.10)

where

\[ p_n = 2 d_0 p Q_n \]  \hspace{1cm} (3.11)

For the plate elements the loading of Eq. (3.8) reduces to:

\[ \bar{p}_{ln} = p_0 dy_1 Q_n \sin \alpha_n x \]  \hspace{1cm} (3.12)

where p, Q and \( \alpha_n \) are given in Eqs. (3.1), (3.4) and (3.5) respectively. The deflected surface of the plate elements given by Eq. (3.9) yields:

\[ w_{ln} = Y_n (y) \sin \alpha_n x \]  \hspace{1cm} (3.13)

From the form of deformations of the beam and plate elements, it
can be observed that the four stress resultants along the joint-lines should be in the following form:

\[ S = S_n \cos \alpha_n x \]  
(3.14a)

\[ H = H_n \sin \alpha_n x \]  
(3.14b)

\[ R = R_n \sin \alpha_n x \]  
(3.14c)

\[ M = M_n \sin \alpha_n x \]  
(3.14d)

where \( S_n, H_n, R_n \) and \( M_n \) are the unknown coefficients yet to be determined.

3.3 Plate Elements Under Vertical Edge Forces

Fig. 8 shows the plate element (i-1) subjected to the vertical edge forces \( R^j \) and \( R^{j-1} \). The \( R^j \) and \( R^{j-1} \) forces are acting on the plate element at \( y=a \) and \( y=0 \), respectively. The solution is obtained for the \( R^j \) and \( R^{j-1} \) forces separately.

3.3.1 Vertical Edge Force at \( y=a \)

The edge force \( R^j \) acting along the line \( y=a \) of the plate element (i-1) has the following form:

\[ R^j = R_n^j \sin \alpha_n x \]  
(3.15)

where \( R_n^j \), the unknown coefficient associated with joint-line \( j \), is independent of the \( x \) and \( y \) variables. The governing differential equation for the small-deflection bending theory of the plate is:

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0 \]  
(3.16)
The deflected surface of the plate can be expressed by:

\[ w = R_n^j Y_n(y) \sin \alpha_n x \]  \hspace{1cm} (3.17)

where \( Y_n(y) \) is a function of \( y \) and yet to be determined. The coefficient \( R_n^j \) in Eq. (3.17) has been introduced for the sake of simplicity in the final results.

Substituting the corresponding derivatives of Eq. (3.17) into Eq. (3.16) results in an ordinary differential equation:

\[ Y_n^{IV} - 2 \alpha_n^2 Y_n^{II} + \alpha_n^4 Y_n = 0 \]  \hspace{1cm} (3.18)

The solution to Eq. (3.18) yields:

\[ Y_n(y) = A_n \cosh \alpha_n y + B_n \alpha_n y \sinh \alpha_n y + C_n \sinh \alpha_n y + D_n \alpha_n y \cosh \alpha_n y \]  \hspace{1cm} (3.19)

where \( A_n, B_n, C_n \) and \( D_n \) are constants, to be determined by consideration of boundary conditions. Substituting Eq. (3.19) into Eq. (3.17) results in:

\[ w = R_n^j (A_n \cosh \alpha_n y + B_n \alpha_n y \sinh \alpha_n y + C_n \sinh \alpha_n y + D_n \alpha_n y \cosh \alpha_n y) \sin \alpha_n x \]  \hspace{1cm} (3.20)

The boundary conditions at the simply-supported ends are:

\[ w = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \ x = 0 \ \text{and} \ L \]  \hspace{1cm} (3.21)

The above boundary conditions are automatically satisfied, as expected. The boundary conditions along the free lateral ends are:
\[-D \left[ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] = \begin{cases} 0 & \text{at } y=0 \\ \mathbf{R}^j & \text{at } y=a \end{cases} \tag{3.22a}\]

\[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } y=0 \text{ and } a \tag{3.22b}\]

where

\[D = \frac{E t^3}{12(1-\nu^2)} \tag{3.23}\]

D is the flexural rigidity and t is the thickness of the plate. \(\nu\) is the Poisson's ratio and E is the modulus of elasticity of the material. Substituting the corresponding derivatives of Eq. (3.20) into the four boundary conditions given by Eqs. (3.22a) and (3.22b) results in four linear simultaneous equations in terms of the coefficients \(A_n\), \(B_n\), \(C_n\) and \(D_n\). The solution to these equations is:

\[A_n = \frac{-2}{1-\nu} B_n \tag{3.24a}\]

\[B_n = \frac{-1}{\alpha_n^3 D} \frac{\nu+3}{\alpha_n^2} \frac{\sinh \alpha_n a + (1-\nu) \alpha_n^a \cosh \alpha_n a}{(\nu+3)^2 \sinh^2 \alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \tag{3.24b}\]

\[C_n = \frac{1+\nu}{1-\nu} D_n \tag{3.24c}\]

\[D_n = \frac{1-\nu}{\alpha_n^3 D} \frac{\alpha_n a \sinh \alpha_n a}{(\nu+3)^2 \sinh^2 \alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \tag{3.24d}\]
Therefore, Eq. (3.20) together with Eqs. (3.24a, b, c and d) completely define the deflected surface of the plate.

### 3.3.2 Vertical Edge Force at y=0

The plate element (i-1) is now considered under the edge force $R_{j-l}$ acting along the joint-line (j-l). The solution to this case can in general be obtained directly by following the same procedure as outlined in Section 3.3.1. However, the solution of Section 3.3.1 can be used in this case by a simple transformation of the reference axes. Substituting $(a-y)$ for $y$, $R_{j-l}^n$ for $R^n_j$ and $(-w)$ for $w$ in Eq. (3.20) results in the solution for this case:

$$w = -R_{j-l}^n \left[ A_n \cosh \alpha_n (a-y) + B_n \alpha_n (a-y) \sinh \alpha_n (a-y) + C_n \sinh \alpha_n (a-y) + D_n \alpha_n (a-y) \cosh \alpha_n (a-y) \right] \sin \alpha_n x$$

(3.25)

where $A_n$, $B_n$, $C_n$ and $D_n$ are given by Eqs. (3.24a, b, c and d).

### 3.4 Plate Elements Under the Edge Moments

As shown in Fig. 9, the plate element (i-1) is subjected to edge moments $M^j$ and $M^{j-1}$ along the joint-lines $y=a$ and $y=0$, respectively. The analysis of the plate under the edge moments is similar to that in Section 3.3.

#### 3.4.1 Edge Moment at y=a

The element (i-1) is first considered under the edge moment:

$$M^j = M^j_n \sin \alpha_n x$$

(3.26)
The deflected surface of the plate is similar to Eq. (3.20):

\[ w = M_n \left( A_n \cosh \alpha_n y + B_n \alpha_n y \sinh \alpha_n y + C_n \sinh \alpha_n y + D_n \alpha_n y \cosh \alpha_n y \right) \sin \alpha_n x \]  

(3.27)

The boundary conditions at the free lateral sides are

\[ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad \text{at } y = 0 \text{ and } a \]  

(3.28a)

\[ -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = \begin{cases} 
0 & \text{at } y = 0 \\
M_n & \text{at } y = a 
\end{cases} \]  

(3.28b)

Substitution of the corresponding derivatives of Eq. (3.27) into Eqs. (3.28a and b) results in four linear simultaneous equations in terms of \( A_n' \), \( B_n' \), \( C_n' \) and \( D_n' \). The solution to these equations yields:

\[ A_n' = \frac{-2}{1-\nu} B_n' \]  

(3.29a)

\[ B_n' = -\alpha_n D_n \]  

(3.29b)

\[ C_n' = \frac{1+\nu}{1-\nu} D_n' \]  

(3.29c)

\[ D_n' = \frac{1}{\alpha_n^2 D} \frac{(\nu+3) \sinh \alpha_n a - (1-\nu) \alpha_n a \cosh \alpha_n a}{(\nu+3)^2 \sinh^2 \alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \]  

(3.29d)

where \( D_n \) is given by Eq. (3.24d).
3.4.2 **Edge Moment at y = 0**

The edge moment $M_{j-1}^i$ is applied to the plate element $(i-1)$, along the line $y = 0$. The solution to this case is obtained by substituting $(a-y)$ for $y$, $(-M_{n}^{j-1})$ for $M_{n}^{j}$ and $(-w)$ for $w$ in Eq. (3.27):

$$w = M_{n}^{j-1} \left[ A'_n \cosh \alpha_n (a-y) + B'_n \alpha_n (a-y) \sinh \alpha_n (a-y) + C'_n \sinh \alpha_n (a-y) + D'_n \alpha_n (a-y) \cosh \alpha_n (a-y) \right] \sin \alpha_n x$$

(3.30)

where $A'_n$, $B'_n$, $C'_n$ and $D'_n$ are given by Eqs. (3.29a, b, c and d).

3.5 **Reciprocal Relations for the Plate Elements**

For any structure with linear load-deformation characteristics, the reciprocal theorems, known as "Betti's Law", are applicable. However, the linear load-deformation relationship can also be dependent on the loading. "Betti's Law" can be expressed as follows:

The work done by a set of loads acting through the displacements produced by a second set of loads is equal to the work done by the second set of loads acting through the displacements produced by the first set of loads.

Betti's Law can be used in the analysis presented here, since the load-deformation relationship was assumed linear. In addition, there are some specific reciprocal relations which can be applied.
to plates. In this section, with the use of Betti's Law, some particular reciprocal relations are derived for the plates simply-supported along two ends.

Consider a plate simply-supported along two ends, and with any boundary conditions along the other two sides. As shown in Fig. 10, two arbitrary lines $y = y_1$ and $y = y_2$, are chosen perpendicular to the simply-supported ends. Points 1 and 2 are located on the lines $y = y_1$ and $y = y_2$, some distance $x_1$ from the $y$ axis. Referring again to Fig. 10, the following reciprocal relations are valid:

1. Deflection of the plate along the line $y = y_2$ due to a load $P$ at point 1, is identical to the deflection along the line $y = y_1$ due to the load $P$ at point 2.

2. The slope $\frac{\partial w}{\partial y}$ along the line $y = y_2$ due to a load $P$ at point 1, is identical to the deflection along the line $y = y_1$, due to a moment $M$ equal to $P$ at point 2.

To prove the validity of the above reciprocal relations, two equal loads $P$, designated by $P_1$ at point 1 and $P_2$ at point 2, are expanded by the Fourier series:

$$\tilde{P}_1 = \tilde{P}_2 = \sum_{n=1}^{\infty} P_n \sin \alpha_n x$$  \hspace{1cm} (a)

where

$$P_n = \frac{2P}{L} \sin \frac{n \pi \alpha_n x_1}{L}$$  \hspace{1cm} (b)

$$\alpha_n = \frac{n \pi}{L}$$  \hspace{1cm} (c)
Since the Fourier series expansion is identical for the loads $P_1$ and $P_2$, the reciprocal relations will be proven for a general term of the loading series. Thus, Eq. (a) reduces to:

$$\bar{P}_{1n} = \bar{P}_{2n} = P_n \sin \alpha_n x$$  \hspace{1cm} (d)

Designating the deflection of the plate due to the loads $\bar{P}_{1n}$ and $\bar{P}_{2n}$ by $w^{(1)}$ and $w^{(2)}$ respectively, then:

$$w^{(1)} = Y_n(y) \sin \alpha_n x$$  \hspace{1cm} (e)

$$w^{(2)} = Z_n(y) \sin \alpha_n x$$  \hspace{1cm} (f)

where $Y_n(y)$ and $Z_n(y)$ are functions of $y$ and independent of the variable $x$. The deflection of the plate along the line $y = y_2$ due to the load $\bar{P}_{1n}$, and along the line $y = y_1$ due to the load $\bar{P}_{2n}$ can be found from Eqs. (e) and (f). The results are:

$$w^{(1)} \bigg|_{y = y_2} = Y_n(y_2) \sin \alpha_n x$$  \hspace{1cm} (g)

$$w^{(2)} \bigg|_{y = y_1} = Z_n(y_1) \sin \alpha_n x$$  \hspace{1cm} (h)

Applying Betti's Law, the work done by the load $\bar{P}_{1n}$ through the deflection produced by the load $\bar{P}_{2n}$, is equal to the work done by the load $\bar{P}_{2n}$ through the deflection produced by the load $\bar{P}_{1n}$. Thus:

$$\int_0^L (P_n \sin \alpha_n x) Z_n(y_1) \sin \alpha_n x \, dx = \int_0^L (P_n \sin \alpha_n x) Y_n(y_2) \sin \alpha_n x \, dx$$  \hspace{1cm} (i)
where \( P_n \) is given in Eq. (b) and is independent of the variable \( x \). Thus, Eq. (i) reduces to:

\[
Z_n(y_1) = Y_n(y_2) \quad (j)
\]

Substituting Eq. (j) into Eqs. (g) and (h) results in the following:

\[
w^{(1)}y = y_2 \quad ; \quad w^{(2)}y = y_1 \quad (3.31)
\]

The proof is thus complete. The second reciprocal relation stated above can be proven in a similar fashion.

The above-stated reciprocal relations are also valid for line loadings, since a line loading can be thought of as a series of concentrated loads. For the analysis of the plate elements under the wheel-loads, the above-stated reciprocal relations will be used in order to avoid the complicated and lengthy direct analysis.

3.6 **Plate Elements Under the Applied Wheel Loads**

The plate element, shown in Fig. 6, is subjected to a wheel load uniformly distributed over a rectangular area. The direct solution to this case involves the determination of a general solution to the governing differential equation:

\[
\nabla^4 w = q \quad (3.32)
\]

where \( \nabla^4 w \) is defined by the left-hand side of Eq. (3.16), and \( q \)
is the distributed load on the plate. The general solution is made up of two parts. First a complementary solution satisfying Eq. (3.16) and second, a particular solution satisfying Eq. (3.32). The complementary solution is given by Eq. (3.20), in which $R_{jn}$ would be omitted. The particular solution can be selected as the solution to an infinitely long plate, simply-supported along its two limited sides. This selection can be made since it is not necessary for the particular solution to satisfy all of the boundary conditions. The general solution, obtained by the summation of the complementary and particular solutions, should satisfy all of the boundary conditions.

The direct solution as outlined above is mathematically very complicated and lengthy. On the other hand, the reciprocal relations derived in Section (3.5) make it possible to obtain the solution indirectly and very simply. Fig. 7 shows the strip of the loaded area with an infinitesimal width of $dy_1$, as was introduced in Section (3.2). The general term of the Fourier series expansion for this strip loading is given in Eq. (3.12). The displacements of the plate elements along the joint-lines due to the strip loading can be obtained indirectly by applying the reciprocal relations of Section (3.5). To include the effect of the complete loaded area, the results of the strip loading will be integrated with respect to $y_1$ between the limits given in Eq. (3.7).

The deflection of the plate shown in Fig. 7 along the edge $y=a$, due to the strip loading of Eq. (3.12), is obtained by

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substituting \((p \frac{dy}{d\gamma} Q_n)\) for \(R^j\), and \(y_1\) for \(y\) in Eq. (3.20):

\[
w_{y=a} = p \frac{dy}{d\gamma} Q_n (A_n \cosh \alpha_n y_1 + B_n \alpha_n y_1 \sinh \alpha_n y_1
\]

\[
+ C_n \sinh \alpha_n y_1 + D_n \alpha_n y_1 \cosh \alpha_n y_1 \) \sin \alpha_n x \tag{3.33}
\]

where \(A_n, B_n, C_n\) and \(D_n\) are given in Eqs. (3.24a, b, c and d).

The integration of Eq. (3.33) with respect to \(y_1\) between the limits given by Eq. (3.7) yields:

\[
w_{y=a} = K_n \sin \alpha_n x \tag{3.34}
\]

where

\[
K_n = p \frac{Q}{n} (A_n a_n + B_n b_n + C_n c_n + D_n d_n) \tag{3.35}
\]

and

\[
a_n = \frac{2}{\alpha_n} \sinh \alpha_n d \cosh \alpha_n \gamma \tag{3.36a}
\]

\[
b_n = 2d \cosh \alpha_n d \cosh \alpha_n \gamma \frac{c_n - a_n}{c_n} \tag{3.36b}
\]

\[
c_n = \frac{2}{\alpha_n} \sinh \alpha_n d \sinh \alpha_n \gamma \tag{3.36c}
\]

\[
d_n = 2d \cosh \alpha_n d \sinh \alpha_n \gamma + \alpha_n \gamma \frac{a_n - c_n}{a_n} \tag{3.36d}
\]

The slope \(\frac{\partial w}{\partial y}\) along the joint-line \(y=a\), due to the wheel load, is obtained by first substituting \(y_1\) for \(y\) and \((-p \frac{dy}{d\gamma} Q_n)\)
for $M_n^j$ in Eq. (3.27), and then integrating it with respect to $y$, between the limits given by Eq. (3.7). The result is the following:

$$\frac{\partial w}{\partial y} \bigg|_{y=a} = K_{ln} \sin \alpha_n x$$  \hspace{1cm} (3.37)

where

$$K_{ln} = p Q_n \left( A_n \ a_n + B_n \ b_n + C_n \ c_n + D_n \ d_n \right)$$  \hspace{1cm} (3.38)

$A_n', B_n', C_n'$ and $D_n'$ are given in Eqs. (3.29a, b, c and d).

$A_n$, $B_n$, $C_n$ and $D_n$ are given in Eqs. (3.36a, b, c and d).

The deflection and the slope $\frac{\partial w}{\partial y}$ along the joint-line $y=0$ can be found in a similar way by applying the reciprocal relations and using Eqs. (3.25) and (3.30). However, the solution for the previous case given by Eqs. (3.34) and (3.37) can be applied to this case by a simple transformation of the reference axes. Substituting $(a-\eta)$ for $\eta$, $(-w)$ for $w$ and $(-p)$ for $p$ in Eqs. (3.34), (3.35), (3.36a, b, c and d), (3.37) and (3.38) results in the following solution for the deflection and the slope:

$$w \bigg|_{y=0} = G_n \sin \alpha_n x$$  \hspace{1cm} (3.39)

$$\frac{\partial w}{\partial y} \bigg|_{y=0} = G_{ln} \sin \alpha_n x$$  \hspace{1cm} (3.40)

where

$$G_n = p Q_n \left( A_n \ a_n' + B_n \ b_n' + C_n \ c_n' + D_n \ d_n' \right)$$  \hspace{1cm} (3.41)
\[ G_{jn} = P Q_n (A_n a_n + B_n b_n + C_n c_n + D_n d_n) \]  
(3.42)

and

\[ a_n' = \frac{2}{\sigma_n} \sinh \alpha_n d \cosh \alpha_n (a-n) \]  
(3.43a)

\[ b_n' = 2d \cosh \alpha_n d \cosh \alpha_n (a-n) c_n' - a_n' \]  
(3.43b)

\[ c_n' = \frac{2}{\sigma_n} \sinh \alpha_n d \sinh \alpha_n (a-n) \]  
(3.43c)

\[ d_n' = 2d \cosh \alpha_n d \sinh \alpha_n (a-n) + \alpha_n (a-n) a_n' - c_n' \]  
(3.43d)

### 3.7 Plate Elements Under the In-Plane Edge Forces

For the analysis of the plate elements under the in-plane edge forces, a plane-stress elasticity solution is employed. The plane-stress condition is defined by assuming that only the stress components \( \sigma_x, \sigma_y \) and \( \tau_{xy} = \tau_{yx} \) exist. Furthermore, these stress components are a function of the \( x \) and \( y \) variables and are independent of \( z \). The notations \( \sigma_x \) and \( \sigma_y \) denote the stress components normal to the planes perpendicular to the \( x \) and \( y \) axes respectively. The notation \( \tau_{xy} \) denotes the shearing stress on a plane perpendicular to the \( x \) axis and directed toward the \( y \) axis. The in-plane displacements \( u \) along the \( x \) axis, and \( v \) along the \( y \) axis are also a function of the \( x \) and \( y \) variables and independent of \( z \). In this section a typical plate element (i-1) subjected to the normal edge forces \( H^1 \) and \( H^{1-1} \), and the tangential edge forces \( S^1 \) and \( S^{1-1} \) will be analyzed.
3.7.1 Plane-Stress Solution

The differential equations of equilibrium for the plane-stress condition and in the absence of the body-forces are in the following form:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (3.44a)
\]
\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad (3.44b)
\]

Eqs. (3.44a and b) provide the necessary and sufficient condition for the existence of a function \( \psi(x,y) \) such that:

\[
\sigma_x = \frac{\partial^2 \psi}{\partial y^2} \quad (3.45a)
\]
\[
\sigma_y = \frac{\partial^2 \psi}{\partial x^2} \quad (3.45b)
\]
\[
\tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y} \quad (3.45c)
\]

The function \( \psi \) is called the "Airy stress function" in honor of G. B. Airy, who first introduced the concept of the stress function in 1862. When the stress function \( \psi \) is used for the solution to the plane-stress problems, the evaluation of stress components given in Eqs. (3.45a, b and c) insures the satisfaction of the equilibrium requirements given in Eqs. (3.44a and b).

The stress-strain relationships for the material, which was assumed homogeneous, isotropic and linearly elastic, are in
the following form:

\[ \epsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) \]  
\[ (3.46a) \]

\[ \epsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \]  
\[ (3.46b) \]

\[ \gamma_{xy} = \frac{\tau_{xy}}{G} \]  
\[ (3.46c) \]

where

\[ G = \frac{E}{2(1+\nu)} \]  
\[ (3.47) \]

The relationship between the strains and displacements are given as:

\[ \epsilon_x = \frac{\partial u}{\partial x} \]  
\[ (3.48a) \]

\[ \epsilon_y = \frac{\partial v}{\partial y} \]  
\[ (3.48b) \]

\[ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \]  
\[ (3.48c) \]

where the displacements are assumed to be small so that the second order terms can be neglected. From Eqs. (3.48a, b and c), it can be noted that the three strain components \( \epsilon_x \), \( \epsilon_y \) and \( \gamma_{xy} \) cannot be chosen independently, since they are completely defined by the two displacements \( u \) and \( v \). Therefore, there must exist one relation between these three strain components. This relation, which is normally referred to as the compatibility equation, can be written as:
The substitution of Eqs. (3.45a, b and c), (3.46a, b and c) and (3.47) into Eq. (3.49) yields:

\[
\nabla^4 \psi = 0
\]  

(3.50)

where \( \nabla^4 \) is the differential operator defined as follows:

\[
\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}
\]  

(3.51)

Eq. (3.50) is the compatibility condition in terms of the function \( \psi \). Thus, by the use of the stress function \( \psi \), the solution to the plane elasticity problems reduces to determination of a solution to Eq. (3.50), which satisfies the boundary conditions.

To proceed with the solution of Eq. (3.50), the stress function \( \psi \) is selected as follows:

\[
\psi = Y_n(y) \sin \alpha_n x
\]  

(3.52)

where \( Y_n(y) \) is a function of \( y \). The governing differential equation given in Eq. (3.50) and the selected stress function given in Eq. (3.52) are analogous to Eqs. (3.16) and (3.17) respectively. Thus, the function \( Y_n(y) \) is in the same form as that given in Eq. (3.19). The stress function \( \psi \) given in Eq. (3.52) can be written as:
\[
\psi = (A_{1n} \cosh \alpha_{n}y + B_{1n} \alpha_{n}y \sinh \alpha_{n}y \\
+ C_{1n} \sinh \alpha_{n}y + D_{1n} \alpha_{n}y \cosh \alpha_{n}y) \sin \alpha_{n}x
\]  

(3.53)

where \(A_{1n}, B_{1n}, C_{1n}\) and \(D_{1n}\) are constants, to be determined by consideration of the boundary conditions. Substituting the corresponding derivatives of Eq. (3.53) into Eqs. (3.45a, b and c), the stress components are found as follows:

\[
\sigma_x = \alpha_n^2 \psi + 2 \alpha_n^2 (B_{1n} \cosh \alpha_n y + D_{1n} \sinh \alpha_n y) \sin \alpha_n x 
\]  

(3.54a)

\[
\sigma_y = -\alpha_n^2 \psi 
\]  

(3.54b)

\[
\tau_{xy} = -\alpha_n^2 [A_{1n} \sinh \alpha_n y + B_{1n} (\sinh \alpha_n y + \alpha_n y \cosh \alpha_n y) \\
+ C_{1n} \cosh \alpha_n y + D_{1n} (\cosh \alpha_n y + \alpha_n y \sinh \alpha_n y)] \cos \alpha_n x
\]  

(3.54c)

The boundary conditions along the simply-supported sides can be expressed in the following manner:

\[
\sigma_x = 0 \quad \text{at } x = 0 \quad \text{and } L \quad \text{(3.55a)} \\
v = 0 \quad \text{at } x = 0 \quad \text{and } L \quad \text{(3.55b)}
\]

The boundary condition of Eq. (3.55a) is automatically satisfied. The boundary condition given in Eq. (3.55b) will be considered when the in-plane displacements \(u\) and \(v\) are determined.

For the case in which the plate elements are subjected to the edge forces along the lateral side \(y=a\), the boundary conditions along the side \(y=0\) are the following:
\[ \sigma_y = 0 \quad \text{at } y = 0 \]  
(3.56a)

\[ \tau_{xy} = 0 \quad \text{at } y = 0 \]  
(3.56b)

Satisfaction of the boundary conditions of Eqs. (3.56a and b) results in:

\[ A_{ln} = 0 \]  
(3.57a)

\[ B_{ln} = - C_{ln} \]  
(3.57b)

Thus, for the case in which the lateral side \( y = 0 \) is free of edge forces, the stress components given in Eqs. (3.54a, b and c) reduce to:

\[ \sigma_x = \alpha_n^2 \left[ B_{ln} \left( 2 \cosh \alpha_n y + \alpha_n y \sinh \alpha_n y \right) - C_{ln} \left( \sinh \alpha_n y + \alpha_n y \cosh \alpha_n y \right) \right] \sin \alpha_n x \]  
(3.58a)

\[ \sigma_y = - \alpha_n^2 \left[ C_{ln} \alpha_n y \sinh \alpha_n y + B_{ln} \left( \sinh \alpha_n y - \alpha_n y \cosh \alpha_n y \right) \right] \sin \alpha_n x \]  
(3.58b)

\[ \tau_{xy} = - \alpha_n^2 \left[ B_{ln} \left( \sinh \alpha_n y + \alpha_n y \cosh \alpha_n y \right) - C_{ln} \alpha_n y \right] \cos \alpha_n x \]  
(3.58c)

where \( B_{ln} \) and \( C_{ln} \) are yet to be determined by consideration of the boundary conditions along the side \( y = a \).

The in-plane displacements \( u \) and \( v \) are found by substituting Eqs. (3.58a, b and c) and (3.46a, b and c) into Eqs. (3.48a, b and c) and integrating the results in Eqs. (3.48a) and (3.48b)
with respect to the x and y variables respectively. The results are:

\[
\begin{align*}
u &= \frac{-\alpha_n^2}{E} \left\{ B_{ln} \left[ 2 \cosh \alpha_n y + (1+\nu) \alpha_n y \sinh \alpha_n y \right] \\
&\quad - C_{ln} \left[ (1-\nu) \sinh \alpha_n y + (1+\nu) \alpha_n y \cosh \alpha_n y \right] \right\} \cos \alpha_n x \\
v &= \frac{-\alpha_n^2}{E} \left\{ B_{ln} \left[ (l+\nu) \alpha_n y \cosh \alpha_n y - (1-\nu) \sinh \alpha_n y \right] \\
&\quad - C_{ln} \left[ -2 \cosh \alpha_n y + (1+\nu) \alpha_n y \right] \right\} \sin \alpha_n x
\end{align*}
\] (3.59a,b)

When the displacements u and v are first obtained from the integration of the results in Eqs. (3.48a) and (3.48b), they will contain some unknown functions \( Z_n(y) \) and \( X_n(x) \) respectively. These two unknown functions can be proven to be linear functions corresponding to the rigid-body motion of the plate by satisfying Eq. (3.48c). This rigid-body motion is then eliminated by the application of the boundary condition of Eq. (3.55b).

3.7.2 Normal Edge Force at \( y=a \)

The plate element (i-1), shown in Fig. 11, is subjected to the in-plane normal edge forces \( H_j \) and \( H_j \) along the edges \( y=a \) and \( y=0 \) respectively. The solution is first obtained for a situation in which only the edge force \( H_j \) is applied and thus, the lateral side \( y=0 \) is free of any edge forces. The edge force \( H_j \) is in the following form:

\[
H_j = H_n^j \sin \alpha_n x
\] (3.60)
The stress components and the in-plane displacements for this case are those given in Eqs. (3.58a, b and c) and (3.59a and b) respectively. Therefore, the solution to this case reduces to the determination of the constants $B_{ln}$ and $C_{ln}$.

The boundary conditions along the lateral side $y=a$ can be written as:

$$\sigma_y = \frac{H_j}{t} \sin \alpha_n x \quad \text{at } y = a \quad (3.61a)$$

$$\tau_{xy} = 0 \quad \text{at } y = a \quad (3.61b)$$

Substituting Eq. (3.58b) into Eq. (3.61a) and Eq. (3.58c) into Eq. (3.61b) results in two simultaneous equations in terms of the constants $B_{ln}$ and $C_{ln}$. Solution to these equations yields:

$$B_{ln} = - \frac{H_j}{\alpha^2 b t} \frac{\alpha_n a \sinh \alpha_n a}{\sinh^2 \alpha_n a - \alpha_n^2 a^2} \quad (3.62a)$$

$$C_{ln} = - \frac{H_j}{\alpha_n^2 t} \frac{\sinh \alpha_n a + \alpha_n a \cosh \alpha_n a}{\sinh^2 \alpha_n a - \alpha_n^2 a^2} \quad (3.62b)$$

### 3.7.3 Normal Edge Force at $y=0$

The normal edge force $H_{j-1}^n$, which has the same form as that of Eq. (3.60), is now applied to the plate element $(i-1)$ along the side $y=0$. The direct solution for this case can be obtained by applying the boundary conditions along the lateral sides $y=0$ and $y=a$ to determine the constants $A_{ln}, B_{ln}, C_{ln}$ and $D_{ln}$ in...
Eqs. (3.54a, b and c). The in-plane deformation can then be obtained by following the same procedure as outlined in Section 3.7.1. However, by a simple transformation of the reference axes, the solution of Section 3.7.2 can be applied in this case. Therefore, the stress components for this case are obtained from Eqs. (3.58a, b and c) in which (a-y) is substituted for y. The in-plane displacements are those given in Eqs. (3.59a and b) in which (a-y) and (-v) are substituted for y and v respectively. The constants $B_{ln}$ and $C_{ln}$ are the same as those given in Eqs. (3.62a and b), except that $H_{jn}^j$ in those equations should be replaced by $H_{jn}^{j-1}$.

### 3.7.4 Tangential Edge Force at $y=a$

As shown in Fig. 12, the element (i-1) is subjected to the tangential edge forces $S^j$ along the side $y=a$, and $S^{j-1}$ along the side $y=0$. The solution is given in this section for the case in which only the tangential edge force $S^j$ is applied to the plate element (i-1). The edge force $S^j$ is in the following form:

$$S^j = S_n^j \cos \alpha_n x$$  \hspace{1cm} (3.63)

Since for this case the lateral side $y=0$ is free of edge forces, the solution obtained in Section 3.7.1 applies. The stress components are those given in Eqs. (3.58a, b and c) and the displacements are those given in Eqs. (3.59a and b). The constants $B_{ln}$ and $C_{ln}$ in these equations are determined by the application of the boundary conditions along the lateral side $y=a$. To distinguish the constants $B_{ln}$ and $C_{ln}$ for this case from those given in Eqs. (3.62a,
and b), the constants in this case will be designated \( B_{ln}' \) and \( C_{ln}' \).

The boundary conditions along the lateral side \( y=a \) are the following:

\[
\sigma_y = 0 \quad \text{at } y = a \quad (3.64a)
\]

\[
\tau_{xy} = \frac{s_j}{t} \cos \alpha_n x \quad \text{at } y = a \quad (3.64b)
\]

Substituting Eq. (3.58b) into Eq. (3.64a) and Eq. (3.58c) into Eq. (3.64b) results in two simultaneous equations in terms of the constants \( B_{ln}' \) and \( C_{ln}' \). The solutions to these equations are:

\[
B_{ln}' = \frac{s_j}{t} \frac{n}{\alpha_n} \frac{\alpha_n a \cosh \alpha_n a - \sinh \alpha_n a}{\sinh^2 \alpha_n a - \alpha_n^2 a^2} \quad (3.65a)
\]

\[
C_{ln}' = \frac{s_j}{t} \frac{n}{\alpha_n} \frac{\alpha_n a \sinh \alpha_n a}{\sinh^2 \alpha_n a - \alpha_n^2 a^2} \quad (3.65b)
\]

### 3.7.5 Tangential Edge Force at \( y=0 \)

The element (i-1) is now considered to be subjected to the tangential edge force \( s_j^{i-1} \) along the lateral side \( y=0 \). The edge force \( s_j^{i-1} \) is in the same form as that given in Eq. (3.63). The solution for this case is the same as that given in Section 3.7.3, except that the constants \( B_{ln} \) and \( C_{ln} \) are replaced by \( B_{ln}' \) and \( C_{ln}' \) as given in Eqs. (3.65a and b). The coefficient \( s_j \) in Eqs. (3.65a and b), on the other hand, is replaced by \( s_j^{i-1} \).
4. ANALYSIS OF BEAM ELEMENTS

4.1 General

In the analysis of the beam elements presented in this chapter, the beam elements are assumed to be simply-supported at the ends. The method of analysis employed here is a small-deflection beam theory in which a combined action of bi-axial bending, axial force and torsion will be considered.

The interior beam elements have symmetrical cross-sections with respect to the z-axis and are subjected to the joint-line forces along both lateral sides. The exterior beams, on the other hand, have unsymmetrical cross-section and are subjected to the joint-line forces along only one side. Therefore, the beam analysis will be developed here for a typical beam element with unsymmetrical cross-section and subjected to the joint-line forces along both lateral sides. The results of this development can then be applied to both the interior and exterior beams. The applied wheel loads on the beam elements are given in Eqs. (3.10) and (3.11) in terms of the Fourier series. The joint-forces are in the form given in Eqs. (3.14a, b, c and d).

4.2 Differential Equations of Equilibrium

An infinitesimal block of beam element i, between the two cross-sections with the longitudinal coordinates of x and \((x + dx)\), is subjected to the type of forces shown in Fig. 13. The reference axis system shown in this figure is a right-handed
coordinate system passing through the centroid of the cross-section, which is designated by \( c \). The shear-center of the cross-section is designated by \( s \). The external forces on this infinitesimal block are the joint-line forces \( S^j, H^j, R^j \text{ and } M^j \); the joint-line forces \( S^{j+1}, H^{j+1}, R^{j+1} \text{ and } M^{j+1} \); and the applied wheel load \( P_{bn} \). The joint-line forces are applied to the beams at the level of the middle plane of the plate elements. The shear-center and the centroid of the cross-section are located some distance \( z_s \) and \( z_c \) below the level of the joint-lines, respectively. The \( y \)-coordinate of the shear-center is designated by \( y_s \) and the \( y \)-coordinate of the joint-lines are designated by \( y^{j+1} \) and \( y^j \), where \( y^{j+1} \) and \( y^j \) are always positive and negative quantities, respectively.

The internal stress resultants in the infinitesimal element shown in Fig. 13 consist of the following:

1. Two shearing forces \( V_y \) and \( V_z \) acting in the direction of the \( y \) and \( z \) axes, respectively, and passing through the shear-center.

2. A twisting moment \( M_x \) about an axis parallel to the \( x \)-axis and passing through the shear-center.

3. Two bending moments \( M_y \) and \( M_z \) about the centroidal axes \( y \) and \( z \).

4. An axial force \( F_x \) through the centroid of the cross-section.

The infinitesimal element shown in Fig. 13 is in equilibrium when the summation of all the external and internal forces
along the reference axes, as well as the summation of their moments about these axes, or some parallel axes, vanish. The summation of all the forces along the x, y and z axes yields:

\[
\frac{dF_x}{dx} = S^j - S^{j+1} \tag{4.1}
\]

\[
\frac{dV_y}{dx} = H^j - H^{j+1} \tag{4.2}
\]

\[
\frac{dV_z}{dx} = R^j - R^{j+1} - \tilde{p}_{bn} \tag{4.3}
\]

Summing up the moments of all the forces about an axis parallel to the x-axis and passing through the shear-center results in the following:

\[
\frac{dM_x}{dx} = M^{j+1} - M^j + (H^j - H^{j+1}) (z_s - (y^{j+1} - y_s) R^{j+1})

- (y_s - y^j) R^j + (y_s - \eta) \tilde{p}_{bn} \tag{4.4}
\]

Summing up the moments of all the forces about the y and z axes results in:

\[
\frac{dM_y}{dx} = V_y + z_c (S^{j+1} - S^j) \tag{4.5}
\]

\[
\frac{dM_z}{dx} = - V_y + y^{j+1} S^{j+1} - y^j S^j \tag{4.6}
\]

Eqs. (4.1) and (4.6) are the differential equations of equilibrium.
4.3 Deformations

A point on any cross-section of the beam will undergo a displacement which can be resolved into three linear displacements $u$, $v$ and $w$ along the reference axes $x$, $y$ and $z$ respectively. The distortion of the cross-section is assumed to be negligible compared to the deformations of the whole cross-section as a unit. Thus, the displacement of a point on any cross-section is completely defined by the displacement of the centroid of the cross-section and the angle of twist, $\phi$. However, due to the torsion of the beam, the cross-section will twist about the shear-center and thus, the centroid of the cross-section will undergo some displacement due to this twist, unless the centroid happens to coincide with the shear-center. Therefore, it would be more appropriate to consider the displacement $u$ of the centroid of the cross-section, and the displacement $v$ and $w$ of the shear-center.

4.3.1 Longitudinal Displacement

The displacement of the centroid of the cross-section along the $x$-axis is designated by $u_c$. The stress-displacement relationship can be written as:

$$\frac{du_c}{dx} = \frac{F_x}{EA}$$  (4.7)

where $E$ is the modulus of elasticity of the material, and $A$ is the net area of the cross-section. Differentiating Eq. (4.7) and substituting Eq. (4.1) into the results yield the following:
The longitudinal displacement, \( u_c \), of the centroid is found from Eq. (4.8) by substituting Eq. (3.14a) for \( S^j \) and \( S^{j+1} \) and integrating the results:

\[
\frac{d^2 u_c}{dx^2} = \frac{1}{E A} \left( S^j - S^{j+1} \right) \tag{4.8}
\]

\[
u_c = \frac{1}{2} \frac{\alpha_n}{E A} \left( S_n^{j+1} - S_n^j \right) \cos \alpha_n x \tag{4.9}
\]

4.3.2 Deflections due to Bending

The deflections \( v \) along the \( y \)-axis and \( w \) along the \( z \)-axis, produced by the bending of the beam, are the same for all the points on a cross-section. However, in the presence of torsion, the deflections \( v \) and \( w \) due to bending are the same as those of the shear-center and are designated by \( v_s \) and \( w_s \).

The moment-curvature relationship for the bi-axial bending can be written as:

\[
M_y = \frac{EI}{\rho_z} - \frac{EI_{yz}}{\rho_y} \tag{4.10a}
\]

\[
M_z = \frac{EI}{\rho_y} - \frac{EI_{yz}}{\rho_z} \tag{4.10b}
\]

where \( \rho_y \) and \( \rho_z \) are the radii of curvatures. \( I_y \) and \( I_z \) are the moments of inertia with respect to the \( y \) and \( z \) axes respectively. \( I_{yz} \) is the product of inertia. With the employed sign convention,
as shown in Fig. 13, the curvatures are related to the deflections in the following manner:

\[
\frac{1}{\rho_y} = \frac{d^2v_s}{dx^2} \quad (4.11a)
\]

\[
\frac{1}{\rho_z} = -\frac{d^2w_s}{dx^2} \quad (4.11b)
\]

Substituting Eqs. (4.11a and b) into Eqs. (4.10a and b) results in the following:

\[
M_y = -EI_y \frac{d^2w_s}{dx^2} - EI_{yz} \frac{d^2v_s}{dx^2} \quad (4.12a)
\]

\[
M_z = EI_z \frac{d^2v_s}{dx^2} + EI_{yz} \frac{d^2w_s}{dx^2} \quad (4.12b)
\]

Eqs. (4.12a and b) can be solved simultaneously for \( \frac{d^2v_s}{dx^2} \) and \( \frac{d^2w_s}{dx^2} \).

The results are:

\[
\frac{d^2v_s}{dx^2} = \frac{1}{E} \left( I'_y M_z + I'_{yz} M_y \right) \quad (4.13)
\]

\[
\frac{d^2w_s}{dx^2} = -\frac{1}{E} \left( I'_z M_y + I'_{yz} M_z \right) \quad (4.14)
\]
where

\[ I'_y = \frac{I_y}{I_y I_z - I_{yz}} \]  

(4.15a)

\[ I'_z = \frac{I_z}{I_y I_z - I_{yz}} \]  

(4.15b)

\[ I'_{yz} = \frac{I_{yz}}{I_y I_z - I_{yz}} \]  

(4.15c)

Differentiating Eqs. (4.13) and (4.14) twice, and substituting the corresponding derivatives of Eqs. (4.5) and (4.6) together with Eqs. (4.2) and (4.3) results in the following:

\[
\frac{d^4 v_s}{dx^4} = \frac{1}{E} [I'_y (H^{j+1} - H^j + y^{j+1} \frac{ds^{j+1}}{dx} - y^j \frac{ds^j}{dx}) \\
+ I'_{yz} (R^j - R^{j+1} - \bar{p}_{bn} + z_c \frac{ds^{j+1}}{dx} - z_c \frac{ds^j}{dx})] 
\]  

(4.16)

\[
\frac{d^4 w_s}{dx^4} = \frac{1}{E} [I'_z (R^{j+1} - R^j + \bar{p}_{bn} + z_c \frac{ds^j}{dx} - z_c \frac{ds^{j+1}}{dx}) \\
+ I'_{yz} (H^j - H^{j+1} + y^j \frac{ds^j}{dx} - y^{j+1} \frac{ds^{j+1}}{dx})] 
\]  

(4.17)

The boundary conditions at the simply-supported ends can be expressed in the following form:

\[ v_s = w_s = \frac{d^2 v_s}{dx^2} = \frac{d^2 w_s}{dx^2} = 0 \]  

at \( x = 0 \) and \( L \)  

(4.18)
The deflections $v_s$ and $w_s$ can be obtained by substituting Eqs. (3.14b and c) and (3.10) and the first derivative of Eq. (3.14a) into Eqs. (4.16) and (4.17), and integrating the results four times. Satisfaction of the boundary conditions of Eq. (4.18) will eliminate the constants of integration and the final results for the deflections $v_s$ and $w_s$ are as follows:

$$v_s = \frac{1}{\alpha_n E} \left[ I' \left( H_n^{j+1} - H_n^j + y_n^j \alpha_n S_n^j - y_n^{j+1} \alpha_n S_n^{j+1} \right) \right]$$

$$w_s = \frac{1}{\alpha_n E} \left[ I' \left( R_n^j - p_n + z_n \alpha_n S_n^j - z_n \alpha_n S_n^{j+1} \right) \right]$$

$$+ I'_{yz} \left( R_n^j - p_n + z_n \alpha_n S_n^j - z_n \alpha_n S_n^{j+1} \right) \sin \alpha_n x \quad (4.19)$$

$$+ I'_{yz} \left( H_n^j - v_n^y \alpha_n S_n^j - v_n^y \alpha_n S_n^{j+1} \right) \sin \alpha_n x \quad (4.20)$$

4.3.3 Angle of Twist

The relation between the twisting moment $M_x$ and the angle of twist $\phi$, can be written as:

$$M_x = G J \frac{d\phi}{dx} - E C \frac{d^3\phi}{dx^3} \quad (4.21)$$

$G J$ is the torsional rigidity and $E C$ is the warping rigidity. $G$ is the shearing modulus of the material, $J$ is the torsional constant and $C$ is the warping constant. $J$ and $C$ are properties
of cross-section and thus, for prismatic beams, they are constant. The warping rigidity of the box-beams is very small, and negligible in comparison with their torsional rigidity. Therefore, in the remainder of this development, the relation between the twisting moment $M$ and the angle of twist $\varphi$, will be assumed to be as follows:

$$M_x = G J \frac{d\varphi}{dx}$$

(4.22)

Differentiating Eq. (4.22) and substituting Eq. (4.4) into the results yield the following:

$$\frac{d^2\varphi}{dx^2} = \frac{1}{G J} \left[ M^{i+1} - M^i + z_s H^i - z_s H^{i+1} - (y^{i+1} - y_s) R^{i+1} - (y_s - y^i) R^i + (y_s - \eta) \bar{p}_{bn} \right]$$

(4.23)

The boundary conditions at the ends can be written as:

$$\varphi = 0 \quad \text{at} \quad x = 0 \text{ and } L$$

(4.24)

The angle of twist $\varphi$ is found by substituting Eqs. (3.14b, c, and d) and (3.10) into Eq. (4.23) and integrating the results twice. Applying the boundary condition of Eq. (4.24) will eliminate the linear expression resulting from the integration, and the final answer will be as follows:
\[
\varphi = \frac{1}{2\alpha_n G} \left[ M_n^j \right. - M_n^{j+1} + z_s H_n^{j+1} - z_s H_n^j + (y_s^{j+1} - y_s) R_n^{j+1}
\]

\[
+ (y_s - y_s^j) R_n^j + (\gamma - y_s) p_n \right] \sin \alpha_n x
\]  

(4.25)

The development presented here is applicable to beams with other types of cross-sections. In the derivation of Eqs. (4.1) through (4.21), no reference was made to the specific shape of the beam cross-section, and thus these equations are valid for beams with other types of cross-sections as well. However, Eq. (4.22) is valid only for beams with negligible warping rigidity. For beams with appreciable warping rigidity, Eq. (4.21) should be used instead of Eq. (4.22).

4.4 Internal Stress Resultants

The internal stress resultants in the beams can be obtained from the force-displacement relationships. Substituting the first derivative of Eq. (4.25) for \( \frac{d\varphi}{dx} \) into Eq. (4.22) results in the following:

\[
M_x = \frac{1}{\alpha_n} \left[ M_n^j - M_n^{j+1} + z_s H_n^{j+1} - z_s H_n^j + (y_s^{j+1} - y_s) R_n^{j+1}
\]

\[
+ (y_s - y_s^j) R_n^j + (\gamma - y_s) p_n \right] \cos \alpha_n x
\]  

(4.26)

The moments \( M_y \) and \( M_z \) are obtained from Eqs. (4.12a) and (4.12b) by substituting the corresponding derivatives of \( v_s \) and \( w_s \) from
Eqs. (4.19) and (4.20):

\[ M_y = \frac{1}{\alpha_n^2} \left( R_{n+1}^j - R_n^j + p_n + z_\text{e} \alpha_n S_{n+1}^j - z_\text{c} \alpha_n S_n^j \right) \sin \alpha_n x \]  
(4.27)

\[ M_z = \frac{1}{\alpha_n^2} \left( H_n^j - H_{n+1}^j + y_{n+1} \alpha_n S_{n+1}^j - y_n \alpha_n S_n^j \right) \sin \alpha_n x \]  
(4.28)

The axial force \( F_x \) can be found from Eq. (4.7) by substituting Eq. (4.9) for \( u_c \). The result is as follows:

\[ F_x = \frac{1}{\alpha_n^2} \left( S_n^j - S_{n+1}^j \right) \sin \alpha_n x \]  
(4.29)

The shearing forces \( V_y \) and \( V_z \) can easily be determined from the equilibrium conditions given by Eqs. (4.5) and (4.6). Substituting the corresponding derivatives of \( M_y \) and \( M_z \) from Eqs. (4.27) and (4.28) into Eqs. (4.5) and (4.6) results in the following:

\[ V_y = \frac{1}{\alpha_n} \left( H_{n+1}^j - H_n^j \right) \cos \alpha_n x \]  
(4.30)

\[ V_z = \frac{1}{\alpha_n} \left( R_{n+1}^j - R_n^j + p_n \right) \cos \alpha_n x \]  
(4.31)

4.5 Application to the Interior and Exterior Beam Elements

The developments presented in Sections 4.2, 4.3 and 4.4 were made in such a way that they could conveniently be applied to the interior and exterior beam elements. In this section the use of the results of these developments is discussed, and the deformations of the beam elements along the joint-lines are
4.5.1 Interior Beam Elements

The interior beam elements have symmetrical cross-section with respect to the z-axis and as a result, the product of inertia $I_{yz}$ and the y-coordinate of shear-center $y_s$, will vanish. Therefore, for the interior beam elements the following relations hold:

\[ y_s = 0 \]  \hspace{1cm} (4.32a)
\[ y_{j+1} = -y_j = \frac{b}{2} \]  \hspace{1cm} (4.32b)
\[ I_{y}^{'} = \frac{1}{I_z} \]  \hspace{1cm} (4.32c)
\[ I_{z}^{'} = \frac{1}{I_y} \]  \hspace{1cm} (4.32d)
\[ I_{yz}^{'} = 0 \]  \hspace{1cm} (4.32e)

The above relations should be substituted in the expressions derived for deformations and internal stress resultants in Sections 4.3 and 4.4. Although the substitution of these relations will substantially reduce the expressions for deformations and internal stress resultants, they are not repeated here in order to save space.

4.5.2 Exterior Beam Elements

The exterior beam elements are considered to have unsymmetrical cross-section in order to make it possible to take into account the effect of curb and parapets when desired. On
the other hand, the exterior beam elements are under the influence of joint-line forces only along one lateral side. Therefore, the deformations and the internal stress resultants of the exterior beam elements are obtained from those derived in Sections 4.3 and 4.4, by eliminating the joint-line force coefficients of one of the lateral sides. The joint-line force coefficients designated by the superscript \((j+1)\) or those designated by the superscript \(j\) will be omitted depending upon whether the exterior beam element is located at the right side or at the left side of the bridge.

4.5.3 Deformations Along the Joint-Lines

The deformations of the beam element \(i\), along the joint-line \(j\), can be determined from the following relations:

\[
\begin{align*}
    u^j &= u_c - y^j \frac{dv_s}{dx} + z_c \frac{dw_s}{dx} \\
    v^j &= v_s + z_c \varphi \\
    w^j &= w_s + y^j \varphi \\
    \varphi^j &= \varphi
\end{align*}
\]  

(4.33a) (4.33b) (4.33c) (4.33d)

The deformations along the joint-line \((j+1)\) are in the same form and can be obtained by substituting \((j+1)\) for the superscript \(j\) in the above equations. The expressions for the deformations \(u_c, v_s, w_s\) and \(\varphi\), derived in Section 4.3, together with the modifications of Sections 4.5.2 and 4.5.3 should be substituted in Eqs. (4.33a, b, c, d).
c and d). These joint-line deformations should be determined in order to satisfy the compatibility conditions along the joint-lines.

4.6 Properties of the Cross-Section

The cross-sectional properties of the beam elements affecting the analysis are discussed in this section. Although the beam elements contain some reinforcing and prestressing steel bars, the effect of these bars on the cross-sectional properties is very small and can be ignored. Therefore, the cross-sectional properties are determined based on the gross area of the concrete.

The moments of inertia $I_y$ and $I_z$, and the location of the centroid of the cross-section for the interior and exterior elements, together with the product of inertia $I_{yz}$ for the exterior beams are found by the usual straightforward procedure. The other cross-sectional properties affecting the analysis are the coordinates of the shear-center and the torsional constant $J$.

The location of the shear center and the torsional constant are determined by assuming that the beam elements have thin-walled sections. The thin-walled section assumption for the beam elements corresponds to the assumption that the shearing stresses due to bending and torsion are constant through the thickness. This assumption is fairly good for the interior beam elements. For the exterior beam elements the curb section and the beam section, added together, will produce a thick rectangular block on the top of the beam. The analysis for the shear-center of the exterior beam as a
thick-walled member will reduce to the determination of the solution to a governing second-order partial differential equation. This differential equation is analogous to those of the torsion and membrane theories. However, this analysis is quite involved and thus, an approximate solution will be obtained based on the analysis of thin-walled members.

Figs. 14 and 15 show the actual and the idealized cross-section of the interior and exterior beams, together with the profile of the deflected surface of the analogous membrane. The solid curves in the profile sections shown in Fig. 15 represent the deflected surface of the analogous membrane around the curb and parapet for the actual cross-section of the exterior beam. The areas under the solid curves can be approximated by the areas under the dotted curves. Therefore, as shown in Fig. 15, the cross-section of the exterior beam is considered to be composed of the cross-section of the interior beam plus two rectangular sections for the curb and parapet. The thickness of the rectangular section representing the curb is considered to be equal to the thickness of the curb and that of the top of the interior beam element.

4.6.1 Location of Shear-Center

Consider the bending of a thin-walled single-cell member shown in Fig. 16a. The reference axis system shown in this figure is a right-handed coordinate system passing through the centroid of the cross-section. An infinitesimal element of
this member is subjected to the internal forces shown in Fig. 16b. The s-coordinate is measured along the perimeter of the cross-section. $q_s$ is the shear-flow per unit length of $s$ and is positive when directed toward the positive direction of $s$. The equilibrium differential equation is obtained by setting the summation of the forces acting along the $x$-axis equal to zero. The result is in the following form:

$$\frac{\partial q_s}{\partial s} + t \frac{\partial \sigma_x}{\partial x} = 0$$

where $t$ is the wall thickness of the cross-section and is a function of the variable $s$. Integrating Eq. (4.34) results in the following:

$$q_s = q_o - \int_0^s t \frac{\partial \sigma_x}{\partial x} \, ds$$

$q_o$ is the shear-flow at the origin $s = 0$.

Fig. 16c shows the location of the shear-center and the reference axes of the cross-section. The resultant shearing forces $V_y$ and $V_z$ should pass through the shear-center in order for the cross-section to remain untwisted. To obtain the $y$-coordinate of the shear-center, it can be assumed that $M_z = 0$, and thus $V_y = 0$. Taking moments about any convenient point, say point $0$ in Fig. 16c, results in:

$$Y_0 V_z = \int_0^t r q_s \, ds$$

where $Y_o$ is the $y$-coordinate of the shear-center relative to
point 0. \( r \) is the moment arm of the shearing force \( (q_s \, ds) \) about point 0, and is dependant on \( s \). The moment arm \( r \) is positive in the same sense as is the moment itself. In Fig. 16c the positive direction of moment is indicated. The upper limit of the integral in Eq. (4.36) is designated by \( t \), to indicate the integration over the entire periphery of the cross-section. The determination of \( Y_o \) from Eq. (4.36) requires the explicit form of the shear-flow \( q_s \).

The expression for normal stress \( \sigma_x \) in bi-axial bending can be written as follows:

\[
\sigma_x = (z \, I_z' - y \, I_y' \, M_y - (y \, I_y' - z \, I_y') \, M_z \tag{4.37}
\]

where \( I_y', \, I_z' \) and \( I_y' \) are given in Eqs. (4.15a, b and c). Assuming \( M_z = 0 \), as mentioned above, and differentiating Eq. (4.37) with respect to the \( x \)-variable, result in the following:

\[
\frac{\partial \sigma_x}{\partial x} = (z \, I_z' - y \, I_y') \, V_z \tag{4.38}
\]

The shear-flow \( q_s \) can now be obtained by substituting Eq. (4.38) into Eq. (4.35). The result is as follows:

\[
q_s = q_o - [I_z' \, Q_y(s) - I_y' \, Q_z(s)] \, V_z \tag{4.39}
\]

where

\[
Q_y(s) = \int_0^s z \, t \, ds \tag{4.40a}
\]

\[
Q_z(s) = \int_0^s y \, t \, ds \tag{4.40b}
\]
\( Q_y(s) \) and \( Q_z(s) \) are the moments of area, from zero (origin of \( s \)) to \( s \), about the centroidal axes \( y \) and \( z \), respectively.

Eq. (4.39) shows that the shear-flow \( q_s \) is indeterminate, since the value of \( q_o \) is unknown. In case of thin-walled open-sections, \( s \) can be measured from the free end of the cross-section, resulting in a zero value for \( q_o \), and thus the shear-flow is determinate. For the closed-section under consideration, one could introduce a cut at any convenient point on the cross-section, and measure \( s \) from the cut-point, in order to reduce the case to that of the open sections. The unknown shear-flow \( q_o \) is then superimposed on the open-section. Therefore, the resulting shear-flow is still in the same form as that of Eq. (4.39).

The compatibility requirements at the cut-point are ensured by enforcing the condition that the integral of the shearing strains over the periphery of the closed-loop of the cross-section must vanish. Hence:

\[
\int \frac{q_s}{G} \ ds = 0 \tag{4.41}
\]

Assuming a constant shearing modulus \( G \), and substituting Eq. (4.39) for \( q_s \) into Eq. (4.41) result in the following:

\[
q_o = V_z Q_o \tag{4.42}
\]

where

\[
Q_o = \frac{1}{\mathcal{J}} \frac{ds}{d \mathcal{J}} \left[ I_z \int \frac{Q_y(s)}{t} \ ds - I_{yz} \int \frac{Q_z(s)}{t} \ ds \right] \tag{4.43}
\]
Q_o is a constant and can be thought of as a cross-sectional property. The shearing-flow q_s for the closed-loop portion of the cross-section can now be determined explicitly by substituting Eq. (4.42) for q_o into Eq. (4.39):

\[ q_s = [Q_o - I_z^r Q_y(s) + I_{yz}^r Q_z(s)] V_z \]  \hspace{1cm} (4.44)

The y-coordinate of the shear-center relative to point 0 is given in Eq. (4.36) in terms of the shear-flow q_s. Therefore, substituting Eq. (4.44) for q_s into Eq. (4.36) yields the final result as follows:

\[ Y_0 = 2 A_o Q_o - I_z^r \int_0^t Q_y(s) r \ ds + I_{yz}^r \int_0^t Q_z(s) r \ ds \]  \hspace{1cm} (4.45)

where \( Q_o \) is given in Eq. (4.43) and \( A_o \) is average of the areas enclosed by the outer and inner boundaries of the closed-loop portion of the cross-section. Similarly, one can obtain the z-coordinate of the shear-center, relative to point 0, as the following:

\[ Z_0 = -2 A_o Q_1 + I_y^r \int_0^t Q_z(s) r \ ds - I_{yz}^r \int_0^t Q_y(s) r \ ds \]  \hspace{1cm} (4.46)

where \( Q_1 \) is similar to \( Q_o \) as follows:

\[ Q_1 = \frac{1}{t} \int \frac{Q_z(s)}{t} ds - \int \frac{Q_y(s)}{t} ds \]  \hspace{1cm} (4.47)

Eqs. (4.45) and (4.46) represent the location of the shear-center relative to any convenient point of the cross-section.
The location of the shear-center of the exterior beam elements is obtained from Eqs. (4.45) and (4.46). For the interior beam-elements, because of the symmetry with respect to the z-axis, the shear-center lies on the z-axis, and thus only the z-coordinate of the shear-center need be found. Furthermore, the relations given in Eqs. (4.32c, d and e) for the interior beam will reduce the expression for the z-coordinate of the shear-center, given in Eqs. (4.46) and (4.47).

4.6.2 Torsional Constant

The torsional constant $J$ for the interior beam elements are determined from the following expression:

$$
J = \frac{4A_o^2}{\int ds} t
$$

(4.48)

$A_o$ is average of the areas enclosed by the outer and inner boundaries of the cross-section. $t$ is the wall-thickness of the cross-section, and $s$ is measured along the perimeter of the cross-section.

For the exterior beam elements, as described before, the torsional constant $J$ is taken as the summation of the torsional constant of the interior beam given in Eq. (4.48) and the torsional constants of two rectangular sections representing the curb and parapets. The torsional constants of the rectangular sections are found as the following:

$$
J = \frac{1}{3} b t^3 \left( 1 - 0.63 \frac{t}{b} \right)
$$

(4.49)
b and t are the dimensions of the rectangular section and t is the smaller.

Eq. (4.48) is derived for thin-walled sections based on the membrane analogy, and Eq. (4.49) is derived for narrow rectangular sections. The complete derivation of these two equations can be found in Reference (29).
5. DEVELOPMENT OF SOLUTION

5.1 General

In this chapter the results of the analysis of all of the beam and plate elements are combined in order to arrive at the complete solution. The method of solution employed here is a flexibility type which is mostly referred to as the method of influence coefficients. The unknown stress resultants along the joint-lines, as well as all the internal stress resultants and deformations of the beam and plate elements, are in terms of trigonometric functions of $x$. Therefore, as can be seen from the form of Eqs. (3.14a, b, c and d), the unknowns are actually the coefficients $S_n$, $H_n$, $R_n$ and $M_n$ of the stress-resultants along the joint-lines. The continuity requirements are met by equating the deformations of the adjacent beam and plate elements along the joint-lines. The deformations of the beam and plate elements along the joint-lines are in the compatible form of the trigonometric functions. This fact can easily be examined from the results of the analysis of the beam and plate elements presented in chapters 3 and 4. Therefore, the continuity requirements along each joint-line provide four equations in terms of the unknown stress-resultant coefficients, since the trigonometric functions of $x$ can be cancelled out.

The fact that the trigonometric functions of $x$ can be cancelled out of the continuity equations would mean the following:
If the continuity requirements are met at
\textbf{one point} along the joint-line, then the con-
tinuity requirements are automatically met
along the \textbf{entire} joint-line.

Hence, the matrix formulation of the solution will be developed
based on the coefficients of the stress resultants and deformations,
since the trigonometric functions are known.

5.2 \textbf{Flexibility Matrices of the Beam and Plate Elements}

The displacements of the beam and plate elements along
the joint-lines produced by the joint-line stress resultants, are
represented by the column matrices $\theta_b$ and $\theta_p$. Thus:

$$
\theta_b = \begin{bmatrix}
\vdots \\
\theta_{i-2} \\
\theta_i \\
\theta_{i+2} \\
\vdots \\
k,l
\end{bmatrix} \\
\theta_p = \begin{bmatrix}
\vdots \\
\theta_{i-3} \\
\theta_{i-1} \\
\theta_{i+1} \\
\vdots \\
k,l
\end{bmatrix}
$$

(5.1a) (5.1b)

where \(k\), the order of these matrices, is the number of the
joint-line displacements and is equal to \(8 (N_B - 1)\). Each term
in the matrices given in Eqs. (5.1a and b) is a sub-matrix de-
fining the displacements of one beam or plate element. $\theta_i$ and
$\theta_{i-1}$, for example, are in the following form:
Each term of these two sub-matrices is a sub-matrix itself. \( \theta_i \) and \( \theta_{i-1} \), for example, are the displacements of the beam element \( i \) and the plate element \( (i-1) \) along the joint-lines \( j \) and \( (j-1) \) respectively. Hence:

\[
\begin{align*}
\theta_i &= \begin{ \{ \}
\theta_j \\
\theta_{j+1}
\end{ \{ \}
\}
8,1

\theta_{i-1} &= \begin{ \{ \}
\theta_{j-1} \\
\theta_{j-1}
\end{ \{ \}
\}
8,1
\end{align*}
\]

(5.2a)  (5.2b)

Considering the beam element \( i \) and the plate element \( (i-1) \), the force-displacement relationship can be written as:

\[
\begin{align*}
\theta_i &= f_b N_i \
\theta_{i-1} &= f_p N_{i-1}
\end{align*}
\]

(5.4a)  (5.4b)

The square matrices \( f_b \) and \( f_p \) are the flexibility matrices of the individual beam and plate elements respectively. The column matrices \( N_i \) and \( N_{i-1} \) represent the unknown joint-line stress-resultant
coefficients and are in the following form:

\[
N_1 = \begin{bmatrix} N^j_1 \\ N^{j+1} \end{bmatrix}_{8,1} \quad N_{i-1} = \begin{bmatrix} N^{j-1} \\ N^j \end{bmatrix}_{8,1} \tag{5.5a} \tag{5.5b}
\]

where each term of these two matrices is a sub-matrix associated with the joint-lines designated by the superscripts. The sub-matrix \(N^j\), for example, is in the following form:

\[
N^j = \begin{bmatrix} s^j_n \\ H^j_n \\ R^j_n \\ M^j_n \end{bmatrix}_{4,1} \tag{5.5c}
\]

The flexibility matrices \(f_b\) and \(f_p\) can each be partitioned into four sub-matrices, in the following form:

\[
f_b = \begin{bmatrix} f_{bl,t} & f_{bl,r} \\ \cdot & \cdot & \cdot \\ f_{br,t} & f_{br,r} \end{bmatrix}_{8,8} \tag{5.6a}
\]

\[
f_p = \begin{bmatrix} f_{pt,t} & f_{pt,r} \\ \cdot & \cdot & \cdot \\ f_{pr,t} & f_{pr,r} \end{bmatrix}_{8,8} \tag{5.6b}
\]

\(f_{pt,r}\), for example, is a sub-matrix representing the flexibility...
coefficients of the plate elements associated with the left side joint-line and produced by the right side joint-line stress resultants. The flexibility matrices \( f_b \) and \( f_p \) are obtained based on the analysis of the beam and plate elements presented in Chapters 3 and 4. Each term of the flexibility matrices \( f_p \) is an expression in terms of hyperbolic functions. The details of the flexibility matrices \( f_b \) and \( f_p \) are given in Sections 8.1 and 8.2 of the Appendix, in order to save space in the content of the text.

The force-displacement relationship for all of the beam and plate elements along the joint-lines can be expressed as follows:

\[
\begin{align*}
\theta_b &= F_b N \\
\theta_p &= F_p N
\end{align*}
\]  

(5.7a)  

(5.7b)

where \( N \) is a column matrix representing the unknown stress-resultant coefficients of all the joint-lines. \( F_b \) and \( F_p \) are the overall flexibility matrices of the beam and plate elements respectively. Hence:

\[
N = \begin{bmatrix}
N^j & N^{j+1} & N^{j+2} & \cdots & N_{k-1} & N_k
\end{bmatrix}
\]  

(5.8)
The overall flexibility matrices $F_b$ and $F_p$ are in the following form:

$$F_p = \begin{bmatrix} f_p & & & \end{bmatrix}$$

(5.9a)

$$F_b = \begin{bmatrix} f_{br,r} & & & \end{bmatrix}$$

(5.9b)

5.3 Displacements of the Beam and Plate Elements Due to the Wheel Loads

The displacements of the beam and plate elements along the joint-lines, due to the applied wheel loads, are represented
by the following column matrices:

\[
\delta_b = \begin{pmatrix}
\delta_{i-2} \\
\delta_i \\
\delta_{i+2} \\
\end{pmatrix}_{k,l} \quad \delta_p = \begin{pmatrix}
\delta_{i-3} \\
\delta_{i-1} \\
\delta_{i+1} \\
\end{pmatrix}_{k,l}
\]

(5.10a) \hspace{1cm} (5.10b)

The typical terms \(\delta_i\) and \(\delta_{i-1}\) in the above two matrices can be written in the following form:

\[
\delta_i = \begin{pmatrix}
\delta^j_i \\
\delta^{j+1}_i \\
\end{pmatrix}_{8,1} \quad \delta_{i-1} = \begin{pmatrix}
\delta^{j-1}_{i-1} \\
\delta^j_{i-1} \\
\end{pmatrix}_{8,1}
\]

(5.11a) \hspace{1cm} (5.11b)

\(\delta^{j-1}_{i-1}\) and \(\delta^j_{i-1}\), given in Eq. (5.11b) are the sub-matrices representing the displacements of the plate element \((i-1)\) along the joint-lines \((j-1)\) and \(j\) produced by the wheel loads. Applying Eqs. (3.34), (3.37), (3.39) and (3.40), results in the following:

\[
\delta^{j-1}_{i-1} = \begin{pmatrix}
0 \\
0 \\
G_{n,(i-1)} \\
G_{ln,(i-1)} \\
\end{pmatrix}_{4,1} \quad \delta^j_{i-1} = \begin{pmatrix}
0 \\
0 \\
K_{n,(i-1)} \\
K_{ln,(i-1)} \\
\end{pmatrix}_{4,1}
\]

(5.12a) \hspace{1cm} (5.12b)
For other plate elements only the superscript \( j \) and the subscript \( i \) in Eqs. (5.12a and b) will be changed by 2.

\[ \delta_{ij} \text{ and } \delta_{ij+1} \text{ in Eq. (5.11a) are the sub-matrices representing the displacements of the beam element } i \text{ along the joint-lines } j \text{ and } (j+1) \text{ due to the wheel loads respectively. To determine these two sub-matrices, Eqs. (4.9), (4.19), (4.20) and (4.25) are substituted in Eqs. (4.33a, b, c and d). The results are as follows:} \]

\[
\delta_{ij} = \frac{p_{n,i}}{\alpha_n^2} \begin{Bmatrix}
\frac{y_j I_{yj} + z_c I_z}{E \alpha_n^2} \\
\frac{z_s \eta_i \left( \frac{I_{yj}}{G J} - \frac{I_{yz}}{E \alpha_n^2} \right)}{G J} \\
\frac{\eta_i \left( y_j - y_s \right)}{G J} + \frac{I_z}{E \alpha_n^2} \\
\frac{\eta_i}{G J}
\end{Bmatrix}_{4,1}
\]

\( \delta_{ij+1} \) is in the same form as given in Eq. (5.13), except that the superscript \( j \) is changed to \( (j+1) \). For the interior beam elements, Eq. (5.13) reduces to a simpler form since the relations given in Eqs. (4.32a, b, c, d and e) should be applied. For other interior beam elements, only the superscript \( j \) and the subscript \( i \) in
Eq. (5.13) will be changed by 2. For the exterior beam elements \( \delta_i \) given in Eq. (5.11a) reduces to either \( \delta_i^j \) or \( \delta_i^{j+1} \), depending on whether the exterior beam is located at the right side or at the left side of the bridge.

5.4 Continuity Equations

The unknown joint-line stress-resultant coefficients are represented by the matrix \( N \) given in Eq. (5.8). The relative displacements along the joint-lines, produced by the joint-line stress resultants, are represented by the column matrix \( \theta \). Hence:

\[ \theta = \theta_b - \theta_p \]  \hspace{1cm} (5.14)

\( \theta_b \) and \( \theta_p \) are defined in Eq. (5.1a and b). Substituting Eqs. (5.7a and b) into Eq. (5.14) results in the following:

\[ \theta = FN \]  \hspace{1cm} (5.15)

where

\[ F = F_b - F_p \]  \hspace{1cm} (5.16)

The matrix \( F \) is the overall flexibility matrix of the bridge superstructure. \( F_b \) and \( F_p \), the overall flexibility matrices of the beam and plate elements, are given in Eqs. (5.9a and b). The relative displacements along the joint-lines, due to the applied wheel loads, are designated by the column matrix \( \delta \). Thus:

\[ \delta = \delta_b - \delta_p \]  \hspace{1cm} (5.17)
\( \delta_b \) and \( \delta_p \) are known column matrices given in Eqs. (5.10a and b). The continuity requirements along the joint-lines are expressed as follows:

\[
\delta + F N = 0 \tag{5.18}
\]

The solution to the above continuity equation yields:

\[
N = -F^{-1} \delta \tag{5.19}
\]

Therefore, the unknown joint-line stress-resultant coefficients are found from Eq. (5.19).

5.5 Internal Stress Resultants in the Beam Elements

The internal stress resultants in the beam elements were derived in Chapter 4, and are given in Eqs. (4.26) through (4.31). The matrix formulation of these internal stress resultants is made simply by putting Eqs. (4.26) through (4.31) into one matrix equation.

The internal stress resultants in beam element \( i \), designated by the column matrix \( T_i \), can be written in the following form:

\[
T_i = \begin{bmatrix} F_{x,i} \\ V_{y,i} \\ V_{z,i} \\ M_{x,i} \\ M_{y,i} \\ M_{z,i} \end{bmatrix} = B N_i + B_0,i P_n \tag{5.20}
\]
where the matrix $N$ is given in Eq. (5.59) and $p_n$ is given in Eq. (3.11). The matrix $B$ can be partitioned in the following form:

$$B = [B_1 \quad B_2]$$  \hspace{1cm} (5.21a)

where $B_1$ and $B_2$ are the sub-matrices representing the coefficients associated with $N_j$ and $N_{j+1}$ respectively. From Eqs. (4.26) through (4.31) the matrices $B_1$, $B_2$ and $B_{0,i}$ can be found as follows:

$$B = [B_1 \quad B_2] = \frac{1}{\alpha_n} \begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -z_s (y_s - y_i) & 1 & 0 & z_s (y_{j+1} - y_s) & -1 \\
-z_c & 0 & -\frac{1}{\alpha_n} & 0 & z_c & 0 & \frac{1}{\alpha_n} & 0 \\
y_j & 0 & 0 & y_{j+1} & -\frac{1}{\alpha_n} & 0 & 0 \\
\frac{1}{\alpha_n} & 0 & 0 & \frac{1}{\alpha_n} & 0 & 0 & 0 \\
\end{bmatrix}$$  \hspace{1cm} (5.21b)

$$B_{0,i} = \frac{1}{\alpha_n} \begin{bmatrix}
0 \\
0 \\
1 \\
(\eta_i - y_s) \\
\frac{1}{\alpha_n} \\
0 \\
\end{bmatrix}$$  \hspace{1cm} (5.21c)
For the interior beam elements, the relations given in Eqs. (4.32a and b) are substituted in the matrices $B$ and $B_{0,i}$ given in Eqs. (5.21b and c). For the exterior beam elements the matrices $N_i$ and $B$ will reduce to either $N_i^j$ and $B_1$ or $N_i^{j+1}$ and $B_2$, depending on whether the exterior beam is located at the right side or at the left side of the bridge.

5.6 Effective Width of Slab

The effective width of slab is defined as the portion of slab acting compositely with the beams. It is the usual practice to define such an effective width in order to determine the maximum bending stresses simply by applying the elementary beam theory of bending. Therefore, the effective width of slab is considered here in such a way that the resulting composite beams will be free of any internal axial force. The normal stress in the effective portion of slab is taken as the average of normal stresses in the beam elements at the level of the middle plane of slab, in order to comply with the simple beam theory. Hence:

$$
\sigma_{x,i} = -z_i \left( I'_y M_y,i + I'_{yz} M_z,i \right) - \frac{1}{2} \left( y^j + y^{j+1} \right) \left( I'_y M_z,i + I'_{yz} M_y,i \right)
+ \frac{F_{x,i}}{A}
$$

The effective width of slab for the interior beams can be determined from the following expression:
E.W. = b - \frac{E_b F_{x,i}}{E_s \sigma_{x,i}} \quad (5.23)

For the exterior beams, the effective width is found as follows:

E.W. = b + b_s - \frac{E_b F_{x,i}}{E_s \sigma_{x,i}} \quad (5.24)

where \( b_s \) is the width of slab extended beyond the exterior beams (see Fig. 15). \( \sigma_{x,i} \) is first obtained from Eq. (5.22) and is then substituted in Eqs. (5.23) and (5.24). The total moment about the y-axis on the composite section, designated by \( M_{yc,i} \), can be found as follows:

\[ M_{yc,i} = M_{y,i} + z_c F_{x,i} \quad (5.25) \]

Eq. (5.22) is subjected to the relations given in Eqs. (4.32b, c, d and e) when used for the interior beams.
6. DISCUSSION OF RESULTS

6.1 General

An extensive field study of the actual load distribution in spread box-beam bridges has been under way at Lehigh University since 1964. In Section 6.2 of this chapter the validity of the theoretical analysis presented in this paper is verified by comparison with the results of the field tests. The theoretical analyses should not always be accepted with complete confidence without verification by comparison with the results of tests on real structures. On the other hand, field tests cannot give the full picture of the behavior of structures since the effect of all existing variables cannot be studied. In Section 6.3 of this chapter an extensive study of all variables involved in the analysis is presented. Over three hundred different bridge superstructures are analyzed under different patterns of truck loading in order to present the full picture of the behavior of spread box-beam bridges. Based upon the results, a simplified design procedure for the determination of lateral load distribution is developed and recommended.

6.2 Comparison of Field Test Results and Theory

A comparison of the field test results with the theoretical analysis for the Drehersville Bridge, the Berwick Bridge, the White Haven Bridge and the Philadelphia Bridge are depicted in Figs. 17 through 26. In all of these figures, the points marked
by squares and triangles show the results of the theory when the curbs, and the curbs and parapets together, are considered to act compositely with the exterior beams, respectively. The points marked by circles represent the field test results. The plots in Figs. 17 through 26 show the influence lines for moments in the exterior and interior beams as the center of the test truck is located at different positions across the width of the bridge. The moment percentages, plotted in these figures, are those of the section of maximum moment as the drive axle of the test vehicle passes over this section. The beams are designated by the letters A, B, C, etc., from the left to the right side of the cross-section of the bridge.

6.2.1 Drehersville Bridge

The test span, which was the northwest span of the three-span bridge, was simply supported with a length of 61 feet 6 inches. The bridge superstructure contained five identical precast, prestressed box-beams of 48 inches width and 33 inches depth, which were equally spaced at 86 inches, center-to-center. The reinforced concrete deck providing a roadway 30 feet in width, was cast in place compositely with the beams. The specified minimum slab thickness was 7-1/2 inches. The curbs and parapets had the standard dimensions given in the PDH standard specifications. The strain distribution at the outer face of the exterior beams, curbs, and parapets indicated full composite action between the exterior beam, the slab, the curb and the parapet sections.
Although the full composite action of the curbs and parapets was observed in the field test, the data reduction was based on some effective width for the parapet:

"The sequence of calculation for the exterior girder was (1) to check whether maximum slab width of 84 inches was required; if so (2) to check whether the maximum curb width of 33 inches was required, and then; if so (3) to calculate the required width of parapet."

Therefore, the results of the field test are expected to fall somewhere between the theoretical values designated by the square and triangular points in Figs. 17, 18 and 19. A study of the influence lines shown in Figs. 17, 18 and 19 reveals an excellent agreement between the results of the field test and the theory.

6.2.2 Berwick Bridge

The test span which was the center span of the three-span bridge, had a length of 65 feet 3 inches. Four identical precast, prestressed box-beams 48 inches wide and 39 inches deep, were spaced at 8 feet 9-3/8 inches, center-to-center. The slab, which provided a 28-foot roadway, was cast in place compositely with the beams. The specified minimum thickness of the slab was 7-1/2 inches. The 33-inch wide curbs and the 15-inch wide parapets were cast monolithically. The parapets were constructed with 1/2-inch wide vertical gaps at four equally spaced positions along the span.
The reduction of the test data was based on a similar procedure as described in Section 6.2.1. Thus, the test results are again expected to fall somewhere between the results of the theory with curb, and with both curb and parapet, as designated by square and triangular points in Figs. 20 and 21.

6.2.3 White Haven Bridge

The test span had almost identical properties with the Berwick Bridge, except that the four box-beams in this case were 36 inches wide and 42 inches deep. This field test was conducted in order to study the influence of beam width. The test span had a length of 66 feet 1-5/8 inches. The beams were spaced at 9 feet, center-to-center. The slab had a specified minimum thickness of 7-1/2 inches, and provided a 28-foot roadway. The curb and parapet sections were identical to those of the Berwick Bridge, except that the parapets contained 1/2-inch wide vertical gaps at three equally spaced positions along the span.

Figs. 22 and 23 show the comparison of the results of the field test and the theory. The field test results fall somewhere between the theoretical values designated by the square and triangular points, as expected.

6.2.4 Philadelphia Bridge

The purpose of the Philadelphia Bridge field study was to investigate the effect of the mid-span diaphragm on load distribution. The bridge was first tested with the mid-span diaphragm...
in place, and then the same tests were repeated after the diaphragm had been removed. The test span had a length of 71 feet, 9 inches. The bridge superstructure consisted of five identical precast, prestressed box-beams, covered with a cast-in-place reinforced concrete deck. The box-beams, which were 48 inches wide and 42 inches deep, were equally spaced at 9 feet 6 inches, center-to-center. The mid-span diaphragm was 10 inches thick. The reinforced concrete deck provided a 40-foot roadway. The specified minimum thickness of slab was 7-1/2 inches.

The results of this field test showed that the effect of mid-span diaphragm was not of particular significance, and when all of the design traffic lanes were loaded, this effect became negligible. Another important finding in this test was that the parapets were not fully effective in composite action with the curbs. Therefore, based on this field test, the full effectiveness of parapets in composite action with the curbs is questionable. With regard to the effect of mid-span diaphragms, the following conclusion was made, based on the results of this field test:

"The diaphragms did transmit loads laterally, but owing to compensating effects when various lanes were loaded, the experimentally determined distribution factors were not appreciably affected. Based on the testing of the Philadelphia Bridge, the necessity of the use of mid-span diaphragms is questionable."

-77-
Figs. 24, 25 and 26 show the comparison between the results of the field test and the theory, in the form of influence lines for moments in the exterior beam A, the adjacent interior beam B, and the center beam C. In these figures the results of the Guyon-Massonnet theory \(^{15,16}\) are also depicted by solid triangular points. The comparison of the results of the field test and the Guyon-Massonnet theory was made in Reference 8. As can be observed from Figs. 24, 25 and 26, the results of the Guyon-Massonnet theory do not agree with the test results as nearly as do the results of the theory presented in this paper.

6.3 Study of the Variables

The comparison of the results from the field test with those derived from the theory presented in Section 6.2 proved the validity of the theory. The next step in this investigation was to study the effect of all variables involved in the analysis. These variables are composed of the material properties and the bridge geometry. The material properties involved in the analysis are Poisson's ratio \(ν\) and the ratio of the modulus of elasticity of beam concrete to that of the slab. The parameters concerning the geometry of the bridge which affect the results of the analysis are:

1. Dimensions of curbs and parapets
2. Thickness of slab
3. Dimensions of precast beams
4. Span length of bridge
5. Spacing of beams
6. Total width of bridge
7. Number of beams

6.3.1 Material Properties

Two material properties, namely Poisson's ratio $\nu$ and the ratio of modulus of elasticity of beam concrete to that of the slab ($\frac{E_b}{E_s}$) are involved in the analysis. Poisson's ratio $\nu$ for concrete varies widely, depending on the age of the concrete, type of aggregates, and other factors. To observe the effect of Poisson's ratio, a high and a low limiting value of 0.25 and 0.05 were chosen for the comparison. Fig. 27 shows the effect of these two limiting values of $\nu$ on the load distribution in a 40-foot wide bridge with 5 beams. The span length was taken as 70 feet. The results shown in Fig. 27 clearly indicate the insignificance of the variation of $\nu$ on the load distribution. For this reason, an average value of 0.15 was considered as a fixed value for Poisson's ratio.

The modulus of elasticity for precast prestressed concrete beams is higher than that of the cast-in-place reinforced concrete slab. The ratio of the two, however, is normally smaller than 1.5 and greater than 1.0. A comparison is shown in Fig. 28 for the limiting values of 1.5 and 1.0. Since the influence of this variation on the load distribution is insignificant, the modulus of elasticity of concrete will be considered to be the same for both the prestressed beams and the slab.
6.3.2 Effect of Curbs and Parapets

Figs. 29, 30 and 31 show the influence lines for beams A, B and C of the Philadelphia Bridge. These influence lines are plotted in order to observe the effect of curbs and parapets. The points marked by circles in these figures show the results from the theoretical analysis when the effect of curbs and parapets are not taken into account. Basically, curbs and parapets are not installed as load-carrying members. The results of field tests showed the full effectiveness of curbs in composite action with the exterior beams, with partial effectiveness of the parapets, at least in the test conducted on the Philadelphia Bridge. Therefore, while it does not seem safe and reasonable to account for full effectiveness of the parapets, it certainly would be appropriate to consider the curbs to act compositely with the exterior beams. On the other hand, the practicing engineer apparently would rather not account for the curbs in the design in order to simplify the design and construction of bridges.

From Figs. 29, 30 and 31, it can be observed that the effect of curbs on the load distribution is not very significant, and it would be on the conservative side, at least for the interior beams, to disregard this effect. Therefore, the effect of curbs and parapets will be omitted in the remainder of this study for the sake of simplicity in the design and construction of bridges.

6.3.3 Variation in Thickness of Slab

The effect of variation in slab thickness is demonstrated
in Fig. 32. As shown in this figure, the slab thickness of the Philadelphia Bridge is changed from 7.5 inches to 9.0 inches, and the effect of this change is not particularly significant. On the other hand, based on the design of slab, the Pennsylvania Department of Highways applies the following rules to the variation of slab thickness:

<table>
<thead>
<tr>
<th>Clear Spacing Between Beams</th>
<th>Slab Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>From 2' - 9&quot; to 5' - 7&quot;</td>
<td>7-1/2&quot;</td>
</tr>
<tr>
<td>From 5' - 10&quot; to 6' - 10&quot;</td>
<td>8&quot;</td>
</tr>
<tr>
<td>From 7' - 0&quot; to 7' - 9&quot;</td>
<td>8-1/2&quot;</td>
</tr>
</tbody>
</table>

Since the load distribution is not very sensitive to the variation of slab thickness, the above PDH rules are considered to be adequate, and will be used in the remainder of this study.

6.3.4 Variation in the Dimensions of Precast Beams

The only variable dimensions in the cross-section of precast beams are the width and the depth. The width is either 3 or 4 feet. Since the majority of bridges are constructed with 4-foot wide beams, in the following development, only 4-foot wide beams will be considered. The same development can be repeated with 3-foot wide beams if desired.

The effect of the variation of beam depth on the load distribution is demonstrated in Fig. 33. The effect seems to be significant, at least for the beam located directly under the load.
6.3.5 **Effect of Span Length**

Figs. 34 and 35 show the effect of span length of the bridge on the load distribution. The total width of the bridge is taken as 40 feet, and the span length from 40 feet to 115 feet with intervals of 15 feet. Fig. 34 shows the influence of the span length, when the standard truck loading is located in the center of the bridge, which will produce the maximum response in the center beam. Fig. 35 shows the same effect for the maximum response of the exterior beam. A study of these two figures reveals the significant influence of span length on the load distribution. When the span length of the bridge is changed from 115 feet to 40 feet, the moment percentage is doubled in the center beam and increased by 50% in the exterior. On the other hand, the effect of span length on the exterior beam moments of Fig. 34, and on the center beam moments of Fig. 35 is not only less pronounced, but it is in the opposite direction.

In order to observe the complete picture of the span length effect, the influence lines for the moments in the exterior beam A, the adjacent interior beam B, and the center beam C of a 40-foot wide bridge are shown in Figs. 36, 37, and 38 respectively. The influence lines in these figures are shown for different span lengths of 40, 70 and 100 feet. From the examination of these influence lines, it can be concluded that the more uniformly the bridge is loaded, the less pronounced is the effect of span length. This would mean that the effect of span length is dependent upon
the number of design traffic lanes and the total width of the bridge. Furthermore, as will be seen later, the number of beams is another factor affecting this behavior.

Since one of the major factors influencing the load distribution is the span length, and as can be seen from Figs. 34 and 38, the load distribution factor is a decreasing function of span length. To observe the type of this decreasing function, the moment percentages in the center beam of Fig. 34 are plotted against the corresponding span lengths in Fig. 39. The curve passing through the plotted points in this figure has an approximately hyperbolic shape with the horizontal line of 20% moment as an asymptote. The fact that the 20% value is an asymptote can readily be observed from Fig. 34. As the span length increases, the load is distributed more uniformly in all of the beams. When the span length increases sufficiently, all of the beams will take the same share of the load.

In summary, from the study of Fig. 39, it can be concluded that the load distribution is likely to be equal to a function inversely proportional to span length plus a constant which is equal to the total load divided by the number of beams.

6.3.6 Spacing of Beams

Another important factor influencing the load distribution is the spacing of beams. In fact, the load distribution factors listed in the AASHO specifications are in the form of spacing divided by a constant number. For a given width of bridge, the
maximum load distribution in the beams increases as the spacing of beams increases. Therefore, it would be natural to suspect that the load distribution is directly proportional to the spacing of beams. By combining the effects of the span length and the spacing of beams, it can be concluded that the load distribution is likely to be a linear function of $S/L$, where $S$ is the spacing of beams, and $L$ is the span length of the bridge.

From the study of influence lines for moments in different interior beams, it was found that when all of the design traffic lanes are loaded, the maximum moment will occur in the center beam or in the beam closest to the center line of the bridge. The standard HS20 truck was assumed to occupy a width of 10 feet within each design traffic lane. Fig. 40 shows plots of the maximum load distribution factors against different values of $S/L$ for a 44-foot wide bridge with five beams. The load distribution factors in this figure are given in fraction of wheel loads of the HS20 standard truck. To observe the effect of the beam depth, the plots of load distribution factors in the figure are shown for the beam depths of 48, 36 and 21 inches.

According to the AASHO specifications, a 44-foot wide bridge should be designed for four traffic lanes. On the other hand, according to the PDH, the new bridges constructed in this state will contain a 10-foot full shoulder width at each side of the bridge for class 2 and 3 highways. This would mean that a 44-foot wide bridge will carry only two traffic lanes. The top
three curves in Fig. 40 were obtained by considering four design traffic lanes \((N_L = 4)\) and the bottom two curves for two traffic lanes \((N_L = 2)\). Therefore, another important factor is the number of design traffic lanes, \(N_L\). To continue with the remainder of this study, it was decided that it would be more appropriate to follow the AASHO specifications in regard to the number of design traffic lanes.

In Section 6.3.5 it was mentioned that the effect of span length on the load distribution is influenced by the number of design traffic lanes. This important point can be observed from Fig. 40. Therefore, the number of design traffic lanes, \(N_L\), not only affects the load distribution directly, but also influences the effect of span length on the load distribution.

As shown in Fig. 40, the load distribution factors for a 44-foot wide bridge with five beams can be approximated by the linear formula:

\[
D.F. = 1.6 + 1.4 \frac{S}{L} \tag{6.1}
\]

The value of 1.6 in the above formula was obtained by dividing the number of wheel loads on the bridge by the number of beams. This value will be reached at the hypothetical point of \(\frac{S}{L} = 0\). Since all of the design traffic lanes, \(N_L\), should be loaded for maximum response, the number of trucks would be equal to \(N_L\) and the number of wheel loads would be equal to \(2N_L\). Thus, the value of 1.6 can be formulated as \(\frac{2N_L}{N_B}\), where \(N_B\) is the number of beams.
6.3.7 **Total Width of Bridge**

When the number and the spacing of beams are known, the width of the bridge is almost fixed, except for a small variation in the distance from the center line of the exterior beams to the face of the curbs. However, an important influence of the width of the bridge on the load distribution is the fact that according to the AASHO specifications, the width of the bridge will determine the number of design traffic lanes, $N_L$. The number of design traffic lanes influences the load distribution factors significantly, as observed from Fig. 40. To continue with the study, the following rules listed in the AASHO specifications will be applied to determine the design traffic lanes:

<table>
<thead>
<tr>
<th>Width of Bridge in Feet</th>
<th>$N_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 to 30 inc.</td>
<td>2</td>
</tr>
<tr>
<td>over 30 to 42 inc.</td>
<td>3</td>
</tr>
<tr>
<td>over 42 to 54 inc.</td>
<td>4</td>
</tr>
<tr>
<td>over 54 to 66 inc.</td>
<td>5</td>
</tr>
</tbody>
</table>

6.3.8 **Number of Beams**

To study the influence of the number of beams on the pattern of load distribution, a maximum and a minimum clear spacing of 7'-9" to 2'-9", respectively, are considered. These limits on clear spacing are listed in the PDH standards for spread box-beam bridges. Considering the above limits, a 56-foot wide bridge can be designed with six, seven, eight or nine beams. In Fig. 41, the
load distribution factors for the 56-foot wide bridge with six, seven, eight and nine beams are plotted against different values of S/L. The plots are given in this figure for 48-and 21-inch beam depths. A study of Fig. 41 reveals that the load distribution factors are in the following linear form:

\[
D.\ F. = \frac{2NL}{NB} + k \frac{S}{L}
\]  

(6.2)

where k, the slope of the lines, varies with the number of beams. The effect of the beam depths decreases as the number of beams increases. In fact, as far as the practical range of the beam depths is concerned, the 48-inch deep beams are normally used for long span bridges and the 21-inch deep beams for short span. Therefore, the linear load distribution formula given in Fig. 41 for the 48-inch deep beams is quite accurate for long and moderate span length bridges, and would be slightly on the conservative side for short span bridges.

Next, a 66-foot wide bridge was considered, since it was the widest bridge that would be designed for five traffic lanes. The 66-foot bridge can be designed with seven, eight, nine or ten beams. The results are shown in Fig. 42. In Figs. 43 and 44, the same type of behavior for a 44-and 54-foot wide bridge is observed. The widths of 44 and 54 feet were considered in order to study the pattern of the load distribution in bridges with four design traffic lanes.

Figs. 45 and 46 show the plots of load distribution
factors against different values of $\frac{S}{L}$ for 32-and 42-foot wide bridges. A 32-or 42-foot wide bridge is designed based on three design traffic lanes. The 32-foot wide bridge can be designed with five, six or seven beams. The plots of load distribution factors in Figs. 42 through 46 are also in the same linear form.

Fig. 47 shows the basic arrangement used for the above developments, with a list of the beam spacing and slab thickness for the bridge superstructures analyzed. The width of slab extended beyond the exterior beams, designated by $b_s$ in Fig. 47, was taken as 1' - 6", except in two cases where it was necessary to reduce this dimension in order not to decrease the PDH specified minimum beam spacing of 6' - 9". The span length and the beam depth in each of the basic bridge superstructures listed in Fig. 47 were varied as indicated in Figs. 40 through 46.

6.4 Load Distribution Factors for Interior Beams

From the developments presented in Section 6.3, it is evident that the load distribution factors for the interior beams have the following linear form:

$$D.F. = \frac{2N_L}{N_B} + k \frac{S}{L} \quad (6.2)$$

where $k$ varies with the number of beams, the number of design traffic lanes, and the total width of the bridge. After a study of different values of $k$ given in Figs. 41 through 46, the following simple formulation was developed:
\[
k = 0.07 W - N_L (0.10 N_L - 0.26) - 0.20 N_B - 0.12 \quad (6.3)
\]

where \(W\) is the total width of bridge in feet.

In the development of Eq. (6.3) special attention was given to the simplicity of the result. An absolute error \(e\) in the evaluation of the value of \(k\) will result in an absolute error \((e \frac{S}{L})\) in the load distribution factors evaluated from Eq. (6.2). The value of \((\frac{S}{L})\), for the majority of spread box-beam bridges, ranges from \(1/4\) to \(1/15\). Therefore, any error in the evaluation of the value of \(k\) will result in an error, reduced by a factor ranging from \(1/4\) to \(1/15\), in the load distribution factors.

6.5 Load Distribution Factors for Exterior Beams

The effect of span length on the exterior beam moments does not seem appreciable when all design traffic lanes are loaded. This fact can be seen from the influence lines given in Fig. 36. However, as was observed in the developments presented for the interior beams, the total width of the bridge, the number of design traffic lanes, and finally, the number of beams could influence the behavior. Therefore, the same bridge superstructures considered in the developments for the interior beams are analyzed again under the type of loading which produces maximum response in the exterior beam.

Figs. 48 through 53 show the plots of load distribution factors against different values of \(\frac{S}{L}\) for the same bridge superstructures considered in Section 6.3.8. A study of the plots given
in these figures reveals that the load distribution factors for the exterior beams are not appreciably influenced by the span length of the bridge. The present provisions in the PDH specifications consider the load distribution factors for the exterior beams as the reaction of the wheel loads obtained by assuming the flooring to act as a simple span between the exterior and the adjacent interior beams. The distribution factors calculated, based on the PDH provisions, are also given in Figs. 48 through 53 for the sake of comparison.

The load distribution factors given in Figs. 48 through 53 can be approximated by either the following formula:

\[
D.F. = \frac{2N_L}{N_B} \quad (6.4)
\]

or by the PDH provisions. In Fig. 48, for example, the plot of the distribution factors for \(N_B = 7\) can be approximated as \(D.F. = 1.43\), whereas based on the present PDH provisions the corresponding distribution factor is 1.33. Therefore, for bridges with five design traffic lanes, as can be seen in Figs. 48 and 49, the load distribution factors given in Eq. (6.4) will govern, and the PDH provisions yield very low values.

The plots of load distribution factors for a 54-foot wide bridge with six, seven and eight beams are shown in Fig. 50. For six-and seven-beam bridges, the load distribution factors given in Eq. (6.4) will goven, and for the eight-beam bridge,
the PDH provisions will govern. For the 44-foot wide bridge with five, six or seven beams, shown in Fig. 51, again Eq. (6.4) yields more accurate values. The same pattern of behavior can be observed from Figs. 52 and 53 for the 42-and 32-foot wide bridges. Therefore, based upon the above developments, the load distribution factors for the exterior beams should be taken as the greater of the results obtained from Eq. (6.4) and the present PDH provisions.

6.6 Effective Width of Slab

The distribution factors given in Sections 6.4 and 6.5 were obtained based on the moments in the composite beams given in Eq. (5.25). The composite beams, as described in Section 5.6, are formed with consideration of the effective width of slab. The effective width of slab predicted by this theory was found to be very close to the center-to-center beam spacing. The present provisions of the PDH specifications specify that the effective width of slab shall not exceed the beam spacing, or twelve times the thickness of slab plus the width of the beams. An examination of PDH criteria for the determination of slab thickness, given on page 81, reveals that actually the beam spacing will govern the determination of effective width. Therefore, based on the results of the theory, the present PDH provisions for the determination of the effective width of slab are found to be adequate.

6.7 Design Recommendations

Based upon the developments presented in this chapter,
the following simplified procedures are recommended for the
determination of the lateral load distribution factors in spread
box-beam bridges:

1. The live-load bending moments in interior
   beams shall be determined by applying to
   the beams the fraction of wheel loads speci-
   fied by the following formula:

   \[ D.F. = \frac{2N_L}{N_B} + k \frac{S}{L} \]

   where

   \[ k = 0.07 W - N_L (0.10 N_L - 0.26) - 0.20 N_B - 0.12 \]

   \( W \) = Roadway width between curbs in feet

   \( N_L \) = Number of design traffic lanes

   \( N_B \) = Number of beams

   \( S \) = Average beam spacing

   \( L \) = Span length

2. The live-load bending moments in exterior
   beams shall be determined by applying to
   the beams the reaction of wheel loads ob-
   tained by assuming the flooring to act as
   a simple span between the exterior and the
   adjacent interior beams, but shall not be
   less than \( \frac{2N_L}{N_B} \).
7. SUMMARY AND CONCLUSIONS

7.1 Summary

A method of analysis is developed for the beam-slab bridges in order to investigate the load distribution pattern in spread box-beam bridges. In this method of analysis, the bridge superstructure is reduced to an articulated system by introducing a series of joint-lines between the beams and slab. The method of solution employed is a flexibility type. The analysis of plate and beam elements are presented in Chapters 3 and 4. In Chapter 5, the matrix formulation of the flexibility type of solution is developed. A general computer program was prepared to apply the analysis.

The validity of this theoretical analysis is verified by comparison with field test results. The comparison of the results of four different field tests with the theory is included in Chapter 6. An extensive study of the parameters affecting the analysis is also presented in Chapter 6. Over three hundred different bridge superstructures were analyzed under different patterns of HS20 truck loading, in order to present the full picture of the behavior of spread box-beam bridges. Based upon the results obtained, a simplified design procedure for the determination of lateral load distribution is developed and recommended in Chapter 6, Section 6.6.
7.2 **Conclusions**

The method of analysis presented in the text is a general method applicable to any beam-slab bridge superstructure. In the formulation of the method, special consideration was given to box-beams, but modifications which should be made for other types of beams are also specified. Excellent agreement between the results of the field tests and the theory was observed.

The following conclusions are made, based upon the results of over three hundred bridges analyzed under different patterns of truck loadings:

1. The effect of curbs and parapets acting compositely with the slab tends to reduce the vehicular load carried by the interior girders, and to increase the load carried by the exterior girders, in comparison with loads carried by a beam-slab system without curbs and parapets.

2. The span length of the bridge significantly influences the lateral distribution of vehicular loads, while beam depth influences the distribution to a lesser degree. The effects of (1) slab thickness and (2) the modular ratio between beam and slab concrete have little effect on the distribution.

3. The live-load distribution factors for both exterior and interior beams in a particular superstructure cross-section can be accurately represented as a
linear function, of the form indicated in Section 6.7.

4. The number of design traffic lanes is a very important factor in establishing load distribution factors.

5. The expressions for distribution factors presented in this report were developed for beam-slab superstructures without curbs and parapets. In general, these values are less than values based on current design provisions. Based on the findings from a number of field tests, it is felt that strong consideration should be given to changes in design philosophy and construction practice to permit consideration of the effect of the curbs and parapets on the structural behavior of the superstructure. These changes would definitely result in a further reduction in distribution factors, and possibly reduce the overall cost of the superstructure.

7.3 Recommendations for Future Studies

The analysis developed in this paper is applicable to any type of beam-slab simple-span bridge loaded within the elastic limit. However, the accompanying computer program was prepared for box-beam bridges. Therefore, the following areas of study are recommended for possible future research, with some of these areas being simple extensions of the study presented in this paper:
1. The problem of load distribution in any other type of beam-slab bridge loaded within the elastic limit can be investigated, using the analysis presented in this paper and modifying the accompanying computer program accordingly.

2. The analysis presented in this paper can be extended in order to develop an analysis for the problem of load distribution in continuous multi-span bridges.

3. The analysis of skewed bridges is another potential area of study, and is recommended here since it will yield information regarding the effect of skew on the pattern of load distribution.

4. The problem of load distribution in bridges loaded beyond the elastic limit is another important area of interest. Future studies in this area will make it possible to obtain some insight into the behavior of bridges under any possible over-load conditions, and to predict the factor of safety against the collapse conditions.
8.1 **Flexibility Matrix of Plate Elements**

The coefficients of the flexibility matrix $f_p$ are determined based on the analysis of the plate elements presented in Chapter 3. As an example, consider the flexibility coefficients associated with vertical edge force $R_j$. These coefficients are obtained from Eq. (3.20) and the first derivative of Eq. (3.20) with respect to $y$, by setting once $y = 0$ and then $y = a$. The flexibility coefficients associated with other edge forces and edge moments are obtained in a similar fashion. The results are as follows:

$$
\begin{align*}
\begin{bmatrix}
  -E_{1n} & E_{1n} & 0 & 0 & A'_{2n} & A_{2n} & 0 & 0 \\
  E_{1n} & -F_{1n} & 0 & 0 & -A_{2n} & A_{1n} & 0 & 0 \\
  0 & 0 & -E_n & -F_n & 0 & 0 & A_n & A'_n \\
  0 & 0 & F_n & -F'_n & 0 & 0 & A'_n & E'_n \\
\end{bmatrix}
\end{align*}
$$

$$f_p = \begin{bmatrix}
  -E'_{1n} & E_{1n} & 0 & 0 & E'_{1n} & E_{1n} & 0 & 0 \\
  -A'_{2n} & A_{2n} & 0 & 0 & E_{1n} & F_{1n} & 0 & 0 \\
  -A_{2n} & -A_{1n} & 0 & 0 & E_{1n} & F_{1n} & 0 & 0 \\
  0 & 0 & -A_n & A'_n & 0 & 0 & E_n & -F_n \\
  0 & 0 & A'_n & -E'_n & 0 & 0 & F_n & F'_n
\end{bmatrix}
$$

(8.1)
where

\[ F_n = \frac{1}{(1-\nu)\alpha_n^2 D} \frac{(1+\nu)(\nu+3)\sinh^2\alpha_n a - (1-\nu)^2 \alpha_n^2 a^2}{(\nu+3)^2 \sinh^2\alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \]  \hspace{1cm} (8.2)

\[ F'_n = \frac{-2}{(1-\nu)\alpha_n^2 D} \frac{(\nu+3)\sinh\alpha_n \cosh\alpha_n a - (1-\nu)\alpha_n a}{(\nu+3)^2 \sinh^2\alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \]  \hspace{1cm} (8.3)

\[ E_n = \frac{2}{(1-\nu)\alpha_n^2 D} \frac{(\nu+3)\sinh\alpha_n \cosh\alpha_n a + (1-\nu)\alpha_n a}{(\nu+3)^2 \sinh^2\alpha_n a - (1-\nu)^2 \alpha_n^2 a^2} \]  \hspace{1cm} (8.4)

\[ E'_n = \frac{2\alpha_n}{1-\nu} D'n \]  \hspace{1cm} (8.5)

and \( D'_n \) is given in Eq. (3.29d).

\[ A_{2n} = -\frac{2\alpha_n}{E} B_{1n} \]  \hspace{1cm} (8.6)

\[ A_{1n} = -\frac{2\alpha_n}{E} C_{1n} \]  \hspace{1cm} (8.7)

where \( B_{1n} \) and \( C_{1n} \) are those given in Eqs. (3.62a and b) in which \( h_n^j \) should be omitted.

\[ A'_{2n} = -\frac{2\alpha_n}{E} B'_{1n} \]  \hspace{1cm} (8.8)

\( B'_{1n} \) is that given in Eq. (3.65a) in which \( s_n^j \) will be omitted.

\[ E_{1n} = \frac{-1}{\alpha_n^2 E} \frac{(1-\nu)\sinh^2\alpha_n a + (1+\nu)\alpha_n^2 a^2}{\sinh^2\alpha_n a - \alpha_n^2 a^2} \]  \hspace{1cm} (8.9)

\[ E'_{1n} = \frac{2}{\alpha_n^2 E} \frac{\sinh\alpha_n \cosh\alpha_n a - \alpha_n a}{\sinh^2\alpha_n a - \alpha_n^2 a^2} \]  \hspace{1cm} (8.10)
\[ F_{1n} = \frac{2}{\alpha_n t} \frac{\sinh \alpha a \cosh \alpha a + \alpha a}{\sinh^2 \alpha a - \alpha a^2} \]  

(8.11)

and finally \( A_n \) and \( A_n^t \) are those given in Eqs. (3.24a) and (3.29a).

8.2 Flexibility Matrix of Beam Elements

The coefficients of the flexibility matrix \( f_b \) are determined from Eqs. (4.33a, b, c and d). The results are as follows:

\[
\begin{bmatrix}
  f^j_{11} & f^j_{12} & f^j_{13} & 0 & f^j_{15} & -f^j_{12} & -f^j_{13} & 0 \\
  -f^j_{15} & f^j_{12} & f^j_{13} & 0 & -f^j_{11} & f^j_{12} & f^j_{13} & 0 \\
  f^j_{12} & f^j_{22} & f^j_{23} & f^j_{24} & f^j_{12} & -f^j_{22} & f^j_{23} & -f^j_{24} \\
  f^j_{13} & f^j_{23} & f^j_{33} & f^j_{34} & -f^j_{13} & f^j_{23} & f^j_{37} & -f^j_{34} \\
  -f^j_{24} & -f^j_{34} & f^j_{44} & 0 & f^j_{24} & -f^j_{34} & -f^j_{44} \\
  -f_{15} & f_{12} & f_{13} & 0 & -f_{11} & f_{12} & f_{13} & 0 \\
  -f_{12} & f_{22} & -f_{23} & f_{24} & f_{12} & -f_{22} & -f_{23} & -f_{24} \\
  f_{13} & f_{23} & f_{37} & f_{34} & -f_{13} & f_{23} & -f_{33} & -f_{34} \\
  0 & f_{24} & f_{34} & f_{44} & 0 & f_{24} & f_{34} & -f_{44}
\end{bmatrix}
\]

(8.12)
where

\[ f_{11}^j = \frac{-1}{E \alpha_n} \left[ y^j \left( z_c I_{yz}^i + y^j I_y^j \right) \right] + z_c \left( y^j I_{yz}^i + z_c I_z^i \right) + \frac{1}{\bar{A}} \]  

\[ f_{12}^j = \frac{1}{E \alpha_n} \left( z_c I_{yz}^i + y^j I_y^j \right) \]  

\[ f_{13}^j = \frac{-1}{E \alpha_n} \left( y^j I_{yz}^i + z_c I_z^i \right) \]  

\[ f_{22}^j = - \left( \frac{I_{yz}^i}{E \alpha_n} + \frac{z_s^2}{G J \alpha_n^2} \right) \]  

\[ f_{23}^j = \frac{I_{yz}^i}{E \alpha_n} + \frac{z_s \left( y_s^i - y^j \right)}{G J \alpha_n^2} \]  

\[ f_{24}^j = \frac{z_s}{G J \alpha_n^2} \]  

\[ f_{33}^j = - \frac{I_z^i}{E \alpha_n} - \frac{(y_s^i - y^j)^2}{G J \alpha_n^2} \]  

\[ f_{34}^j = \frac{y^j - y_s^i}{G J \alpha_n^2} \]  

\[ f_{44}^j = \frac{1}{G J \alpha_n^2} \]  

\[ f_{15}^j = \frac{1}{E \alpha_n^2} \left[ z_c \left( y^j I_{yz}^i + z_c I_z^i \right) + \frac{1}{\bar{A}} \right] - y^j \left( z_c I_{yz}^i + y^j I_y^j \right) \]  

\[ - y^j \left( z_c I_{yz}^i + y^j I_y^j \right) \]
\[ f^j_{37} = \frac{I_z}{E \alpha_n} - \frac{(y_s - y^j)^2}{G J \alpha_n^2} \] (8.23)

and the coefficients designated by the superscript \((j+1)\) are obtained from Eqs. (8.13) through (8.23) by substituting \(y^{j+1}\) for \(y^j\). The coefficients of the flexibility matrix \(f_b\), given in Eq. (8.12), are subjected to the relations given in Eqs. (4.32a, b, c, d and e) when applied to the interior beam elements.
9. FIGURES
Fig. 1 Transverse Cross-Section of Spread Box-Beam Bridge
Fig. 2 Standard Precast Prestressed Box-Beam
Simple Supports At Both Ends
(typical)

Fig. 3 Elements and Joint-Lines
Fig. 4 Reference Axes and Positive Direction of Forces
Fig. 5 Wheel Load Dimensions
Fig. 6 Wheel Load Location on a Typical Plate Element

Fig. 7 A typical Strip of the Loaded Area
Fig. 8 Vertical Edge Forces on Element (i-1)

\[ R_{n}^{j-1} = R_{n}^{j-1} \sin \alpha_n x \]

Fig. 9 Edge Moments on Element (i-1)

\[ M_{n}^{j-1} = M_{n}^{j-1} \sin \alpha_n x \]

\[ M_{n}^{j} = M_{n}^{j} \sin \alpha_n x \]
Fig. 10 Reciprocal Relations

\[(w)^1_{y=y_2} = (w)^2_{y=y_1}\]
Fig. 11 In-Plane Normal Edge Forces on Element (i-i)

\[ H^j_i = H^j_i \sin \alpha_n x \]

Fig. 12 In-Plane Tangential Edge Forces on Element (i-i)

\[ S^j_i = S^j_i \cos \alpha_n x \]
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Fig. 15 Idealized and Actual Cross-Section of Exterior Beam Elements
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Fig. 17 Influence Line for Beam A - Drehersville Bridge

Span Length: 61' - 6"
Beam Spacing: 7' - 2"
Roadway Width: 30' - 0"
Fig. 18 Influence Line for Beam B - Drehersville Bridge

Span Length: 61'-6"
Beam Spacing: 7'-2"
Roadway Width: 30'-0"
Fig. 19 Influence Line for Beam C - Drehersville Bridge
Fig. 20  Influence Line for Beam A - Berwick Bridge

Span Length:  65' - 3"
Beam Spacing:  8' - 9 3/8"
Roadway Width:  28' - 0"
Fig. 21 Influence Line for Beam B - Berwick Bridge

Span Length: 65'-3"
Beam Spacing: 8'-9 3/8"
Roadway Width: 28'-0"

Δ Theory with Curb & Parapet
□ Theory with Curb
○ Test
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Fig. 24 Influence Line for Beam A - Philadelphia Bridge
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- Span Length: 71'-9"
- Beam Spacing: 9'-6"
- Roadway Width: 40'-0"
Fig. 30 Influence Line for Beam B - Philadelphia Bridge
Fig. 31 Influence Line for Beam C - Philadelphia Bridge

Span Length: 71' - 9"
Beam Spacing: 9' - 6"
Roadway Width: 40' - 0"
Fig. 32 Effect of Slab Thickness
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Fig. 34 Effect of Span Length on Center Beam Moments

W = 40'-0"
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Fig. 35 Effect of Span Length on Exterior Beam Moments
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66-Ft. Wide Bridge ($N_L = 5$)
Fig. 43 Distribution Factors for Interior Beam
44-Ft. Wide Bridge ($N_L = 4$)

D.F. = $\frac{2N_L}{N_B} + 1.40 \frac{S}{L}$

D.F. = $\frac{2N_L}{N_B} + 1.00 \frac{S}{L}$

D.F. = $\frac{2N_L}{N_B} + 1.20 \frac{S}{L}$
Fig. 44 Distribution Factors for Interior Beam
54-Ft. Wide Bridge ($N_L = 4$)
Fig. 45 Distribution Factors for Interior Beam
32-Ft. Wide Bridge ($N_L = 3$)

D.F. = \[ \frac{2N_L}{N_B} + 1.20 \frac{S}{L} \]

D.F. = \[ \frac{2N_L}{N_B} + 0.80 \frac{S}{L} \]

$N_B = 4$

$N_B = 5$

48" Deep Beam

21" Deep Beam
Fig. 46 Distribution Factors for Interior Beam
42-Ft. Wide Bridge ($N_L = 3$)
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Fig. 50 Distribution Factors for Exterior Beam
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Fig. 51 Distribution Factors for Exterior Beam
44-Ft. Wide Bridge \( (N_L = 4) \)
Fig. 52 Distribution Factors for Exterior Beam
42-Ft. Wide Bridge (N_L = 3)
Fig. 53 Distribution Factors for Exterior Beam
32-Ft. Wide Bridge ($N_L = 3$)
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