Strength of Rectangular Composite Box Girders

ANALYSIS OF COMPOSITE BOX GIRDERS

by

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong> INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Objective and Scope</td>
<td>3</td>
</tr>
<tr>
<td><strong>2.</strong> FLEXURAL ANALYSIS</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Review of Bending Theory</td>
<td>6</td>
</tr>
<tr>
<td>2.2 Elastic Moduli and Stress–Strain Relationships of the Deck and the Bottom Flange</td>
<td>10</td>
</tr>
<tr>
<td>2.2.1 Reinforced Concrete Deck</td>
<td>10</td>
</tr>
<tr>
<td>2.2.2 Orthotropic Bottom Flange</td>
<td>16</td>
</tr>
<tr>
<td>2.3 Differential Equations of Stress Function and Solutions</td>
<td>19</td>
</tr>
<tr>
<td>2.4 Stresses in Flanges and Webs</td>
<td>22</td>
</tr>
<tr>
<td>2.5 Results and Comparisons</td>
<td>26</td>
</tr>
<tr>
<td><strong>3.</strong> TORSIONAL ANALYSIS</td>
<td>33</td>
</tr>
<tr>
<td>3.1 Torsional Sectional Properties of Single-celled Box</td>
<td>33</td>
</tr>
<tr>
<td>3.2 Location of Twisting Center</td>
<td>40</td>
</tr>
<tr>
<td>3.3 Differential Equations and Solutions</td>
<td>42</td>
</tr>
<tr>
<td>3.4 Results and Comparisons</td>
<td>46</td>
</tr>
<tr>
<td><strong>4.</strong> DISTORTIONAL STRESSES</td>
<td>48</td>
</tr>
<tr>
<td><strong>5.</strong> ELASTIC STRESSES AND DEFLECTIONS</td>
<td>55</td>
</tr>
<tr>
<td>5.1 Stresses</td>
<td>55</td>
</tr>
<tr>
<td>5.2 Effect of Cracks in Concrete Deck Due to Negative Moment</td>
<td>58</td>
</tr>
<tr>
<td>5.3 Deflections</td>
<td>59</td>
</tr>
<tr>
<td><strong>6.</strong> SUMMARY AND CONCLUSIONS</td>
<td>61</td>
</tr>
<tr>
<td><strong>7.</strong> ACKNOWLEDGMENTS</td>
<td>63</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS (continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>TABLES</td>
<td>64</td>
</tr>
<tr>
<td>FIGURES</td>
<td>70</td>
</tr>
<tr>
<td>APPENDIX A - FLEXURAL STRESSES IN SINGLE CELL</td>
<td>121</td>
</tr>
<tr>
<td>COMPOSITE BOX GIRDERS</td>
<td></td>
</tr>
<tr>
<td>APPENDIX B - ROTATIONS AND DERIVATIVES OF WARPING</td>
<td>149</td>
</tr>
<tr>
<td>FUNCTION</td>
<td></td>
</tr>
<tr>
<td>NOTATION</td>
<td>154</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>160</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

1.1 Background

Steel and composite steel-concrete box girders have become increasingly popular as bridge superstructures in the last two decades. The main reasons are that the box girders are (1) structurally efficient because of their high torsional rigidity, (2) aesthetically pleasing because of their long span with shallow depth, and (3) highly economical in fabrication and in maintenance because of their segmental type of construction and their interior space sealed to provide a noncorrosive atmosphere.

It was not until the four unfortunate erection failures of steel box girder bridges in Austria, the United Kingdom, Australia, and Germany that this type of structure received extensive research. In particular, the Merrison Committee was established by the United Kingdom Department of the Environment to inquire into the basis of design and method of erection of steel box girder bridges. The committee issued the "Interim Design and Workmanship Rules". The content of the rules is mainly the elastic, prebuckling stress analysis taking into consideration the effects of shear lag, torsional warping, cross-sectional distortion, residual stresses, and plate initial imperfections.
The methods of analysis and design of box girders have been surveyed several times to date (1.4, 1.5, 1.6). The prismatic folded plate theory by Goldberg and Leve (1.7) considers the box girder to be made up of an assemblage of folded plates. This method uses two-dimensional elasticity theory for determining membrane stresses and classical plate theory for analyzing bending and twisting of the component plates. The analysis is limited to straight, prismatic box girders composed of isotropic plates with no interior diaphragms and with simply supported end conditions. Scordelis (1.8) later presented a folded plate analysis for simply supported, single-span box girder bridges with or without intermediate diaphragms.

The thin-walled elastic beam theory developed by Vlasov (1.9) has been refined and extended to treat simple or continuous single-cell girders with longitudinally or transversely stiffened plate elements and with rigid or deformable interior diaphragms (1.10, 1.11). The complexity of the refined analytical methods, the "plate element" method and the "generalized coordinate" method, tends to obscure the effects of the major design parameters. A simplified version of the refined methods for determining the stresses induced from cross-sectional distortion of a single-celled box girder has been developed based on an analogy with the theory of a beam on an elastic foundation (1.12).

The finite element method is the most general of the methods utilized. It can treat any loading and boundary conditions, varying
girder dimensions and material properties, and interior diaphragms. However, more computer time is required than with the other methods. The main problem in the finite element procedures has been to seek a more sophisticated displacement field so that the resulting stresses and node displacements can represent the actual conditions more realistically. Scordelis (1.13) has used rectangular elements with six degrees of freedom per node in his concrete box girder studies. Lim and Moffat (1.14) have developed third order extensional-flexural rectangular elements. These elements were extensively used in conducting parametric studies of shear lag and cross-sectional distortion for the preparation of the Merrison Interim Design Rules (1.3).

In addition, there are modified methods such as Finite Segment Method (1.13) and Finite Strip Method (1.15). The finite segment method, using matrix progression procedures, is based on simplified folded plate theory. The finite strip method is the extension of the finite element method to the case of finite strips.

1.2 Objective and Scope

Most of the methods of analysis mentioned above are primarily used for the examination of stresses and cross-sectional proportioning of steel or concrete box girders. The main objective of this work is the development of a procedure for stress analysis of steel-concrete composite box girders on the basis of classical elastic theory. From the procedure, information can be derived for proportioning of box girder cross-sections.
The common configurations of composite box girder cross-sections are illustrated in Fig. 1.1. The top flange of a box girder may be a cast-in-place or precasted reinforced concrete slab. The bottom flange may be a plane or an orthotropic steel plate. The webs may be plain steel plates stiffened transversely or transversely and longitudinally. A composite section with a single box and a concrete deck is chosen for this study. A combination of single boxes form a multi-box with the webs carrying the flexural shear and the top deck serving as the roadway. This probably is one of the most efficient and economical arrangements for continuous span composite steel-concrete bridges. The procedure developed here for single cell boxes can be applied to multi-cell composite box girders.

In this study a load eccentric to the shear center, as shown in Fig. 1.2, is decomposed into bending and torsional systems. The torsional system is further decomposed into pure torsional and distortional systems. The bending system considering shear lag effect is examined in Chapter 2. The concrete deck and the bottom flange can be either isotropic or orthotropic. The expressions for the stress distribution in and the equivalent widths of the flanges are deduced. The pure torsional system is considered in Chapter 3. A unified, consistent method for the evaluation of torsional cross-sectional properties of composite box sections is presented. In Chapter 4 the distortional system is discussed. The torsional and the distortional warping normal
stresses are compared. The deflection caused by cross-sectional
distortion is also discussed. In Chapter 5 the computed and
the experimental stresses and deflections are compared within the
elastic, prebuckling range. Good agreement has been observed.
2. FLEXURAL ANALYSIS

2.1 Review of Bending Theory

The Bernoulli-Navier hypothesis states that plane cross sections of a flexural member remain plane after bending. This requires that the longitudinal strain of a fiber is proportional to its distance from the neutral axis. For box girders with high ratios of flange width to span length, shear lag effect in the flanges cannot be ignored. This hypothesis is therefore not valid. A more rigorous solution is required.

If the equivalent elastic constants, which are to be discussed in Section 2.2.1, are obtained for the deck of a composite box girder as shown in Fig. 2.1, the moment of inertia about the centroidal principal axes of a cross section can be computed in the same manner as for conventional composite sections. The normal stresses are given by

\[
\sigma_z = \frac{M_y}{I_x} \quad (2.1a)
\]

\[
\sigma_z = \frac{M_x}{I_y} \quad (2.1b)
\]

in which \(\sigma_z\) is the longitudinal stress, and \(M_x, M_y, I_x, I_y\) are the bending moments and moments of inertia of the transformed section about the centroidal principal x- and y-axes, respectively.
The shearing stress at a point in the cross section due to a shear force $V_y$ acting in the y-direction is given by

$$\tau = \frac{V_y Q_x}{I_x t_i} \quad (2.2a)$$

To compute the static moment of area, $Q_x$, a cut must first be introduced to make the cross section of a box girder as shown in Fig. 2.2, determinate. (2.1) The shear flow in the cut section is

$$q_o = \frac{V_y}{I_x} \int_0^s y \left( \frac{E_i}{E_r} t_i \right) ds = \frac{V_y Q_x}{I_x} \quad (2.2b)$$

The compatibility condition requires that the longitudinal relative movement at the cut be zero, thus requiring a shear flow $q_1$.

$$q_1 = -\frac{\int q_o ds}{\int \frac{G_i}{t_i} ds} \quad (2.2c)$$

The resultant shear flow is $q$,

$$q = q_o + q_1 \quad (2.2d)$$

and the resultant shearing stress is $\tau$.

$$\tau = \frac{q}{t_i} \quad (2.2e)$$

Substituting Eqs. 2.2b, 2.2c and 2.2d into Eq. 2.2e will result in Eq. 2.2a, with

$$Q_x = Q_x - \frac{\int \frac{Q_x ds}{G_i t_i}}{\int \frac{G_i ds}{t_i}} \quad (2.2f)$$
In Eqs. 2.2, the symbols are:

\[ y = \text{y-coordinate of a point considered,} \]
\[ E_i = \text{Young's modulus of an individual element,} \]
\[ E_r = \text{reference Young's modulus used in the cross-sectional transformation,} \]
\[ t_i = \text{thickness of an individual element,} \]
\[ Q_x = \text{transformed statical moment of area of the cut section, about x-axis, at a point considered,} \]
\[ G_i = \text{shear modulus of an individual element,} \]
\[ Q_x = \text{adjusted statical moment of area for the closed cross section, about x-axis, at a point considered,} \]
\[ \oint = \text{integral extending over the entire closed perimeter.} \]

If the shear modulus of reference material, \( G_r \), is divided through both the numerator and denominator of the fractional terms of Eq. 2.2f, the following equation will result.

\[
\bar{Q}_x = Q_x - \frac{\oint Q_x \, ds}{\oint \left( \frac{G_i}{G_r} \right) t_i} \quad \text{(2.2g)}
\]
The term, \( \frac{G_i}{G_r} \, t_i \) in Eq. 2.2g can be considered as a thickness transformation.

Similar to the case of Eqs. 2.2, the shearing stress at a point in the cross section due to a shear force \( V_x \) acting in the \( x \)-direction is given by

\[
\tau = \frac{V_x \, Q_y}{I_y \, t_i}
\]

(2.3a)

and

\[
\bar{Q}_y = Q_y - \frac{\int \frac{Q_y \, ds}{G_i/G_r \, t_i}}{\int \frac{G_i}{G_r} \, t_i}
\]

(2.3b)

It is to be noted that the equivalent longitudinal modulus of elasticity of the deck, \( E_t \) (Section 2.2.1) should be used in the calculation of the moments of inertia and the statical moments of area in Eqs. 2.1, 2.2, and 2.3 because bending strains take place in the longitudinal direction.

As mentioned previously, shear lag effect in the flanges must be considered in box girders with high flange width to span ratios. Under flexural loading, the flanges sustain in-plane shearing forces along the lines of connection with the webs. These in-plane shearing forces cause shearing deformations and result in nonuniform longitudinal strains. This results in nonuniform stresses across the
widths of the flanges. This characteristic is termed as shear lag. To evaluate the effects of shear lag, the stress-strain relations of the deck and the bottom flange must first be formulated.

2.2 Elastic Moduli and Stress-Strain Relationships of the Deck and the Bottom Flange

Figure 2.3 shows reinforcing bars in a concrete deck and stiffeners on a steel bottom flange plate which contribute respectively to the overall or equivalent elastic moduli of the deck and to the stress distribution in the bottom flange. By treating the deck and the bottom flange of a box girder as problems of plane stress elasticity, the stress-strain relationship for each of these structural components may be established.

2.2.1 Reinforced Concrete Deck

The elastic properties of the concrete deck may be evaluated in a manner analogous to that for composite materials (2.2). Figure 2.4 depicts the model used by Ekvall (2.3, 2.4) for a one-layer composite, in which t is the thickness of the one-layer composite and \( d_f \) the fiber diameter. The matrix material is reinforced with unidirectional, equally spaced round fibers which are securely bonded to the matrix. Both materials are assumed to obey Hooke's Law. The modulus of elasticity in the longitudinal (\( x_1 \)) direction is estimated by the classical law of mixtures.

\[
E_1 = A_f E_f + A_m E_m \quad (2.4a)
\]
where

\[ E_1 = \text{longitudinal (x_1-direction) modulus of elasticity of the one-layer composite,} \]

\[ A_f = \text{ratio of the area of reinforcing fibers to the cross-sectional area of composite,} \]

\[ E_f = \text{Young's modulus of reinforcing fibers,} \]

\[ A_m = \text{ratio of the area of matrix to the cross-sectional area of composite, and} \]

\[ E_m = \text{Young's modulus of the matrix.} \]

The Poisson's ratio of the composite is computed similarly.

\[ \nu_{12} = A_f \nu_f + A_m \nu_m \]  \hspace{1cm} (2.4b)

where

\[ \nu_{12} = \text{Poisson's ratio of the composite in the longitudinal (x_1) direction due to stresses} \]

\[ \nu_f = \text{Poisson's ratio of reinforcing fibers, and} \]

\[ \nu_m = \text{Poisson's ratio of the matrix.} \]

It is to be noted that in Eqs. 2.4, \( A_f + A_m = 1 \)

The transverse modulus of elasticity and the shear modulus of elasticity are expressed in Refs. 2.2 and 2.3 in the form of sine function integrals. Execution of the integrations results in the following:

\[ E'_2 = \frac{E_f E_m}{b_1} \left[ \frac{\pi}{2} - \frac{2}{\sqrt{a_1^2 - b_1^2}} \tan^{-1} \frac{a_1 - b_1}{\sqrt{a_1^2 - b_1^2}} \right], (a_1 > b_1) \]  \hspace{1cm} (2.5a)
where

\[
E_{2}' = \text{transverse (x}_2\text{-direction) modulus of elasticity of the one-layer composite with one-fiber diameter thickness}
\]

\[
G_{12}' = \text{shear modulus corresponding to } E_{2}',
\]

\[
a_1 = E_f (1 - \nu_m^2),
\]

\[
b_1 = R [E_m - E_f (1 - \nu_m^2)],
\]

\[
R = \text{the ratio of fiber diameter to fiber spacing (center-to-center) in the transverse (x}_2\text{) direction.}
\]

\[
H = \frac{1}{b_2} \left[ \pi - \frac{a_2}{\sqrt{b_2 - a_2}} \ln \frac{1 + \sqrt{b_2 - a_2}}{1 - \sqrt{b_2 - a_2}} \right], (b_2 > a_2),
\]

\[
a_2 = \frac{G_m}{G_f},
\]

\[
b_2 = 1 - \frac{G_m}{G_f},
\]

\[
G_f = \text{shear modulus of reinforcing fibers, and}
\]

\[
G_m = \text{shear modulus of the matrix.}
\]

Equations 2.4 and 2.5 are for one-layer composites with thickness, \(t\), equal to one fiber diameter, \(d_f\) (Fig. 2.4). If \(t\) is larger than \(d_f\), Eqs. 2.4 remain the same for evaluating the longitudinal direction properties. In the transverse (x2) direction, the moduli are again estimated by applying the law of mixtures.
\[ E_2 = \frac{d_f}{t} E_2' + \frac{t - d_f}{t} E_m \]  

(2.6a)

and

\[ G_{12} = \frac{d_f}{t} G_{12}' + \frac{t - d_f}{t} G_m \]  

(2.6b)

After the evaluation of the four elastic constants, \( E_1, E_2, \nu_{12}, \) and \( G_{12}, \) the fifth, \( \nu_{21}, \) can be obtained by the orthotropic identity.

\[ \nu_{12} E_2 = \nu_{21} E_1 \]  

(2.7)

The stress-strain relationship with respect to the principal axes \( x_1 \) and \( x_2 \) (Fig. 2.4) are as follows:

\[
\sigma = \begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{pmatrix} = \begin{pmatrix}
\frac{E_1}{\Lambda} & \frac{E_2 \nu_{12}}{\Lambda} & 0 \\
\frac{E_2 \nu_{12}}{\Lambda} & \frac{E_2}{\Lambda} & 0 \\
0 & 0 & G_{12}
\end{pmatrix} \begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{pmatrix}
\]  

(2.8)

where \( \Lambda = 1 - \nu_{12} \nu_{21}. \) The orthotropic identity, Eq. 2.7, has been used to make the stiffness matrix of Eq. 2.8 symmetrical.

Practically all reinforcing bars in concrete decks are either parallel or perpendicular to the longitudinal (z) axis of the girder. Individual layers of reinforcing bars and concrete can be considered as unidirectional composites as shown in Fig. 2.5. For a layer with bars parallel to the z-axis,
\[
\mathbf{\sigma} = \begin{bmatrix}
\sigma_z \\
\sigma_x \\
\tau_{zx}
\end{bmatrix} = \begin{bmatrix}
\frac{E_1}{\Lambda} & \frac{E_2 \nu_{12}}{\Lambda} & 0 \\
\frac{E_2 \nu_{12}}{\Lambda} & \frac{E_2}{\Lambda} & 0 \\
0 & 0 & G_{12}
\end{bmatrix} \begin{bmatrix}
\varepsilon_z \\
\varepsilon_x \\
\gamma_{zx}
\end{bmatrix} \quad (2.9a)
\]

and for a layer with bars perpendicular to the z-axis,

\[
\mathbf{\sigma} = \begin{bmatrix}
\sigma_z \\
\sigma_x \\
\tau_{zx}
\end{bmatrix} = \begin{bmatrix}
\frac{E_2}{\Lambda} & \frac{E_2 \nu_{12}}{\Lambda} & 0 \\
\frac{E_2 \nu_{12}}{\Lambda} & \frac{E_1}{\Lambda} & 0 \\
0 & 0 & G_{12}
\end{bmatrix} \begin{bmatrix}
\varepsilon_z \\
\varepsilon_x \\
\gamma_{zx}
\end{bmatrix} \quad (2.9b)
\]

or in simple matrix form,

\[
\{\mathbf{\sigma}\} = [\mathbf{c}] \{\mathbf{\varepsilon}\} \quad (2.9c)
\]

with the stiffness matrix \([\mathbf{c}]\) described in Eqs. 2.9a and 2.9b.

The elastic stresses of the total deck can be obtained by
staking analysis(2.2,2.4).

\[
\{\mathbf{\sigma}\} = (\sum_{i=1}^{n} \frac{t_i}{t_c}) \{\mathbf{\varepsilon}\} \quad (2.10)
\]

where \(\mathbf{\sigma}\) and \(\mathbf{\varepsilon}\) are respectively the stresses and strains in the deck, \([\mathbf{c}]_i\) the stress-strain relationship matrix of layer \(i\), \(t_i\) the thickness of layer \(i\), \(t_c\) the total deck thickness, and \(n\) the number of layers. Inversely, the strains are obtained from the stresses,

\[
\{\mathbf{\varepsilon}\} = (\sum_{i=1}^{n} \frac{t_i}{t_c})^{-1} \{\mathbf{\sigma}\} \quad (2.11a)
\]
or simply
\[ \{e\} = [S] \{\sigma\} \]  \hspace{1cm} (2.11b)

where
\[
\begin{bmatrix}
\epsilon_z \\
\epsilon_x \\
\gamma_{zx}
\end{bmatrix}
\]
\[ \{e\} = \begin{bmatrix}
\epsilon_z \\
\epsilon_x \\
\gamma_{zx}
\end{bmatrix} \]  \hspace{1cm} (2.11c)

\[
\begin{bmatrix}
\sigma_z \\
\sigma_x \\
\tau_{zx}
\end{bmatrix}
\]
\[ \{\sigma\} = \begin{bmatrix}
\sigma_z \\
\sigma_x \\
\tau_{zx}
\end{bmatrix} \]  \hspace{1cm} (2.11d)

and
\[
[S] = \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{bmatrix} \]  \hspace{1cm} (2.11e)

It is to be noted that the continuity condition requires that the strains in the individual layers be equal to the overall composite strains \(\{e\}\):
\[ \{e\}_1 = \{e\}_2 = \ldots = \{e\} \]  \hspace{1cm} (2.12)

The effect of the eccentricity of reinforcing bars in each layer has been neglected in the above derivations.

The \([S]\) matrix of Eq. 2.11e is symmetric, with \(S_{13} = S_{23} = 0\). From this matrix, the equivalent elastic constants of the total deck plate are obtained.

\[ E_z = \frac{1}{S_{11}} \]  \hspace{1cm} (2.13a)
The stress-strain relationship for the total deck is expressed conventionally.

\[ \varepsilon_z = \frac{1}{E_z} (\sigma_z - \nu_{zx} \sigma_x) \]  
\[ \varepsilon_x = \frac{1}{E_x} (\sigma_x - \nu_{xz} \sigma_z) \]  
\[ \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}} \]

2.2.2 Orthotropic Bottom Flange

For the bottom flange plate which is orthotropically reinforced by stiffeners (Fig. 2.3), the stress-strain relationship is derived using the same procedures as those employed by Abdel-Sayed (2.5). Two assumptions are made: (1) the shear forces are carried by the plate only; and (2) the stiffeners are considered as bar elements taking only axial forces. Hence, the stress-strain relations for the stiffeners and the plate in the longitudinal (z) direction are simplified.

\[ (\sigma_z) = E_s \varepsilon_z \]
where

\[
\sigma_z = \frac{E_s}{1 - \nu_s^2} (\varepsilon_z + \nu_s \varepsilon_x)
\]

(2.14b)

and

\[
\sigma_z = \text{longitudinal stress in the reinforcing stiffeners},
\]

\[
\sigma_p = \text{longitudinal stress in the plate},
\]

\[E_s = \text{Young's modulus of elasticity of steel, and}
\]

\[\nu_s = \text{Poisson's ratio of steel}.
\]

The axial force in a stiffener is assumed to be smeared uniformly over the stiffener spacing, thus the equivalent longitudinal force per unit length of the flange plate is as shown in Fig. 2.6 and represented by

\[
N_z = E_s \varepsilon_z \left( -\frac{t_f}{1 - \nu_s^2} + \frac{(a_z)_r}{s_x} \right) + \frac{E_s t_f}{1 - \nu_s^2} \nu_s \varepsilon_x
\]

(2.15a)

Similarly,

\[
N_x = E_s \varepsilon_x \left( -\frac{t_f}{1 - \nu_s^2} + \frac{(a_z)_r}{s_z} \right) + \frac{E_s t_f}{1 - \nu_s^2} \nu_s \varepsilon_z
\]

(2.15b)

and

\[
N_{zx} = N_{xz} = G_s \gamma_{zx} t_f
\]

(2.15c)

where

\[
N_z = \text{equivalent longitudinal (z) force per unit length of plate},
\]

\[
N_x = \text{equivalent transverse (x) force per unit length of plate},
\]
\( N_{zx} \) = shear force per unit length of plate,

\( t_f \) = thickness of plate,

\( (a_z)_r \) = cross-sectional area of stiffeners in z-direction,

\( (a_x)_r \) = cross-sectional area of stiffeners in x-direction,

\( s_x \) = spacing of longitudinal stiffeners,

\( s_z \) = spacing of transverse stiffeners, and

\( G_s \) = shear modulus of elasticity of steel, \( \frac{E_s}{2(1 + \nu_s)} \).

Solving Eqs. 2.15 for the strains results in

\[
\varepsilon_z = \frac{1}{E_s (a_3 c_3 - b_3)} \left( a_3 N_z - b_3 N_x \right) \tag{2.16a}
\]

\[
\varepsilon_x = \frac{1}{E_s (a_3 c_3 - b_3)} \left( c_3 N_x - b_3 N_z \right) \tag{2.16b}
\]

\[
\gamma_{zx} = \frac{N_{zx}}{G_s t_f} \tag{2.16c}
\]

in which

\[
a_3 = \frac{t_f}{1 - \nu_s^2} + \frac{(a_x)_r}{s_x} \tag{2.16d}
\]

\[
b_3 = \frac{t_f \nu_s}{1 - \nu_s^2} \tag{2.16e}
\]

\[
c_3 = \frac{t_f}{1 - \nu_s^2} + \frac{(a_z)_r}{s_z} \tag{2.16f}
\]
2.3 Differential Equations of Stress Function and Solutions

With elastic stress-strain relations established, the differential equations of plane stress elasticity can be formulated to solve for the shear lag in top deck and bottom flange. For a plane stress element (Fig. 2.6), the equilibrium equations are (2.6):

\[
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} = 0 \quad (2.17a)
\]

\[
\frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \sigma_x}{\partial x} = 0 \quad (2.17b)
\]

and the compatibility equation is

\[
\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \quad (2.18)
\]

The stresses are related to the Airy's stress function by

\[
\sigma_z = \frac{\partial^2 F}{\partial x^2} \quad (2.19a)
\]

\[
\sigma_x = \frac{\partial^2 F}{\partial z^2} \quad (2.19b)
\]

\[
\tau_{zx} = -\frac{\partial^2 F}{\partial z \partial x} \quad (2.19c)
\]

Substitution of Eqs. 2.13 and 2.19 into 2.18 gives the differential equation of stress function for the concrete deck.

\[
a \frac{\partial^4 F}{\partial x^4} + b \frac{\partial^4 F}{\partial x^2 \partial z^2} + c \frac{\partial^4 F}{\partial z^4} = 0 \quad (2.20a)
\]

where

\[
a = S_{11} \quad (2.20b)
\]
If the deck is isotropically reinforced, then \(a = c\) and \(b/a = 2\), and Eq. 2.20a reduces to the conventional form.

\[
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial z^4} = 0 \tag{2.21}
\]

For an orthotropic bottom flange, the forces per unit length (instead of the stresses) are related to the stress function by

\[
N_z = \frac{\partial^2 F}{\partial x^2} \tag{2.22a}
\]

\[
N_x = \frac{\partial^2 F}{\partial z^2} \tag{2.22b}
\]

\[
N_{zx} = -\frac{\partial^2 F}{\partial z \partial x} \tag{2.22c}
\]

By substituting Eqs. 2.16 and 2.22 into Eq. 2.18, the differential equation of the stress function for the bottom flange is obtained.

\[
p \frac{\partial^4 F}{\partial x^4} + 2q \frac{\partial^4 F}{\partial x^2 \partial z^2} + r \frac{\partial^4 F}{\partial z^4} = 0 \tag{2.23a}
\]

in which

\[
p = a_3 \tag{2.23b}
\]

\[
q = \frac{(1 + \nu_s)(a_3 c_3 - b_3^2)}{t_f} - b_3 \tag{2.23c}
\]

\[
r = c_3
\]

If the bottom flange has only longitudinal ribs, the \(a_{kr}\) terms in Eqs. 2.15b and 2.16d are equal to zero. If the bottom
flange consists of a plane isotropic plate, Eq. 2.23a will reduce to the conventional Eq. 2.21.

The solution to the governing differential equations, Eqs. 2.20a, 2.21 and 2.23a can be expressed as a Fourier's series (2.7).

\[ F = \sum_{n=1}^{\infty} \frac{1}{2} X_n \sin \alpha_n z \]  

(2.24)

where \( \alpha_n = \frac{n\pi}{\ell} \), \( n \) = an integer, \( X_n \) = a function of \( x \) only, and \( \ell \) = a characteristic length between two points along the girder span at which the moments are zero. For simple beams, \( \ell \) is the span length.

By substituting Eq. 2.24 into Eq. 2.21 for an isotropic concrete deck, a linear ordinary differential equation in \( X_n \) is obtained, to which the solution is expressed as

\[ X_n = A_n \cosh \alpha_n x + B_n \sinh \alpha_n x + C_n x \cosh \alpha_n x + D_n x \sinh \alpha_n x \]  

(2.25)

For an orthotropic concrete deck, Eq. 2.20a gives

\[ X_n = A_n \cosh r_1 \alpha_n x + B_n \sinh r_1 \alpha_n x + C_n \cosh r_2 \alpha_n x + D_n \sinh r_2 \alpha_n x \]  

(2.26a)

where

\[ r_1 = \left( \frac{b + \sqrt{b^2 - 4ac}}{2a} \right)^{1/2} , \quad (b^2 > 4ac) \]  

(2.26b)
\[ r_2 = \frac{b - \sqrt{b^2 - 4ac}}{2a} \frac{1}{2}, \quad (b^2 > 4ac) \] 

(2.26c)

The coefficients \( A_n, B_n, C_n \) and \( D_n \) are to be determined by the boundary conditions at the edges of the deck at proper values of \( x \).

Similarly, the solution for an isotropic bottom flange is:

\[
X_n = E_n \cosh \alpha_n x + F_n \sinh \alpha_n x \\
+ G_n x \cosh \alpha_n x + H_n x \sinh \alpha_n x
\]

(2.27)

For an orthotropic bottom flange:

\[
X_n = E_n \cosh r_3 \alpha_n x + F_n \sinh r_3 \alpha_n x \\
+ G_n \cosh r_4 \alpha_n x + H_n \sinh r_4 \alpha_n x
\]

(2.28a)

where

\[
r_3 = \left( \frac{q + \sqrt{q^2 - pr}}{p} \right) \frac{1}{2}, \quad (q^2 > pr)
\]

(2.28b)

\[
r_4 = \left( \frac{q - \sqrt{q^2 - pr}}{p} \right) \frac{1}{2}, \quad (q^2 > pr)
\]

(2.28c)

The integration constants \( E_n \) through \( H_n \) are again to be determined by the boundary conditions.

2.4 Stresses in Flanges and Webs

Because the solutions to the stress functions have been represented by the sine series, the external moment must be correspondingly expressed in the same series in order to obtain the stresses in the flanges and webs. Between two points of zero moment on a
beam or box girder, the moment can be expressed by the following equation:

\[ M_x = \sum_{n=1}^{\infty} M_n \sin \alpha_n z \]  

(2.29a)

where \( \alpha_n = \frac{n\pi}{\ell} \), and \( \ell \) is the length between two adjacent points of inflection or points of zero moment.

For a simple beam subjected to a concentrated load as shown in Fig. 2.7a,

\[ M_n = \frac{2}{n^2} \frac{P \ell}{\pi} \sin \alpha_n \pi \quad n = 1, 2, 3, \ldots \]  

(2.29b)

and for a simple beam subjected to uniform load throughout the span (Fig. 2.7b),

\[ M_n = \frac{4}{n^3} \frac{w \ell^2}{\pi} \quad n = 1, 3, 5, \ldots \]  

(2.29c)

For continuous girders, the origin of the \( z \)-coordinate is always selected at a point where the moment is zero. The length of the half period, \( \ell \), is terminated at an adjacent point where the moment is also zero. To evaluate the stresses, the following general assumptions are made.

(a) The box girder and loading are symmetrical about the centroidal, principal \( y \)-axis.

(b) The thickness of the concrete deck and the bottom flange is small compared with the box girder depth. Thus the bending stiffness of the flanges may be neglected, and the flanges are treated as plane stress problems as done in Sections 2.2 and 2.3.
(c) The ordinary beam theory of bending is applicable to webs with depth-to-span ratio less than 1/4 (2.8).

(d) The existence of diaphragms is ignored in computing the flange stresses and the equivalent widths, which are not sensitive to the diaphragm rigidity nor greatly affected by the presence of diaphragms (2.8).

(e) Complete interaction develops between the concrete deck and the steel portion.

In addition, the following boundary conditions and strain compatibilities are adopted:

(a) The longitudinal normal stresses and equivalent forces are equal at the web to deck and web to bottom flange junctions due to symmetry of load and cross section.

\[
(\sigma_z)_x = \frac{b_c}{2} = (\sigma_z)_x = -\frac{b_c}{2} \quad (2.30a)
\]

and

\[
(N_z)_x = \frac{b_f}{2} = (N_z)_x = -\frac{b_f}{2} \quad (2.30b)
\]

(b) The shearing stresses in the deck and the shearing forces in the bottom flange are zero at the vertical axis of symmetry.

\[
(\tau_{zx})_x = 0 = 0 \quad (2.30c)
\]

\[
(N_{zx})_x = 0 = 0 \quad (2.30d)
\]
(c) The transverse displacement of the deck, \( u \), is assumed to be zero at the deck-to-web junctions.

\[
(u)_x = \pm \frac{b}{c} / 2 = 0 \quad (2.31a)
\]

and

\[
u = \int_{c_x} \ dx \quad (2.31b)
\]

(d) The in-plane transverse normal stresses are assumed to be zero at the bottom flange-to-web junctions by neglecting the small protruding lips of the bottom flange as shown in Fig. 2.8.

\[
(N_x)_x = \pm \frac{b_f}{2} = 0 \quad (2.31c)
\]

(e) The normal stresses in the transverse direction and the in-plane shearing stresses are zero at the edges of the deck.

\[
(\sigma_x)_x = \pm \frac{w_c}{2} = 0 \quad (2.32a)
\]

\[
(\tau_{zx})_x = \pm \frac{w_c}{2} = 0 \quad (2.32b)
\]

(f) For simplicity, the small protruding portions of the bottom flange outside the webs are assumed fully effective as shown in Fig. 2.9. That is, the distribution of longitudinal forces \((N_z)\) is uniform throughout these portions (Fig. 2.8).

(g) The longitudinal strains of the flanges and the webs are equal at their junctions.

By using the above conditions and formulating the equilibrium of external and internal bending moments, the coefficients \( A_n \) to \( H_n \) of Eqs. 2.25 to 2.28 can be determined. The solution of these
coefficients and the subsequent substitution into the equations for stresses are given in Appendix A. The resulting distribution of longitudinal stresses in a cross section is sketched in Fig. 2.9. The normal stresses are highest at the flange-to-web junctions, and are lower in other parts of the flanges as the result of the shear lag effect.

In practical application, an equivalent width of a flange is often used within which the normal stress is assumed to be uniform and equal to the maximum normal stress at the flange-to-web junction (Fig. 2.9). The total resulting force in the equivalent width is equal to the computed stress resultant in the flange \( Z_c \) or \( Z_g \) in Fig. 2.8. The equivalent widths of the concrete deck and the bottom flange are listed in Appendix A.

2.5 Results and Comparisons

In order to examine the effects of deck reinforcement and of deck orthotropic characteristics on the stresses, an arbitrary box girder with low concrete strength and different deck reinforcement arrangements is considered. The details of the box girder are shown in Fig. 2.10. The girder span to box width ratio is equal to 4. The elastic constants are:

Concrete deck:

\[
\begin{align*}
\sigma_c' &= 17.25 \text{ MN/m}^2 \ (2.5 \text{ ksi}) \\
\nu_c &= 0.17 \\
E_c &= 19,880 \text{ MN/m}^2 \ (2881 \text{ ksi}) \\
G_c &= 8496 \text{ MN/m}^2 \ (1231 \text{ ksi})
\end{align*}
\]
Reinforcing bars and steel plates:

\[ \nu_f = \nu_s = 0.3 \]
\[ E_f = E_s = 203,550 \text{ MN/m}^2 \text{ (29,500 ksi)} \]
\[ G_f = G_s = 78,315 \text{ MN/m}^2 \text{ (11,350 ksi)} \]

Orthotropic Deck (computed for reinforcement 8/4 - 4/4):

\[ E_z = 29,223 \text{ MN/m}^2 \text{ (4235 ksi)} \]
\[ E_x = 23,425 \text{ MN/m}^2 \text{ (3395 ksi)} \]
\[ G_{zx} = 9,194 \text{ MN/m}^2 \text{ (1332 ksi)} \]
\[ \nu_{zx} = 0.1635, \frac{A_{fz}}{A_c} = 4.91\% \]
\[ \nu_{xz} = 0.1311, \frac{A_{fx}}{A_c} = 1.23\% \]

The results of the stress computations for orthotropic 8/4 - 4/4 deck are sketched in Fig. 2.11 and listed in Table 2.1. Also listed in the table are the results for two other cases of the same box girder geometry: one with deck reinforcing bars rotated 90 degrees so that less reinforcement is in the longitudinal (z) direction (orthotropic 4/4 - 8/4 deck), and the other with no reinforcement at all (plain 0-0 deck).

The decrease in deck longitudinal reinforcement leads to a reduction in longitudinal in-plane stiffness of the deck, hence to a decrease of \( \sigma_z \) in the gross deck as listed in Table 2.1. However, the longitudinal stresses in the concrete itself increase with the decrease in the reinforcement. The maximum stress \( \sigma_z \) in the concrete increases 22% from orthotropic 8/4 - 4/4 deck to plain 0-0 deck.
The influence of the deck properties on the longitudinal stresses in the bottom flange is indirect. These stresses are influenced by the magnitude of the moment of inertia, $I_x$, and the position of the neutral axis. The decrease in reinforcement, from a steel ratio of 4.91% to 1.23% to zero, causes only a very small (1%) increase in the maximum longitudinal stress in the bottom flange.

For all three cases shown in Table 2.1, the effect of shear lag on the longitudinal stresses is quite prominent, regardless of reinforcement arrangement, as indicated by the maximum-to-average stress ratio as well as by the longitudinal normal stress distribution as shown in Fig. 2.11a.

By considering the above results, a plain concrete deck alone can, with due consideration of the shear lag effects, be used for normal stress computations for practical purposes.

The shear stress distribution is shown in Fig. 2.11b. Values computed by considering orthotropic characteristics and by considering a plain concrete deck with either a shear lag analysis or ordinary beam theory are all very close to each other. Therefore, for practical purposes, a plain concrete deck can be assumed and shearing stress computations can be carried out using only ordinary beam theory.

To examine shear lag effects further, some experimental results on aluminum beams by Tate (2.10) are compared with the stresses computed by the theory presented herein. The dimensions of the aluminum alloy beams are shown in Fig. 2.12. The elastic constants are:
Since Tate assumed no transverse displacement at the flange to web junctions, as is assumed in this study for the top flange, the experimental results are compared with the computed top flange stresses as shown in Fig. 2.13. Good agreement between the experimental (dots) and computed stresses (solid lines) is evident. The computed bottom flange stresses are about 5 to 6% lower than those computed for the top flanges.

The use of an equivalent flange width to circumvent a cumbersome shear lag analysis has been a design practice. Equivalent width charts are usually plotted as a function of the span to actual width ratio \((2.9, 2.11)\). Moffatt and Dowling \((2.12)\) used rectangular third order extensional-flexural finite elements \((1.14, 2.13)\) to obtain the equivalent widths. Cross section 4 as shown in Fig. 2.14 was examined. The results from their computation and from this study are compared in Fig. 2.15. Good agreement is observed, even in the region of \(\ell/b_f < 2\) \((h_w/\ell > 1/4)\) where ordinary beam theory is considered not applicable to the webs, but is still used in developing the curves.

In addition to the span-to-width ratio and the loading conditions, the material and geometrical properties of a cross section also affect the equivalent width. Hildrebrand and Reissner \((2.11)\) used the least work method to investigate the box beams and
concluded that the amount of shear lag depends on the G/E ratio and the stiffness parameter \( m = \frac{3I_w + I_s}{I_w + I_s} \), where \( I_w \) and \( I_s \) are moments of inertia of webs and of flange sheets, respectively, about the neutral axis of the box beam. The \( m \) parameter is a measure of the stiffness of webs relative to that of flanges. Its value lies between 1 for zero web areas, and 3 for zero flange areas. The equivalent widths of Tate's specimen B2, meeting the conditions \( m = 2 \) and \( G/E = 3/8 \) used in developing the curves by Hildebrand and Reissner, are computed by this study. The two results are compared in Fig. 2.16. Again, good agreement is observed.

It is to be noted that the computed equivalent widths of the bottom flange, instead of the top flange, are used in Fig. 2.15 for comparison. Since the webs of the cross section are relatively thinner as compared to the flanges, the transverse displacement at the web-to-flange junctions is considered as completely free, as is assumed in this study for the bottom flange. For the top flange where no transverse displacement at the web-to-flange junctions is assumed, the computed equivalent widths are only slightly greater, with a maximum difference less than 3% from the bottom flange values. In Fig. 2.16, for Tate's specimen B2, the equivalent width of the top flange are plotted because the webs are relatively more stocky and the assumption of no transverse displacement at the web-to-flange junctions appears to be more applicable. The equivalent widths of the bottom flange, however, are only 0.5% smaller than those for the top flange.
Figures 2.15 and 2.16 show Winter's\(^{(2.9)}\) curve which accounts for the influence of the material constants \((G/E)\), but not the geometrical properties \((m)\) of a cross section. The \(m\) values of the cross sections in Figs. 2.15 and 2.16 are 1.15 and 2.0, respectively, indicating that the latter cross section has relatively heavier webs. From both figures, it can be concluded that a cross section with heavier webs relative to the flanges has greater flange equivalent widths, as it should be.

The equivalent widths of composite box girders with projecting deck portions are computed for a \(3810\text{ mm} \times 2540\text{ mm}\) \((150\text{ in.} \times 100\text{ in.})\) box section with the web slenderness ratio equal to 200 and the bottom flange width to thickness ratio of 150. Table 2.2 lists the equivalent width ratios for this cross section with deck projecting widths \(w_c/b_c = 3.0\) and 2.2. The latter ratio is the maximum allowed by AASHTO\(^{(2.14)}\). In both cases, the effects of span-to-actual width ratio is predominant.

The influence of deck projecting widths on the equivalent widths of the box flanges is examined in Fig. 2.17, in which the equivalent widths are plotted as a function of the projecting widths. The near-horizontal lines show that the influence is minor. The equivalent widths of the projecting deck itself, on the other hand, decreases with the increase of projecting width, as is listed in Table 2.2.

Most of the studies on shear lag are concerned with noncomposite stiffened or unstiffened box sections without projecting top flange\((2.9,2.10,2.11,2.15)\), or concerned with I-girders with
orthotropic steel decks$^{(2.5)}$. This study provides a procedure to evaluate the shear lag effects on composite box sections with projecting deck portions. Although for the sample composite box girders of this study the orthotropic characteristics of the concrete deck were found insignificant. The influence of these characteristics may be important for other structures such as box and π sections with metal-formed composite deck. These structures can be analyzed by using the equations developed in this chapter.
3. **TORSIONAL ANALYSIS**

3.1 **Torsional Sectional Properties of Single-celled Box**

The mathematical expressions of thin-walled elastic beam theories for evaluating the torsional sectional properties of homogeneous and composite open cross sections have been developed and verified by many investigators\(^{(1.9,3.1,3.2,3.3,3.4,3.5,3.6)}\). For closed cross sections composed of single material, methods for evaluation were developed by Benscoter\(^{(3.7)}\) and Dabrowski\(^{(3.3)}\). For composite, closed sections the evaluation of the St. Venant (uniform) torsional constant was treated by Kollbrunner and Basler\(^{(3.1)}\). The non-uniform torsion of composite closed sections is considered here.

The following fundamental assumptions are made:

1. The component parts of the cross-section are thin-walled and can be treated as membranes.
2. The cross-sectional shape is preserved by sufficiently spaced rigid diaphragms which are free to warp out of their plane.
3. The box girder is prismatic. The thickness of the component elements may vary along the profile of a cross section, but not along the length of the girder.
4. The materials forming the cross section satisfy Hooke's law. The longitudinal normal stresses will be evaluated disregarding the Poisson's effect\(^{(1.9,3.1,3.2,3.3)}\).
5. The connection between one material and another is monolithic so that no slippage or separation will occur.

Figure 3.1 shows a thin-walled composite closed cross section. From the uniform torsion analysis, the shear deformation in the middle surface due to St. Venant torsion is given by

\[ \gamma_i = \frac{\partial w}{\partial s} + \frac{\partial u_s}{\partial z} = \frac{q_{sv}}{G_i t_i} \]  

(3.1)

where

- \( \gamma_i \) = shear strain of an individual material,
- \( w \) = displacement in the axial z-direction,
- \( u_s \) = displacement in the tangential direction,
- \( q_{sv} \) = St. Venant shear flow,
- \( G_i \) = elastic shear modulus of an individual material, and
- \( t_i \) = thickness of an individual material.

For small angle of rotation and on the basis of the assumption that cross sections maintain their shape, the quantity \( \partial u_s / \partial z \) in Eq. 3.1 is equal to \( \rho_o \phi' \), in which \( \rho_o \) is the distance from the shear center to contour tangent and \( \phi' \) the angle of twist per unit length. After substitution of \( \partial u_s / \partial z \) the axial displacement \( w \) can be obtained by integrating Eq. 3.1 with respect to \( s \).

\[ w = w_o + \int_0^s \frac{q_{sv}}{G_i t_i} ds - \phi' \int_0^s \rho_o ds \]  

(3.2)

in which \( w_o \) is the longitudinal displacement at \( s = 0 \). The first integral in Eq. 3.2 applies only to the cell walls but not to the open projecting parts of a cross section as shown in Fig. 3.2.
Integrating Eq. 3.2 around the closed perimeter gives rise to the relationship

\[ \oint \frac{q_{sv}}{G_i t_i} \, ds - \oint \phi' \oint \rho_o \, ds = 0 \]  

(3.3)

From membrane analogy of torsion (3.1, 3.3)

\[ q_{sv} = \frac{M_T}{2 A_o} \]  

(3.4)

in which \( M_T \) is the section torque and \( A_o \) the enclosed area of the closed part. The rate of twist, \( \phi' \), can be expressed as

\[ \phi' = \frac{M_T}{G_r K_T} \]  

(3.5)

where \( G_r \) is the elastic shear modulus of the reference material and \( K_T \) the St. Venant torsional constant. The second integral expression in Eq. 3.3 can be represented by

\[ \oint \rho_o \, ds = 2 A_o \]  

(3.6)

Substituting Eqs. 3.4, 3.5 and 3.6 into Eq. 3.3 and solving for \( K_T \) results in

\[ K_T = \frac{4 A_o^2}{\oint \frac{G_i}{G_r} t_i} \]  

(3.7)

The term \( \frac{G_i}{G_r} t_i \) in Eq. 3.7 can be considered as a transformed thickness.
By expressing \( q_{sv} \) in terms of \( \varphi' \) through the combination of Eqs. 3.4 and 3.5 and substituting it into Eq. 3.2, the following expression for warping displacement is obtained.

\[
w = w_o - \varphi' \frac{w_o}{w_o} \tag{3.8a}
\]

in which

\[
\frac{w_o}{w_o} = \int_o^s \rho_o \, ds - \frac{2 A_o}{\int_o^s \rho_i \, ds} \int_o^s \rho_i \, ds
\]

\[
\frac{G_i}{G_r} \int_i^{t_i} \tag{3.8b}
\]

is called the double sectorial area or the unit warping function with respect to the shear center of the cross section. To evaluate \( w_o \) at points in a closed cross section with open components (Fig. 3.2), one can first proceed around the closed perimeter to find the values for points on the perimeter by using Eq. 3.8b. This can be executed by assuming the unit warping to be zero at an arbitrary point of origin of integration (for example, point 1 in Fig. 3.2). The \( w_o \) value for a point on the open projecting part can be obtained by performing the integration, \( \int \rho_o \, ds \), starting from the junction of the open projecting element with the closed perimeter, where the \( w_o \) value has been found, to the point in question. Values of \( w_o \) are dependent on the path of integration.

When the warping displacement is constrained, then normal and corresponding shearing stresses are created in addition to the uniform torsional shear flow. If Poisson's effect is ignored (assumption 4), the warping normal stress is represented by

\[
\sigma_w = E_i \cdot \frac{\partial w}{\partial z} = E_i (\varphi'_o - \varphi'' w_o) \tag{3.9}
\]
The total normal force due to warping on a cross section can be obtained by summing the warping normal stresses over the entire cross sectional area, $A$, and this is equal to zero.

$$ N = \int_A \sigma_w t \, ds = 0 \quad (3.10) $$

From Eqs. 3.9 and 3.10, $\sigma_w$ is deduced.

$$ \sigma_w = E_i \phi'' \bar{w}_n \quad (3.11a) $$

where

$$ \bar{w}_n = \frac{1}{A^*} \int_A \bar{w}_o \, dA^* - \bar{w}_o \quad (3.11b) $$

is defined as normalized unit warping with respect to the shear center, and

$$ A^* = \int_A \, dA^* = \int_A \frac{E_i}{E_x} t \, ds \quad (3.11c) $$

is the transformed area of the section. The values of normalized unit warping, $\bar{w}_n$, are independent of the integration path and represent the warping distribution in a cross section.

In addition to the St. Venant torsional shear, the effect of warping torsional shear on the warping displacement can be also taken into consideration. Benscoter\(^{(3.7)}\) assumed that the distribution of warping displacement in a section is still akin to $\bar{w}_n$ which is deduced from considering only the St. Venant torsional shear.

For the span-wise distribution, a warping function $f(z)$ is introduced. The warping displacement is expressed as

$$ w = f' \bar{w}_n \quad (3.12) $$
The warping normal stress is obtained by
\[ \sigma_w = E_i w' = E_i f'' w_n \] (3.13)

The bimoment is defined as
\[ B = \int_A \sigma_w w_n \, dA \] (3.14)

Substitution of Eq. 3.13 into 3.14 results in
\[ B = E_i f'' L \] (3.15a)

where
\[ L = \int_A \frac{2}{w_n} \, dA^* \] (3.15b)

is called the warping moment of inertia.

The equilibrium of the longitudinal forces on a differential element (Fig. 3.1) requires that
\[ \frac{\partial \sigma_w}{\partial z} t + \frac{\partial q}{\partial s} = 0 \] (3.16)

where \( q \) = total shear flow, including both St. Venant and warping shears. Substituting Eq. 3.13 into 3.16 and performing integration leads to the expression for the shear flow.
\[ q = q_o - E_i S_w f''' \] (3.17a)

in which
\[ S_w = \int_0^s w_n \, dA^* \] (3.17b)

The sum of moment of shear flow about the shear center over the cross section is the section torque:
\[ M_T = q_o \int_\delta \rho_o \, ds - E_i f'' \int_A S_w \rho_o \, ds \] (3.18a)
from which the integration constant, $q_o$, can be determined:

$$q_o = \frac{M_T}{2 A_o} + \frac{E_i}{2 A_o} \int_A S \rho_o ds$$  \hspace{1cm} (3.18b)

On substitution of Eq. 3.18b into 3.17a, it is obtained

$$q = \frac{M_T}{2 A_o} - E_i \bar{S}_{\bar{w}} f''$$  \hspace{1cm} (3.19a)

in which $\bar{S}_{\bar{w}}$ is the warping statical moment,

$$\bar{S}_{\bar{w}} = S_{\bar{w}} - \frac{1}{2 A_o} \int_A S \rho_o ds$$  \hspace{1cm} (3.19b)

The distribution of $\bar{S}_{\bar{w}}$ and of $\bar{w}_n$ on a cross section is dependent on the dimensions of component parts. A typical distribution of each is shown in Fig. 3.3. The total shear flow expressed in Eq. 3.19a comprises two parts. The primary or St. Venant shear flow

$$q_{sv} = \frac{M_T}{2 A_o}$$  \hspace{1cm} (3.20a)

has been given by Eq. 3.4 and occurs only in the closed part of a box section. The secondary shear flow

$$q_w = - E_i \bar{S}_{\bar{w}} f'''$$  \hspace{1cm} (3.20b)

is induced by the warping torsion and is in self-equilibrium.

Equation 3.20b can be used to evaluate the secondary shear flow in any part of the cross section.

The St. Venant shearing stress in the open projecting parts of a cross section (Fig. 3.2) can be approximated by

$$\Delta T = \frac{M_T}{K_T} \cdot \frac{G_i}{G_r} t_i$$  \hspace{1cm} (3.21)
with $K_T$ value computed from Eq. 3.7. This shearing stress varies linearly across the thickness of the projecting parts. For the walls of the closed perimeter, $\Delta \tau$ is added algebraically to the shear from Eq. 3.20a to form the total St. Venant shearing stress.

$$\tau_{sv} = \frac{M_{T}}{2A_o t_i} \pm \frac{M_T}{K_T} \cdot \frac{G}{r} \cdot t_i$$  

(3.22)

For composite steel-concrete box girders, the concrete deck is much thicker than the steel plates and has much lower allowable shearing stress, therefore the consideration of $\Delta \tau$ may be of importance.

From Eqs. 3.8b, 3.11b and 3.13, it can be deduced that for a section consisting of straight plate elements with constant thickness along the s-direction, the $\sigma_w$ distribution is linear along the plate profile. This implies that the warping stresses are evaluated by the Navier hypothesis. Unlike the shear lag in the flanges of a cross section under bending moment, the warping stresses are in effect resisted by the plate inplane bending. Thus, as long as the width to length ratio of the component plates is small (less than 1/2 by Ref. 3.8 or 1/4 as suggested in Ref. 2.8) the assumption of linear distribution of warping normal stresses is applicable.

### 3.2 Location of Twisting Center

The location of the shear or twisting center is essential for the evaluation of section properties and torsional rotations. As depicted in Fig. 3.4, the relationship between $\rho$ and $\rho_o$ is

$$\rho_o = \rho + y_o \frac{dx}{ds} - x_o \frac{dy}{ds}$$  

(3.23)
where $\rho$ is the distance from the centroid of the cross section to the tangent to a point $P$ in question. Integrating Eq. 3.23 from an arbitrary origin $O$ to point $P$ on the perimeter,

$$\int_0^s \rho \, ds = \int_0^s \rho \, ds + y_0 \, x - y_0 \, x_1 - x_0 \, y + x_0 \, y_1$$  \hspace{1cm} (3.24)$$

and substituting into Eq. 3.8b gives

$$\bar{w}_0 = \bar{w} + y_0 \, x - y_0 \, x_1 - x_0 \, y + x_0 \, y_1$$  \hspace{1cm} (3.25a)$$

where

$$\bar{w} = \int_0^s \rho \, ds - \frac{2 \, A_0}{\int_0^s G_i \, ds} \int_0^s \frac{ds}{G_r \, t_i}$$  \hspace{1cm} (3.25b)$$

is defined as double sectorial area or unit warping with respect to the centroid.

By substituting Eq. 3.25a into 3.11b and noting that $x$- and $y$- axes are centroidal, but may not be principal axes, it is obtained

$$\bar{w}_n = \frac{1}{A_x} \int_A \bar{w} \, dA^* - \bar{w} - y_0 \, x + x_0 \, y$$  \hspace{1cm} (3.26)$$

The $\bar{w}_n$ values can be calculated by either Eq. 3.13b or Eq. 3.26. The moments about $x$- and $y$- axes produced by the warping normal stresses are zero, or

$$M_x = \int_A \sigma_w \, y \, t \, ds = 0$$  \hspace{1cm} (3.27a)$$

and

$$M_y = \int_A \sigma_w \, x \, t \, ds = 0$$  \hspace{1cm} (3.27b)$$
Equations 3.13a, 3.26, 3.27a and 3.27b are combined to give

\[ x_0 \frac{I_x}{I} - y_0 \frac{I_y}{I_{xy}} = \frac{I_0}{I_{wy}} \]  
(3.28a)

\[ x_0 \frac{I_{xy}}{I} - y_0 \frac{I_y}{I_{xy}} = \frac{I_0}{I_{wx}} \]  
(3.28b)

where

\[ I_x = \int_A y^2 \, dA^*, \text{ the moment of inertia about } x\text{-axis}, \]  
(3.28c)

\[ I_y = \int_A x^2 \, dA^*, \text{ the moment of inertia about } y\text{-axis}, \]  
(3.28d)

\[ I_{xy} = \int_A xy \, dA^*, \text{ the product moment of inertia}, \]  
(3.28e)

\[ I_{wx} = \int_A \bar{wx} \, dA^*, \text{ the warping product of inertia}, \]  
(3.28f)

\[ I_{wy} = \int_A \bar{wy} \, dA^*, \text{ the warping product of inertia}. \]  
(3.28g)

The location of torsion center is obtained by solving Eqs. 3.28a and 3.28b.

\[ x_0 = \frac{I_y I_{-wy} - I_{xy} I_{-wx}}{I_x I_y - I_{xy}^2} \]  
(3.29a)

\[ y_0 = -\frac{I_x I_{-wx} - I_{xy} I_{-wy}}{I_x I_y - I_{xy}^2} \]  
(3.29b)

3.3 Differential Equations and Solutions

The differential equations of torsion have been derived by Benscoter (3.7) and Dabrowski (3.3). Benscoter used Galerkin's method to solve the differential equation of displacements and...
developed the differential equations relating the spanwise warping displacements and the angle of rotation to the applied torsional load. Dabrowski obtained the same equations by enforcing the shear flow obtained from the equilibrium condition (Eqs. 3.18 and 3.19) to satisfy the connectivity requirement of the closed perimeter, and by enforcing the shear flow obtained from the displacement relationships (Eqs. 3.1 and 3.12) to satisfy the equilibrium condition. The differential equations are as follows:

\[
E_T \frac{I}{w} f'''' - G_T K_T \theta' = -M_T \tag{3.30a}
\]

\[
\theta' = \mu f' + \frac{M_T}{G_T I_c} \tag{3.30b}
\]

in which the coefficient

\[
\mu = 1 - \frac{K_T}{I_c} \tag{3.30c}
\]

is called the warping shear parameter by Dabrowski \((3.3)\) and is a measure of cross-sectional slenderness \((3.7)\). For a very thin section \(\mu\) approaches unity. For a fairly thick section it lies in the neighborhood of one half. And

\[
I_c = \int_A \rho_o^2 \, dA^* \tag{3.30d}
\]

is the central second moment of area, or central moment of inertia.

The rate of the angle of twist, \(\theta'\), can be eliminated from Eqs. 3.30a and 3.30b to obtain a differential equation in terms of warping function \(f\) only.

\[
E_T \frac{I}{w} f'''' - \mu G_T K_T f' = -\mu M_T \tag{3.31}
\]
Equations 3.30b and 3.31 are applicable to longitudinal segments of a girder subjected to concentrated torsional load as shown in Fig. 3.5. The solutions to $\phi$ and $f$ are

$$\phi = C_1 + \mu(C_2 \cosh \lambda z + C_3 \sinh \lambda z) + \frac{M_T z}{G_r K_T}$$  \hspace{1cm} (3.32a)$$

$$f = C_4 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_T z}{G_r K_T}$$  \hspace{1cm} (3.32b)$$

where

$$\lambda = \sqrt{\frac{\mu G_r K_T}{E_r I_w}}$$  \hspace{1cm} (3.32c)$$

Differentiating Eqs. 3.30b and 3.31 with respect to $z$ results in the following equations for girder segments subjected to distributed torsional load $m_z$ (Fig. 3.5).

$$\phi'' = \mu f'' - \frac{m_z}{G_r I_c}$$  \hspace{1cm} (3.33a)$$

$$E_r I_w f^{iv} - \mu G_r K_T f'' = \mu m_z$$  \hspace{1cm} (3.33b)$$

The particular solutions to Eqs. 3.33 depend on the loading pattern of $m_z$. For uniformly distributed twisting moment $m_t$, the solutions are

$$\phi = C_1 + C_2 z + \mu(C_3 \cosh \lambda z + C_4 \sinh \lambda z) - \frac{m_t z^2}{2G_r K_T}$$  \hspace{1cm} (3.34a)$$

$$f = C_4 + C_2 z + C_3 \cosh \lambda z + C_4 \sinh \lambda z - \frac{m_t z^2}{2G_r K_T}$$  \hspace{1cm} (3.34b)$$
The coefficients in Eqs. 3.32 and 3.34 are determined from the boundary conditions. At the fixed end the rotation and the warping displacement (Eq. 3.12) are completely restrained,

\[ \theta = 0 \quad (3.35a) \]

and

\[ f' = 0 \quad (3.35b) \]

At simply supported and free ends the warping is entirely free. Thus the warping normal stress is zero, and from Eq. 3.13

\[ f'' = 0 \quad (3.36) \]

The angle of rotation is assumed to be completely restrained at a simple support (Eq. 3.35a). At the interior support of a continuous member or at a location where the applied torsional moment changes, the continuity conditions between the left and the right require

\[ \phi_L = \phi_R \quad (3.37a) \]
\[ f'_L = f'_R \quad (3.37b) \]
\[ f''_L = f''_R \quad (3.37c) \]

As examples of solution, four different cases of loading and boundary conditions as shown in Fig. 3.5 are solved and the results listed in Appendix B. In all four cases, the rotation of beam section, the second derivative of longitudinal warping function, \( f'' \), and the section torque are given. The warping torsional shear flow can be obtained by taking differentiation of \( f'' \)
and using Eq. 3.20b. Illustrative plots of section torque, rotation \( \phi, f'' \) and \( f''' \) are shown in Fig. 3.6. Plots of this kind covering frequently encountered \( \lambda \) values of composite box girders will facilitate the design procedures.

### 3.4 Results and Comparisons

Test results from two model composite box girders\(^{(3.9)}\) are compared with computed values to examine the effectiveness of the thin-walled torsional theory as applied to the composite closed sections. The details of the girders are shown in Figs. 3.7 and 3.8. The loads were applied vertically (upward and downward) at diaphragms and combined into a theoretically pure torsional load without longitudinal bending.

The experimental shearing stresses to be discussed herein are computed from measured strains. The shearing stresses in the middle of webs and of bottom flanges are identified in Fig. 3.9 by symbols. For the webs, the test results agree very well with the computed values (by Eq. 3.19a). In the bottom flanges, however, the computed shearing stresses are higher than the experimental values. In Fig. 3.10, the measured rotations of the overhanging ends are compared with the calculated results (by Eqs. B.1a and B.2d). The computations underestimate the rotations slightly. This could be partially due to the calculation of rotations from the measured vertical and horizontal deflections\(^{(3.9)}\). An examination of the effects of cross-sectional distortion is made in Chapter 4.
Overall, the proposed procedures estimate the stresses and rotations with sufficient accuracy. The experimental results confirm the validity of the extension of thin-walled torsional theory to the composite closed sections.
4. DISTORTIONAL STRESSES

The thin-walled elastic beam theory as utilized in the last chapter assumes no cross-sectional deformation. In actual box girders only a limited number of interior diaphragms are provided to help maintain the cross-sectional shape. Thus, distortion of cross section may occur, particularly between diaphragms, under the distortional load components as shown in Fig. 1.2. Figure 4.1 depicts the cross-sectional distortion. Distortion induces, in the component plates of the box, transverse bending moments as shown in Fig. 4.2 and longitudinal in-plane bending as depicted in Fig. 4.3, which causes warping of the cross section.

There exist many methods of analyzing distortional stresses of box sections. The similarity of the governing differential equation of a single-celled box section to that of a beam on elastic foundation was noted by Vlasov\textsuperscript{(1.9)}. The beam-on-elastic-foundation (BEF) analogy was later developed by Wright, Abdel-Samad and Robinson\textsuperscript{(1.12)}. They consider that the total resistance to an applied torsional load is given by the sum of the box section frame action and the longitudinal warping action. Dabrowski\textsuperscript{(3.3)} also arrived at the BEF analogy for the analysis of cross-sectional distortion of curved box girders. A displacement method which neglected the frame action of the box was employed by Dalton and Richmond\textsuperscript{(4.1)}.
The transfer matrix method applied to folded-plate theory was developed by Sakai and Okumura\(^\text{(4.2)}\). A finite element procedure using extensional-flexural elements was developed for box girder analysis by Lim and Moffat\(^\text{(1.14,2.13)}\). The results are reported in Ref. 4.3.

The purpose of this chapter is to reiterate some of the results obtained on distortional analyses.

Two factors, the number and the rigidity of interior diaphragms, influence the distortion of a box section. It has been shown\(^\text{(4.3)}\) that, on the basis of weight, plate diaphragms are the most efficient in reducing distortional stresses. It has also been shown\(^\text{(4.2)}\) that relatively thin plates may be considered rigid as long as no yielding or buckling takes place. For box girders with rigid diaphragms, distortional stresses are minimal if loads are applied at diaphragms, and comparatively large if loads are located between diaphragms\(^\text{(1.12,4.2)}\).

To examine the effects of the spacing of rigid diaphragms in a simply supported box girder, a vertical, concentrated load which is eccentric to the shear center may be applied at various locations along the span of the girder. The maximum total normal stress (sum of the flexural, torsional and distortional parts) occurs at the cross section under the load. The stress versus load location plots thus are the stress envelopes. By comparing the stress envelopes of box girders having different numbers of equally spaced interior diaphragms, the effects of diaphragm spacing can be examined.
Two rectangular cross sections examined for distortion are as shown in Fig. 4.4. Normal stress envelopes for these two cross sections have been computed \(^{(4.4)}\) by a plane stress finite element procedure \(^{(4.5)}\) and are plotted in Figs. 4.5 and 4.6. The elastic moduli of steel and concrete are 200,100 MN/m\(^2\) (29,000 ksi) and 25,530 MN/m\(^2\) (3700 ksi), and the Poisson's ratios are 0.3 and 0.15, respectively. For each box girder, rigid diaphragms spaced at L/2, L/4, and L/6 are considered, with end diaphragms always present at the supports of the 36,576 mm (1440 in.) span.

Figure 4.5 is for the box section with a width-to-depth ratio of 2.0. Straight lines connecting the solid geometrical symbols are the normal stress envelopes, obtained using plane stress finite elements, for the bottom flange-to-web junction under the load (point 3). The solid curve gives the longitudinal stresses for the same point computed by thin-walled elastic beam theory considering flexure and torsion, but not distortion. The distance between the straight line envelopes and the solid curve are the normal stresses corresponding to cross-sectional distortion. For comparison, the distortional warping normal stresses at the mid-distances between two adjacent diaphragms are computed by the BEF analogy and are added to the thin-walled elastic beam theory values. The total normal stresses are plotted using open geometrical symbols in Fig. 4.5. These normal stresses compare very well with those of the envelopes obtained by the finite element procedure.
Figure 4.6 shows similar stress envelopes for the box section with a width-to-depth ratio of 1/2. Again, the total normal stresses from the two procedures compare well. This indicates that the classical thin-walled elastic beam theory plus the BEF analogy can be utilized for the evaluation of total stresses in the box girders.

From Figs. 4.5 and 4.6 it can be concluded that, regardless of the diaphragm spacing, when loads are applied at a diaphragm, the distortional stresses are very small. The envelopes of total stresses are practically in contact with the solid curve obtained by thin-walled elastic beam theory. Therefore, distortion of cross-section need not be considered when loads are applied at rigid diaphragms. On the other hand, distortional stresses may be quite high when loads are applied between diaphragms, as indicated by the distance between the finite element stress envelopes and the solid curves. These conclusions further confirm the characteristics pointed out by others (1.12,4.2).

For both box girders shown in Figs. 4.5 and 4.6, the distortional stresses are reduced as the number of diaphragms is increased. Theoretically, these stresses can be reduced to negligible values if more diaphragms are used, resulting in the solid curve obtained by the thin-walled beam theory for point 3. However, for these two box sections with no distortion, the highest normal stress due to flexure and torsional warping occurs at point 4, the bottom flange-to-web junction opposite from the concentrated load. This
stress is determined by the box girder geometry and can not be reduced without changing the cross section dimensions. The magnitudes of the normal stress at point 4 along the half span are plotted in Figs. 4.5 and 4.6 as dashed curves. The distance between the solid and the dashed curves equals twice the magnitude of the torsional warping normal stresses, and represents the influence of warping torsion on the stresses at the two corners of the bottom flange.

Since the total normal stress, including distortional effects, at point 3 can be reduced by adding diaphragms, it is suggested herein that the interior diaphragms be spaced such that the maximum total normal stress at point 3 at mid-distance between diaphragms is equal to or smaller than the inherent, unreducible maximum normal stress at point 4 at midspan where a diaphragm exists. In this way, the maximum normal stress for design is that at midspan as computed using thin-walled elastic beam theory.

How the torsional and distortional warping normal stresses relate to each other depends on the geometrical shape and dimensions of the box section. In Fig. 4.5, the maximum total normal stress including distortional effects for diaphragms spaced at L/8 (open square) is still higher than the maximum normal stress obtained by the thin-walled beam theory for point 4 at midspan. For the box girder of Fig. 4.6, a diaphragm spacing of L/6 brings the two maximum normal stresses to about the same level.

To examine further the effects of cross-sectional geometry on the torsional and distortional warping normal stresses, five box
girders of the same component plate thicknesses and same span length are studied. Two of the five have cross sections as shown in Fig. 4.4. The other three have box width-to-depth ratios of 5, 1, and 1/5, respectively. The load magnitudes are such that the St. Venant torsional shear flow is the same for all five cross-sections. The results are listed in Table 4.1 for torsional warping normal stresses at point 3 at midspan and for distortional warping normal stresses at the same point under the load at the mid-distance between diaphragms. For all five sections, the distortional stresses decrease with increasing number of diaphragms. For the section with $b_f/h_w$ equal to 5, the torsional and distortional warping normal stresses are of the same sign, thus add to each other. For the remaining four sections, the torsional and distortional warping normal stresses are of opposite sign, and only one of the four has distortional warping normal stress less than twice the torsional warping normal stress when diaphragm spacing is $L/6$; only two of the four when $L/8$. These results point out the necessity of evaluating the distortional warping normal stresses between diaphragms after the selection of box section geometry and dimensions.

The maximum transverse distortional bending stresses computed by the BEF analogy for the five box girders of Table 4.1 are listed in Table 4.2. These bending stresses are for loads applied at mid-distance between diaphragms. The magnitude of these stresses are high when the diaphragms are far apart, but decrease rapidly with increasing number of diaphragms. When loads are applied at
diaphragms, these stresses are practically zero. For example, for diaphragms spaced at L/2 and loads applied at mid-span diaphragm, the maximum transverse distortional bending stresses are 0.41, 0.35, and 0.21 MN/m$^2$ (0.6, 0.05 and 0.03 ksi) for the box girders with $b_f/h_w$ ratios of 2, 1, and 1/2, respectively. Such magnitudes can well be ignored.

The distortion of cross section also affects the box girder deflection (Fig. 4.1). The deflection profile of the girders of Fig. 4.5 and 4.6 are plotted in Figs. 4.7 and 4.8. Also plotted in the figures as dashed curves are the deflection profiles from thin-walled elastic beam theory. For both box girders, the deflections caused by the cross-sectional distortion are greatly reduced when the diaphragms are at a spacing of L/4. When the spacing is at L/6, the deflection profiles including the distortional effects are practically coincident to those by the thin-walled elastic beam theory.

In summary, the effects of cross-sectional distortional of box girders on the stresses and deflections are relatively unimportant when rigid diaphragms are closely spaced, or when loads are applied at rigid diaphragms. Only when loads are between far-apart diaphragms is it necessary to consider the distortional effects of cross sections.
5. ELASTIC STRESSES AND DEFLECTIONS

5.1 Stresses

The superposition of stress and deflection induced by flexure, torsion and cross-sectional distortion provides the total elastic stress and deflection. The total longitudinal normal stress at a point of a composite box girder is thus given by

\[ \sigma_T = \sigma_B + \sigma_W + \sigma_D \] (5.1)

where \( \sigma_B \), \( \sigma_W \) and \( \sigma_D \) are bending, warping torsional, and distortional normal stresses computed according to Chapters 2, 3, and 4, respectively. The corresponding total shearing stress is the sum of flexural, St. Venant torsional, warping torsional, and distortional shears.

\[ \tau_T = \tau_B + \tau_{SV} + \tau_W + \tau_D \] (5.2)

Four model composite box girders were tested at Fritz Engineering Laboratory (3.9, 5.1). The results of experimental stresses were compared with computed values so as to evaluate the validity of the theories. Two of the box girders were 3658 mm (144 in.), and the other two 12,192 mm (480 in.) in overall length. The details of these box girders are shown in Figs. 5.1, 5.2, 3.7 and 3.8. The material properties are listed in Table 5.1 and the cross-sectional properties summarized in Table 5.2. The shear lag
effect was found to be small for these box girders, therefore, the entire cross section of each box girder was considered effective in flexure. The effects of the reinforcing bars in the concrete deck were found to be insignificant. The bars were, therefore, not included in calculating the cross-sectional properties listed in Table 5.2. In each pair of the four box girders, one (D2 and L2) had thinner webs, thus was flexurally and torsionally weaker than the other (D1 and L1).

In the tests of the model box girders all loads were applied at the diaphragms. Therefore the effects of cross-sectional distortion were theoretically negligible. For a load of 44.5 KN (10 k) applied over a web at the midspan of girder D1, where a diaphragm existed, the computed maximum distortional warping normal stress and distortional transverse bending stress were only 0.676 MN/m² (0.098 ksi) and 0.172 MN/m² (0.025 ksi), respectively, as compared to a total longitudinal normal stress of 49.7 MN/m² (7.2 ksi). Consequently in the computation of total normal and shearing stresses, the distortional terms in Eqs. 5.1 and 5.2 were omitted for these specimens. The experimental results, however, included the effects of all the contributing factors.

The experimental stresses to be discussed in this chapter are converted from measured strains, which are assumed as elastic deformations. The computed and experimental normal stresses in two cross sections are shown in Figs. 5.3 and 5.4. For the cross section of girder D1 in Fig. 5.3, the shear lag effect was not prominent.
The computed and tested stresses agree quite well. For the cross-section of girder D2 in Fig. 5.4, the torsional warping caused the total longitudinal normal stress on one side of the bottom flange to be 46.7% higher than that on the other side. Good agreement between the computed and experimental stresses is evident.

Along the length of the specimens, the comparison of stresses are shown in Fig. 5.5 for box girder D2. In the figure stresses were computed at the measured points and the results connected by straight lines to form normal and shearing stress diagrams. At the two bottom flange points close to the webs (D2-A and D2-B), the computed and experimental normal stresses agreed fairly well. Along the middle of the bottom flange (D2-C), the flexural shear was zero, thus the shearing stresses were due to torsional effects only. The computed values were higher than the recorded magnitudes at some of the points.

The validity of Eqs. 5.1 and 5.2 is dictated by the onset of nonlinear behavior of the box girder. Figure 5.6 is a load versus normal stress plot for two bottom flange points of specimen D2. The strain gages were located in box panel 3 (Fig. 5.2). The theoretical and experimental stresses increased linearly with the increase of load. The stresses at gage D2-A started to deviate from the prediction line above the buckling load of web panel 3N. The stresses in gage D2-B started to deviate from the straight line prediction at higher loads. Beyond the buckling load, box girder panel behavior is nonlinear and linear-elastic predictions become invalid. The strength of composite box girders beyond buckling load will be discussed in a separate report.
5.2 Effect of Cracks in Concrete Deck Due to Negative Moment

It has been difficult to evaluate accurately the stiffness of reinforced concrete beams for stress and deflection calculations\(^{(5.2,5.3)}\). In the negative moment region of composite box girders, tensile cracks develop in the concrete deck. The stiffness of the girder is variable along its length, being largest between cracks where the concrete contributed to the stiffness, and smallest directly at a crack. To facilitate the computation of stresses in the steel plates and the deflection of the girder in the elastic, prebuckling stage, the girder stiffness may be approximated by an average value computed using a partial deck thickness together with the steel U-section.

An idealized schematic diagram of load-deflection relationship in the negative moment region is shown in Fig. 5.7. For initially uncracked concrete, the full deck is effective until the development of tension cracks (OA). Then, a partial deck thickness is assumed effective (AB). Further increase in load enlarges the existing cracks and causes additional cracks in the deck. Hence only the reinforcing bars plus the steel U-section remains effective in resisting the additional load. The onset of nonlinearity (C) due to yielding of the steel or buckling of plates may occur before or after point B, depending on the cross-sectional proportioning of the girder. Since reinforced concrete decks often have hairline cracks in the negative moment region due to shrinkage and dead load, the condition of uncracked deck (OA) seldom exists. A partial deck
thickness can therefore be assumed initially for practical purposes. In this case the load-deflection relationship is represented by the line 0B'C'D'.

From the tests on the four composite box girder specimens (Figs. 3.7, 3.8, 5.1 and 5.2), it was found that in the elastic, prebuckling stage a partial deck thickness equal to the distance from the center of the bottom layer longitudinal reinforcing steel to the bottom of the concrete deck can be satisfactorily considered as effective for stress and deflection computations.

To examine this assumption of partial deck thickness, the diagrams of load versus stress for various locations in the box girder specimens are plotted in Figs. 5.8, 5.9, 5.10 and 5.11. Figures 5.8 and 5.9 are for total normal stresses at web points near the composite deck, where the computed normal stresses are influenced most by the assumption of partial deck thickness. It is seen that the partial deck assumption gives good prediction of normal stresses for these specimens. The similarly good prediction is observed for the shearing stresses shown in Figs. 5.10 and 5.11.

5.3 Deflections

Similar to stresses, the deflections caused by bending, torsion and distortion can be superimposed to obtain the total values. Figure 5.12 compares the calculated and measured deflection profiles of two box girder specimens. Partial deck thickness was employed in
computing the deflection for the case of negative bending. Good agreement is observed.

Three load versus deflection plots are given in Figs. 5.13 to 5.15. Figure 5.13 is for positive bending plus torsion with the predicted linear-elastic deflection computed using full deck thickness. Figures 5.14 and 5.15 are for negative bending plus torsion, for which partial deck thickness has been used in deflection computation. In all three cases the experimental deflections agree well with the calculated values, confirming further the validity of the assumption of partial deck thickness.

The good correlation between the computed and experimental stresses and deflections indicated that the method of analysis proposed and discussed in Chapters 2, 3 and 4, and the partial deck thickness discussed in this chapter are valid for the composite box girders.
6. SUMMARY AND CONCLUSIONS

This report presents a procedure of stress analysis for composite box girders. Although the primary concern is on steel-concrete composite box girders, the procedure is applicable to box girders of any materials which follow the Hooke's Law.

Within the linear-elastic range of behavior of box girders, the shear lag effects on the flexural stresses and the warping effects on the torsional stresses are included in the analysis procedure. The behavior of cross-sectional deformation is discussed briefly. The total stress or deflection at a point of a box girder is the sum of those due to flexure (Chapter 2 and Appendix A), torsion (Chapter 3 and Appendix B), and distortion (Chapter 4). For the cases examined, the computed stresses and deflections are in good agreement with those from experimental studies.

From the results of this study, a number of conclusions can be drawn:

1. Shear lag effects are prominent for box girders with small span-to-width ratios. For these box girders the normal stresses have to be evaluated by the shear lag analysis, whereas the shearing stresses may be computed by the ordinary beam theory.
2. The heavier the webs are relative to the flanges, the greater are the equivalent widths of the flanges.

3. The projecting widths of the deck beyond the webs have little effect on the equivalent widths of the flanges.

4. The thin-walled torsional theory for box sections with homogeneous components can be applied to composite box sections by performing proper transformations in evaluating the torsional sectional properties.

5. The orthotropic properties of concrete deck due to reinforcing bars have little effect on the flexural and the torsional stresses.

6. The effects of cross-sectional distortion can be reduced by adding interior diaphragms. If the loads are applied at sufficiently rigid diaphragms, the distortional effects are practically negligible.

7. For box girders subjected to negative bending with or without torsion in the elastic prebuckling stages, a partial deck thickness can be used for estimating the stresses and deflections.

The results of flexural and torsional analyses in this study and the conclusions above can be employed as a basis for generating information for box girder design. Work in this respect can and should be carried out.
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Thanks are due the coworkers of the project, including J. A. Corrado, R. E. McDonald, C. Yilmaz and Professor A. Ostapenko for their assistance and suggestions. Thanks are also due Mrs. Dorothy Fielding for typing and processing this report. Appreciation is extended to Mr. J. M. Gera and his staff for tracing the drawings.
TABLE 2.1 COMPARISONS OF MAXIMUM AND AVERAGE LONGITUDINAL NORMAL STRESSES ($\sigma_z$)
OF AN ILLUSTRATIVE BOX GIRDER

<table>
<thead>
<tr>
<th></th>
<th>Orthotropic 8/4 - 4/4</th>
<th>Orthotropic 4/4 - 8/4</th>
<th>Plain 0 - 0 Deck</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gross Deck</td>
<td>In Concrete</td>
<td>Bottom Flange</td>
</tr>
<tr>
<td>Maximum $\sigma_z$, MN/m$^2$ (ksi)</td>
<td>-2.14 (-0.310)</td>
<td>-1.46 (-0.211)</td>
<td>34.36 (4.976)</td>
</tr>
<tr>
<td>Average $\sigma_z$, MN/m$^2$ (ksi)</td>
<td>-1.17 (-0.172)</td>
<td>-0.81 (-0.117)</td>
<td>22.43 (3.246)</td>
</tr>
<tr>
<td>Maximum/Average $%$</td>
<td>180</td>
<td>180</td>
<td>153</td>
</tr>
</tbody>
</table>
### TABLE 2.2 EQUIVALENT WIDTH RATIOS

<table>
<thead>
<tr>
<th>( \frac{w_c}{b_c} )</th>
<th>( \frac{\lambda_1}{b_c} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12.5</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>.307</td>
<td>.511</td>
<td>.622</td>
<td>.686</td>
<td>.735</td>
<td>.768</td>
<td>.779</td>
<td>.819</td>
<td>.835</td>
<td>.851</td>
<td>.880</td>
<td>.900</td>
<td>.927</td>
<td>.944</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>.148</td>
<td>.277</td>
<td>.401</td>
<td>.493</td>
<td>.568</td>
<td>.620</td>
<td>.639</td>
<td>.700</td>
<td>.727</td>
<td>.752</td>
<td>.798</td>
<td>.830</td>
<td>.873</td>
<td>.900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.263</td>
<td>.464</td>
<td>.580</td>
<td>.655</td>
<td>.710</td>
<td>.749</td>
<td>.773</td>
<td>.805</td>
<td>.825</td>
<td>.842</td>
<td>.874</td>
<td>.896</td>
<td>.925</td>
<td>.943</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.309</td>
<td>.516</td>
<td>.631</td>
<td>.703</td>
<td>.751</td>
<td>.788</td>
<td>.815</td>
<td>.837</td>
<td>.854</td>
<td>.869</td>
<td>.895</td>
<td>.914</td>
<td>.938</td>
<td>.952</td>
<td></td>
</tr>
</tbody>
</table>

**b_c = b_f = 3810 mm (150 in.), \( t_c = 102 (4), t_f = 25 (1), h_w = 2540 (100, t_w = 13 (1/2), w_{tf} = 305 (12), t_{tf} = 25 (1), E_s/E_c = 10, E_c/G = 2.34, E_s = .203,550 \text{ MN/m}^2 (29,500 \text{ ksi})**
TABLE 4.1 WARPING NORMAL STRESSES, MN/m² (ksi) AT BOTTOM CORNER OF BOX BENEATH LOAD

<table>
<thead>
<tr>
<th>$\frac{b_f}{h_w}$</th>
<th>Torsional $2\sigma_w$</th>
<th>Distortional $\sigma_w$ Due to Diaphragms Spaced at</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L/2</td>
<td>L/4</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.96</td>
<td>25.58</td>
<td>12.82</td>
</tr>
<tr>
<td>(1.30)</td>
<td>(3.71)</td>
<td>(1.86)</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-6.48</td>
<td>54.95</td>
<td>27.51</td>
</tr>
<tr>
<td>(-0.94)</td>
<td>(7.97)</td>
<td>(3.99)</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-16.14</td>
<td>84.60</td>
<td>50.75</td>
</tr>
<tr>
<td>(-2.34)</td>
<td>(12.27)</td>
<td>(7.36)</td>
</tr>
<tr>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-22.76</td>
<td>82.33</td>
<td>41.16</td>
</tr>
<tr>
<td>(-3.30)</td>
<td>(11.94)</td>
<td>(5.97)</td>
</tr>
<tr>
<td>1/5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-28.54</td>
<td>58.06</td>
<td>29.86</td>
</tr>
<tr>
<td>(-4.14)</td>
<td>(8.42)</td>
<td>(4.33)</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>( \frac{b_f}{h_w} )</th>
<th>Diaphragms Spaced at</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L/2</td>
<td>L/4</td>
<td>L/6</td>
<td>L/8</td>
</tr>
<tr>
<td>5</td>
<td>12.14 (1.76)</td>
<td>(1.45)</td>
<td>(0.21)</td>
<td>0.41</td>
</tr>
<tr>
<td>2</td>
<td>59.16 (8.58)</td>
<td>8.89</td>
<td>(2.41)</td>
<td>1.03</td>
</tr>
<tr>
<td>1</td>
<td>188.30 (27.31)</td>
<td>28.26</td>
<td>(1.30)</td>
<td>3.79</td>
</tr>
<tr>
<td>1/2</td>
<td>52.82 (7.66)</td>
<td>7.93</td>
<td>(2.14)</td>
<td>0.90</td>
</tr>
<tr>
<td>1/5</td>
<td>8.83 (1.28)</td>
<td>1.03</td>
<td>0.28</td>
<td>0.07</td>
</tr>
</tbody>
</table>

**TABLE 4.2**

TRANSVERSE DISTORTIONAL BENDING STRESSES, MN/m² (ksi)
AT THE BOTTOM OF WEBS
<table>
<thead>
<tr>
<th>Properties</th>
<th>Specimens</th>
<th>D1</th>
<th>D2</th>
<th>L1</th>
<th>L2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steel</strong></td>
<td>Small Top Flanges</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yield Stress*</td>
<td></td>
<td>213.9</td>
<td>207.0</td>
<td>253.2</td>
<td>253.2</td>
</tr>
<tr>
<td></td>
<td>Webs</td>
<td>(31.0)</td>
<td>(30.0)</td>
<td>(36.7)</td>
<td>(36.7)</td>
</tr>
<tr>
<td></td>
<td>Bottom Flange</td>
<td></td>
<td></td>
<td>262.2</td>
<td>259.4</td>
</tr>
<tr>
<td>Young's Modulus of Elasticity</td>
<td></td>
<td></td>
<td></td>
<td>203,550</td>
<td>(29,500)</td>
</tr>
<tr>
<td>Shear Modulus of Elasticity</td>
<td></td>
<td></td>
<td></td>
<td>78,315</td>
<td>(11,350)</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td></td>
<td></td>
<td></td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td><strong>Concrete</strong></td>
<td>Compressive Strength*</td>
<td>34.5</td>
<td>38.0</td>
<td>38.0</td>
<td>29.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.0)</td>
<td>(5.5)</td>
<td>(5.5)</td>
<td>(4.21)</td>
</tr>
<tr>
<td>Young's Modulus of Elasticity</td>
<td></td>
<td>25,530</td>
<td>25,806</td>
<td>27,462</td>
<td>23,391</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3700)</td>
<td>(3740)</td>
<td>(3980)</td>
<td>(3390)</td>
</tr>
<tr>
<td>Shear Modulus of Elasticity</td>
<td></td>
<td>10,902</td>
<td>11,040</td>
<td>11,730</td>
<td>10,005</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1580)</td>
<td>(1600)</td>
<td>(1700)</td>
<td>(1450)</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td></td>
<td></td>
<td></td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td><strong>Deck</strong></td>
<td>Yield Stress of Deck Reinforcement*</td>
<td>483</td>
<td></td>
<td>331</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(70)</td>
<td></td>
<td>(48)</td>
<td></td>
</tr>
<tr>
<td>Specimens Properties</td>
<td>D1</td>
<td>D2</td>
<td>L1</td>
<td>L2</td>
<td></td>
</tr>
<tr>
<td>----------------------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td>----------</td>
<td></td>
</tr>
<tr>
<td>Transformed Area of Section, cm² (in.²)</td>
<td>106.13 (16.45)</td>
<td>104.26 (16.16)</td>
<td>554.77 (85.99)</td>
<td>492.26 (76.30)</td>
<td></td>
</tr>
<tr>
<td>Distance from N.A. to mid-line of Bottom Flange, cm (in.)</td>
<td>25.27 (9.95)</td>
<td>25.55 (10.06)</td>
<td>74.88 (29.48)</td>
<td>73.51 (28.94)</td>
<td></td>
</tr>
<tr>
<td>Moment of Inertia about x-axis, Iₓ, cm⁴ (in.⁴)</td>
<td>2.097 x 10⁴ (503.8)</td>
<td>2.061 x 10⁴ (495.2)</td>
<td>1.084 x 10⁶ (2.603 x 10⁴)</td>
<td>1.006 x 10⁶ (2.416 x 10⁴)</td>
<td></td>
</tr>
<tr>
<td>Moment of Inertia about y-axis, Iᵧ, cm⁴ (in.⁴)</td>
<td>4.878 x 10⁴ (1171.9)</td>
<td>4.828 x 10⁴ (1160.0)</td>
<td>1.235 x 10⁶ (2.966 x 10⁴)</td>
<td>1.074 x 10⁶ (2.580 x 10⁴)</td>
<td></td>
</tr>
<tr>
<td>Shear Center above N.A., cm (in.)</td>
<td>2.512 (0.989)</td>
<td>2.845 (1.120)</td>
<td>3.647 (1.436)</td>
<td>3.472 (1.367)</td>
<td></td>
</tr>
<tr>
<td>St. Venant Torsional Constant Kₜ, cm⁴ (in.⁴)</td>
<td>1.425 x 10⁴ (342.3)</td>
<td>1.206 x 10⁴ (289.7)</td>
<td>8.171 x 10⁵ (1.963 x 10⁴)</td>
<td>6.918 x 10⁵ (1.662 x 10⁴)</td>
<td></td>
</tr>
<tr>
<td>Warping Moment of Inertia, Iₗ, cm⁶ (in.⁶)</td>
<td>1.588 x 10⁶ (5915.2)</td>
<td>1.862 x 10⁶ (6934.5)</td>
<td>3.555 x 10⁸ (1.324 x 10⁶)</td>
<td>4.090 x 10⁸ (1.523 x 10⁶)</td>
<td></td>
</tr>
<tr>
<td>Central Moment of Inertia, Iᶜ, cm⁴ (in.⁴)</td>
<td>2.351 x 10⁴ (564.8)</td>
<td>2.277 x 10⁴ (547.0)</td>
<td>1.170 x 10⁶ (2.810 x 10⁴)</td>
<td>1.084 x 10⁶ (2.605 x 10⁴)</td>
<td></td>
</tr>
<tr>
<td>Warping Shear Parameter, µ</td>
<td>0.3940</td>
<td>0.4704</td>
<td>0.3017</td>
<td>0.3619</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 1.1 Types of Box Girder Cross Section
<table>
<thead>
<tr>
<th>Specimens Properties</th>
<th>D1</th>
<th>D2</th>
<th>L1</th>
<th>L2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transformed Area of Section, ( \text{cm}^2 ) (in.(^2))</td>
<td>106.13</td>
<td>104.26</td>
<td>554.77</td>
<td>492.26</td>
</tr>
<tr>
<td>Distance from N.A. to mid-line of Bottom Flange, ( \text{cm} ) (in.)</td>
<td>25.27</td>
<td>25.55</td>
<td>74.88</td>
<td>73.51</td>
</tr>
<tr>
<td>Moment of Inertia about x-axis, ( I_x ), ( \text{cm}^4 ) (in.(^4))</td>
<td>2.097 ( \times 10^4 ) (503.8)</td>
<td>2.061 ( \times 10^4 ) (495.2)</td>
<td>1.084 ( \times 10^6 ) (2.603 ( \times 10^4 ))</td>
<td>1.006 ( \times 10^6 ) (2.416 ( \times 10^4 ))</td>
</tr>
<tr>
<td>Moment of Inertia about y-axis, ( I_y ), ( \text{cm}^4 ) (in.(^4))</td>
<td>4.878 ( \times 10^4 ) (1171.9)</td>
<td>4.828 ( \times 10^4 ) (1160.0)</td>
<td>1.235 ( \times 10^6 ) (2.966 ( \times 10^4 ))</td>
<td>1.074 ( \times 10^6 ) (2.580 ( \times 10^4 ))</td>
</tr>
<tr>
<td>Shear Center above N.A., ( \text{cm} ) (in.)</td>
<td>2.512 (0.989)</td>
<td>2.845 (1.120)</td>
<td>3.647 (1.436)</td>
<td>3.472 (1.367)</td>
</tr>
<tr>
<td>St. Venant Torsional Constant ( K_T ), ( \text{cm}^4 ) (in.(^4))</td>
<td>1.425 ( \times 10^4 ) (342.3)</td>
<td>1.206 ( \times 10^4 ) (289.7)</td>
<td>8.171 ( \times 10^5 ) (1.963 ( \times 10^4 ))</td>
<td>6.918 ( \times 10^5 ) (1.662 ( \times 10^4 ))</td>
</tr>
<tr>
<td>Warping Moment of Inertia, ( I_w ), ( \text{cm}^6 ) (in.(^6))</td>
<td>1.588 ( \times 10^6 ) (5915.2)</td>
<td>1.862 ( \times 10^6 ) (6934.5)</td>
<td>3.555 ( \times 10^8 ) (1.324 ( \times 10^6 ))</td>
<td>4.090 ( \times 10^8 ) (1.523 ( \times 10^6 ))</td>
</tr>
<tr>
<td>Central Moment of Inertia, ( I_c ), ( \text{cm}^4 ) (in.(^4))</td>
<td>2.351 ( \times 10^4 ) (564.8)</td>
<td>2.277 ( \times 10^4 ) (547.0)</td>
<td>1.170 ( \times 10^6 ) (2.810 ( \times 10^4 ))</td>
<td>1.084 ( \times 10^6 ) (2.605 ( \times 10^4 ))</td>
</tr>
<tr>
<td>Warping Shear Parameter, ( \mu )</td>
<td>0.3940</td>
<td>0.4704</td>
<td>0.3017</td>
<td>0.3619</td>
</tr>
</tbody>
</table>
Fig. 1.1 Types of Box Girder Cross Section
Fig. 1.2 Decomposition of Box Girder Loading
Fig. 2.1 Single Celled Composite Box Girder
Fig. 2.2 Flexural Shear in a Single Celled Box
Orthotropic Bottom Flange

Fig. 2.3 Configuration of Deck and Bottom Flange of Composite Box Girder
Fig. 2.4 Unidirectionally Fiber-Reinforced Composite
Fig. 2.5 Reinforced Concrete Deck and Unidirectional Layers
Fig. 2.6 Plane Stresses (Forces) on an Element
Fig. 2.7 Moment Diagrams of a Simple Beam

\[ M_x(z) = \sum_{n=1}^{\infty} \frac{2P l}{n^2 \pi^2} \sin \alpha n \pi \sin \frac{n \pi z}{l} \]

\[ M_x(z) = \sum_{n=1,3,5}^{\infty} \frac{4w l^2}{n^3 \pi^3} \sin \frac{n \pi z}{l} \]
Fig. 2.8 Components of a Box Section and Normal Stresses and Forces
Fig. 2.9 Longitudinal Normal Stress Pattern and Equivalent Widths of a Box Section
445 KN (100 K)

Fig. 2.10 An Illustrative Box Girder

Deck Reinforcement
(Orthotropic 8/4-4/4)
Fig. 2.11 Stresses at Midspan of the Illustrative Box Girder (Orthotropic 8/4 - 4/4 Deck)
Fig. 2.12 Tate's Specimens Bl and B2 (Ref. 2.10)
Fig. 2.13 Flange Stresses at Midspan of Tate's Specimens B1 and B2 (Ref. 2.10)
Fig. 2.14 Cross Section 4 of Moffat and Dowling
(Ref. 2.12)
Fig. 2.15 Comparison of Equivalent Widths of Cross Section
4 from Moffat and Dowling (Ref. 2.12)
Fig. 2.16 Comparison of Equivalent Widths of Tate's Specimen B2 from Hildebrand and Reissner (Ref. 2.9)
Fig. 2.17 Influence of Deck Projecting Widths on the Equivalent Widths
C: Centroid
SC: Shear Center

(a) Composite Closed Cross-Section

(b) Mid-Line Contour

(c) Differential Element

Fig. 3.1 Thin-walled Composite Closed Cross Section
Fig. 3.2 Composite Closed Cross Section with Open Projecting Elements

(a) Pure Torsional Shear

(b) Integration Paths
Fig. 3.3 Typical Distributions of $\bar{w}$ and $\bar{S}_{\bar{w}}$ on a Cross Section
Fig. 3.4 Geometrical Relationships Between $\rho$ and $\rho_0$
Fig. 3.5 Illustrative Loading and Boundary Conditions
Fig. 3.6 Typical Diagrams of Section Torque, $\phi$, $f''$ and $f'''$
Fig. 3.7 Specimens L1 and L2 (All Units in mm (in.))
Fig. 3.8 Cross Section at Plate Diaphragms of L1 and L2
Fig. 3.9 Shearing Stresses in Specimens L1 and L2
Fig. 3.10 Torque versus Angle of Rotation
Fig. 4.1 Cross-sectional Distortion

Fig. 4.2 Diagram of Transverse Bending Moment Due to Cross-sectional Distortion
Fig. 4.3 Distribution of Distortional Warping Stresses
Fig. 4.4 Cross Sections Examined for Distortion
Fig. 4.5 Stress Envelope for Cross Section with $b_f/h_w = 2$
Fig. 4.6 Stress Envelope for Cross Section with $b_f/h_w = 1/2$
Fig. 4.7 Deflection Profile for Bottom Flange Corner 3,
\( \frac{h_f}{h_w} = 1/2 \)
Fig. 4.8 Deflection Profile for Bottom Flange Corner 3,
\[ \frac{b_f}{h_w} = 2 \]
Fig. 5.1 Specimens D1 and D2 (all units in mm (in.))
Fig. 5.2 Cross Section at Plate Diaphragms of D1 and D2
(All units in mm (in.))
Fig. 5.3 Normal Stresses in the Cross Section of Girder D1 at Z = 1429 mm (56.25 in.)
Fig. 5.4 Normal Stresses in the Cross Section of Girder D2 at \( z = 1397 \text{ mm} \) (55 in.)
Fig. 5.5 Distribution of Normal and Shearing Stresses in the Bottom Flange Along the Span of Girder D2
Fig. 5.6 Load versus Normal Stresses in the Bottom Flange of Girder D2 at Z = 1397 mm (55 in.)
By Reinforcing Bars + Steel U-Shape

Fig. 5.7 Schematic Diagram of Load-Deflection Relationship in the Negative Moment Region
Fig. 5.8 Load versus Normal Stress in the Web of Girder D1
at $Z = 3099$ mm (122 in.)
Fig. 5.9 Load versus Normal Stress in the Webs of Girder D2 at Z = 3099 mm (122 in.)
Fig. 5.10 Load versus Shearing Stresses in the Webs of Girder D1 at Z = 3099 mm (122 in.)
Fig. 5.11 Load versus Shearing Stresses in the Webs of Girder DI at $Z = 3353$ mm (132 in.)
Fig. 5.12 Deflections along the Span of Box Girders D1 and D2
Fig. 5.13 Load versus Deflection at Midspan of Girder D2
Fig. 5.14 Load versus Deflection at Overhanging End of Girder D2
Fig. 5.15 Load versus Deflection at Overhanging End, L1
A.1 Orthotropic Deck and Bottom Flange

The stress function expressions for the top and bottom flanges are Eqs. 2.26 and 2.28, respectively.

For the flanges, combination of the conditions of symmetry of Eqs. 2.30 with Eqs. 2.19, 2.22, 2.24, 2.26 and 2.28 gives

\[ B_n = D_n = 0 \] \hspace{1cm} (A.1a)

and

\[ F_n = H_n = 0 \] \hspace{1cm} (A.1b)

By substituting the expression of transverse normal strain as given by Eq. 2.13g into Eq. 2.31b, performing the integration, and utilizing Eqs. 2.19, 2.24, 2.26 and A.1a, the following equation is obtained.

\[
u = - \frac{1}{E_x} \sum_{l=1}^{\infty} \frac{1}{l \alpha_n} \left[ A_n \left( \frac{1}{r_2} + \nu_{xz} r_2 \right) \sinh r_2 \alpha_n x \right. \\
+ \left. C_n \left( \frac{1}{r_2} + \nu_{xz} r_2 \right) \sinh r_2 \alpha_n x \right] \sin \alpha_n z \]

\hspace{1cm} (A.2)

This equation and the boundary condition of Eq. 2.31a combine to give the following relationship

\[ C_n = - \zeta_n A_n \] \hspace{1cm} (A.3a)
where
\[ \zeta_n = \frac{r_2 (1 + \nu_{xz} r_1^2) \sinh \frac{r_1 \alpha_n b_c}{2}}{r_1 (1 + \nu_{xz} r_2^2) \sinh \frac{r_2 \alpha_n b_c}{2}} \]  
\hspace{12cm} (A.3b)

For the bottom flange, from Eqs. 2.22b, 2.24, 2.28 and A.1b and the boundary condition of Eq. 2.31c, the following relationship is established.
\[ G_n = - \eta_n E_n \]  
\hspace{12cm} (A.4a)

where
\[ \eta_n = \frac{\cosh \frac{r_3 \alpha_n b_f}{2}}{\cosh \frac{r_4 \alpha_n b_f}{2}} \]  
\hspace{12cm} (A.4b)

For the projecting portions of the concrete deck, the longitudinal strain \((\varepsilon_z)\) at the connection line \((x = \pm b_c/2)\) is equal to that of the deck between two webs. At this line, the transverse displacement \(u\) is zero as is expressed in Eq. 2.31a. Since rigid body translation and rotation at the deck to web junctions do not occur, it follows that \(\frac{\partial^2 u}{\partial z^2} x = \pm b_c/2 = 0\), from which the following boundary condition is deduced by Winter(2.9).

\[ \frac{1}{E_z} \left( \frac{\partial^3 F}{\partial x^3} - \nu_{xz} \frac{\partial^3 F}{\partial z \partial x^2} \right) x = \frac{b_c}{2} = - \frac{1}{G_{zx}} \left( \frac{\partial^3 F}{\partial x^2 \partial x} \right) x = \frac{b_c}{2} \]  
\hspace{12cm} (A.5)

These conditions and Eqs. 2.32 give rise to the following relationships:
\[ A_n = p_n A_n + q_n^2 C_n \] (A.6a)

\[ C_n = s_n A_n + t_n C_n \] (A.6b)

\[ B_n = -s_n A_n + \rho_n C_n \] (A.6c)

\[ D_n = -\theta_n A_n + \kappa_n C_n \] (A.6d)

where

\[ p_n' = \frac{\cosh \frac{\alpha_n b_c}{2}}{1 + \frac{(r_2^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) s_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) p_n]}{1 + \frac{(r_1^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) t_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) q_n]}{1 + \frac{(r_2^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) s_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) p_n]}{1 + \frac{(r_1^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) t_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) q_n]}}}} \] (A.6e)

\[ q_n' = \frac{\cosh \frac{\alpha_n b_c}{2}}{1 + \frac{(r_2^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) s_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) p_n]}{1 + \frac{(r_1^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) t_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) q_n]}{1 + \frac{(r_2^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) s_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) p_n]}{1 + \frac{(r_1^2 + \nu_{zx})[r_1(r_1^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) t_n + r_2(r_2^2 + \nu_{zx} - \frac{E_{zx}}{G_{zx}}) q_n]}}}} \] (A.6f)
\[ s_n' = \frac{\frac{r_1^2 + \nu_{zx}}{r_2 + \nu_{zx}} \cosh \frac{1}{2} \alpha_n b_c}{c} \]
\[ + \frac{(r_1^2 + \nu_{zx}) [r_1 (r_1^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} t_n + r_2 (r_2^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} q_n]}{1 + (r_2^2 + \nu_{zx}) [r_1 (r_1^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} s_n + r_2 (r_2^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} p_n]} \]

(A.6g)

\[ t_n' = \frac{\frac{r_2 \alpha_n b_c}{\cosh \frac{1}{2}}}{\cosh \frac{1}{2} \alpha_n b_c} \]
\[ + \frac{(r_1^2 + \nu_{zx}) [r_1 (r_1^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} t_n + r_2 (r_2^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} q_n]}{1 + (r_2^2 + \nu_{zx}) [r_1 (r_1^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} s_n + r_2 (r_2^2 + \nu_{zx}) - \frac{E_z}{G_{zx}} p_n]} \]

(A.6h)

\[ p_n = \frac{1}{\text{Denominator}} \]  

(A.6i)

\[ q_n = \frac{r_2 \sinh \frac{1}{2} \alpha_n (w_c - b_c)}{\sinh \frac{1}{2} \alpha_n (w_c - b_c)} \]
\[ \frac{r_1 \alpha_n (w_c - b_c)}{cosh \frac{1}{2}} c \]
\[ \frac{r_2 \alpha_n (w_c - b_c)}{cosh \frac{1}{2}} c \]

(Denominator)

(A.6j)
\[
 s_n = \frac{r_1 \alpha_n (w_c - b_c)}{2 \sinh \frac{2}{2}} \sinh \frac{r_2 \alpha_n (w_c - b_c)}{2} \quad \text{Denominator}
\]

\[
 - \frac{r_2 \cosh \frac{r_1 \alpha_n (w_c - b_c)}{2}}{2 \cosh \frac{r_2 \alpha_n (w_c - b_c)}{2}} \quad \text{Denominator}
\]

(A.6k)

\[
 t_n \quad \text{Denominator}
\]

Denominator = \[
 r_1 \sinh \frac{r_2 \alpha_n (w_c - b_c)}{2} \cosh \frac{r_1 \alpha_n (w_c - b_c)}{2}
\] - \[
 r_2 \sinh \frac{r_1 \alpha_n (w_c - b_c)}{2} \cosh \frac{r_2 \alpha_n (w_c - b_c)}{2} \tag{A.6m}
\]

\[
 e_n = s_n p_n' - t_n s_n' \tag{A.6n}
\]

\[
 \rho_n = t_n t_n' - s_n q_n' \tag{A.6o}
\]

\[
 \theta_n = p_n p_n' - q_n s_n' \tag{A.6p}
\]

\[
 \kappa_n = q_n t_n' - p_n q_n' \tag{A.6q}
\]

Equations A.6 relate the stress function coefficients of the projecting deck to those of the deck between webs. There are also compatibility conditions between the longitudinal strains of the flanges and those of the webs along their lines of connection.

Let \( Z_c \) and \( Z_s \) be the total resultant forces of the longitudinal stresses in the concrete deck and bottom flange, respectively.
Substitution of Eqs. 2.19a, 2.24, 2.26, A.1a, and A.6 into A.7a, and Eqs. 2.22a, 2.24, 2.28 and A.1b into A.7b, respectively, results in the following.

\[ Z_c = 2 t_c \sum_{n=1}^{\infty} (\phi_n \cdot A_n + \psi_n \cdot C_n) \sin \alpha_n z \]  
(A.7c)

\[ Z_s = 2 \sum_{n=1}^{\infty} (\mu_n \cdot E_n + \omega_n \cdot G_n) \sin \alpha_n z \]  
(A.7d)

where

\[ \phi_n = \frac{1}{\alpha_n} \left[ r_1 \sinh \frac{r_1 \alpha_n b_c}{2} + p_n \cdot r_1 \sinh \frac{r_1 \alpha_n (w_c - b_c)}{2} \right. \]

\[ - s_n \cdot r_1 (\cosh \frac{r_1 \alpha_n (w_c - b_c)}{2} - 1) \]

\[ + s_n \cdot r_2 \sinh \frac{r_2 \alpha_n (w_c - b_c)}{2} \]

\[ - \theta_n \cdot r_2 (\cosh \frac{r_2 \alpha_n (w_c - b_c)}{2} - 1) \]  
(A.7e)

-126-
The total internal moment can be considered as composed of two parts: a part $2M'$ generated by stresses in the two webs (plus the small steel top flanges), and the other part $M''$ due to the longitudinal forces $Z_c$ and $Z_s$. From the equilibrium of the internal moment and the external moment of Eq. 2.29, it is obtained

$$M' = \frac{1}{2} \left( \sum_{n=1}^{\infty} M_n \sin \alpha_n z - Z_s e_s + Z_c e_c \right)$$

(A.8)
in which $e_s$ and $e_c$ are the absolute values of the distance from the centroid of the web (tee) to the middle line of the bottom flange and concrete deck, respectively (Fig. 2.8).
The axial strain of the web at the mid-depth level of concrete is equal to that of the concrete deck, as given by Eq. 2.13f, along the line of junction.

\[
\frac{1}{E_s} \left( \frac{M_e}{I^t} - \frac{Z_c + Z_s}{2A^t} \right) = \frac{1}{E_z} \left( \sigma_z - \nu z \sigma_x \right) = \frac{b}{2}
\]  

(A.9a)

Similarly, at the junction of web and bottom flange, with the flange longitudinal strains given by Eq. 2.16a,

\[
\frac{1}{E_s} \frac{M'_e e_s}{I^t} - \frac{Z_c + Z_s}{2A^t} = \frac{1}{E_s} \frac{a_3 N_z - b_3 N_x}{a_3 c_3 - b_3} x = \frac{b}{2}
\]

(A.9b)

In Eqs. A.9, \( I^t \) is the moment of inertia of the tee, consisting of a web and a small steel top flange, about its horizontal centroidal axis, and \( A^t \) the corresponding cross-sectional area.

By substituting Eqs. A.7c, A.7d and A.8 into Eqs. A.9, expressing the right hand sides of Eqs. A.9 in terms of stress functions, and making use of the relationships of stress function coefficients developed previously, coefficients \( A_n \) and \( E_n \) can be solved and are expressed as shown below.

\[
A_n = \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I^t \left( \Theta_n' \eta_n' - \zeta_n \kappa_n' \right)} M_n
\]

(A.10a)

\[
E_n = \frac{e_c \Theta_n' - e_s \zeta_n'}{2E_s I^t \left( \Theta_n' \eta_n' - \zeta_n \kappa_n' \right)} M_n
\]

(A.10b)
where

\[ \kappa_n' = \left( \frac{e_s}{E_s I'} + \frac{1}{E_s A'} \right) (\mu_n - \varphi_n \eta_n) + \]

\[ + \frac{a_3}{E_s (a_3 c_3 - b_3^2)} \left( r_3^2 \cosh \frac{r_3 \alpha_n b_f}{2} \right) \]

\[ - \eta_n r_4^2 \cosh \frac{r_4 \alpha_n b_f}{2} \]  \hspace{1cm} (A.10c)

\[ \eta_n' = \left( \frac{e_s}{E_s I'} - \frac{1}{E_s A'} \right) (\mu_n - \varphi_n \eta_n) \]  \hspace{1cm} (A.10d)

\[ \theta_n' = \left( \frac{e_s}{E_s I'} - \frac{1}{E_s A'} \right) t_c (\phi_n - \zeta_n \psi_n) \]  \hspace{1cm} (A.10e)

\[ \zeta_n' = \left( \frac{e_s}{E_s I'} + \frac{1}{E_s A'} \right) t_c (\phi_n - \zeta_n \psi_n) \]

\[ + \frac{1}{E_z} (v_{zx} + r_2^2) \cosh \frac{r_1 \alpha_n b_c}{2} \]

\[ - \frac{1}{E_z} \zeta_n (v_{zx} + r_2^2) \cosh \frac{r_2 \alpha_n b_c}{2} \]  \hspace{1cm} (A.10f)

Once coefficients \( A_n \) and \( E_n \) are determined, all others can be computed from Eqs. A.3, A.4, A.5 and A.6. The stresses can then be computed through the stress functions and are summarized as follows:

1. For concrete deck between webs

\[ \sigma_z = \sum_{n=1}^{\infty} \frac{e_s \kappa_n' - e_s \eta_n'}{2E_s I'} (\theta_n' - \zeta_n \kappa_n') \left( r_2 \cosh \frac{r_1 \alpha_n x}{2} \right) \]

\[ - \zeta_n r_2^2 \cosh \frac{r_2 \alpha_n x}{2} M_n \sin \alpha_n z \]

\[ - \frac{1}{129} \]  \hspace{1cm} (A.11a)
\[
\sigma_x = - \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I (\theta_n \eta_n - \zeta_n \kappa_n')} (\cosh r_1 \alpha_n x)
- \zeta_n \cosh r_2 \alpha_n x) M_n \sin\alpha_n z
\]
\quad \text{(A.11b)}

\[
\tau_{xz} = - \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I (\theta_n \eta_n - \zeta_n \kappa_n')} (r_1 \sinh r_1 \alpha_n x)
- \zeta_n r_2 \sinh r_2 \alpha_n x) M_n \cos\alpha_n z
\quad \text{(A.11c)}
\]

(2) For the projecting portions of the deck

\[
\sigma_z = \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I (\theta_n \eta_n - \zeta_n \kappa_n')} [p_n'
- \zeta_n q_n') r_1^2 \cosh r_1 \alpha_n (x - \frac{b_c}{2})
- (\xi_n + \zeta_n \rho_n') r_1^2 \sinh r_1 \alpha_n (x - \frac{b_c}{2})
+ (s_n' - \zeta_n t_n') r_2^2 \cosh r_2 \alpha_n (x - \frac{b_c}{2})
- (\Theta_n + \zeta_n \kappa_n') r_2^2 \sinh r_2 \alpha_n (x - \frac{b_c}{2})] M_n \sin\alpha_n z
\]
\quad \text{(A.11d)}
(3) For the bottom flange between webs:

\[
\frac{\lambda_3}{b_f} = \frac{2k}{n b_f} \left[ \sum_{n=1}^{\infty} \frac{e_c \theta_n - e_s \zeta_n}{(\theta_n \eta_n - \zeta_n \kappa_n)_n} \right] \left( r_3 \sinh \frac{r_3 \alpha_a b_f}{2} ight)
- r_4 \eta_n \sinh \frac{r_4 \alpha_a b_f}{2} \cdot M \sin \alpha_n z \left[ \sum_{n=1}^{\infty} \frac{e_c \theta_n - e_s \zeta_n}{(\theta_n \eta_n - \zeta_n \kappa_n)_n} \right]
- \left( r_3^2 \cosh \frac{r_3 \alpha_a b_f}{2} - r_4^2 \eta_n \cosh \frac{r_4 \alpha_a b_f}{2} \right) M \sin \alpha_n z
\]

(A.12d)

(4) The projecting portions of the bottom flange are assumed to be fully effective.
A.2 Orthotropic Concrete Deck and Isotropic Bottom Flange

All the equations in Section A1 which are pertinent to the orthotropic deck remain applicable.

For the isotropic bottom flange, the differential equation of Eq. 2.21 and the stress function expression of Eq. 2.27 are applicable. With \( (a_x)_r = (a_z)_r = 0 \), the strains of Eq. 2.16 reduce to

\[
\varepsilon_z = \frac{1}{E_s} \cdot \frac{1}{t_f} (N_z - \nu_s N_x) \quad (A.13a)
\]

\[
\varepsilon_x = \frac{1}{E_s} \cdot \frac{1}{t_f} (N_x - \nu_s N_z) \quad (A.13b)
\]

\[
\gamma_{zx} = \frac{N_{zx}}{G_{sf} t_f} \quad (A.13c)
\]

By the symmetry of stress pattern it is obtained

\[
F_n = G_n = 0 \quad (A.14)
\]

From the assumed boundary condition of Eq. 2.31c that the transverse stress is zero at the bottom flange-to-web junctions, it is deduced

\[
H_n = - \eta_n E_n \quad (A.15a)
\]

where

\[
\eta_n = \frac{\cosh \frac{\alpha_n b_f}{2}}{\sinh \frac{\alpha_n b_f}{2}} \quad (A.15b)
\]
The small projecting lips of the bottom flange are again assumed fully effective. The total longitudinal resultant force in the bottom flange is given by

\[ Z_s = 2 \sum_{n=1}^{\infty} \left( \mu_n E_n + \omega_n H_n \right) \sin \alpha_n z \]  

(A.16a)

where

\[ \mu_n = \frac{1}{\alpha_n^2} \sinh \frac{\alpha_n b_f}{2} + \frac{w_f - b_f}{2} \cosh \frac{\alpha_n b_f}{2} \]  

(A.16b)

\[ \omega_n = \left[ \frac{1}{\alpha_n^2} + \frac{b_f(w_f - b_f)}{4} \right] \sinh \frac{\alpha_n b_f}{2} \]

\[ + \frac{1}{\alpha_n^2} \left( w_f - \frac{b_f}{2} \right) \cosh \frac{\alpha_n b_f}{2} \]  

(A.16c)

The compatibility of longitudinal strains at the junction lines of bottom flange and webs leads to the following:

\[ \frac{1}{E_s} \left( \frac{M}{I} - \frac{Z_c + Z_s}{2A} \right) = \frac{1}{E_s t_f} (N_z - \nu_s N_x) = b_f/2 \]  

(A.17)

where the right hand side is from Eq. A.13a. After substitution of the terms into Eq. A.17 and its companion Eq. A.9a, the coefficients \( A_n \) and \( E_n \) can be solved. The expressions for \( A_n \) and \( E_n \) are exactly the same as given by Eqs. A.10 except with the values of \( \kappa_n \) and \( \eta_n \) represented by

\[ \kappa_n' = \left( \frac{E_s}{I} \right)^2 + \frac{1}{E_s A} (\mu_n - \omega_n \eta_n) + \frac{1}{E_s t_f} \left[ \cosh \frac{\alpha_n b_f}{2} \right] \]

\[ - \eta_n \left( \frac{2}{\alpha_n} \cosh \frac{\alpha_n b_f}{2} + \frac{b_f}{2} \sinh \frac{\alpha_n b_f}{2} \right) \]  

(A.18a)
In all of these equations for orthotropic concrete deck and isotropic bottom flange, the values of $\phi_n$, $\psi_n$ and $\zeta_n$ are from Eqs. A.7e, A.7f and A.3b, and $\mu_n$, $\omega_n$ and $\eta_n$ are from Eqs. A.16b, A.16c and A.15b.

The stresses in the component parts of the box are summarized as follows:

1. For concrete deck between webs:
   Same as given by Eqs. A.11a, A.11b and A.11c.

2. For the projecting portions of the deck:
   Same as given by Eqs. A.11d, A.11e and A.11f.

3. For bottom flange between webs

\[
N_z = \sum_{n=1}^{\infty} \frac{e_c \Theta_n' - e_s \zeta_n'}{2E_s I'(\Theta_n' \eta_n' - \zeta_n' \kappa_n')} \cdot [\cosh \alpha_n x - \eta_n (\frac{2}{\alpha_n} \cosh \alpha_n x + x \sinh \alpha_n x)] 
\]

\[
M_n \sin \alpha_n z \tag{A.19a}
\]

\[
N_x = -\sum_{n=1}^{\infty} \frac{e_c \Theta_n' - e_s \zeta_n'}{2E_s I'(\Theta_n' \eta_n' - \zeta_n' \kappa_n')} \cdot [\cosh \alpha_n x - \eta_n x \sinh \alpha_n x] M_n \sin \alpha_n z \tag{A.19b}
\]
\[ N_{zx} = - \sum_{n=1}^{\infty} \frac{e_c \frac{\Theta_n}{I_n} - e_s \frac{\zeta_n}{I_n}}{2E_s I_n} \left( \frac{\eta_n}{\alpha_n} - \frac{\zeta_n}{\kappa_n} \right) \cdot \alpha_n \]

\[ [\sinh \alpha_n x - \eta_n \left( \frac{1}{\alpha_n} \right) \sinh \alpha_n x + x \cosh \alpha_n x] \cdot M_n \cosh \alpha_n z \] (A.19c)

(4) For the projecting lips of bottom flange:

\[ N_z = \sum_{n=1}^{\infty} \frac{e_c \frac{\Theta_n}{I_n} - e_s \frac{\zeta_n}{I_n}}{2E_s I_n} \left( \frac{\eta_n}{\alpha_n} - \frac{\zeta_n}{\kappa_n} \right) \cdot \alpha_n \]

\[ [\cosh \frac{\alpha_n b_f}{2} - \eta_n \left( \frac{2}{\alpha_n} \right) \cosh \frac{\alpha_n b_f}{2} \]

\[ + \frac{b_f}{2} \sinh \frac{\alpha_n b_f}{2}] \cdot M_n \sin \alpha_n z \] (A.19d)

(Assumed to be uniform from juncture to tip.)

\[ N_x = 0 \quad \text{(Assumed)} \] (A.19e)

\[ N_{zx} = - \sum_{n=1}^{\infty} \frac{e_c \frac{\Theta_n}{I_n} - e_s \frac{\zeta_n}{I_n}}{2E_s I_n} \left( \frac{\eta_n}{\alpha_n} - \frac{\zeta_n}{\kappa_n} \right) \cdot \alpha_n \]

\[ [\cosh \frac{\alpha_n b_f}{2} - \eta_n \left( \frac{2}{\alpha_n} \right) \cosh \frac{\alpha_n b_f}{2} \]

\[ + \frac{b_f}{2} \sinh \frac{\alpha_n b_f}{2}] \cdot \left( \frac{w_f}{2} - x \right) M_n \cos \alpha_n z \] (A.19f)

(5) For a web:

The moment, shear and axial force taken by a web and a small steel top flange are the same as given by Eqs. A.11m, A.11n and A.11o. The normal stress, \( \sigma_z \), is given by Eq. A.11p.

The shearing stress is

-135-
\[
\sigma_x = - \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I (\theta_n' \eta_n - \zeta_n \kappa_n')} [p_n',
\]
\[
- \zeta_n q_n' \cosh r_1 \alpha_n (x - \frac{b_c}{2})
\]
\[
- (\xi_n + \zeta_n \rho_n) \sinh r_1 \alpha_n (x - \frac{b_c}{2})
\]
\[
+ (s_n' - \zeta_n t_n') \cosh r_2 \alpha_n (x - \frac{b_c}{2})
\]
\[
- (\theta_n + \zeta_n \kappa_n) \sinh r_2 \alpha_n (x - \frac{b_c}{2}) M_n \sin \alpha_n z
\]

(A.11e)

\[
\tau_{zx} = - \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I (\theta_n' \eta_n - \zeta_n \kappa_n')} [p_n',
\]
\[
- \zeta_n q_n' \sinh r_1 \alpha_n (x - \frac{b_c}{2})
\]
\[
- (\xi_n + \zeta_n \rho_n) \cosh r_1 \alpha_n (x - \frac{b_c}{2})
\]
\[
+ (s_n' - \zeta_n t_n') \sinh r_2 \alpha_n (x - \frac{b_c}{2})
\]
\[
- (\theta_n + \zeta_n \kappa_n) \cosh r_2 \alpha_n (x - \frac{b_c}{2}) M_n \cos \alpha_n z
\]

(A.11f)
(3) For the bottom flange between webs

\[
N_z = \sum_{n=1}^{\infty} \frac{e_c \theta_n' - e_s \zeta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left( r_3^2 \cosh r_3 \alpha_n x - \eta_n r_4^2 \cosh r_4 \alpha_n x \right) M_n \sin \alpha_n z \quad (A.11g)
\]

\[
N_x = -\sum_{n=1}^{\infty} \frac{e_c \theta_n' - e_s \zeta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left( \cosh r_3 \alpha_n x - \eta_n \cosh r_4 \alpha_n x \right) M_n \sin \alpha_n z \quad (A.11h)
\]

\[
N_{zx} = -\sum_{n=1}^{\infty} \frac{e_c \theta_n' - e_s \zeta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left( r_3 \sinh r_3 \alpha_n x - \eta_n r_4 \sinh r_4 \alpha_n x \right) M_n \cos \alpha_n z \quad (A.11i)
\]

(4) For the projecting portion of the bottom flange

\[
N_z = \sum_{n=1}^{\infty} \frac{e_c \theta_n' - e_s \zeta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left( r_3^2 \cosh r_3 \alpha_n b_f \right) M_n \sin \alpha_n z
\]

\[
\quad - \eta_n r_4^2 \cosh \frac{r_4 \alpha_n b_f}{2} M_n \sin \alpha_n z \quad \text{(Assumed to be uniform from junction to tip)} \quad (A.11j)
\]

\[
N_x = 0 \quad \text{(Assumed)} \quad (A.11k)
\]
\[ N_{zx} = -\sum_{n=1}^{\infty} \frac{e_c \theta_n' - e_s \zeta_n'}{2E_s I \left( \theta_n \eta_n - \zeta_n \kappa_n \right)} \alpha_n \left( r_3 \cosh \frac{r_3 \alpha_n b_f}{2} \right) \]
\[ - \eta_n r_4^2 \cosh \left( \frac{r_4 \alpha_n b_f}{2} \right) \left( -\frac{w_f}{2} - x \right) M_n \cos \alpha_n z \]
\[ = \frac{1}{2} \sum_{n=1}^{\infty} \left[ 1 - \frac{e_c \left( \mu_n - w_n \eta_n \right) (e_c \theta_n' - e_s \zeta_n')}{E_s I \left( \theta_n \eta_n - \zeta_n \kappa_n \right)} \right] t_c e_c (\varphi_n - \psi_n \zeta_n) (e_c \kappa_n' - e_s \eta_n' ) \]
\[ + \frac{t_c e_c (\varphi_n - \psi_n \zeta_n) (e_c \kappa_n' - e_s \eta_n' )}{E_s I \left( \theta_n \eta_n - \zeta_n \kappa_n \right)} \alpha_n M_n \cos \alpha_n z \]
\[ = \frac{-1}{2E_s I} \sum_{n=1}^{\infty} \left[ \frac{t_c \left( \varphi_n - \psi_n \zeta_n \right) (e_c \kappa_n' - e_s \eta_n')}{\theta_n \eta_n - \zeta_n \kappa_n} \right] \eta_n \sin \alpha_n z \]
\[ + \frac{\left( \mu_n - w_n \eta_n \right) (e_c \theta_n' - e_s \zeta_n')}{\theta_n \eta_n - \zeta_n \kappa_n} \right] M_n \cos \alpha_n z \]
\[ \sigma_z = \frac{N_z'}{A} + \frac{M_y'}{I} \]

(A.11d)

For a web (plus the small steel top flange)

(A.11m)

(A.11n)

(A.11o)

(A.11p)
in which \( y' \) is the vertical distance from the centroid of the web (tee) to the fiber considered, being positive if downward.

\[
\tau_{zy} = \frac{1}{2EsI_t} \sum_{n=1}^{\infty} \left( \frac{e_c}{\Theta_n} \right)' \left( \frac{e_s}{\zeta_n} \right)' \left[ \frac{r_3}{2} \sinh \frac{r_3}{2} \right] \frac{\alpha_n b_f}{r_3} \sinh \frac{r_3}{2} \frac{\alpha_n b_f}{r_3} \\
- \eta_n r_4 \sinh \frac{r_4}{2} \frac{\alpha_n b_f}{r_4} - \frac{w_f}{2} b_f \alpha_n (r_4 \cos \frac{r_4}{2} \cosh \frac{r_4}{2} \alpha_n b_f) \\
- \eta_n r_4 \cosh \frac{r_4}{2} \frac{\alpha_n b_f}{r_4} \right] M_n \cos \frac{\alpha_n z}{r_3} + \frac{V'}{I_t} \\
+ \frac{\langle \frac{dN}{dz} \rangle}{\frac{A_1}{A_t}} \\
(A.11q)
\]

where \( Q' \) is the statical moment of area \( (A_1) \) about the horizontal centroidal axis of the web (tee) \( (x' - \text{axis in Fig. 2.8}) \), \( A_1 \) is taken from the section in consideration to the bottom outermost fiber of the web (tee) and \( \frac{dN}{dz} \) is the derivative of \( N' \) with respect to \( z \). The individual equivalent widths of the flange portions are computed below.

(1) For concrete deck between webs:

By definition, the equivalent width, \( \lambda_1 \) (Fig. 2.9) can be found by

\[
t_c \cdot \frac{\lambda_1}{2} \cdot (\sigma_z) = \frac{b_c}{2} = \int \sigma_z t_c dx \\
(A.12a)
\]
By substituting Eq. A.11a into the above, integrating and nondimensionalizing with $b_c$, it is obtained

$$\frac{\lambda_1}{b_c} = \frac{2\ell}{\pi b_c} \left[ \sum_{n=1}^{\infty} \frac{e_c \kappa_n}{(\theta_n \eta_n - \zeta_n \kappa_n)n} \right] \left( r_1 \sinh \frac{r_1 \alpha_n b_c}{2} \right) - \zeta_n r_2 \sinh \frac{r_2 \alpha_n b_c}{2} \right) M_n \sin \alpha_n z \right]$$

$$\left( r_2 \cosh \frac{r_1 \alpha_n b_c}{2} - \zeta_n r_2 \cosh \frac{r_2 \alpha_n b_c}{2} \right) M_n \sin \alpha_n z \right]$$

For the projecting portion of the deck:

$$\frac{\lambda_2}{w_c - b_c} = \frac{2\ell}{\pi(w_c - b_c)} \left[ \sum_{n=1}^{\infty} \frac{e_c \kappa_n}{(\theta_n \eta_n - \zeta_n \kappa_n)n} \right] \left\{ (r_1 \phi_n - \zeta_n r_1 \phi_n) r_1 \sinh \frac{r_1 \alpha_n b_c}{2} - \zeta_n (r_1 \phi_n - \zeta_n r_1 \phi_n) \right\}$$

$$\left\{ \sum_{n=1}^{\infty} \frac{e_c \kappa_n}{(\theta_n \eta_n - \zeta_n \kappa_n)n} \right\} \left[ (s_n - \zeta_n q_n) r_2^2 + (s_n - \zeta_n q_n) r_2^2 \right] M_n \sin \alpha_n z \right\}$$

(A.12c)
\[
\tau_{zy} = \frac{1}{2E_s} \int \frac{t}{w} \sum_{n=1}^{\infty} \frac{e_{c} \theta_n - e_{s} \zeta_n}{\theta_n \eta_n - \zeta_n \kappa_n} \left[ \sinh \frac{\alpha_n b_f}{2} \right]
\]

\[
- \eta_n \left( \frac{1}{\alpha_n} \sinh \frac{\alpha_n b_f}{2} + \frac{b_f}{2} \cosh \frac{\alpha_n b_f}{2} \right)
\]

\[
- \frac{w_f}{2} \alpha_n \left[ \cosh \frac{\alpha_n b_f}{2} - \eta_n \left( \frac{2}{\alpha_n} \cosh \frac{\alpha_n b_f}{2} \right) \right]
\]

\[
+ \frac{b_f}{2} \sinh \left( \frac{\alpha_n b_f}{2} \right) \right] M_n \cos \alpha_n z + \frac{V}{1} t_w + \frac{\left( \frac{dn}{dz} \right) A_1}{A_1 w}
\]

(A.19g)

where \(Q'\) and \(A_1\) are determined by the same procedure as that for Eq. A.11q.

The equivalent widths are the following:

(1) For concrete deck between webs:

Same as expressed by Eq. A.12b.

(a) For the projecting portion of the deck:

Same as expressed by Eq. A.12c.

(3) For bottom flange between webs:

\[
\lambda_3 \frac{b_f}{b_f} = \frac{2E_s}{n b_f} \left\{ \right. \sum_{n=1}^{\infty} \frac{e_{c} \theta_n - e_{s} \zeta_n}{\theta_n \eta_n - \zeta_n \kappa_n} \left[ \sinh \frac{\alpha_n b_f}{2} \right]
\]

\[
- \eta_n \left( \frac{1}{\alpha_n} \sinh \frac{\alpha_n b_f}{2} + \frac{b_f}{2} \cosh \frac{\alpha_n b_f}{2} \right) M_n \sin \alpha_n z \right) \left. \right/ \left( \right. \sum_{n=1}^{\infty} \frac{e_{c} \theta_n - e_{s} \zeta_n}{\theta_n \eta_n - \zeta_n \kappa_n} \left[ \cosh \frac{\alpha_n b_f}{2} - \eta_n \left( \frac{2}{\alpha_n} \cosh \frac{\alpha_n b_f}{2} \right) \right]
\]

\[
+ \frac{b_f}{2} \sinh \left( \frac{\alpha_n b_f}{2} \right) M_n \sin \alpha_n z \right) \}
\]

(A.20)
(4) The projecting portions of the bottom flange are assumed to be fully effective.

A.3 Isotropic Concrete Deck and Bottom Flange

Equations of Section A.2 which are derived for an isotropic bottom flange are applicable. These are Eqs. A.14, A.15, A.16 and A.18. For an isotropic deck

\[
\frac{E_z}{E_x} = \frac{E_x}{E_z} \quad (A.21a)
\]

and

\[
\nu_{zx} = \nu_{xz} \quad (A.21b)
\]

The governing differential equation is Eq. 2.21 and the stress function expressions are Eqs. 2.24 and 2.25. All the procedures in determining the stress function coefficients are the same as those employed previously. The results are listed in the following:

\[
\zeta_n = \frac{\sinh \frac{\alpha_n b c}{2}}{\frac{bc}{2} \cosh \frac{\alpha_n b c}{2} - \frac{1 - \nu_{xz}}{1 + \nu_{xz}} \cdot \frac{1}{\alpha_n} \sinh \frac{\alpha_n b c}{2}} \quad (A.22)
\]

\[
p_n' = \frac{\cosh \frac{\alpha_n b c}{2}}{1 + \frac{2}{\alpha_n (1 + \nu_{xz})}} \cdot \frac{\alpha_n (1 + \nu_{zx}) s_n - (3 + \nu_{zx} - \frac{E_z}{G_{zx}}) p_n}{\alpha_n (1 + \nu_{zx} - \frac{E_z}{G_{zx}}) t_n - (3 + \nu_{zx} - \frac{E_z}{G_{zx}}) q_n} \quad (A.23a)
\]
\[ q_n' = \frac{2}{\alpha_n (1 + \nu_{zx})} \cosh \frac{\alpha_n b c}{2} + \frac{b c}{2} \sinh \frac{\alpha_n b c}{2} \]

\[ 1 + \frac{2}{\alpha_n (1 + \nu_{zx})} \cdot \frac{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} s_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} p_n}{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} t_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} q_n} \]

(A.23b)

\[ s_n' = \frac{\alpha_n (1 + \nu_{zx})}{2} \cosh \frac{\alpha_n b c}{2} \]

\[ 1 + \frac{2}{\alpha_n (1 + \nu_{zx})} \cdot \frac{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} t_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} q_n}{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} s_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} p_n} \]

(A.23c)

\[ t_n' = \frac{\cosh \frac{\alpha_n b c}{2} + \frac{\alpha_n (1 + \nu_{zx})}{2} \cdot \frac{b c}{2} \sinh \frac{\alpha_n b c}{2}}{1 + \frac{2}{\alpha_n (1 + \nu_{zx})} \cdot \frac{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} t_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} q_n}{\alpha_n (1 + \nu_{zx}) - \frac{E_z}{G_{zx}} s_n - (3 + \nu_{zx}) - \frac{E_z}{G_{zx}} p_n}} \]

(A.23d)

\[ p_n = \frac{1}{\text{Denominator}} \]

(A.23e)

\[ q_n = \frac{1}{\alpha_n} \cosh^2 \frac{\alpha_n (w_c - b_c)}{2} \]

(A.23f)

\[ s_n = \frac{1}{\alpha_n} \cosh^2 \frac{\alpha_n (w_c - b_c)}{2} \]

(A.23g)
\[ t_n = \frac{w - b}{2} \quad (\text{Denominator}) \]

\[ \text{Denominator} = -\frac{w - b}{\alpha_n} \sinh \frac{\alpha_n(w - b)}{2} \cosh \frac{\alpha_n(w - b)}{2} \]

\[ \hat{g}_n = t_n s_n' - s_n p_n' \quad (A.23j) \]

\[ \rho_n = s_n q_n' - t_n t_n' \quad (A.23k) \]

\[ \theta_n = p_n p_n' - q_n s_n' \quad (A.23l) \]

\[ \kappa_n = q_n t_n' - p_n q_n' \quad (A.23m) \]

\[ \tilde{\psi}_n = \frac{\alpha_n w c}{2} + \frac{\alpha_n b c}{2} + \frac{\alpha_n(w - b)}{2} \sinh \frac{\alpha_n(w - b)}{2} \cosh \frac{\alpha_n(w - b)}{2} \]

\[ + \rho_n \left( \frac{\alpha_n(w - b)}{2} - 1 \right) + t_n \left[ \frac{1}{\alpha_n} \sinh \frac{\alpha_n(w - b)}{2} \right] \]

\[ + \frac{w - b}{\alpha_n} \sinh \frac{\alpha_n(w - b)}{2} - 1 \right] + \kappa_n \left[ \frac{1}{\alpha_n} \sinh \frac{\alpha_n(w - b)}{2} \right] \]

\[ + \frac{w - b}{\alpha_n} \sinh \frac{\alpha_n(w - b)}{2} \cosh \frac{\alpha_n(w - b)}{2} \left( \text{A.24b} \right) \]
The expressions for \( \eta_n \), \( \mu_n \), and \( \omega_n \) are identical to Eqs. A.15b, A.16b, and A.16c, respectively.

\[
\theta_n' = \left( \frac{c}{E_s I'} - \frac{1}{E_s A'} \right) t_c (\theta_n - \zeta_n \eta_n) \tag{A.25a}
\]

\[
\zeta_n' = \left( \frac{2}{E_s I'} + \frac{1}{E_s A'} \right) t_c (\theta_n - \zeta_n \eta_n) + \frac{1 + \nu_{zz}}{E_z} \cosh \frac{\alpha_n b c}{2}
- \zeta_n \frac{1}{E_z \alpha_n} \left[ \cosh \frac{\alpha_n b c}{2} + \frac{b c}{2} \left( 1 + \nu_{zz} \right) \sinh \frac{\alpha_n b c}{2} \right] \tag{A.25b}
\]

The expressions for \( \kappa_n' \) and \( \eta_n' \) are the same as Eqs. A.18a and A.18b.

The following is a summary of expressions for stresses:

1. For the deck between webs:

\[
\sigma_z = \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left[ \cosh \alpha_n x
- \zeta_n \frac{1}{\alpha_n} \left( \cosh \alpha_n x + x \sinh \alpha_n x \right) \right] M_n \sin \alpha_n z \tag{A.26a}
\]

\[
\sigma_x = \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left( \cosh \alpha_n x
- \zeta_n x \sinh \alpha_n x \right) M_n \sin \alpha_n z \tag{A.26b}
\]

\[
\tau_{zx} = \sum_{n=1}^{\infty} \frac{e_c \kappa_n' - e_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left[ \sinh \alpha_n
- \zeta_n \frac{1}{\alpha_n} \left( \sinh \alpha_n x + x \cosh \alpha_n x \right) \right] M_n \cos \alpha_n z \tag{A.26c}
\]
(2) For the portions of the deck outside webs:

\[
\sigma_z = \sum_{n=1}^{\infty} \frac{\varepsilon_c \xi_n' - \varepsilon_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left[ (p_n' - \zeta_n q_n') \cosh \alpha_n (x - \frac{b}{2}) 
\right.
\]

\[
- (\xi_n + \zeta_n \rho_n) \sinh \alpha_n (x - \frac{b}{2}) - (\theta_n + \zeta_n \kappa_n) \left[ \frac{2}{\alpha_n} \sinh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
+ (x - \frac{b}{2}) \cosh \alpha_n (x - \frac{b}{2})] + (s_n' - \zeta_n t_n') \left[ \frac{2}{\alpha_n} \cosh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
+ (x - \frac{b}{2}) \sinh \alpha_n (x - \frac{b}{2}) \right] M_n \sin\alpha_n z
\]  
(A.26d)

\[
\sigma_x = - \sum_{n=1}^{\infty} \frac{\varepsilon_c \xi_n' - \varepsilon_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left[ (p_n' - \zeta_n q_n') \cosh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
- (\xi_n + \zeta_n \rho_n) \sinh \alpha_n (x - \frac{b}{2}) - (\theta_n + \zeta_n \kappa_n) \left( x - \frac{b}{2} \right) \cosh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
[ (s_n' - \zeta_n t_n') (x - \frac{b}{2}) \sinh \alpha_n (x - \frac{b}{2}) \right] M_n \sin\alpha_n z \]  
(A.26e)

\[
\tau_{zx} = - \sum_{n=1}^{\infty} \frac{\varepsilon_c \xi_n' - \varepsilon_s \eta_n'}{2E_s I'(\theta_n' \eta_n' - \zeta_n' \kappa_n')} \left[ (p_n' - \zeta_n q_n') \sinh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
- (\xi_n + \zeta_n \rho_n) \cosh \alpha_n (x - \frac{b}{2}) - (\theta_n + \zeta_n \kappa_n) \left[ \frac{1}{\alpha_n} \cosh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
+ (x - \frac{b}{2}) \sinh \alpha_n (x - \frac{b}{2}) \right]
\]

\[
+ (s_n' - \zeta_n t_n') \left[ \frac{1}{\alpha_n} \sinh \alpha_n (x - \frac{b}{2}) \right.
\]

\[
+ (x - \frac{b}{2}) \cosh \alpha_n (x - \frac{b}{2}) \right] M_n \cos\alpha_n z
\]  
(A.26f)
(3) For the bottom flange between webs:
Same as given by Eqs. A.19a, A.19b and A.19c.

(4) For the projecting lips of bottom flange:
Same as given by Eqs. A.19d, A.19e and A.19f.

(5) For the webs:
The expressions for moment, shear and axial force taken by a web and a small steel top flange are the same as Eqs. A.11m, A.11n and A.11o, respectively.

The normal stress is computed from Eq. A.11p and the shearing stress from Eq. A.19g.

The equivalent widths are summarized as follows:

(1) For the deck between webs:

\[
\frac{\lambda}{b_c} = \frac{2A}{\pi b_c} \left\{ \sum_{n=1}^{\infty} \left[ \frac{e_c \kappa_n' - e_s \eta_n'}{\eta_n' - \zeta_n \kappa_n'} \right] \left[ \sinh \frac{\alpha_n b_c}{2} - \zeta_n \left( \frac{b_c}{2} \cosh \frac{\alpha_n b_c}{2} + \frac{1}{\alpha_n} \sinh \frac{\alpha_n b_c}{2} \right) M_n \sin \alpha_n z \right] \right\}
\]

\[
\left\{ \sum_{n=1}^{\infty} \left[ \frac{e_c \kappa_n' - e_s \eta_n'}{\eta_n' - \zeta_n \kappa_n'} \right] \left[ \cosh \frac{\alpha_n b_c}{2} - \zeta_n \frac{\alpha_n b_c}{2} \cosh \frac{\alpha_n b_c}{2} \right] \right\}
\]

\[
+ \frac{b_c}{2} \sinh \frac{\alpha_n b_c}{2} \right\} M_n \sin \alpha_n z
\]
(2) For the projecting portion of the deck:

\[
\frac{\lambda_2}{v_c - b_c} = \frac{2L}{n(w_c - b_c)} \sum_{n=1}^{\infty} \frac{e_c \kappa_n}{\theta_n \eta_n - \zeta_n \kappa_n} \left\{ \alpha_n \varphi_n - \sinh \frac{\alpha_n b_c}{2} \right\} M_n \sin \alpha_n z/
\]

\[
- \zeta_n \left[ \alpha_n \psi_n - \left( \frac{1}{\alpha_n} \sinh \frac{\alpha_n b_c}{2} + \frac{b_c}{2} \cosh \frac{\alpha_n b_c}{2} \right) \right] M_n \sin \alpha_n z/
\]

\[
\left\{ \sum_{n=1}^{\infty} \frac{e_c \kappa_n}{\theta_n \eta_n - \zeta_n \kappa_n} \left[ (\rho'_n - \zeta_n q'_n) \right] + (s'_n - \zeta_n t'_n) \left( \frac{2}{\alpha_n} \right) \right\} M_n \sin \alpha_n z
\]

(A.27b)

(3) For the bottom flange between webs:

Same as given by Eq. A.20.

(4) The projecting portions of the bottom flange are assumed to be fully effective.
APPENDIX B

ROTATIONS AND DERIVATIVES OF WARPING FUNCTION

B.1 Simple Beam with Concentrated Torsional Moment (Fig. 3.5a)

\[ 0 \leq Z \leq \alpha L \]

\[ \phi = \frac{TL}{GrKT} \left[ \frac{1}{\lambda L} \left( \sinh \alpha \lambda L \cth \lambda L - \cosh \alpha \lambda L \right) \sinh \lambda Z \
+ (1 - \alpha) \frac{Z}{L} \right] \]  

\[ (B.1a) \]

\[ \phi'' = \frac{T}{GrKT} \lambda \left( \sinh \alpha \lambda L \cth \lambda L - \cosh \alpha \lambda L \right) \sinh \lambda Z \]  

\[ (B.1b) \]

Section torque = \((1 - \alpha) T\)  

\[ \text{(B.1c)} \]

\[ \alpha L \leq Z \leq L \]

\[ \phi = \frac{TL}{GrKT} \left[ \frac{1}{\lambda L} \sinh \alpha \lambda L \left( \cth \lambda L \sinh \lambda Z - \cosh \lambda Z \right) \
+ \alpha (1 - \frac{Z}{L}) \right] \]  

\[ (B.1d) \]

\[ \phi'' = \frac{T}{GrKT} \lambda \sinh \alpha \lambda L \left( \cth \lambda L \sinh \lambda Z - \cosh \lambda Z \right) \]  

\[ \text{(B.1e)} \]

Section torque = \(- \alpha T\)  

\[ \text{(B.1f)} \]
B.2 Concentrated Torsional Moment at Overhanging End of Simple Beam (Fig. 3.5b)

0 ≤ Z ≤ L

\[ \theta = \frac{\mu T L}{G_r K_T} \cdot \frac{1}{D} \left[ (\sinh \lambda L \cosh \lambda L_1 - \cosh \lambda L) \sinh \lambda Z + \sinh \lambda L (\cosh \lambda L - \sinh \lambda L \cosh \lambda L_1) \frac{Z}{L} \right] \] (B.2a)

\[ f'' = \frac{TL}{G_r K_T} \lambda^2 \frac{\sinh \lambda L \cosh \lambda L_1 - \cosh \lambda L}{D} \sinh \lambda Z \] (B.2b)

Section torque =

\[ \frac{-\mu \sinh \lambda L (\sinh \lambda L \cosh \lambda L_1 - \cosh \lambda L)}{D} \cdot T \] (B.2c)

L ≤ Z ≤ L₁

\[ \theta = \frac{TL}{G_r K_T} \cdot \frac{1}{D} \left[ \lambda L - \mu \sinh \lambda L \left( \cosh \lambda Z - \cosh \lambda L_1 \sinh \lambda Z \right) \right] + \frac{TZ}{G_r K_T} \] (B.2d)

\[ f'' = \frac{TL}{G_r K_T} \lambda^2 \cdot \frac{\sinh \lambda L}{D} \left( \cosh \lambda L_1 \sinh \lambda Z - \cosh \lambda Z \right) \] (B.2e)

Section torque = T \hspace{1cm} (B.2f)

where

\[ D = -\lambda L + \mu \sinh \lambda L \left( \cosh \lambda L - \sinh \lambda L \cosh \lambda L_1 \right) \] (B.2g)
B.3 Simple Beam with Uniformly Distributed Torsional Moment in Part of Span (Fig. 3.5c)

\[ \phi \leq Z \leq \alpha L \]

\[ \phi = \frac{\mu m_t}{\lambda^2 G_r K_T} \{ \cosh \lambda Z - [\sinh \alpha \lambda L + (1 - \cosh \alpha \lambda L) \sinh \lambda L] \sinh \lambda Z - 1 \} \]

\[ + \frac{\alpha L m_t}{2 G_r K_T} (2 - \alpha - \frac{Z}{\alpha L}) Z \]

(B.3a)

\[ \phi'' = \frac{m_t}{G_r K_T} \{ \cosh \lambda Z - [\sinh \alpha \lambda L + (1 - \cosh \alpha \lambda L) \sinh \lambda L] \sinh \lambda Z - 1 \} \]

(B.3b)

Section torque = \( (1 - \frac{\alpha}{2}) \alpha L m_t - m_t Z \) \( (B.3c) \)

\[ \alpha L \leq Z \leq L \]

\[ \phi = \frac{\mu m_t}{\lambda^2 G_r K_T} (1 - \cosh \alpha \lambda L)( \cosh \lambda Z - \sinh \lambda Z) - \sinh \lambda Z + \frac{\alpha^2 L^2}{2 G_r K_T} (1 - \frac{Z}{L}) \cdot m_t \]

(B.3d)

\[ \phi'' = \frac{m_t}{G_r K_T} (1 - \cosh \alpha \lambda L)( \cosh \lambda Z - \sinh \lambda L \sinh \lambda Z) \]

(B.3e)

Section torque = \( - \frac{\alpha^2 L}{2} \cdot m_t \) \( (B.3f) \)

In the case where \( m_t \) extends over the whole span, Eqs. B.3a through B.3c can be used by setting \( \alpha \) equal to 1.
B.4 Uniformly Distributed Torsional Moment Throughout Overhanging Portion of Simple Beam (Fig. 3.5d)

0 \leq Z \leq L

\phi = \frac{\mu m_t}{D} \cdot \frac{\lambda (L_1 - L) \sinh \lambda (L_1 - L) - \cosh \lambda (L_1 - L) + 1}{L \sinh \lambda Z - Z \sinh \lambda L}.

(B.4a)

f'' = \frac{m_t}{G r K_T} \cdot \frac{\lambda L}{D} \cdot \left[ \lambda (L_1 - L) \cdot \sinh \lambda (L_1 - L) - \cosh \lambda (L_1 - L) + 1 \right] \sinh \lambda Z

(B.4b)

Section torque = - \mu m_t \cdot \frac{\sinh \lambda L}{\lambda D} \cdot \left[ \lambda (L_1 - L) \sinh \lambda (L_1 - L) - \cosh \lambda (L_1 - L) + 1 \right]

(B.4c)

L \leq Z \leq L_1

\phi = \frac{\mu m_t}{\lambda^2 G r K_T} \cdot \frac{1}{D} \cdot \left[ \lambda L (L_1 - L) - \mu \sinh \lambda L \left[ \sinh \lambda (L_1 - Z) - \sinh \lambda (L_1 - L) \right] \right.

+ \lambda L \cosh \lambda L \left[ \sinh \lambda (L_1 - Z) - \sinh \lambda (L_1 - L) \right] \right.

+ \mu \sinh \lambda L \sinh \lambda (L - Z) + \lambda L (\sinh \lambda Z - \sinh \lambda L) \left] \right.

+ \frac{m_t}{G r K_T} \left( Z - L \right) \left( L_1 - \frac{Z + L}{2} \right)

(B.4d)
\[ f'' = \frac{m_t}{G_rK_T} \cdot \frac{1}{D} \left[ \lambda^2 L (L_1 - L) - \mu \right] \sinh \lambda L \sinh \lambda (L_1 - Z) \]

\[ + \lambda L \cosh \lambda L \sinh \lambda (L_1 - Z) + \mu \sinh \lambda L \sinh \lambda (L - Z) \]

\[ + \lambda L \sinh \lambda Z \right] - \frac{m_t}{G_rK_T} \]

\[ \text{Section torque} = (L_1 - Z) m_t \]

\[ \text{where} \]

\[ D = \sinh \lambda L_1 \left[ \lambda L + \mu \sinh \lambda L (\sinh \lambda L \cosh \lambda L_1 - \cosh \lambda L) \right] \]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Area</td>
</tr>
<tr>
<td>$A_f$</td>
<td>Ratio of reinforcing fiber area to cross-sectional area of composite</td>
</tr>
<tr>
<td>$A_m$</td>
<td>Ratio of matrix area to cross-sectional area of composite</td>
</tr>
<tr>
<td>$A_N$ to $H_N$</td>
<td>Coefficients in expression for $X_n$</td>
</tr>
<tr>
<td>$A_o$</td>
<td>Enclosed area of box section</td>
</tr>
<tr>
<td>$A^*$</td>
<td>Transformed area of cross section</td>
</tr>
<tr>
<td>$a, b, c$</td>
<td>Coefficients of differential equation of stress function for concrete deck</td>
</tr>
<tr>
<td>$(a_x)_r$, $(a_z)_r$</td>
<td>Cross-sectional area of bottom flange transverse stiffener(x) and longitudinal stiffener (z)</td>
</tr>
<tr>
<td>$B$</td>
<td>Bimoment (due to warping)</td>
</tr>
<tr>
<td>$b_c$</td>
<td>Concrete deck width between webs of box girder</td>
</tr>
<tr>
<td>$b_f$</td>
<td>Bottom flange width between webs of box girder</td>
</tr>
<tr>
<td>$C_1$ to $C_5$</td>
<td>Coefficients in the equation of angle of twist, $\Theta$, and warping function, $f$</td>
</tr>
<tr>
<td>$d$</td>
<td>Diameter</td>
</tr>
<tr>
<td>$d_f$</td>
<td>Diameter of fiber</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>$E_f$</td>
<td>Young's modulus of reinforcing fiber in composite</td>
</tr>
<tr>
<td>$E_m$</td>
<td>Young's modulus of matrix of composite</td>
</tr>
<tr>
<td>$E_r$</td>
<td>Young's modulus of reference material</td>
</tr>
<tr>
<td>$E_s$</td>
<td>Young's modulus of steel</td>
</tr>
</tbody>
</table>
NOTATIONS (continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_t$</td>
<td>Equivalent modulus of elasticity of reinforced concrete deck in the longitudinal (z) direction</td>
</tr>
<tr>
<td>$F$</td>
<td>Airy's stress function</td>
</tr>
<tr>
<td>$f(z)$</td>
<td>Warping function</td>
</tr>
<tr>
<td>$G$</td>
<td>Shear modulus; G's with subscript correspond to E's</td>
</tr>
<tr>
<td>$h_w$</td>
<td>Height of web</td>
</tr>
<tr>
<td>$I$</td>
<td>Moment of inertia</td>
</tr>
<tr>
<td>$I_c$</td>
<td>Central moment of inertia</td>
</tr>
<tr>
<td>$I_s$</td>
<td>Moment of inertia of flange with respect to neutral axis of box</td>
</tr>
<tr>
<td>$I_{x}$, $I_{y}$, $I_z$</td>
<td>Moment of inertia with respect to x, y and z axis, respectively</td>
</tr>
<tr>
<td>$I_{xy}$</td>
<td>Product moment of inertia</td>
</tr>
<tr>
<td>$I_w$</td>
<td>Moment of inertia of web with respect to neutral axis of box</td>
</tr>
<tr>
<td>$I_\omega$</td>
<td>Warping moment of inertia</td>
</tr>
<tr>
<td>$I_{\omega x}$, $I_{\omega y}$</td>
<td>Warping moment of inertia with respect to x and y axis, respectively</td>
</tr>
<tr>
<td>$K_T$</td>
<td>St. Venant torsional constant</td>
</tr>
<tr>
<td>$L$</td>
<td>Span length</td>
</tr>
<tr>
<td>$M$</td>
<td>Bending moment</td>
</tr>
<tr>
<td>$M_n$</td>
<td>Coefficient (moment) in sine series expression of moment</td>
</tr>
<tr>
<td>$M_T$</td>
<td>Torque</td>
</tr>
<tr>
<td>$M_x$, $M_y$, $M_z$</td>
<td>Bending moment with respect to x, y, z axis, respectively</td>
</tr>
</tbody>
</table>
### NOTATION (continued)

- **m_t**  
  Uniformly distributed torsional moment

- **m_z**  
  Distributed torsional moment

- **N**  
  Forces per unit length of plate

- **N_x, N_z**  
  Axial force in x and z direction, respectively

- **N_{zx}**  
  Shear force in zx plane

- **p**  
  Load

- **p, q, r**  
  Coefficient of differential equation of stress function for bottom flange

- **Q**  
  Static moment of area of cut cross-section

- **\overline{Q}**  
  Adjusted state moment of area of cut cross-section

- **q**  
  Shear flow

- **q_o**  
  Shear flow in cut cross-section

- **q_l**  
  Shear flow to impose compatibility at cut of cross-section

- **q_{sv}**  
  St. Venant shear flow

- **r_n**  
  Coefficient in expression for X_n

- **S**  
  Element of matrix relating stress to strain, with subscripts

- **S_{w}**  
  Warping static moment

- **s**  
  Distance along wall of box section; spacing of stiffeners

- **s_x**  
  Spacing of longitudinal stiffener in x direction of bottom flange

- **s_z**  
  Spacing of transverse stiffener in z direction of bottom flange
NOTATION (continued)

\( t \)  
Thickness

\( t_c \)  
Thickness of concrete deck

\( t_f \)  
Thickness of bottom flange

\( t_i \)  
Thickness of element \( i \)

\( t_{tf} \)  
Thickness of steel top flange

\( u \)  
Displacement along \( x \)-direction

\( u_s \)  
Displacement in the tangential direction of cross-section

\( V \)  
Shear force

\( w \)  
Width; displacement along \( z \)-direction (axial displacement)

\( w_b \)  
Overall width of bottom flange

\( w_c \)  
Overall width of concrete deck

\( w_o \)  
Axial displacement at \( s = o \)

\( w_{tf} \)  
Width of steel top flange

\( X_n \)  
Function of \( x \) in Fourier's series

\( x, y, z \)  
Centroidal principal axes

\( Z_c, Z_s \)  
Total axial force in concrete deck and bottom (steel) flange, respectively

\( \alpha_n \)  
Coefficient in Fourier's series

\( \gamma \)  
Shearing strain

\( \delta \)  
Deflection

\( \varepsilon \)  
Normal strain
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>Nondimensional coefficient in warping function $f(z)$ and angle of twist $\theta(z)$; effective width</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>Effective width of concrete deck between webs</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>Effective width of overhanging portion of concrete deck</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>Effective width of bottom flange between webs</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Warping shear parameter</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>Poisson's ratio of composite in direction 1 due to stresses applied in direction 2</td>
</tr>
<tr>
<td>$\nu_f$</td>
<td>Poisson's ratio of reinforcing fiber of composite</td>
</tr>
<tr>
<td>$\nu_m$</td>
<td>Poisson's ratio of matrix of composite</td>
</tr>
<tr>
<td>$\nu_s$</td>
<td>Poisson's ratio of steel</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Distance from centroid to tangent on cell wall</td>
</tr>
<tr>
<td>$\rho_o$</td>
<td>Distance from shear center to tangent on cell wall</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Normal stress</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>Bending normal stress</td>
</tr>
<tr>
<td>$\sigma_D$</td>
<td>Distortional normal stress</td>
</tr>
<tr>
<td>$\sigma_W$</td>
<td>Warping normal stress</td>
</tr>
<tr>
<td>$\sigma_x, \sigma_y, \sigma_z$</td>
<td>Stress along $x, y, z$ direction, respectively</td>
</tr>
<tr>
<td>$(\sigma_z)_p$</td>
<td>Longitudinal stress in the plate of the bottom flange</td>
</tr>
<tr>
<td>$(\sigma_z)_r$</td>
<td>Longitudinal stress in the reinforcing stiffeners of the bottom flange</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Shear stress</td>
</tr>
<tr>
<td>$\tau_B$</td>
<td>Flexural shearing stress</td>
</tr>
</tbody>
</table>
NOTATION (continued)

\( \tau_D \)  
Distortional shearing stress

\( \tau_W \)  
Warping tortional shearing stress

\( \tau_{sv} \)  
St. Venant shear stress

\( \phi(z) \)  
Angle of twist

\( \phi' \)  
Angle of twist per unit length

\( \bar{\omega} \)  
Unit warping with respect to centroid

\( \bar{\omega}_o \)  
Unit warping function with respect to shear center

\( \bar{\omega}_n \)  
Normalized unit warping with respect to shear center
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