"BUCKLING TESTS ON WF COLUMNS UNDER OBLIQUE LOADING" by F. Campus and C. Massonnet

Translated by Michael D. Grigoriadis
Translation of the paper by F. Campus and C. Massonnet entitled

"Recherches sur le flambement de colonnes en acier A37, a profil en double T, sollicitées obliquement"

by

Michael D. Grigoriadis

This report has been compiled as a Special Problem in Civil Engineering (C.E.406) under the direction of Prof. T. V. Galambos as a partial fulfillment of study towards the degree of Master of Science in Civil Engineering.

Lehigh University
May 1960
205 A 29
ACKNOWLEDGEMENTS

The translator is greatly indebted to Dr. Theodore V. Galambos, who supervised the study and made valuable suggestions.

This problem was prepared in order to furnish an otherwise inaccessible reference to current research at Fritz Engineering Laboratory, Lehigh University. Professor William J. Eney is Director of Fritz Laboratory.

Acknowledgement is also due to Miss Grace E. Mann who typed the problem with great care. Her cooperation is sincerely appreciated.
In 1955 Professors F. Campus and C. Massonnet conducted a large number of column tests (95) under eccentric compressive loads with unequal and opposite eccentricities.

These tests were accompanied by a very comprehensive and quite complete paper which gives an insight to European column testing procedures and analysis. The numerous test results as well as their interpretation given in this paper constitute a useful source of reference for the column research at Fritz Engineering Laboratory, Lehigh University.

The influence of residual stresses and varying material properties on the flexural and flexural-torsional buckling of eccentrically loaded columns is considered. However, for the case of flexural-torsional buckling theory of obliquely \( (e_2 / e_1) \) loaded columns only a procedure is mentioned without going into any numerical verification. Nevertheless, attempts of establishing a simple interaction type formula for the design of such columns are made.
The translator believes he has achieved quite a complete picture of European column testing and a thorough review of the elastic as well as the elastic-plastic buckling theory of eccentrically loaded columns.
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The tests described in this study were performed on A37 mild steel members of regular wide flange (DIE) or I shaped (PNS) cross sections. All columns were loaded with an oblique compressive force whose line of action was in the plane of the web and which had end eccentricities $e_1$ and $e_2$. The tests were performed for the following values of parameters:

$$
\frac{e_2}{e_1} = +1; 0; -1;
$$

slenderness ratios for buckling normal to the plane of the web of 40, 60, 80, 100, 130, 175; and ratios of end eccentricity $e_1$ to the core radius $m = 0.5, -1, -3$.

Prior to the column buckling tests, the material properties of the member were thoroughly investigated. Numerous coupons, taken from various places of the cross section were tested in tension as well as in compression. The residual stress distribution was also determined. According to these preliminary tests there exists an appreciable variation in the yield point of the steel depending on the location of the point under consideration; the residual stresses present in each of the rectangles have parabolic distribution. Furthermore, the flange tips are under the effect of high compressive residual stresses, thus reducing the resistance of the member against flexural buckling.
It was possible to arrive at an average stress-strain diagram for one flange, with the aid of a theory taking into account the distribution of the residual stresses in the flanges. This diagram is in accordance with the experimental diagram obtained from stub column tests. The proportional limit is quite low ($R_p = 13 \text{ kg/mm}^2$) compared with the value obtained from coupon tests, with a quite extended plastic range.

The important feature of these buckling tests is the use of oil pressured spherical supports which simulate perfectly the simple support condition in the plane of the web as well as in a plane perpendicular to the web. This type of end support prevented warping almost entirely.

Most of the columns failed in the following manner: A smaller or larger plastic deformation due to combined bending and axial force in the plane of the web was followed by lateral-torsional buckling.

Only small number of bars, in addition to those with opposite eccentricities, failed by excessive plastic bending at one or both ends followed in the late loading stages by local buckling of the flanges.
For the analysis of the test results it was decided to establish simple design formulas applicable in practice which represent the actual case as closely as possible.

A detailed study of the most rational form of interaction formulas has been performed and it has been shown that the necessary safety with respect to the experimental failure loads is always obtained by calculating the failure load \( P \) using the linear interaction formula:

\[
\frac{P}{P_o} + \frac{M_{\text{equ}}}{M_o (1 - \frac{P}{P_o})} = 1. \tag{1}
\]

In this equation:

- \( P_o \) is the ENGESSER-SHANLEY column buckling load,
- \( M_o \) is the critical bending moment which is equal to
  \[ M_o = \alpha \frac{l}{r}, \]
  where \( \alpha \) is a reduction coefficient dependent to only one geometric parameter \( \frac{l}{r} \).
- \( M_{\text{equ}} \) is an equivalent bending moment, related to the magnitude of the end moments \( M_1 \) and \( M_2 \) by the expression:
  \[ M_{\text{equ}} = \sqrt{0.3(M_1^2 + M_2^2) + 0.4M_1 M_2}; \]

Finally, \( 1 - \frac{P}{P_o} \) is a factor of proportionality of taking into account the increase of the bending moment due to the deflection of the column. This expression \( P_o \) is the critical elastic Euler load for buckling in the loading plane.
The member can also fail by excessive flexural plastic deformation, when the classical condition
\[ \frac{P}{\Omega R_e} + \frac{M}{I R_e} = 1 \] (2)
is fulfilled. Therefore, the correct value of the failure load is always the smaller one of the two given by (1) or (2).

Design formulas may be derived from these two formulas by introducing a safety factor proportional to the slenderness ratio in order to take into account the imperfections present in rolled shapes used in industry. Such formulas have been presented in this report and it was shown that an appreciable economy in steel may be achieved when compared with the present Belgian specifications.

This economy may be improved more by increasing the admittable critical stresses at buckling, modifying the safety factors and slenderness ratios adopted in Belgium. The propositions made are proved to be safe enough when compared with the experimental failure loads, taking into account the effect of the imperfections in industrial members.

Finally, two methods providing an exact analysis of columns under oblique compression have been developed.
Both of these methods take into account the variation of the material properties from point to point of the cross section, and the residual stress distribution, as determined from the preliminary tests. In order to simplify the computations without any effect on the degree of accuracy, I and WF sections have been approximated to consist simply of only two flanges which provide the same area and moments of inertia, thus the same slenderness ratio and centroid with the actual cross-section.

Using Mohr's classical method of the funicular polygon, a graphical method of determining the elastic-plastic deformations of the bar in its plane have been developed. The latter method enables us to compute graphically the maximum load carried by the column prior to its failure by excessive plastic deformation. It is necessary, however, to know the relation of the reduction in flexural rigidity due to the partial plastification of the section, in terms of the parameters $M$ and $P$.

On the other hand, the flexural-torsional buckling theory has been extended into the plastic range. The formulas obtained for the latter case, are of the same type as those of the elastic theory, but the geometric quantities involved
are no longer constants but functions of the plastification of the section caused by the deflection of the column. The above theory has been found to agree closely with the results of the 9 cases compared.

It has been shown that the approximate method recommended by the German Code DIN 4114 for the calculation of the critical flexural-torsional buckling load leads to unconservative answers.

Finally, suggestions have been made for the application of the above theory to columns under oblique loading with ends free to warp. The corresponding computations become quite complicated.
NOTATIONS

\(a, b, c\) Coefficients of a quadratic equation

\(b\) Width of an I section

\(e\) Subscript used to indicate the exterior flange of an I section

\(e_1\) Eccentricities of the line of action of the compressive force \(P\) at the ends of the member

\(e_2\) Eccentricities of the line of action of the compressive force \(P\) at the ends of the member

\(e_{equ}\) Equivalent eccentricity

\(e_y\) Constant eccentricity in the plane \(yz\)

\(e\) Flange thickness

\(f\) Elastic-plastic deformation of the member

\[
i_x = \frac{I_x}{\Omega} \quad \text{radii of gyration of the member's cross section}
\]

\[
i_y = \frac{I_y}{\Omega}
\]

\(K\) Shape factor

\(k\) Reduction coefficient of bending rigidity \(EI\) for partial plastification of the section

\(h\) Distance between the centroids of the flanges of an I section.

\(H\) Total height of the section

\(h\) Distance between the fictitious fibers of the equivalent section

\(i\) Subscript used to indicate the interior flange of an I section

\(l\) Length of the member

\[
m = \frac{e_i}{r_x} = \frac{e_i \Omega}{(\frac{I}{V})_x} \quad \text{Maximum relative eccentricity}
\]
Fictitious torsional couples produced by flexural-torsional buckling

\[ m_z = \frac{e_{\text{equ}}}{c_1} \]

Reduced equivalent eccentricity

\[ q_x, q_y \]

Fictitious lateral forces produced by flexural-torsional buckling.

\[ r_x = \frac{I_x^2}{2I_x} \]

Core radius in the \( xx \) axis.

\[ r_y \]

Core radius in the \( yy \) axis.

\[ r_0 = \sqrt{r_x^2 + r_y^2 + \frac{I_X^2}{I_x} + \frac{I_Y^2}{I_y}} \]

Radius of polar moment of the inertia of the cross section with respect to the shear center \( O \).

\[ s \]

Safety factor

\[ t \]

Shear force parallel to axis \( xx \) in a flange of the \( I \) section

\[ a \]

Displacements of the center of gravity \( G \) of the cross section in the plane \( xz \) and \( yz \) respectively

\[ v = \frac{b}{2} \]

The distance of the extreme fiber from the \( xx \) axis.

\[ x, y, z \]

Cartesian coordinates of a point on the member

\[ y_E \]

Ordinate of the elastic center

\[ x_0, y_0 \]

Coordinates of shear center

\[ A, A_1, A_2, A_3 \]

Numerical constants

\[ B = EI \]

Bending rigidity of the member

\[ C \]

Torsional rigidity of the member

\[ C_1 \]

Warping rigidity of the member

\[ E \]

Longitudinal modulus of elasticity
G, G₀  Transverse modulus of elasticity
Gᵢ  Instantaneous transverse modulus of elasticity of a fiber which has been subjected to plastic deformation.
Iₓ  Moments of inertia of the cross section whose web coincides with the y axis.
Iᵧ  Reduced equivalent eccentricity
K = \frac{e_{equ}}{e₁}  Reduced equivalent eccentricity
M₀  Critical bending moment or failure moment due to excessive plastic deformation
M, Mᵧ  Bending moments
M₁ = Pe₁  End bending moments at the ends of a member under oblique compression
M₂ = Pe₂
M_{equ}  Equivalent bending moment
P₀  Compressive force causing failure of the column by buckling or by excessive plastic deformation
P  Compressive force
\frac{P₀}{\Pi^2 E Iₓ} \frac{E Iₓ}{I²}  The critical Euler load for elastic buckling in the plane yy
P₁  Critical load for flexural buckling in the plane xx
P_{cr}  Critical flexural-torsional buckling load
P₂  Critical torsional buckling load
S (S₁, Sₑ)  Secant modulus
\overline{S} (\overline{S₁}, \overline{Sₑ})  Osgood's average secant modulus
T  Shearing force
Tₕ(T₁, Tₑ) = \frac{dσ}{dε}  Tangent modulus
\overline{T}_{th}(\overline{T₁}, \overline{Tₑ})  Osgood's average tangent modulus
R  Allowable tensile stress
Re  Apparent lower elastic limit in tension
Res  Apparent upper elastic limit in tension
R  Ultimate stress in tension
Re'  Apparent elastic limit in compression
α  Reduction coefficient taking into account the danger of buckling
βx  Constant used in the flexural-torsional buckling theory.
\[ \varepsilon, \varepsilon_1, \varepsilon_e \]  Fiber strains
\[ \varepsilon_r \]  Residual strains
η  Poisson's ratio
\[ \theta = \frac{\varepsilon_2}{\varepsilon_1} \]  Ratio of end eccentricities
\[ \lambda = \frac{l}{r}, \lambda_x = \frac{l}{l_x}, \lambda_y = \frac{l}{l_y} \]  Slenderness ratios
\[ \lambda_i \]  Ideal slenderness ratio
\[ \rho \]  Radius of curvature
\[ \sigma, \sigma_i, \sigma_e \]  Normal stress in a fiber
\[ \sigma_m = \frac{P}{\Omega} \]  Average compressive stress in the member
\[ \sigma_r \]  Residual stress
\[ \sigma_E \]  Compressive stress at the elastic center
\[ \psi = \frac{\pi}{2} \sqrt{\frac{P}{P_0}} \]  A parameter used in the elastic analysis for the bending of an eccentrically loaded column.
\[ \phi \]  Angle of rotation of the cross section around the shear center
\[ \frac{1}{\phi_{fl}} \] Reduction coefficient for buckling (I.B.N.)

\[ \Omega \] Area of the cross section

\[ \Omega_{fl} \] Reduced area for buckling

\[ \omega \] Omega factor of the DIN 4114 specifications
INTRODUCTION

At present, our knowledge about the buckling strength of members subjected to combined axial compression and bending is quite limited. The only case which has been studied theoretically as well as experimentally is that of a column under the action of an eccentric compressive force, which, unfortunately, is a rare case in practice.

Loading by an oblique compression force is more frequent; it is encountered in:

a) all columns of rigid jointed frames under the action of vertical and wind loads,

b) all girders in compression and under the action of end moments due to the rigidity of the joints.

In both cases, more than often the end moments are of opposite sign, and the line of action of the compressive force intersects the axis of the bar.

Our lack of knowledge on the subject is mostly due to the fact that generally tests were carried out for the sake of verifying an already established theory. However, this theoretical analysis is extremely complicated for the case of members subjected to oblique compression, and there has not been a satisfactory solution to it up to date.
The Belgian specifications related to the design of members under such loading are empirical and lead to an erroneous waste of steel; as a matter of fact, it is specified that the maximum stress is bending should be added to the maximum stress in axial buckling in the weak direction. These stresses, however, do not exist at the same place; the bending stress is generally maximum at one of the ends, whereas the buckling stress has its maximum around the mid-point.\(^{(1)}\)

The main objective of the tests is to reduce from the experimental results simple design rules for columns under oblique compression, which would be more rational and economical than those given in the existing specifications.

The entire design procedure should be based upon a thorough and complete scientific understanding of the involved phenomena, without which the proposed economy may prove dangerous when applied outside the area covered by the tests. Because of this it is necessary to establish an analytical method, besides the simple design formula to be discussed, enabling one to determine with sufficient accuracy the behaviour of obliquely loaded columns when all aspects of the problem are taken into account.

\(^{(1)}\)The buckling tests on columns under oblique loading have been included in the research program of the "Commission Belge pour l'Etude de la Construction Métallique" (C.E.C.M.), for the improvement of our knowledge on the subject.
These aspects are:

1) The flexural and torsional deformations of the member, taking into account the prevention of warping at the ends.

2) The plastic deformations of the bar, modified by the presence of residual stresses in rolled members.

The establishment of such a method of scientific computation requires extensive research based on a study of the existing literature and on the deformation of the specimen during testing; it involves the elaboration of a new complex theory. For this reason the present paper will be divided into two parts. The first part is devoted to the test program; it includes the general test set-up (Chapter I), the preliminary tests on the material properties of the members (Chapter 2) and the description of the column tests (Chapter 3).

The second part of the report contains all the theoretical research, including the derivation of a simple design method based on an interaction formula, as well as an exact scientific explanation of all the observed experimental phenomena.
PART I

THE TESTS

CHAPTER I - GENERAL TEST PROGRAM

The test program was established in such a manner as to explore the practically important range without the need of an excessive number of tests. To define the range one must specify the sections of the members to be tested, their slenderness ratio, the ratio of the end eccentricities $e_2/e_1$, and finally the type of end supports.

A. Selected sections:

It should be noted that buckling occurs under two completely different ways according to the ratio $I_x/I_y$ of the principal moments of inertia of the cross-section. If this ratio is small, as in the case of wide flange sections, the buckling generally occurs by excessive bending in the plane of the applied compressive force. If, on the other hand, this ratio is large, as in the case of an I shape, the buckling will have a tendency to occur by bending perpendicular to the loading plane followed by torsion of the cross section. Due to the reasons given above, it is necessary to perform the tests on two different types of cross section: Wide flange and I sections (Grey and FN).
One section was selected from each type. In view of the tendency to buckling which actually takes place, one (Grey) shape of the D/E series was especially chosen because it was desirable to study the buckling by bending and torsion. Such thin walled sections, as the D/E series, are particularly sensitive to this phenomenon due to their small torsional rigidity.

The member size was chosen according to the available vertical clearance in the testing machine (about 5 meters), and the largest slenderness ratio to be tested.

Adopted sections:

a) Members of wide flange sections

For slenderness ratios up to 100 Section D/E 10.

For slenderness ratios equal or less than 100 Section D/E 20.

b) Members of standard I section

The standard FN 22 section was chosen.

B. Selected slenderness ratios.

The largest value of slenderness ratio to be used in the tests was chosen as \( \lambda = 175 \), a maximum figure allowed by the Belgian IBN specifications. On the other hand, values of below 40 did not present any interest due to the fact that such values are not encountered in any type of building. Furthermore,
columns in general are short ($\lambda < 105$) and it is important to have test data in the neighborhood of these values rather than for longer columns. Therefore, it was decided to select the following six values of slenderness ratios: $\lambda = 40, 60, 80, 100, 130, 175$.

C. Ratio of end eccentricities $e_2/e_1$.

In the tests the compressive force was applied with unequal end eccentricities $e_1$ and $e_2$ respectively, in the plane of symmetry of the member, which is also the plane of maximum rigidity. This loading is similar to an axial compressive force plus two equal end moments $M_1 = Pe_1$ and $M = Pe_2$ (Fig. 1.1). As it was noted in paragraph 1, the columns used in industry are frequently subjected to end moments of opposite sign. In order to avoid an increase in the number of tests to be performed only the cases:

$$e_2/e_1 = +1, 0 \text{ and } -1$$

(See Fig. 1.2 a, b, c) were chosen.

D. Values of the Relative Eccentricity $m$.

The most convenient unit for measuring the eccentricities $e_1$ and $e_2$ is the core radius $r_x$ of the cross section. As known, the normal stress distribution due to combined bending and axial force is given by:
The core radius $r_{x}$ has the value:

$$r_{x} = \frac{2 i \Delta}{h}$$

(1)

Therefore the above equation may be written as:

$$\sigma = \frac{P}{\Omega} \left(1 + \frac{2 \epsilon y}{h r_{x}}\right)$$

(2)

By letting

$$e = m r_{x}$$

(3)

we find as extreme \((y = \pm \frac{1}{2})\) fiber stresses:

$$\sigma_{\text{max}} = \frac{P}{\Omega} (1 \pm m)$$

(4)

"The relative eccentricity \(m\) will be defined as the eccentricity measured in core radius units \(r_{x}\)." Based on previous research work, values of \(m = 0.5, 1, \) and \(3\) were chosen for the present experimental investigation.

The corresponding stress distribution along the height of
the cross section is shown in Figures 1.3 a, b, c.

![Figure 1.3](image)

**E. End supports of the columns.**

Simple supports have been chosen in both ends, as most of the previous experiments here used.

The choice of simple supports should not be interpreted as an effort to simulate the real condition, but as the only supporting condition which is well defined and achieved with a great precision in experimental work. The theory involved is particularly simple and leads to an absolutely correct interpretation of the experimental results. We also should decide on the type of column supports for bending in a plane perpendicular to the loading plane.

Evidently, the simplest solution is the use of knife edges which permit rotation around x-x axis but provide a complete fixity for any rotations around the y-y axis. Such a solution was adopted by various American researchers, whose
tests are actually in progress. [1], [2]. Their experience with this type of supports shows that the fixity achieved in the direction of the strong axis was not perfect. First of all, due to the fact that the end forces of the column are not exactly parallel, the upper knife edge should be adjusted to prevent any transverse initial eccentricities. This adjustment, which is to be done for each column prior to testing, is quite painful and may introduce some errors.

For our tests, spherical oil pressured supports have been adopted. (Fig. 1.5). These supports are lubricated with oil under the same pressure as the pump of the testing machine. The oil lubricates the spherical surfaces in contact, as to allow the spheres to slide on a thin fiber of oil, thus preventing friction. As will be seen later, these supports are superior to the knife edge type of supports. The center 0 of the sphere is situated in the top cast-iron plate, permitting the column to rotate at its ends in both perpendicular directions. The supports even possess a third degree of freedom, rotation around a vertical axis through 0. This undesirable degree of freedom is adequately discussed in the detailed description of the supports.

Let us note, that the current practice in Lehigh University (Ref. 1) is to apply end moments by special hydraulic jacks
acting on the ends of lever arms. In our tests these end moments are produced by the eccentricity of the applied compressive force. Evidently the American solution is more flexible than ours, but considerably more complicated; it is possible either to vary $P$ and $M$ proportionally, or keeping consistent moments vary $P$, or keeping $P$ consistent vary the end moments.

Fig. 1.4

Fig. 1.5
Finally, the support conditions regarding the freedom of warping at the end sections of the columns, will be discussed.

Theoretically the simplest condition is the case of ends "free to warp" where the flanges are free to rotate in their plane at each end. Such a condition is never encountered in practice; the continuity of the columns, and connections with the girders of the frame produce quite complete prevention of warping.

In our tests the columns are supplied with end beam-plates welded to the end sections by fillet welding. These plates themselves are bolted on the top cast-iron plate of the spherical supports, which are considered as rigid. Also, they provide some kind of a complete prevention of warping. However, this cannot be said with absolute certainty. It does not imply any undesirable complications because, as we will see in the detailed description of the tests, the warping phenomena have a secondary effect on the buckling mechanism.

The spherical supports do not allow a rotation more than six degrees in each sense. It results that the inclination of the column in the testing machine cannot exceed the above value. This presents some difficulty in testing short columns
with large eccentricities as:

\[ \frac{e_2}{e_1} = 0 \text{ or } -1. \]

Fig. 1.6

In summary, the sections with variables listed below were chosen:

1 wide flange section (WFE 10 or 20)
1 normal I section (PN 22 S)

with \( \lambda = 40, 60, 80, 100, 130, 175 \)

\[ \frac{e_2}{e_1} = +1, 0, -1 \]

\[ m = 0.5, 1, 3 \]

and with the exclusion of some tests because of the limitation of the inclination to 6°, the test program consists of 94 tests, distributed as shown in Table I.

Every cross in the table represents one experiment. The dashes represent tests which could not be performed because of the inclination condition.
In addition to the tests of Table I, an axial buckling test was performed on a PN22 column of slenderness ratio 40.

In general, the 95 tests are distributed as shown in Table 2.

**TABLE 1**

<table>
<thead>
<tr>
<th>Profile</th>
<th>$\lambda$</th>
<th>$\epsilon_0$</th>
<th>$\epsilon_0 = 1$</th>
<th>$\epsilon_0 = 0$</th>
<th>$\epsilon_0 = -1$</th>
<th>$\epsilon_0 = 1$</th>
<th>$\epsilon_0 = 0$</th>
<th>$\epsilon_0 = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profil DIE 20 $\left[ \frac{L_x}{1_y} = 2.76 \right]$ or DIE 10 $\left[ \frac{L_x}{1_y} = 2.52 \right]$</td>
<td>$m = 0.5$</td>
<td>$\epsilon_0 = 1$</td>
<td>$\epsilon_0 = 0$</td>
<td>$\epsilon_0 = -1$</td>
<td>$\epsilon_0 = 1$</td>
<td>$\epsilon_0 = 0$</td>
<td>$\epsilon_0 = -1$</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>60</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>80</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
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<tr>
<td>100</td>
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<td>$\times$</td>
<td>$\times$</td>
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<td>$\times$</td>
<td>$\times$</td>
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<tr>
<td>130</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>175</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

| Profil PN 22 $\left[ \frac{L_x}{1_y} = 18.89 \right]$ | $m = 0.5$ | $\epsilon_0 = 1$ | $\epsilon_0 = 0$ | $\epsilon_0 = -1$ | $\epsilon_0 = 1$ | $\epsilon_0 = 0$ | $\epsilon_0 = -1$ |
| 40 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 60 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 80 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 100 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 130 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 175 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

**TABLE 2**

<table>
<thead>
<tr>
<th>Profile</th>
<th>Number of Tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIE 10</td>
<td>18</td>
</tr>
<tr>
<td>DIE 20</td>
<td>32</td>
</tr>
<tr>
<td>PN 22 S</td>
<td>45</td>
</tr>
<tr>
<td>Total</td>
<td>95</td>
</tr>
</tbody>
</table>
The lengths in meters of the tested members are given in Table 3.

<table>
<thead>
<tr>
<th>Profil</th>
<th>0,5</th>
<th>1</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>DIE 10</td>
<td>16,71</td>
<td>33,42</td>
<td>100,26</td>
</tr>
<tr>
<td>DIE 20</td>
<td>35,73</td>
<td>71,47</td>
<td>214,41</td>
</tr>
<tr>
<td>PN 22</td>
<td>35,20</td>
<td>70,40</td>
<td>211,20</td>
</tr>
</tbody>
</table>

Finally, the corresponding end eccentricities, in millimeters, are presented in Table 4.

<table>
<thead>
<tr>
<th>Élancement</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>130</th>
<th>175</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profil: DIE 10</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>3,250</td>
<td>4,375</td>
</tr>
<tr>
<td>DIE 20</td>
<td>1,984</td>
<td>2,976</td>
<td>3,968</td>
<td>4,960</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>PN 22</td>
<td>0,808</td>
<td>1,212</td>
<td>1,616</td>
<td>2,020</td>
<td>2,626</td>
<td>3,535</td>
</tr>
</tbody>
</table>

In most of the previous research, the properties of the material, were exclusively determined by coupon tension tests. This procedure proves to be rather illogical due to the fact that in a buckling test the material is primarily stressed by compression.

The resistance of materials laboratory has proposed an improved technique for performing compression tests on coupons. It was decided, therefore, to investigate the variation of the elastic limit along the cross section of different shapes, in tension as well as in compression.
The recent research conducted in the United States, indicated the important role of the residual stresses on the buckling of rolled shapes. Therefore, it was decided to determine with accuracy the distribution of such stresses for the three cross sections used.

Finally, in order to establish a correlation between the coupon tests and the behavior of the entire column, taking into account the effect of the residual stresses, a stub column test was performed for each profile used.
CHAPTER II

2.1 Operational Techniques Used

A. Tension Coupon Tests

The coupons were prepared according to the standard dimensions in Belgium. They were taken from pieces manufactured from the same mill which produced the test columns. In the case of the PN 22 sections, the coupons were taken from the column itself.

The stress-strain diagram for each coupon was drawn using a Baldwin type MD2 electronic register. The lower and upper elastic limits (R_{el} and R_{es} respectively), the ultimate stress (R_{ut}), and the elongations on 5.65\sqrt{\sigma} and 8.16\sqrt{\sigma} for the three sections to be tested are given in paragraphs 2 through 4.

B. Compression Coupon Tests

The compression coupons are small parallelepiped blocks cut in such a way as to have the length dimension along the direction of rolling. Their thickness is the same as the thickness of the section, width equal or larger than their thickness, and length to thickness ratio of about 4. The two faces corresponding to the walls of the section were left rough; the other four faces were machined to parallel and exact plane surfaces.
The compression tests were carried out in a 10 ton Amsler machine, with the use of a 'subpress' which eliminates any relative lateral movements of the heads of the machine. The "subpress" (1) is a device essentially consisting of a steel guide block and a steel plunger which slides in this guide block with a clearance of only a few microns. The plunger is connected to a push rod by a spherical connection which permits relative lateral movements of the heads of the machine without interfering with the function of the device.

Figure 2.1.1 presents a section through the axis of the position. It should be noted that a decrease in the clearance between the guide block and the plunger due to lateral expansion of the latter under the action of the test load, was prevented by the application of the push rod. The load is applied through the push rod to the portion of the plunger between the guide block and the specimen.

For the compression tests, a spherical bearing plate with small bearing balls is attached to the plunger so that the upper face is free to rotate providing a uniform pressure distribution on the end sections of the coupon.

(1) The "subpress" was devised by the Meteorological Laboratory of the Liege University, after an American publication (Ref.2).
The stress-strain diagram for each coupon was obtained using Philips type strain gages glued on two of the lateral faces of the coupon. The centering of the coupons in the "subpress" necessitated the use of a third strain-gage on one of the lateral faces. The centering procedure consisted of loading slightly the specimen and progressively correcting its position until equal strains were recorded for all three strain gages.

In spite of the relatively large slenderness ratio of the coupons \( \lambda = 4 \sqrt{12} \approx 15 \), it was possible for all cases to observe plastic compressive deformations up to the point of yielding and up to the point of buckling. This is due to the excellent arrangement of guidance that the "subpress" provides.

The elastic modulus \( E \) as well as the elastic limit at 0.2% were determined from the \((\sigma, \varepsilon)\) diagrams drawn for each coupon. Fig. 2.1.2 presents one of those diagrams obtained in this manner.

C. Evaluation of the Residual Stresses

The residual stresses due to rolling were evaluated for three sections and used (DIE 10, 20, PN22) on a stretch of 600 m.m. (Fig. 2.1.3)
The principle of the method used is the following: Distances of 300 mm are measured along the lateral surface and set by small steel balls 1/8" in diameter placed in holes drilled on the member. The difference, to the nearest micron, of the distance between these base points and the length of a standard base \((a_1)\) is measured at the same temperature. These measurements were recorded using a movable manual deformeter shown in Fig. 2.1.4. This apparatus was devised by the "Laboratoire d'essais des Constructions du Génie Civil" after a detailed publication.\(^{(1)}\)

Longitudinal strips of 10-20 mm. width are cut so that they contain one measured base. Length measurements of these bases \((2)\), and the standard base \((a_2)\) are recorded.

Comparison of the readings before and after the release of the residual stresses gives the elongation of each strip as:

\[
\Delta l = (2 - 1) - (a_2 - a_1).
\]

Application of Hooke's Law \(\sigma = \frac{E \Delta l}{l}\) leads to the determination of the longitudinal residual stress distribution existing in the member.

The stress given in paragraphs 2 to 4 correspond to the mid-thickness of any part of the cross section obtained by averaging residual stresses at each face. The residual stresses were also measured in the exterior part of the flange tips. (Points A in Fig. 2.1.5). Finally the residual stress at point M was determined by interpolating the results obtained for points B and C, on the residual stress distribution diagram along the center line of the web. (Fig. 2.1.5)

Fig. 2.1.4

Fig. 2.1.5

The adopted method of measuring residual stresses is justified by the following reasoning. These stresses are assumed to be unaxial and parallel to the longitudinal axis of the member. The measurement base of 300 mm was chosen
as to obtain average residual stress values over this stretch. On the other hand, if these stresses were determined for small fragments, as to obtain the direction and the magnitude of the principal stresses, the values that would have been obtained would not have a statistical meaning. They would have a purely local and accidental significance. It is believed that the chosen method furnishes values which have a statistical character. The same thing holds true for the directions. This assumption is justified by favorable results to be presented below.

**Final Remark:** It is obvious that the residual stresses vanish at the ends of the member (Fig. 2.1.3). Since these stresses form a system of forces having a zero resultant force and zero resultant moment, they should, according to Saint Venant's principle, gain their full intensity at a distance from the support equal to the depth of the section. The distance between the ends of the base length and the ends of the member was 150 mm which is a little less from what it was necessary but over-ruled by the available length of the specimens (600 mm.).

**D. Stub Column Tests**

In order to establish a relation between the tension and
compression test results, axial compression tests were performed on stub columns of each cross-section used.

The end sections of these stub columns were carefully machined in such a way as to provide parallel and perfect surfaces. A large number of strain gages were placed vertically at mid-height.

The specimens were aligned by several trials till all the recorded strains were identical for a small load. Then, the stub was loaded at regular load intervals and readings of all gages were recorded for each load stage.

The load intervals were proportionally diminished in the neighborhood of the average elastic limit of the material.

Although the section is partially plastified under the combined effect of the applied compressive and residual stresses, these tests have indicated that the measured strains at the strain gages are not uniform which proves the invalidity of the basic assumption "Plane sections remain plane after loading."

Though, the concept of the "average strain" may be introduced as:

$$\bar{\varepsilon} = \frac{\int_{\Omega} \varepsilon \, d\Omega}{\Omega}$$
which may be calculated by averaging all of the strain gage readings. A \((\sigma_m, \varepsilon_m)\) diagram may be drawn for the entire cross-section. The average stress \(\sigma_m\) may be computed by
\[
\sigma_m = \frac{P}{\Omega}.
\]

Fig. 2.1.6

The experimental \((\sigma_m, \varepsilon_m)\) curve obtained as explained above, is in good agreement with the theoretical \((\sigma_m, \varepsilon_m)\) curve constructed according to paragraph 2.2. Both curves are presented in paragraph 2.4.

Note that the stub column tests were interrupted due to local buckling of the flanges. Consequently, the \((\sigma_m, \varepsilon_m)\) diagram is incomplete.

Fig. 2.1.6 shows the three stub columns after testing.
The maximum compressive stresses attained were:

- DIE 10 = 36.2 kg/mm²
- DIE 20 = 23.9 kg/mm²
- PN 22 = 25.5 kg/mm²

2.2 Determination of the Average Stress-Strain Diagram for a Flange Under Tension or Compression.

In order to analyze the composite bending deformations of a column subjected to a buckling test where the material becomes partially plastic, it is necessary to know the behavior of the member in tension as well as one flange in compression. It is also of importance to take into account the variation of the elastic limit along the width of that flange, (refer to part C of paragraph 2.1), and the residual stress distribution (refer to part C of part 2.1).

The method of establishing the average \((\varepsilon, \varepsilon)\) diagram for one flange in compression will be presented below. Fig. 2.2.1 b represents the residual stresses corresponding to the flange shown in Fig. 2.2.1 a. Let us consider a small element of area \(d\Omega\) in the flange of the cross section where the residual stress is \(\varepsilon_r\).
Suppose now that this material fiber of cross sectional area $d\Omega$ is isolated and thus relieved from residual stresses. If this element is subjected to a pure compression test the resulting $(\sigma, \varepsilon)$ diagram will be similar to the one given in Fig. 2.2.2.

Fig. 2.2.1

Fig. 2.2.2
Now let us investigate the influence of this element on the overall deformation of the column.

If the element is under compressive residual stress $\sigma_r$, (as assumed in Fig. 2.2.1 b), this compressive stress will be added to the stress $\sigma$ due to the overall deformation of the column. The elastic limit will be reduced in compression since

$$\sigma + \sigma_r = R_{eo}$$

Consequently, the apparent elastic limit of the element is:

$$R_e = R_{eo} - \sigma_r$$

and will be attained for a lower stress than in the compression test of the isolated element. It is seen, therefore, that for the case of overall deformation of the member, the exact stress and strain that any element undergoes must be:

$$\sigma = \sigma_o - \sigma_r \quad \text{and} \quad \varepsilon = \varepsilon_o - \varepsilon_r$$

Therefore, instead of the $(\sigma_o, \varepsilon_o)$ diagram a new $(\sigma, \varepsilon)$ diagram may be obtained by transposing the origin $O_o$ to the coordinates $(\sigma_r, \varepsilon_r)$ at 0. (Fig. 2.2.2)

The $(\sigma, \varepsilon)$ diagrams of all fibers of the flange may be obtained by using the transformation of origin on the
previously determined \((\sigma_0, \varepsilon_0)\) diagrams and knowing the residual stress distribution \(\sigma_r\) in the flange.

In order to draw the average stress-strain diagram for one flange, it is assumed that the section does not damage its shape even while undergoing plastic deformations. This means that all of the fibers of the flange undergo the same strain \(\varepsilon\). The value of \(\sigma_{\text{average}}\) for a strain value \(\varepsilon\), will be defined by the relation:

\[
\sigma_m = \sigma_{\text{ave}} = \frac{1}{\Omega \text{ flange}} \int \sigma \, d\Omega \text{ over flange}
\]

In order to construct the \((\sigma_m, \varepsilon)\) diagram the strain \(\varepsilon\) may be given equally spaced values. For all practical purposes, the above integral may be substituted by its sum. The flange may be divided into five sections, at the center lines of the compression coupons.

In the case of a Grey section, the thickness of the flange is constant and it can be written as:

\[
\sigma_m = \frac{1}{l} \sum \sigma_i \, l_i
\]

where \(l\) is the half of the flange width, and \(l_i\) the width of the strips.

Evidently an analogous diagram for tension in the
flange may be easily constructed. The same procedure may be repeated but noting that the stresses will change sign, since the residual stresses $\varepsilon_r$ remain the same in both cases.

2.3 Results for the DIE 10 Section.

Table No. 5 below gives the material properties as determined by four tension coupon tests; Fig. 2.3.1 has been constructed in connection to this table. This figure represents the distribution of the elastic limit, in a cross section at 0.2% compression, as well as some lower and upper limits which have been determined by tension tests. Center lines of the tension or compression coupons are indicated in the same figure. It can be shown that there exists a relation between the material properties in tension and in compression.
TABLE NO. 5

<table>
<thead>
<tr>
<th>Numéro de l'éprouvette</th>
<th>Kg/mm²</th>
<th>Allongem. % sur $S_o \sqrt[2]{5,88}$</th>
<th>Allongem. % sur $S_o \sqrt[2]{8,16}$</th>
<th>Striction en %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{el}$</td>
<td>$R_{ef}$</td>
<td>$R_e$</td>
<td>$R_{el}$</td>
</tr>
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<td>1</td>
<td>24,0</td>
<td>24,0</td>
<td>39,0</td>
<td>39,0</td>
</tr>
<tr>
<td>2</td>
<td>27,9</td>
<td>28,8</td>
<td>41,4</td>
<td>38,0</td>
</tr>
<tr>
<td>3</td>
<td>27,9</td>
<td>28,5</td>
<td>41,2</td>
<td>22,7</td>
</tr>
<tr>
<td>4</td>
<td>27,2</td>
<td>27,8</td>
<td>41,3</td>
<td>37,3</td>
</tr>
</tbody>
</table>

The variation of $R_e$, which changes from 23.5 kg/mm$^2$ at the flange tips to 33.5 kg/mm$^2$ at the mid-point of the web, should also be emphasized as being important.

The residual stress distribution determined by averaging the readings on both faces of the profile is represented in Fig. 2.3.2. It is observed that:

1. the diagram is very regular and practically symmetrical around the xx and yy axes, which justifies the adopted measuring method.

2. the distribution at each face is practically parabolic and in close equilibrium for that face.

It appears that there is no interaction of importance between the flange and the web.

Figure 2.3.3 gives the average residual stress distribution over one quarter of the section. This diagram was
obtained by averaging the experimental results for the four halves of the flanges and the two halves of the web.
Using the method described in paragraph 2, and this diagram, the theoretical \((\sigma, \varepsilon)\) curve representing the average behavior of one flange may be constructed. (Fig. 2.3.4)

It is seen from this figure that the effect of the residual stresses causes a considerable reduction in the value of the elastic limit in the entire flange; its value is reduced to \(R_{pm} = 12.7 \text{ kg/mm}^2\) (Fig. 2.3.4)

![Graph](image)

**Fig. 2.3.4**

The position of the strain gages used in the stub column tests is shown in Fig. 2.3.5.

The diagrams 2.3.4 and 2.3.5 present a satisfactory similarity which confirms the theory of paragraph 2.
2.4 Results for the DIE 20 Section:

The results for the DIE 20 section will be presented the same way without going into detailed explanations.

Table 6 gives the principal mechanical characteristics of the material determined from coupon tests for this particular section; their position is shown in Fig. 2.4.1.

Fig. 2.4.1 shows the elastic limit distribution in a cross section at 0.2% compression. Some values of $R_{ei}$, the elastic limit values determined by tension tests are also given in the same figure. The $R_e$ has the same variation as the case of DIE 10 sections, with a minimum around the flange tips and a maximum at the center of the web.
Fig. 2.3.6

DIE-10
Diagramme de Cours Normale
Début de base de compression sur section rectangulaire.

Fig 2.4.1

DIE-20
Limite elasique à 66% de compression
Re et Rd en traction

Traction

Re kg/cm² Rd kg/cm²
Figure 2.4.2 represents the residual stress distribution in a section; the same conclusions with the ones for the DI10 section may be drawn from this figure. The average residual stress distribution over one quarter of the cross section, is shown in figure 2.4.3. This diagram was obtained by averaging the experimental results for the four halves of the flanges and the two halves of the web. Using the method described in paragraph 2.2, the diagram 2.3.4, the theoretical \((\sigma_m, \varepsilon)\) curve representing the average behavior of one flange in compression (Fig. 2.4.4) and in tension (Fig. 2.4.5) may be constructed.

The elastic limit \((R_e)\) diagram for the web, at 0.2% compression is presented in Fig. 2.4.6. These values were first obtained experimentally by coupon tests, then by stub column tests which include the effect of the residual stresses. It is seen from this diagram that in buckling tests where the web is subjected to compression \((m = 0.5, and 1)\) will behave elastically around its ends up to stresses as high as 30 kg/mm.\(^2\). On the other hand, in the test \(m = 3\), the web will be subjected to tensile strains at one of its ends and will be plastified as soon as the tensile stress equals 18 kg/mm.\(^2\).

Finally, the Fig. 2.4.7 shows the position of the strain gages used in the stub column test. The average
(σ, ε) curve for one flange in compression obtained from this test is shown in Fig. 2.4.8; it is reasonably compatible with the theoretical curve of paragraph 2.2, which is a new justification of the latter method.

2.5 Results for the PN22S Section

Table No. 7 below shows the principal characteristics of the steel as determined by tension tests on 26 coupons whose position in the section is indicated in Fig. 2.5.1. Fig. 2.5.1 shows the elastic limit distribution at 0.2% compression over one quarter of the cross section, as well as the average limit values R_{ei} and R_{es} determined by the 26 tension coupon tests.

<table>
<thead>
<tr>
<th>TABLE NO. 7</th>
</tr>
</thead>
</table>

The variation of R_{e} is not as marked as in case of wide flange shapes and the distribution does not present the same shape; one may observe minima at the middle of the web and/or at the middle of the flanges.
### TABLEAU N° 7 (suite) (cont’d)

<table>
<thead>
<tr>
<th>Profil</th>
<th>Numéro de l'éprouvette</th>
<th>Kg/mm²</th>
<th>Allongem. % sur 5,65√S₀</th>
<th>Allongem. % sur 8,16√S₀</th>
<th>Striction en %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rₑₑ</td>
<td>Rₑₕ</td>
<td>Rₑ</td>
<td></td>
</tr>
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<td>Allongem. % sur 8,16√S₀</td>
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The residual stress distribution is presented in Fig. 2.5.2. Although it seems to have the same parabolic shape as in the case of wide flange sections, it is not any more in equilibrium when flanges and webs are considered individually. It is obvious that there exists a resultant tensile force in every flange and a resultant compressive force in the web. The residual compressive stresses in the web are considerably high and reach the value of 14.3 kg/mm² at the middle of the web.

This fact explains the abrupt relaxation of residual stresses in the web indicated by cracking of the white wash (Fig. 2.5.3); it was observed in all buckling tests on PN22 columns. The length of the relaxation zones may be estimated to be 300 millimeters based on the fact that the distance between two successive yield lines on the white wash was never less than this value.

Fig. 2.5.4 gives the average residual stress values over one quarter of the section.

Using the method described in paragraph 2.2, and this diagram (Fig. 2.5.4), one may construct the theoretical ($\sigma_m, \varepsilon$) curve representing the average behavior of one flange under compression. (Fig. 2.5.5)
Fig. 2.5.6 gives the position of the strain gages used in the stub column test. Fig. 2.5.7 represents the average \((\sigma, \varepsilon)\) diagram for one flange obtained from this test. This diagram should not be considered as an accurate one because a number of strain gages were inactive at the early stages of the tests. The drawn portion is considerably different from the theoretical \((\sigma_m, \varepsilon)\) curve (Fig. 2.5.5), especially in the plastic range.

2.6 Conclusions From Preliminary Tests

The above tests will be considered as being important for the correct interpretation of the buckling tests.

The above preliminary tests are considered as being important for the correct interpretation of the buckling tests.

As a result these tests have indicated that,

a) The elastic limit of the steel presents considerable variations along the cross section of the bar.

b) The elastic limit is affected by the existence of residual stresses due to rolling.

For steel rolled shapes, it is felt that it would be rather unprofitable to base the buckling theory for the elastic-plastic range on an "idealized elastic-plastic!"
stress-strain diagram (Fig. 2.6.1). This suggestion was made by many authors and primarily by Jager (Ref. 34). In connection with the above remark and Jager's calculations one should be able to observe the existing reserve strength in columns under axial compression and bending. The only certain source of analysis for the behavior of the test columns seems to be the \((\varepsilon_m, \varepsilon)\) diagrams, which represent the average behavior of one flange. Actually, the aim of all the preliminary tests in the present chapter, is to provide the necessary data for the construction of these diagrams.

Let us note one more incident in the results obtained from the buckling tests:

The average \((\varepsilon_m, \varepsilon)\) curves for one flange obtained in paragraphs 2.3, 2.4, 2.5 show that, for small plastic strains which are critical for buckling \((\varepsilon < 3 \times 10^{-3})\), the steel of the flanges behaves like a material with a low proportional limit and an extended plastic range; that is, like a brass or duraluminum type of material.

It is obvious that the elastic limit loads should be expected to be quite lower than the loads obtained by putting \(\varepsilon_{\text{max}} = R_e = 25 \text{ kg/mm.}^2\) as it is usually done. Consequently there is not any hope of conforming the Dutheil or any other similar method by the tests presented in this paper.
Fig. 2.4.2

D.I.E-20
Tensions Résiduelles.

Fig. 2.4.3

D.I.E-20
Courbe moyenne des tensions résiduelles dans les semelles et dans l'âme.
TABLE NO. 6

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Fig. 2.4.4

Fig. 2.4.5
Fig. 2.5.7

Fig. 2.6.1
CHAPTER III

BUCKLING TESTS

3.1 Description of test apparatus used.

A detailed description of the test machine, the spherical supports, the abatements of the test columns, and the method of determining stresses and deformations, will be given in the next few paragraphs.

a) Testing Machine

The tests were executed in the 500 ton Amsler testing machine of the Civil Engineering Testing Laboratory of the Liege University. It is essentially composed of the following parts:

1) The press.
2) The independent dynamometer.
3) The pump to produce the oil pressure.

The general set-up of the machine is shown in Fig. 3.1.1 and 3.1.2. The lower part carries the two threaded columns and contains the press cylinder. It is installed in a hollow pit in the ground and rests by its periphery on the edge of this hole.

The test column rests on the piston of the machine by the lower spherical support. It is supported at the top by the upper spherical support and the machine cross-head which is connected to the two vertical screws.
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Fig. 3.4.3
b) Oil Pressured Spherical Supports

The spherical oil bearings are of the type designed by TEMPLIN (Ref. 3). The original American supports were estimated to carry a force of about 135 tons, and have an apparatus attached by suspenders.

It was decided to omit this kind of suspension in our supports for two reasons: savings in cost and to provide enough room for the supports between the vertical screws of the testing machine.

Sectional views of the lower and upper spherical oil bearings are shown in Fig. 3.1.3. Details of the upper support are shown in Fig. 3.1.1 and 3.1.2. A general view of the lower support is presented in Fig. 3.1.4. The suspension of the movable portion is achieved by four long bolts. The clearance provided around these bolts gives enough freedom for small movements of the suspended part.

As mentioned in the first chapter, the supports were designed for an angular movement of at least $6^\circ$ on each side.

The main oil pump of the machine provides the oil pressure for the main cylinder as well as the two spherical bearings. The pump leads to a three pipe connection. Reinforced rubber pipes connected to each part of these connection carry the oil
pressure to the two spherical oil bearings and to the main cylinder. Entrance valves were used in the supports to control the losses, and to limit the quantity of oil in the bearings. Exit valves were equally provided to regulate the counter-pressure in the outer border of the spherical faces in contact. Practice has shown the necessity of letting the exit valves open. The edges were primed to make them water-tight. Although this was the practice in the designing the priming proved unusual.

The top cast-iron plate of the spheres is provided with two special grooves designed to hold the heads of the anchorage bolts from the abutment of the rest column. Adjusting screws provided for these bearing plates help to adjust the position of the column base plate with the required precision. Most of the oil circulating in the supports was collected in
the outside groove and conveyed to the machine reservoir.

The small amount of oil escaping the lower spherical support was collected in a special groove on the fixed part of the support and led to a container on the floor. The oil dripping from the upper support would have to flow down the apparatus and have created extra work and inconvenience to the researchers. In order to prevent this a piece of sheet-iron forming a container is provided under the suspended part of the upper support, and the accumulated oil flows to the machine reservoir by gravity.

Finally, both supports are provided with a special arrangement designed to prevent any rotation of the movable sphere around a vertical axis while permitting rotation about the two other axes. This consists of an extension on the cast-iron top plate of the spherical part and in the plane of symmetry of the machine. A ball bearing is attached to this extension, (See Fig. 3.1.3 and photographs 3.1.2, 3.1.3 and 3.1.4), which is reinforced with a cylindrical seal-ring. This ball bearing, which is provided with a seal-ring, can slide vertically in a slot cut in the web of a heavy WF section bolted to the pedestal of the testing machine (Fig. 3.1.4).

The corresponding pieces for the upper support
bolted at the cross-head of the machine. As mentioned in paragraph 2, the center of the movable sphere is situated at the center of the cast-iron plate covering the sphere, in each support.

Therefore, if the two plates attached to the two spherical parts are brought in contact, a complete sphere will be formed with center common to both pieces.

A perfect working condition of the spherical supports is said to be achieved if the sphere in question can rotate freely about its center with a very small applied effort even when subjected to high compressive loads.

For this reason, the constructor was forced to perform an acceptance test, in which the moment arm of the applied couple to the sphere due to compressive loads applied by the testing machine would not exceed 0.2 mm for any load from 10 to 500 tons.

The assembly of the supports was very laborious. Many retouches and several machine shop operations were performed.

After the completion of all these preparations, the spherical supports showed an excellent working condition between 0 and 300 tons. Measurements have indicated that the
relative eccentricity \( e = \frac{M}{P} \), induced by friction in the
bearings remained between 0.1 and 0.2mm for the load interval
mentioned above.

For compressive load above 300 tons, the deformation
of the spherical pieces becomes considerable and sometimes
produce other implications. This does not concern our case
because all the tests were executed at loads below 100 tons.

c) Abutments of the Test Columns.

As mentioned previously, the test columns are welded
at their ends to bearing plates. Holes are drilled on these
plates, as to match the hole pattern in the cast-iron plates
of the spherical supports.

In the case of relative eccentricities of \( m = 0.5 \) or 1,
the end section of the column is under compressive stresses
only, and is not necessary to reinforce the bearing plate. On
the other hand, in the case of \( m = 3 \), the end sections are
partially in tension and there is a tendency of the end plate
to uplift. The provided anchor bolts certainly cannot resist
this force. The end plate was reinforced by a plate welded
vertically which increases its stiffness considerably. This
arrangement together with the end plate formed a rigid box.
The column end was finally welded to the bottom of this box.
(Fig. 3.1.5 through 3.1.8)
It should be noted that the reinforcements on the end plates do not have any connection with the column itself. The columns are always placed vertically and their behavior is not affected at all by the added reinforcing plates.

d) Measurements of Stresses.

In order to control the centering of the specimen into the machine, stresses were measured at 8 or 12 points in each
column using Philips strain gages, and a Baldwin K type strain gage indicator. At each section four strain gages were placed in the flange tips in 2 or 3 sections along the column.

In the columns with eccentricities \( \frac{e_2}{e_1} \) of 1 or 0, strain gages were located at mid-height and at 200 mm. from the bottom. The only reason was to reduce the inevitable effect of stress concentrations. In the case of the columns with ratio \( \frac{e_2}{e_1} = 1 \), a section at 200 mm. from the top with 4 more gages was added. (Fig. 3.1.9b)

e) Measurements of Displacements.

Transverse deflections were measured by 20 deflection dials of a precision to the nearest 0.01 of a millimeter, and with an appropriate stroke according to its position. Normally for the middle part of the column 30 to 50 mm. stroke dials were used, whereas 10 mm. stroke dials were sufficient for the end parts.

![Fig. 3.1.9](image-url)
As a usual practice in Civil Engineering Testing Laboratories, all the deflection dials were mounted on a support which is independent of the testing machine. This support consisted of a steel pipe square frame resting on the floor. The frame was given the necessary rigidity by diagonal bracing running on all the sides and by tying in three points this frame to the Laboratory frame by steel pipes. For each column, measurements were taken in six equi-distant sections, namely: the ends sections, at mid-height, at quarter and at 3/4 height.

It was necessary to record lateral displacements of the center of gravity \( u \) and \( v \) as well as the rotation of the section in its plane. Although three measurements are sufficient to obtain these results, four different readings for each section were taken utilizing the arrangement given in Fig. 3.1.10.
These dials measure the displacements \( l_1, l_2, l_3, l_4 \) of the ends of the steel rods fastened to the column with clamp fasteners. These rods were used to improve the precision of the measurements for the angle of rotation by increasing the difference between the readings \( l_1, l_2 \), and \( l_3, l_4 \). The use of the above mentioned rods changes the position of the deflection dials 1 and 2 out of the plane of symmetry of the machine. The above arrangement helped in the positioning of the dials because both of the vertical screws of the testing machine happen to lie in its plane of symmetry.

The dials were connected to the test column by thin steel wires of about 0.2 mm. of diameter. Rubber bands from the opposite side of the deflection dials were used to keep the steel wires always stretched.

The wires transmit the displacements of the rods with a sufficient precision (average 0.05mm.). On the other hand the limited resistance of the wire protects the dial from braking in case of a sudden buckling of the specimen.

All the dials were fastened to the steel pipe frame by special clamps permitting easy resetting.
From Fig. 31.10 the displacements of the center of gravity with respect to xx and yy axes are:

and the angle of rotation of the section, in radians is:

\[
\varphi = \frac{l_1 - l_2}{d_1}, \quad \varphi = \frac{l_3 - l_4}{d_2}.
\]

However, the measured values of \[\varphi\] are never precisely the same. The deviation between these values indicates the accuracy of the readings.

In the computations an average value of \[\varphi\] was used:

\[
\varphi = t \left( \frac{l_1 - l_2}{d_1} + \frac{l_3 - l_4}{d_2} \right), \quad u = \frac{l_1 + l_2}{2}, \quad v = \frac{l_3 + l_4}{2}.
\]

As stated above, the dial frame is completely independent of the testing machine, meaning that the column supports undergo small displacements and rotations with respect to the dial.

In order to measure the displacements of the sphere cover plates in their plane four deflection dials were used for each spherical support. These 8 dials were connected to the cast-iron cover plate with the same arrangement utilizing the steel rods and wires described above. Measurements of \(u_i, v_i, \varphi_i\) and \(u_s, v_s, \varphi_s\) for the lower and upper support movements can be taken utilizing these 8 deflection dials.
However, the supports should be considered as fixed in the interpretation of the test results. Therefore values of \( u, v, \) and \( \phi \) obtained for the three intermediate sections (i.e. at 1/4, 1/2, 3/4) should be reduced according to the linear expression of the form:

\[
\begin{align*}
\alpha(x) &= u_i + \frac{u_j - u_i}{l} x; \\
\beta(x) &= v_i + \frac{v_j - v_i}{l} x; \\
\gamma(x) &= \phi_i + \frac{\phi_j - \phi_i}{l} x.
\end{align*}
\]

f) Study of the Plastic Deformations.

In each test, the column was painted with whitewash which cracks as the surface of the steel undergoes deformations of the order of thousandths. The progression of the plastic deformations in the column can be followed by observing the crack pattern of the whitewash.

3.2 General Description of Tests

A. Preliminary Test Calculations

Considering the imperfections present in rolled shapes, it is insufficient and inaccurate to base our computations on the mechanical properties of the section as given in the handbook. It was decided therefore, to compute these properties separately for each of the test columns. The principal moments of inertia \( I_x, I_y \) and the area \( \Omega \) were computed on individual height \( (h) \) and width \( (b) \) measurements of the cross section. The length \( (L) \) was measured directly by tape to the nearest millimeter. Measurements were taken at the middle and two
quarter points. The height $h$ and width $b$ of the cross section were measured using a caliper micrometer whereas the flange thickness was measured with a micrometer at eight points of each section as shown in Fig.'s 3, 2.1 a) and b). All measurements are reduced to average values of $b$, $h$, $e$ (or $b_1$, $h_1$, $e_1$ and $e_2$). Handbook values of the web thickness were used because its measurement is difficult and the contribution of the whole web area in the case of $I_{x}$ is very small (about 15%) and practically zero for $I_{y}$. On the other hand, the contribution of the flanges to the values of $\Omega$, $I_{x}$ and $I_{y}$ was calculated using the actual measurements obtained as explained above.

![Diagram](image)

**Fig. 3.2.1**

In the case of the "Grey" sections, the flanges were approximated with rectangles. For the FM.22 sections, the slope of the lower part of the flange was taken into account. Although average values for the flange width and thickness were determined for each member, the value of the slope was taken from the handbook.
The area was determined by dividing the total weight of the column in kilograms without its base plates, by 7.325 \times 10^{-5}. (1) Upon request the weight of each column was accurately measured by the company which furnished them. Ω

Values determined by this method were found to be in excellent agreement with values obtained using the actual dimensions obtained as explained above. The weight method of determining the cross-sectional area is preferable because it gives more accurate results. The geometrical properties, found as explained above, are used to determine the

a) radius of gyration \( i = \sqrt{\frac{I}{\Omega}} \)

b) slenderness ratio \( \lambda = \frac{l}{i} \)

c) core radius \( r = \frac{2i^2}{h} \)

d) the maximum eccentricity
\[ e = mr_x \text{ with } m = 0.5, 1 \text{ or } 3 \]

for each section.

B. Alignment of Columns in the Testing Machine

Every column was placed in the machine in such a manner that its symmetry coincides with that of the testing machine. The center of the end sections were set at a distance \( a \) and \( (1) 7825 \text{ kg/m}^3 \) is an average value for the specific weight of steel. In the formula used, \( l \) should be in mm and \( \Omega \) in mm².
e₂ from the centers of the spherical supports. Then, the specimen was aligned as follows:

The column was first subjected to an axial compressive load equal to the probable critical load, and all deformations at the eight or twelve strain gages were measured. If the column is exactly linearly aligned and placed properly, the strain gages a₁, a'₁, b and b'₁, should give the same strain values as gages b and b'₁.

Actually though, all these readings are different, due to the critical lateral curvature of the columns.

Then, the column is given, at each end, arbitrary displacements d₁ and d₂ (Fig.3.2.2), in general different for each flange and in the direction of the sought center. Related to these unequal moments d₁ and d₂ the cross section rotates an arbitrary amount. The column is loaded again, and the same procedure is repeated until the conditions \( a'₁ = a''₁ \) and \( b'₁ = b''₁ \) are realized with an error of less than 5%.

The condition \( a'₁ = a''₁ \) and \( b'₁ = b''₁ \) was looked after to hold true especially at the mid-section of the column, because it is the total eccentricity of the line of action at this particular section which causes lateral Euler buckling. The same conditions do not have much influence at the end sections.
Generally, the alignment of each column required 1 to 4 trials. The trial displacements given each time varied from 0.5 mm. to 2 mm. A small number of the tested columns were found to be perfectly aligned at first set.

C. Execution of the Tests

Initial zero readings were taken under an arbitrary load of 1 or 2 tons to avoid any accidental effects due to the mounting of the specimen. Knowing the probable critical load, load increments had been established in advance. In every load stage the load was kept constant while readings of all gages and deflection dials were recorded. The load intervals and the speed of load application were reduced gradually as the applied load approached the computed critical load.

At the plastic range, the load increments were chosen
Fig. 3.2.3

Fig. 3.2.4

Fig. 3.2.5

Fig. 3.2.6
according to all gage readings and the progress of plastic deformations as indicated by the peeling-off of white-wash.

As an example, here is a series of load increments used.

2-15-25-32-36-39-41-43-44-45-1/2 (buckling)  Readings of the various gages were taken only after they were checked to be stable. In most of the tests, it was necessary to wait for 5 or even 10 minutes until the plastic deformations were stable. Due to the slow progress of plastic deformations many of the test columns buckled some time after the last load application.

All 95 tests were performed according to the program given in paragraph 2, with the only exceptions given below:

1) The PN 22 section test ($\lambda = 40$, $e_2/e_1 = -1$, $m = 1$) could not be carried out. The maximum rotation of the supports was not adequate for the alignment of the column. An axial compression test ($m = 0$) was performed instead.

2) For the same reason the test with $\lambda = 60$ $e_2/e_1 = -1$ and $m = 1$, was executed with an eccentricity $m = 0.8$.

3) The first tests on DIE columns, of $\lambda = 175$ were performed before the addition of the rotation preventing apparatus (paragraph 3.1.b). The test No.4 ($\lambda = 175$, $e_2/e_1 = 0$, $m = 1$)
was rejected due to an excessive rotation of the supports around the vertical axis.

4) The test with \( \lambda = 175, e_2/e_1 = -1 \) and \( m = 0.5 \) was modified by increasing the eccentricity to \( m = 0.6 \) and finally

5) Two, out of the 95 tests were rejected, due to improper function of the spherical bearings. Foreign matter, accidentally introduced between the two spherical surfaces, caused improper function of the supports preventing rotation. These tests were: PN22, \( \lambda = 130, e_2/e_1 = 0, m = 0.05 \) and PN22, \( \lambda = 175, e_1/e_2 = -1, m = 0.5 \) which correspond to partially end-restrained columns. Both of these tests were discarded.

D. Test Results

The following results were obtained during the performance of the tests:

1) The maximum load carried by the column.

2) Values of strains \( \varepsilon \), and the displacements \( u, v, \) and \( \phi \), corresponding to different loading stages and measured at \( x = \frac{L}{4}, \frac{L}{2} \) and \( \frac{3L}{4} \).

3) Observations of the kind of the load at which the first white wash cracks appeared, the position of these cracks, etc.

4) Photographs showing the column after buckling, and
finally,

5) A short description on the type of failure of the column. Failure may be a combination of the following types of failure:

A. Elastic lateral Euler buckling: (Fig. 3.2.3)

B. Lateral plastic buckling:
   Symmetrical (s) (Fig. 7.3 and 3.2.6)
   Assymetrical (d) (Fig. 3.1.5)

C. Plastic bending in the plane of the web:
   Symmetrical (s) (Fig. 3.2.6 and 3.2.9)
   Assymetrical (d) (Fig. 3.2.9 and 3.2.10)
   Antisymmetrical (a) (Fig. 3.1.2 and 3.2.8)

D. Additional torsional deformation:
   Symmetrical (s) (Fig. 3.2.9 and 3.2.10)
   Assymetrical (d) (Fig. 3.2.11 and 3.2.12)

E. Local buckling of the compressed flanges:
   (Fig. 3.2.13 and 3.2.14)

F. Local web buckling.

The results of the columns with slenderness ratios $\lambda = 80$ and $\lambda = 60$ are surprising, since buckling loads for most columns of $\lambda = 60$, generally are lower than the buckling loads corresponding to $\lambda = 80$. In order to prove that this controversy is not due to any variation in material properties, tension coupons were taken from the parts of the column remaining
elast. This was done for four columns. The coupons were taken from the flanges as prescribed by the Belgian code.

The results of these 4 tests, which are not presented in this report, were identical. Thus, the previous assumption in variation of the material properties is not justified. The behavior of each column, can be classified qualitatively with a series of letters A, B, C, D, E and F as defined previously. Brief indications regarding the type of buckling etc. will be added to the previous letters.

Values of the column displacement components $u$, $v$, and $\phi$ were used only for drawing certain load-deflection diagrams. For this reason, it did not seem useful to present all measured values for $u$, $v$ and $\phi$ in the present report.

Results which present usefulness are given in Tables 8 and 9. For each row, the table provides a group of columns giving the fundamental parameters of the test, that is the slenderness ratio, $\lambda$, the ratio of the eccentricities ($e_2/e_1$), and the maximum eccentricity expressed as a function of the core radius (m). The geometric properties of the section are given in a second group of columns. (See part A of this paragraph); namely, the column length ($L$) in mm, the average area of its cross section ($\Omega$) in $mm^2$, its two principal moments of inertia, $I_x$ and $I_y$ in $cm^4$, its real slenderness
ratio $\lambda$, which arbitrarily is different from the nominal slenderness ratio ($\lambda$) due to differences in actual and nominal dimensions of the column.

In a third group of columns, the ($P_{\text{max, elast.}}$) for which the first traces of cracking in the white wash were noted, the position of these cracks, and the ($P_{\text{max}}$) are given. One more column is provided for observations where the mode of failure of the column is given by the symbols A, B, C, D, E, F defined previously, and the letters s, a, d indicating the deformation as being symmetric, asymmetric, or antisymmetric.

In the determination of the maximum elastic load, ($P_{\text{max, elast.}}$), the local plastic deformations observed in the vicinity of the welded ends of the column were not taken into consideration.

It should be noted that the values found for $P_{\text{max, elast.}}$ are generally low, which is due to the local yielding caused by the residual stresses present in the bar. This phenomenon was particularly observed in the FM 22 tests, where the residual stresses at the center of the web are as high as $14.5 \text{ kg/m}^2$.

3.3 Test Results.

Values of the average critical buckling stresses $\sigma_{\text{cr}} = \frac{P_{\text{cr}}}{\Omega}$ for all 95 tests, are given in Tables 10 and 11 below.
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Observez (bridage de la roue supérieure).
These tables given all necessary information for the construction of the $9 \sigma_{cr} = f (\lambda)$ curves (Fig. 3.3.1 and 3.3.2), which correspond to new possible combinations of parameters $e_2/e_1$ and $m$.

Figure 3.3.1 represents the same curves for whole flange shapes DIE 10 and DIE 20. The curves present a discontinuity between $\lambda = 100$ and $\lambda = 130$, due to change in cross section.

As it is seen from the previous diagrams, the force carried by the column (shaped or WF) is higher for the loading cases $e_2/e_1 = 0$ and $e_2/e_1 = -1$ than is in the case of $e_2/e_1 = 1$. Therefore, it is worthwhile to differentiate these loading cases in the design of columns. This is precisely what is done in the practical design formula derived in the following chapter.

**TABLE 10**

Average critical stresses for sections DIE10 and DIE 20 in terms of $\lambda, e_2/e_1$, and $m$.

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<th>$\lambda$</th>
<th>$m$</th>
<th>$e_2/e_1 = +1$</th>
<th>$e_2/e_1 = 0$</th>
<th>$e_2/e_1 = -1$</th>
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$m = 6$
### TABLE 11

Average critical stresses for sections PN 22 S in terms of $\lambda$, $e_2/e_1$ and $m$.

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<th>$\lambda$</th>
<th>$e_2/e_1 = +1$</th>
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<th>$e_2/e_1 = -1$</th>
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PART I

DIVISION I

THEORETICAL ANALYSIS ON THE BUCKLING OF ECCENTRICALLY LOADED COLUMNS AND COMPARISON WITH TEST RESULTS.

General

As mentioned in the introduction, we wish to arrive at two distinct types of conclusions based on the test results:

a) A simple design method for columns under oblique compression.

b) A theory, as complete as possible, of the observed instability phenomena, leading to a thorough understanding of these phenomena and enabling us to verify the validity of the simple design method outside the limited region covered by the tests.
DIVISION I
CHAPTER IV
ESTABLISHMENT OF A PRACTICAL INTERACTION FORMULA

4.1 Possible practical solutions of the Buckling Problem for Eccentrically Loaded Columns.

The problem of columns loaded with an oblique compression force is one of a considerable technical importance. In the past, a large number of researchers have tried to establish a practical design method. However, all these methods were developed on the simplest case of a column loaded with an eccentricity in the weak plane. The problem of an eccentrically loaded column in the strong plane is discussed in a very limited number of papers. The instability of the section causes rotation, thus complicating the problem. Finally, the problem of a column loaded with an oblique compressive force is discussed in only two or three papers. As known, this case is the most frequent one, in columns of any framework, as well as in lattice beams, when restraining moments due to the rigidity of the joints are taken into consideration.

The proposed practical solutions are based on:

a) An elastic analysis.

b) An interaction formula.
Both of them will be examined successively:

A. SOLUTIONS BASED ON A PURELY ELASTIC ANALYSIS OF COLUMN BEHAVIOR:

The simplest and the most satisfactory solutions are undoubtedly the ones derived by the classical method of Mechanics of Materials. All of them are based on the fundamental assumption that the strength of a column is exhausted when the maximum combined bending and normal stresses reach the elastic limit of the material.

For eccentrically loaded columns this method leads to the well known secant formula:

$$\sigma_{\text{max}} = \frac{P}{\Omega} \left(1 + \frac{e}{r} \sec \frac{\pi}{2} \sqrt{\frac{P}{P_E}}\right) = \frac{P}{\Omega} \left(1 + \frac{e}{r} \sec \psi\right) = R_e$$

(4.1.1)

The failure load $P$ is directly given by the above formula, if the following values are known:

$R_e$ = Apparent elastic limit of the material,
$P_E$ = Elastic buckling load in the loading plane,
$e$ = Eccentricity of the compressive force,
$\Omega$ = Area of the cross section,
$l$ = Height of the column,
\( i = \frac{l}{r} \) radius of gyration in the plane of buckling,
\( r = \frac{i^2}{v} \) core radius in the plane of buckling.

If the eccentricity \( e \) is zero, the formula (4.1.1) gives the maximum load as:
\[
P = \Omega R_0
\]
corresponding to a complete plastification of the column, without taking into consideration the danger of premature buckling.

To remedy this situation, many authors have proposed applying an additional eccentricity to the load eccentricity \( e \). This additional eccentricity was assumed to be a constant, a function of the length or even a function of the produced maximum stresses. All the past work done in this area has been reviewed by TIMOSHENKO in Ref. 4, pp. 34-32. It will be useful, however, to consider some of Prof. D. H. YOUNG's graphs (Ref. 5 and 6) presented in paragraph 4.3. Among the recent progress we should note the work of DUTHEIL (Ref. 7, 8, 9) who succeeded in developing a method equally applicable to lattice beams, and members buckling by flexure grid systems.
The book by STEPHENSON-CLONINGER (Ref. 11) also contains a great number of graphs to facilitate practical calculations.

The basic assumptions should be made in order to analyze flexural-torsional buckling by this method: The imperfection of the member consists only of an initial curvature and an initial torsion. This hypothesis was advanced by NYLANDER (Ref. 12) and discussed further by MASSONNET (Ref. 13). It is especially included in DUTHEIL's theory of flexural-torsional buckling (Ref. 9).

Finally, it should be mentioned that the above solutions have been taken as reference for some official specifications:

- English Code ESS 449 (PERRY formula)
- French Code for works related with the Ministry of Reconstruction and City Development and for private constructions.
- Swedish Code project (Ref. 10).
- The American A.R.E.A. Specifications, etc.

The following two points can be brought into the discussion about theories of this type: (Ref. 9 and the corresponding discussion)
1) The theory leads to relatively complicated formulas. Its practical application necessitates the use of charts.

2) It does not take into account the plastic reserve of strength which is present in the column after the maximum elastic stresses reach $R_e$.

This reserve is related to the type of loading, the shape of the cross section, and the material. Generally, the reserve strength increases:

a) As the load eccentricity increases (it is very small in the case of axial buckling).

b) As cross sections with increasing plastic capacity are considered, i.e. from an I or boxed section to sections of higher plastic capacity. (Fig. 4.1.1)

\[ \text{Fig. 4.1.1} \]

c) As materials with a large yield plateau (A37 steel for example), and materials without a
plateau but with an extended plastic domain (as the aluminum alloys used in the airplane industry) are considered.

Actually, DUTHIEL (Ref. 7, 8) had proposed to increase the elastic limit of the material artificially, to take care of the plastic reserve resistance. However the correction he proposed did not seem to be sufficiently accurate.

It is interesting to note here the values of plastic reserve strength which have been determined in the United States in comparison of the above theory.

1. The JOHNSTON-CHENEY tests (Ref. 14) on eccentrically loaded columns about the weak axis, show that the deviation between the theoretical and experimental values of the maximum critical loads is 0% for \( \lambda = 120 \) and 43% for \( \lambda = 24 \).

2. The CLARK tests (Ref. 22) showed that values of theoretical critical loads obtained by the secant formula were from 22% higher to 15% lower than the experimental values.
3. The elastically restrained columns tested by FISHER, BJILJARD and WINTER (Ref. 23,24) resisted higher ultimate loads than the ones obtained by $\varepsilon_{\text{max}} = \varepsilon_e$. The difference in values is as high as 100% for large values of the load eccentricity ($m = 2$). There was a considerable deviation even in the case where, as proposed by DUTHEIL, the elastic limit had been raised 15% to take into account the beginning of the plastic stage.

B. SOLUTIONS BASED ON INTERACTION FORMULAS

An eccentric load on a column can always be replaced by an axial force $P$ and a pure bending moment $M$.

For each value of $P$ there is a certain value of $M$ which causes the column to collapse under the combined effect of $P$ and of $M$. These two values $(P,M)$ correspond to a well-determined point $C$ in the $(P,M)$ plot in Fig. 4.1.2. A series of similar points are obtained by varying the value of $P$. The geometrical locus of these points is a curve which cuts the two coordinate axes at points $A$ and $B$, with coordinates $(0,P_0)$ and $(M_0,0)$ respectively. One can say that the curve $A C B$ is an interaction curve between the effects of $M$ and $P$. 
In fact, the interaction formula is considered to be a "mean" of all the experimental values, and does not have any theoretical significance. On the other hand all the theory (presented in paragraph A above) can be put under the form of an interaction formula. Nevertheless, this mean representation would assist in choosing the most rational solution.

Thus, it is necessary that the above described interaction curve represents correctly the buckling phenomenon for the extra values where M or P are zero. Therefore, it is necessary to adopt as value of $P_o$ the axial critical buckling load and as $M_o$ the critical bending moment of the bar.

In order to have a diagram with determined ends A and B, the variables were non-dimensionalized to $\frac{M}{M_o}$ and $\frac{P}{P_o}$. The new non-dimensional interaction curve
has the advantage of passing through points A and B which are located at distance of unity from the origin. Its general equation is:

\[ f \left( \frac{P}{P_0}, \frac{M}{M_0} \right) = 0 \]  

(4.1.2)

![Fig. 4.1.3](image)

It is well agreed that for obliquely loaded columns it is impossible to draw a single interaction curve representing all the buckling phenomena involved. A group of such curves should be drawn, with respect to the slenderness ratio \( \lambda \), the eccentricity ratio \( \theta = \frac{e_2}{e_1} \) etc.

However, whatever the complications may be, it can be said at the present time that the method of interaction curves represents the exact phenomenon for the two extremes of axial buckling and pure bending.
It should be mentioned here that this was not the case in the elastic theories explained in part A above.

A detailed study of such interaction curves for a column under oblique compression is undertaken in paragraphs 4.2 and 4.4 below. Let us note that most of the official codes and specifications have adopted for the design of columns under axial compression and bending a formula of the type:

\[ f\left(\frac{P'}{P'_0}, \frac{M'}{M'_0}\right) = 0. \]

The American AISC Specifications are based on the linear formula

\[ \frac{P'}{P'_0} + \frac{M'}{M'_0} = 1, \]

(4.1.3)

representing the simplest curve, the straight line drawn between points A and B.

It is obvious that the formulas adopted by the Belgian Code (N.B.N.5, Article 245 and 246):¹

\[ \frac{P'}{\Omega} + \frac{M'v}{I \left(1-0.0005 \frac{I}{l^2}\right)} = R \]

(4.1.4)

¹ The primes indicate that the values of P and M are not based on collapse loads but on working loads which are equal to the failure loads divided by the factor of safety.
and the ones given in the German Code:

\[ \frac{\Omega P'}{\Omega} + 0.9 \frac{M' v}{I} = R, \tag{4.1.5} \]

are similar to the linear American AISC interaction formula.

By letting \( M' = 0 \) in these formulas (4.1.4 and 4.1.5) we find:

\[ P' = P' = R \Omega \beta \quad \text{and} \quad P' = P' = \frac{R \Omega}{\rho} \tag{4.1.6} \]

respectively.

By letting \( P' = 0 \) we find:

\[ M' = M' = \frac{R I}{\rho} \left( 1 - 0.0005 \frac{l}{b_0} \right) \quad \text{and} \quad M' = M' = \frac{R I}{\rho} \frac{1}{0.9} \quad \text{respectively.} \tag{4.1.7} \]

Dividing the equations (4.1.4) and (4.1.5) by \( R \) and substituting the values of \( P' \) and \( M' \) given above, we obtain the American formula (4.1.3).

After this brief discussion of the different specifications, we come back to the buckling load obtained from the interaction formulas 4.1.3 and 4.1.5 by equating the safety factors to unity.

The problems in question are the following:
1. What would be the appropriate expressions for the axial buckling load \( P_0 \), and the maximum pure bending moment \( M_0 \)?

2. Is formula 4.1.3 sufficiently representative of the theory and test results? Should its form be improved?

3. What should be the value of the moment \( M \), in formula 4.1.3 if the column is under oblique compression and initial moment values linearly proportional to the applied end moments \( M_1 = P e_1 \) and \( P e_2 \)?

All these questions will be discussed in the next two paragraphs.

4.2 The Best Interaction Formula

A. VALUES OF \( P_0 \) AND \( M_0 \)

The answer to the first of the above listed questions, that is the choice of expressions for \( P_0 \) and \( M_0 \), is quite simple.

In the case of the maximum carried axial load \( P_0 \), the solution that anyone would immediately think is the Engesser-Shanley load given by:

\[
    P_0 = \frac{\pi^4 E I}{L^2},
\]
which has as a particular case the classical Euler formula. However, it seems to be objectionable because the Engesser-Shanley theory applies to perfectly straight bars under ideal loading conditions, whereas all the rolled shapes used in industry have many imperfections and are loaded obliquely.

It is sufficient to include the effect of these imperfections in the factor of safety. Hence, the maximum admissible load, "in service" will be

\[ P' = \frac{\pi^2 E I}{s^{1/2}} \]  

(4.2.1a)

where \( s \) is defined as a function of \( \lambda \). The Belgian Code (N.B.MS) (Ref. 5) makes use of the same concept when \( s \) is varied from 1.3 to 4 when the slenderness ratio changes from 20 to 175.

The German Building Code (DIN 4114, Ref. 16) requirements for this case are based on an ideally straight eccentrically loaded column and assuming ideal material properties. The admissible load \( P' \) is a constant fraction \( \frac{1}{1.71} \) of the maximum load \( P_0 \) carried by the column. However, this load can be put under the form
of the equation (4.2.1a), by introducing a new safety factor.

Now, let us examine the expression for \( M_o \), that is, the maximum moment for pure bending.

If there was not any danger of buckling the formula to be used would have been:

\[
M_o = \frac{R_e I}{v}
\]

or better

\[
M_o = K \frac{R_e I}{v}
\]

where \( K \) is a factor larger than unity, taking into account the partial plastification of the section. As an example, the \( K \) value for an I shape of ordinary dimensions is \( K = 1.17 \). Lately, this factor is taken into consideration in many specifications.

If there is danger for buckling, \( M_o' \) should be taken as the bending moment producing the buckling. A simple expression for this moment will be given in paragraph 4.4.

B. THE BEST FORM TO BE GIVEN TO THE INTERACTION FORMULA

The choice of the form to be given to the interaction formula, has been intensively discussed in the United States
particularly the later years.

In 1942, JOHNSTON and CHENEY, pointed to the fact, that a linear form does not directly take into account the increase in stresses due to the deformation of the member. They also emphasized the fact that columns loaded eccentrically in the plane of the web could fail by flexural-torsional buckling in the plastic range at loads corresponding to the maximum (nominal) stress lower than the apparent elastic limit.

The proposition of the systematic use of interaction curves to solve the problem of eccentrically loaded columns was first made by SHANLEY in 1950. At that time he considered to construct these curves with the aid of tests conducted on different materials and different end cross sections. (Ref. 18)

In 1951, HILL and CLARK published (Ref. 19) the results of buckling tests of I shaped aluminum alloy columns which have been loaded eccentrically in the strong axis.

The test results were found to be in accordance with the theory of flexural-torsional buckling. (See
paragraph 6.2 below). The deviation with respect to this theory was from -2.0% to 46.4%. Test results and theory may be brought even closer by modifying the critical load formula to take into account the effect of the deformations in the plane of the web on the moments. This correction having been applied, the deviation between theory and test results varied from -3.2% to 43.5%.

It should be noted that all the experimental critical loads were higher than the ones given by the linear interaction formula (4.1.3).

HILL and CLARK resumed the analysis of their test results in another paper (Ref. 20). They arrived at the conclusion that the test points form into a group around the curve given by the interaction equation:

$$\frac{P}{P_0} + \frac{M}{M_0 \left(1 - \frac{P}{P_E}\right)} = 1$$

(4.2.6)

where $P_E = \frac{\pi^2 E I}{L^2}$ represents the Euler critical load for elastic buckling in the plane of the applied bending moment and $M_0$ represents the maximum bending moment of the member under pure bending.
In 1951, ZICKEL and DRUCKER (Ref. 21), upon request of the "Column Research Council", undertook a thorough analysis of the different possible interaction formulas. They expressed the opinion that formula (4.2.6) is the best available interaction formula.

In 1953, CLARK (Ref. 17) conducted tests on eccentrically loaded columns, but this time, on full, or boxed rectangular sections which cannot fail neither by local buckling of the walls, nor by flexural-torsional buckling. The principal objective of this research was to obtain experimental interaction formulas and compare them with the proposed ones. The conclusions drawn on this point were the following:

1. The secant formula gives critical loads which vary from 22% lower to 15% higher than the experimental ones.

2. The linear interaction formula (4.1.3) gives critical load values which were up to 39% higher than the experimental values.

3. The interaction formula (4.2.6) gives critical
load values within 10% of the experimental ones.

This formula is established for columns buckling by bending and torsion. However, it can also be recommended as a guide for the design of eccentrically loaded aluminum alloy columns which fail by plastic buckling in the plane of the applied bending moment. According to CLARK the same formula may be used in the case of columns where bending is induced by initial imperfections or transverse loads as long as the maximum bending moment occurs around the mid-height.

In 1953, after all the research undertaken by the Aluminum Company of America (ALCOA), HILL, HARTMANN, and CLARK proposed to adopt formula (4.2.6) for the design of eccentrically loaded, steel or aluminum alloy columns.

![Fig. 4.2.1](image1)
![Fig. 4.2.2](image2)
The figures 4.2.1 and 4.2.2, borrowed from their paper, show the degree of accordance between their tests and this formula. In their conclusions the authors emphasize the need for additional research and tests to determine the limits of applicability of the interaction formula (4.2.6) under the action of the unequal end moments and transverse loads considering the different joint conditions at the column ends.

The effect of the joint conditions on the resistance of eccentrically loaded columns was the object of two studies made by BIJLAARD, FISHER, and WINTER (Ref. 24,25). The authors conducted tests on I shaped steel columns, which were elastically restrained at both ends, and loaded eccentrically in the plane of the weak axis.

The above mentioned authors have indicated that the column has a considerable amount of reserve strength, after the condition $\sigma_{\text{max}} = R_e$ is reached. This reserve depends on the slenderness ratio, the load eccentricity, and the degree of restraint at the ends. Therefore, it is impossible to establish a single constant factor taking into account all those variable parameters.
They have compared their test results with the formula (4.2.6) generalized as:

\[
\frac{P}{P_0} + \frac{M_o}{M_o\left(1 - \frac{P}{P_e}\right)} = 1
\]  

(4.2.7)

In this formula, \(P_o\) is the Engesser-Shanley load calculated for a column with a buckling length dependent on the degree of the elastic restraint. (Tables and charts giving graphical solution to this problem have been published by several authors.)

\(M_o\) is the moment corresponding to a complete plastification of the cross section. It is determined by the usual plastic analysis using the actual dimensions of the cross section and the apparent elastic limit obtained from the compression tests. \(P_e\) is the critical Euler load with a column buckling length, \(l_{fl}'\), determined on the assumption of elastic buckling.

Finally \(P\) is the applied compressive force, and \(M_o\) is the part of the total moment \(P_e\) which, at failure, is taken by the column. The rest is taken by the existing elastic restraint at each end.

Figure 4.2.3 compares test results and formula 4.2.7.
It is seen that formula 4.2.7 is quite safe, and besides the experimental points are relatively well grouped. Use of any other interaction formula resulted in a more severe scattering of the test points. Unfortunately, if the buckling length $l'_0$ can be easily determined, the determination of part $M_c$ of the external moment absorbed by the column necessitates a knowledge of the plastic behavior of this column, which in turn can only be determined by successive approximations. This is the reason why the interaction formula (4.2.7) does not cover but a small part of the exact theoretical analysis made by BIJLAARD, FISHER and WINTER (Ref. 25). Consequently, its use does not present any practical advantage.
The above mentioned authors concluded that elastically restrained columns could be designed only by use of tables and charts, and thus the explained theory would serve as a means of calculating such tables.

They also pointed out that in an elastically restrained column, a portion of the applied moment $M_e$ is resisted by the joint, leaving only the remaining part to be resisted by the column itself. The latter decreases with increasing load, in the elastic domain as much as in the plastic one. Therefore it results that a given eccentricity is less detrimental in the case of an elastically restrained column than it would be in the case of the same column simple supported. The official specifications, which are based on the concept of simple supported columns, tend to be extremely conservative when applied to compression members in rigid frames.

In 1954, SHANLEY (Ref. 26) undertook the problem of the design of compression members, and subjected the notion of interaction curves to a thorough analysis.

SHANLEY noted that the linear interaction formula springs from classical combined bending and axial load
formula for short pieces:

$$\sigma_{\text{max}} = \frac{P}{\Omega} + \frac{M_v}{I}.$$ 

If failure is assumed to occur at the time where

$$\sigma_{\text{max}} = R_e,$$

the failure condition is,

$$\frac{P}{\Omega} + \frac{M_v}{I} = R_e,$$

and by substituting in $P_0 = R_e \Omega$ and $M_0 = R_e \frac{I}{V}$

we get the linear form:

$$\frac{P}{P_0} + \frac{M}{M_0} = 1,$$

(4.1.3)

He observed that this formula should be improved to take into account the following:

1. In axial compression, the column does not fail by a complete plastification of the section, but by buckling in the plastic range. Hence $P_0$ must be taken as the critical Engesser-Shanley load.

2. At the instant of failure the existing maximum moment, $M_{\text{max}}$, in the column, may be larger than the initially applied moment $M = P_0$ due to elastic and plastic deformations of the member. It can be said that:

$$M_{\text{max}} = k M$$

(4.2.8)
where \( k \) is a coefficient of proportionality.

In the case of elastic buckling, \( k \) can be calculated easily. The secant formula, mentioned before can be used:

\[
\sigma_{\text{max}} = \frac{P}{\Omega} \left( 1 + \frac{e}{r} \sec \frac{\pi}{2} \sqrt{\frac{P}{P_{\text{cr}}}} \right),
\]

(4.1.1)

where the initial bending stress, \( \frac{P_{\text{cr}}}{\Omega} \), is multiplied by the "coefficient of proportionality."

\[
k = \frac{1}{\cos \frac{\pi}{2} \sqrt{\frac{P}{P_{\text{cr}}}}}
\]

(4.2.9)

If the ratio \( P/P_{\text{cr}} \) is sufficiently smaller than unity, the cosine term can be replaced by the first two terms of its series:

\[
\cos \psi = 1 - \frac{\psi^2}{2}
\]

and thus

\[
k \approx \frac{1}{1 - \frac{\pi^2 P}{8 P_{\text{cr}}}}
\]

(4.2.10)

Note that this formula is not much different than the one for a member with an initial sinusoidal curvature:

\[
k = \frac{1}{1 - \frac{P}{P_{\text{cr}}}}
\]

(4.2.11)
Since the expression (4.2.11) is much easier in form than (4.2.9), it will be used from now on, even in the case where failure occurs previous to plastic deformation. Although the application of \( k \) into the plastic range is not correct at all, it can be partially plastified by considering the following:

1. In eccentric compression an appreciable part of the cross section remains elastic.

2. In the modern rolled shapes in I or WF sections the plastic deformations of the flanges under compression do not have any significance because they are limited by local buckling.

Formula (4.1.3) may be improved by substituting the above mentioned factor to take the form:

\[
\frac{p}{P_0} + \frac{M}{M_0 \left(1 - \frac{P}{P_0}\right)} = 1
\]

(4.2.6)

This equation is nothing but the empirical interaction form proposed by HILL and CLARK (Ref. 20). SHANLEY's presentation of the formula is also a justification of its essential form.

As the graphical method is concerned, two possible
solutions are present. One would be to carry $P/P_0$ as ordinates and \( \frac{M}{M_0 - \left(1 - \frac{P}{P_0}\right)} \) as abscissas, and was done by HILL, HARTMANN and CLARK (Ref. 23). The figures 4.2.1, 4.2.2 and 4.2.3 show such a representation together with the formula (4.2.6).

The grouping of the test points around the straight line from (1,0) to (0,1), is an indication of the validity of this formula.

Equation (4.2.6) can be written also in this fashion (after SHANLEY):

\[
\frac{P}{P_0} + \frac{M}{M_0 \left(1 - \frac{P}{P_0} \frac{P}{P_0}\right)} = 1.
\]

Introducing the notations:

\[
M/M_0 = x, \quad P/P_0 = y, \quad P_0/P_0 = \eta,
\]

the equation becomes

\[
x + \frac{y}{1 - y} = 1
\]

or

\[
y = (1 - x)(1 - \eta y)
\]

This way of writing has the advantage of showing that the essential parameter is the ratio of the
actual axial buckling load, to the Euler load of the column for elastic buckling in the loading plane.

Fig. (4.2.4) represents a group of interaction curves, all passing through the same extreme points 1.0 and 0.1, and representing equation (4.2.13). As seen in the figure, the curves approach the origin for increasing values of \( \eta \).

\[ \eta = 1 \] corresponds to elastic axial buckling in the loading plane, that is, to a column of a large slenderness ratio.
On the other hand, $\eta = 0$ corresponds to a column whose Engesser-Shanley load is very small compared with its Euler load, therefore to a very short column. It should be emphasized, however, that the parameter $\eta$ is completely independent of the slenderness ratio $\lambda$. $\eta = \text{constant}$ does not necessarily impose that $\eta = \text{constant}$, because of differences in cross sections.\(^1\)

The representation proposed by SHANLEY (Fig. 4.2.4) is superior to the one by HILL, HARTMANN and CLARK (Fig. 4.2.1 and 4.2.2), because it shows clearly the behavior of the parameter $\eta$.

Regarding this fact, the tests run by CHIPMAN upon SHANLEY's request (Ref. 26) are clearly conclusive. The tested members were all aluminum alloys, except one series of low alloy carbon steel. The tested sections included square and rectangular box sections, U, T, I and full rectangular sections.

\(^1\) On the contrary, for pieces having exactly the same cross section, as the PN 22's used in our tests, we have:

$$
\begin{align*}
P_e &= \frac{\pi^2 E I_y}{\mu}, \quad P_x = \frac{\pi^2 E I_x}{\mu}, \\
\text{from where} & \quad \eta = \frac{E_x I_y}{E I_x}, \\
\text{but} & \quad I_y/I_x \quad \text{is a constant, and} \quad E_x/E \quad \text{is a function of} \quad \lambda y.
\end{align*}
$$

Hence $\eta$ becomes a determined function of $\lambda y$. 
A summary of the test results, and the interaction curves corresponding to the two extreme cases \( M = 0 \) and \( M = 1 \) are given in Fig. 4.2.5. The dotted line represents the curve \( y = 1 - x^2 \) which corresponds to a complete plastification of a full rectangular section loaded eccentrically. The effect of the deformations on the loading of the column are neglected.¹

It is seen that the region bounded by \( \eta = 0 \) and \( \eta = 1 \) includes most of the test points. A certain number of points lie in the region bounded by the dotted line of complete plastification and the curve \( \eta = 0 \).

The interval \( 0 < \eta < 1 \) is divided into 5 equal regions. Five diagrams, corresponding to each of these intervals were constructed for each 0.2 increment.

The figures 4.2.6 to 4.2.10 represent the test points selected in five groups according to every interval. Although there is no perfect agreement of theory and experiment the test points tend to fall into the region bounded by the two extremes of \( \eta \).

¹ For the derivation of the equation \( y = 1 - x^2 \) refer to the paper by SHANLEY, or to the one by BAKER (Ref. 27).
RESULTS OF ALL TESTS
Fig. 4.2.5

TEST RESULTS FOR
$0.8 < \eta < 1.0$
Fig. 4.2.6

TEST RESULTS FOR
$0.6 < \eta < 0.8$
Fig. 4.2.7

TEST RESULTS FOR
$0.4 < \eta < 0.6$
Fig. 4.2.8

TEST RESULTS FOR
$0.2 < \eta < 0.4$
Fig. 4.2.9

TEST RESULTS FOR
$0 < \eta < 0.2$
Fig. 4.2.10
From the discussion of the literature on this subject, it is concluded that the simple formula:

\[ \frac{P}{P_0} \frac{M}{M_c} \left( 1 - \frac{P}{P_c} \right) = 1 \]

conveniently represents the buckling phenomenon of an eccentrically loaded column. At present, let us discuss in what way should this formula be improved to equally represent our tests under "oblique" loading. First, let us mention a point which was first considered in 1932 in the design charts by YOUNG (Ref. 6). A column may fail in two entirely different ways:

a) At the instant of failure, the maximum moment (taking into account the deformations of the column) is exerted on the entire member. Failure may occur by plastic buckling in the plane of the web or by flexural-torsional buckling. This phenomenon is represented in a formula of the type (4.2.6). The maximum initial moment \( P_{e1} \) should be replaced by a lower equivalent moment \( M_{equ} \), due to the fact that the tendency of the column for bending depends on the over-all moment diagrams. The expression for this equivalent moment will be presented in the next paragraph.
b) In spite of the elasto-plastic deformations of the member, the most dangerous section is still at the ends where the moment $P_{e1}$ is applied. In this case the column fails by function of a plastic hinge in this section and eventually by local buckling of the compressed flange. Hence, the failure condition is given by the classical combined bending and axial load formula:

$$\frac{P}{\Omega} + \frac{Mv}{I} = R_e$$

The plastic behavior before failure, as observed in this type of test, will be taken care of by a constant of proportionality to be introduced in the above equation. The value of $I/v$ above should be replaced by $K I/v$, where $K$ is the coefficient of proportionality for the sections tested, and is approximately equal to 1.71. \(^1\)

With this modification, it can be shown that the above equation takes the form of the interaction formula:

In the case of DIE 20, where the flanges are very thin, it is nearly impossible to reach a completely plastified section, because failure always occurs by local buckling of the compressed flanges. To induce this consideration into our formula, the value of $K = 1.17$ will be replaced by $K = 1.08$ as proposed by DUTHEIL or evey by unity.
As mentioned above, an expression for the equivalent bending moment should be established. This is a uniformly distributed moment along the full height of the column, causing the same failure load as the experimental one.

This expression will certainly depend on the mode of failure. It could be a flexural-torsional buckling or a buckling due to excessive plastic deformations in the place of the column. These two cases will be investigated separately.

4.3 The Equivalent Bending Moment

A. EQUIVALENT MOMENT EXPRESSION FOR A COLUMN UNDER OBLIQUE COMPRESSION, BUCKLING BY BENDING AND TORSION:

Some years ago, one of the authors has investigated the elastic flexural-torsional buckling of obliquely loaded columns using the energy method. (Ref. 23).
According to the developed calculations, the equivalent moment expression was found to be:

\[ M_{\text{eq}} = \sqrt{0.3 (M_1^2 + M_2^2) + 0.4 M_1 M_2} \]  

(4.3.1)

This formula has as special cases:

for \( M_2 = M_1 \) : \( M_{\text{eq}} = M_1 \)  
for \( M_2 = 0 \) : \( M_{\text{eq}} = 0.543 M_1 \)  
(4.3.2)

for \( M_2 = M_1 \) : \( M_{\text{eq}} = 0.447 M_1 \)

Although this formula is known to be valid only for the elastic range, due to the lack of a better one, it will be used in all cases.

It should be clear that equation (4.3.1) can be written in terms of an "equivalent eccentricity" as a function of the real eccentricities \( e_1 \) and \( e_2 \). In this case

\[ e_{\text{eq}} = \sqrt{0.3 (e_1^2 + e_2^2) + 0.4 e_1 e_2} \]  

(4.3.3)

B. EQUIVALENT ECCENTRICITY EXPRESSION FOR A COLUMN UNDER OBLIQUE COMPRESSION, BUCKLING BY EXCESSIVE PLASTIC DEFORMATIONS IN THE LOADING PLANE.

a) YOUNG's theory.

As will be seen from the complexity of the theory presented in the 5th chapter, it is nearly impossible to arrive at a simple expression for the equivalent
eccentricity, taking into account the plastification of the section.

Nevertheless, an approximate solution for this eccentricity can be found, admitting a failure which occurs when the combined bending and normal stresses reach the apparent elastic limit (Ref. 6). For the sake of multiplication, let:

$$\frac{e_1}{e_1} = 0$$  \hspace{1cm} (4.3.4)

and

$$\frac{1}{2} \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \sqrt{\frac{P}{P_2}} = \phi$$  \hspace{1cm} (4.3.5)

YOUNG indicates in his paper two possible cases:

1. When the critical column load, $P$, is smaller than:

$$P_2 = \left( \arccos \theta \right)^2 \frac{EI}{I^2}$$  \hspace{1cm} (4.3.5)

In this case, the maximum moment occurs at the column end. (Fig. 4.3.1a)

$$M_{max} = M_1 = P_{e1}$$  \hspace{1cm} (4.3.7)

Hence, the maximum combined bending and normal stresses will be:

$$\sigma_m = \frac{P}{\Omega} \left( 1 + \frac{e_1}{r} \right)$$  \hspace{1cm} (4.3.8)

when the axial load $P$ reaches the value $P_2$, the tangent at the column end $A$ becomes parallel to the line of
action of the compressive force.

2. When $P$ gets larger than $P_2$, the location of maximum eccentricity moves from $A$ to $B$ and acts on the full length of the member (Fig. 4.3.1b). YOUNG showed that the maximum eccentricity is equal to:

$$e_{\text{max}} = e_1 \frac{q}{\cos \theta} \quad (4.3.9)$$

where

$$q = \frac{\sqrt{\theta^2 - 20 \cos 2 \psi + 1}}{2 \sin \psi} \quad (4.3.10)$$

Thus, the maximum stress in the column can be written as

$$\sigma_{\text{max}} = \frac{P}{\Omega} \left( 1 + \frac{e_1 - q}{r \cos \psi} \right) \quad (4.3.11)$$

For $P = P_2$ the expression for $e_{\text{max}}$ (4.3.9) reduces to $e_1$, since:

for $P = P_2$ \quad $q = \cos \psi$

On the other hand when the eccentricity is constant, say $\theta = 1$, the formula (4.3.11) simplifies to give the secant formula:

$$\sigma_{\text{max}} = \frac{P}{\Omega} \left( 1 + \frac{e_1 - 1}{r \cos \psi} \right) \quad (4.1.1)$$

It is understood that \quad $q$ should be given the value of unity.
It can be easily verified that the expression for q, \((4.3.10)\), satisfy both of the above conditions.

According to the fundamentals of YOUNG, the critical load is attained when \(\sigma_{\text{max}}\) reaches \(R_e\). It is well defined by the two equations (cf. to 4.3.8 and 4.3.11) given below.

If \(P < P_2\) \(\frac{P}{\omega} (1 + e_1/r) = R_e\) \(\quad (4.3.12)\)

If \(P > P_2\) \(\frac{P}{\omega} (1 + e_1/r \frac{q}{\cos}) = R_e\) \(\quad (4.3.13)\)

b. Computation of the equivalent eccentricity:

The failure load for a column loaded with a constant eccentricity \(e_{\text{equ}}\) is given by

\[
\frac{P}{\omega} \left(1 + \frac{e_{\text{equ}}}{r} \frac{1}{\cos \psi}\right) = R_e
\]
\(\quad (4.3.14)\)

The "equivalent eccentricity" is defined as the eccentricity causing failure of the fictitious column at the same load as the actual failure load of the real column. Therefore, the values of \(P\) in the formulas \((4.3.12), (4.3.13), (4.3.14)\) are the same. Hence, the expressions in parentheses should be equal. It is concluded that the reduced equivalent eccentricity

\[
\eta = \frac{e_{\text{equ}}}{e_1}
\]
\(\quad (4.3.15)\)
will be defined by the following equations:

for \( P < P_2 \) : \( n = \cos \psi \) \hspace{1cm} (4.3.16)

for \( P > P_2 \) : \( n = q = \frac{\sqrt{w^2 - 2 \theta \cos 2 \psi + 1}}{2 \sin \psi} \) \hspace{1cm} (4.3.17)

The case \( P = P_2 \) can be written in a more convenient form. By substituting \( P_2 \) by its expression from (4.3.6) into (4.3.5):

\[
\psi_1 = \frac{\pi}{2} \frac{P_1}{P_2} = \frac{\pi}{2} \frac{(\arccos \alpha)^2 EI}{\pi^2 EI} = \frac{1}{2} \arccos \theta,
\]

or

\[
\theta = \cos 2 \psi_1.
\]

c). Construction of the \( \frac{\varepsilon_{eq}}{\varepsilon_1} = f(\theta, \psi) \) diagram

Now, let us examine the variation of the non-dimensional ratio \( n \) with respect to \( \theta = \frac{\varepsilon_2}{\varepsilon_1} \). Formulas (4.3.16) and (4.3.17) indicate that the ratio \( n \) is a function of \( \theta \) and \( \psi \). Therefore, a diagram with \( \theta \) as abscissas and \( \psi \) as ordinates may be drawn. This diagram will consist of a number of curves corresponding to different values of \( \psi \).

Let us first examine the curve corresponding to

\[
\theta = \cos 2 \psi \quad \text{(4.3.18)}
\]

which is the boundary for the ranges of applicability of formulas (4.3.16) and 4.3.17).
Since, along the curve
\[ n = \cos \psi \]  \hspace{1cm} (4.3.15)
equation 4.3.18 is simplified as:
\[ \theta = (2\cos^2 \psi - 1) = 2n^2 - 1. \]  \hspace{1cm} (4.3.19)

The graphical representation of equation (4.3.19) is a parabola which is shown as dotted in Fig. 4.3.2. The governing formula to the left of this parabola is 4.3.16. It is represented with a series of horizontal lines numbered in values of \( \psi \). Fig. 4.3.2

To the right of the dotted parabola, formula (4.3.17) governs. It is indicated by a series of curves, numbered in values of \( \psi \), which meet the horizontal lines of the same \( \psi \) value along the dotted parabola.

For all values of \( \theta \), \( n = n(\theta) \) reduces to \( n = 1 \) for \( \psi = 0 \). That is, for any value of the ratio \( \frac{\varepsilon_2}{\varepsilon_1} = \theta \), the equivalent eccentricity equals the end eccentricity \( \varepsilon_1 \). This case corresponds to very short columns, whose failure load is well below the Euler load.

On the other hand, for very long columns loaded
Fig. 4.5.1

Fig. 4.6.2.
with small eccentricities, the failure load is very close to the Euler load. Hence, the value of $\psi$ practically becomes $\frac{\pi}{2}$, and consequently equation (4.3.17) becomes:

$$n = \frac{\sqrt{\theta - 2}\theta(-1) + 1 - \theta + 1}{2\times1}.$$

This relation is represented by a straight line joining points (-1,0) and (41, 41).

Considering all the above, the $K, K(\theta, \psi)$ diagram can be established completely.

At present, it should be noted that this diagram can be utilized only by trial and error, since the parameter $\psi = \frac{\pi}{2} \sqrt{\frac{P}{P_s}}$ of a determined curve depends on the failure load $P$, which itself is a function of the equivalent eccentricity $e_{equ} = ne_1$ (ordinate on the diagram).

It would have been much more practical to have a diagram with parameter $\lambda$. Unfortunately though, the relation between $\psi$ and $\lambda$ is not simple. Indeed, by substituting the failure load, as given from equation (4.3.13):

$$P = \frac{\Omega R_s}{1 + q \frac{c_1}{r} \frac{1}{\cos \psi}},$$

(b)
into the defining equation for:

$$\psi^2 = \frac{\pi^2 P}{4 P_s} = \frac{P_l^2}{4 E I}$$  \hspace{1cm} (a)

we find

$$\psi = \frac{\lambda^2 R_e}{4 E} \frac{1}{1 + \frac{e_1 \sqrt{\theta^2 - 2 \theta \cos 2\psi + 1}}{\sin 2\psi}}$$  \hspace{1cm} (c)

which shows that the slenderness ratio $\lambda$ depends not only on $\theta$, but also with respect to the parameters $R_e/E$ and $e_1/r$.

The relations are much simpler on the left side of the dotted parabola ($P < P_2$). Instead the relation (b) we have from formula (4.3.12),

$$P = \frac{\Omega R_e r}{r + e_1}$$

By replacing this value of $P$ into (a) we find

$$\psi^2 = \frac{\lambda^2 R_e}{4 E} \frac{r}{r + e_1}$$  \hspace{1cm} (d)

It is seen that the slenderness ratio $\lambda$ does not any more depend on parameter $\theta$, but only on $\frac{R_e}{E}$ and $\frac{e_1}{r}$.

Thus, the curves $n = f(\lambda)$ are horizontal lines, as in the case of $n = f(\psi)$.

Although the curves $\lambda = \text{constant}$ and $\psi = \text{constant}$ do not coincide, it results from this discussion that
they are similar for the portion of the diagram bounded by the dotted parabola and the 45° line.

The following fundamental conclusion can be drawn from Fig. 4.3.2.

According to the nature of the problem, the equation expressing \( e_{equ} \) as a function of the ratio \( \theta \) of the end eccentricities, can be varied between the extreme values \( n = 1 \) and \( n = \frac{1 + \theta}{2} \). For average values of \( \psi \) \( (\psi \approx \frac{\pi}{2}) \), this rule is not much different from:

\[
e_{equ} = \sqrt{0,3(e_1^2 + e_2^2) + 0,4e_1e_2}
\]

which was presented above, for the case of flexural-torsional buckling failure. It is represented in the diagram by mixed dashed lines.

As a conclusion to this paragraph, it should be mentioned that calculation of equivalent eccentricities for all cases by formula 4.3.3, will result in no serious errors.

4.4 Proposed Interaction Formulas.

The executed tests clearly indicated that an I shaped column under oblique loading may present one of
the three following types of failure:

1) By excessive plastic deformation at one end.
2) By buckling in the plane of the web.
3) By lateral flexural buckling, and torsion.

The first case is not a buckling phenomenon. It obeys the usual combined bending and axial force formula:

\[
\sigma_{\text{max}} = \frac{P}{\Omega} + \frac{M_{\text{max}}}{K} = R_e
\]

(a)

where \(K\) is a factor of plastification of the section, in paragraph 4.2, and later reduced to take into account local buckling of the flanges. The formula (a) can be put into an interaction form:

\[
\frac{P}{P_o} + \frac{M}{M_o} = 1
\]

FORMULA A:

\((4.4.1)\)

The interaction formula to be chosen for the above buckling cases 2 and 3, will be:

FORMULA B:

\[
\frac{P}{P_o} + \frac{M_{\text{equ}}}{M_o \left(1 - \frac{P}{P_o}\right)} = 1
\]

\((4.4.2)\)

where \(P_o\) is the critical Engesser-Shanley load, \(P_E\) the Euler buckling load for the elastic range in the loading plate, \(M_{\text{equ}}\) the equivalent moment as given by the formula \((4.3.1)\), and \(M_o\) the critical bending moment for
the member subjected to pure bending.

The only problem to be solved is to find an expression for $M_o$. The expression for $M_o$ in the elastic range is: (Ref. 4, p. 252)

$$M_o = \frac{\pi \sqrt{BC}}{l} \sqrt{1 + \frac{\pi^2 a^2}{l^2}} \quad (4.4.3)$$

where $C$ is the torsional rigidity of the section, $B=EI_y$ is its flexural rigidity, and $a^2$ is defined by

$$a^2 = \frac{Dh^2}{2C} \quad (4.4.4)$$

where $h$ is the distance between the center lines of the flanges, and $D \approx \frac{B}{Z}$ is the flexural rigidity of the individual flange in its plane.

We have succeeded in generalizing formula (4.4.3) for the case where the elastic limit is exceeded. (paragraph 6.4). However, the formula is too complicated to have practical applications. Thus, it is necessary to approach the problem with approximate methods. On the same line of thinking, DE VRIES (Ref. 17) has shown that the critical bending stress depends practically on the parameter $\frac{1}{b} \frac{h}{e}$, where $l$ is the length of the member, and $h$, $b$, $e$ the dimensions of the cross section defined by Fig. 4.4.1.
The critical pure bending stresses, based on DE VRIES' publication, will be equal to the following expressions:

\[
\sigma_{cr} = \begin{cases} 
25 - 9.3 \times 10^{-4} \left( \frac{lh}{be} \right)^2 \text{ kg/mm}^2 & \text{for } \frac{lh}{be} < 1000 \\
15700 \frac{lh}{be} \text{ kg/mm}^2 & \text{for } \frac{lh}{be} > 1000
\end{cases}
\]  

(4.4.5)

By definition the critical bending moment is:

\[ M_c = \sigma_{cr} I \]

The above formula can be written as a convenience for ultimate applications, in the form:

\[ M_c = R_e \frac{I}{v} \times \frac{\sigma_{cr}}{R_e} = \alpha R_e \frac{I}{v} \]  

(4.4.6)

where \( \alpha \) is the bending coefficient:

\[ \alpha = \frac{\sigma_{cr}}{R_e} = \begin{cases} 
1 - 3.72 \times 10^{-4} \left( \frac{lh}{be} \right)^2 & \text{for } \frac{lh}{be} < 1000 \\
\frac{528}{l} \frac{h}{be} & \text{for } \frac{lh}{be} > 1000
\end{cases} \]  

(4.4.7)

It is proposed that the failure load of the columns be calculated by both formulas A and B, and the lower of the two chosen as the critical load. It is understood that this method by itself gives an indication as to the mode of failure in
each column.

On the other hand, the similarity between the mode of failure predicted by formulas A or B, and the one actually occurred during the tests is a way of judging the accuracy and value of the proposed formulas A and B.

4.5 Comparison with Test Results and Conclusions.

Since interaction formulas are primarily established for use in official specifications, it is advantageous to perform our calculations using the officially specified elastic limit for members subjected to buckling, which in Belgium is $R_e = 25 \text{ kg/mm}^2$.

Elastic limit values (or first yield) obtained from stub column tests are:

<table>
<thead>
<tr>
<th>DIE 10</th>
<th>$R'_e = 27 \text{ kg/mm}^2$ (Fig. 2.3.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIE 20</td>
<td>$R'_e = 23 \text{ kg/mm}^2$ (Fig. 2.4.8)</td>
</tr>
<tr>
<td>PN 22 S</td>
<td>$R'_e = 27 \text{ kg/mm}^2$ (Fig. 2.5.7)</td>
</tr>
</tbody>
</table>

Note that these values include the effect of the variation in material properties and the residual stresses.

The derivation of the above values from the officially accepted $25 \text{ kg/mm}^2$, causes an equal discrepancy in the
in the values of $P_0$ and $M_0$ since these values are pro-
portional to $R_e$. For this reason, in addition to the
straight line equation:

$$\frac{P}{P_0} + \frac{M}{M_0} = 1,$$

the straight lines:

$$\frac{P}{P_0} + \frac{M}{M_0} = \frac{23}{25}, \quad \frac{P}{P_0} + \frac{M}{M_0} = \frac{27}{25},$$

have been drawn on the above mentioned interaction dia-
grams. The reader should note that a correction evalua-
tion of the chosen interaction formula can be achieved
by comparing the plotted test points for each case with
the corresponding straight line.

To avoid any complications, detailed information
has been provided on each diagram in order to have the
reader differentiate for each case. Each point corresp-
doning to section FN 22 3 is designated by a symbol
indicating the entire corresponding test, with reference
to the table below:

<table>
<thead>
<tr>
<th>Slenderness ratio</th>
<th>Ratio of end eccentricities $e_2/e_1$</th>
<th>Maximum eccentricity in core radii</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>$e_2/e_1 = \pm 1$</td>
<td>$m = 0.5$</td>
</tr>
<tr>
<td>80</td>
<td>$e_2/e_1 = 0$</td>
<td>$m = 1$</td>
</tr>
<tr>
<td>100</td>
<td>$e_2/e_1 = -1$</td>
<td>$m = 3$</td>
</tr>
<tr>
<td>130</td>
<td></td>
<td></td>
</tr>
<tr>
<td>175</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Indicated inside the symbol defining the slenderness ratio.

Indicated aside the symbol defining the slenderness.
For sections DIE 20 and DIE 10 the previous symbols are provided with a little white or black flag respectively.

Throughout the treatise the following abbreviation will be adopted.

**EXPRESSION A:** The expression given by formula (4.4.1) and governing the failure by excessive plastid deformation at one end of the column.

**EXPRESSION B:** The expression given by formula (4.4.2) and governing the failure by one or the other buckling mode.

**EXPRESSION C:** The expression

\[
\frac{P}{P_e} + \frac{M_{eq}}{M_e} = 1
\]

(4.5.1)

which is nothing but the expression B with the term \(1 - \frac{P}{P_E}\) left out.

The test results using expressions A or B have already been discussed. As mentioned at the end of paragraph 4.4, for each test the most severe formula of the two and at the same time the most representative of the buckling mode may be determined. Values of the ratios \(P/P_0\) and \(M/M_0\) corresponding to each expression are represented in a cartesian coordinate diagram. Figures 4.5.1 and 4.5.2 have been drawn for columns failed by
expression A, and present test results DIE 20 and PN 22 sections respectively.

The DIE 10 columns did not furnish any points in these diagrams. Expression B is the determining case for these sections.

Figure 4.5.3 shows all columns according to expression B, and respective test points are plotted for cross sections DIE 10 and DIE 20. Fig. 4.5.4 is relative to section PN 22 S.

Finally, Fig. 4.5.5 groups all diagrams 4.5.1 through 4.5.4 for all the 90 test points. In order to give a better picture of the scattering of the test points the coordinates of each point in Fig. 4.5.5 have been multiplied by $R_e/25$. $R_e$ is the corresponding elastic limit value for every column.\(^1\)

Reference to Fig. 4.5.1 and 4.5.2 reveals the fact that formula A gives safe results for 23 out of 28 tests, and that the scattering of test points is satisfactory. Figures 4.5.3 and 4.5.4 shows that the scattering of test points for the columns buckling under case B is

\(^1\) It is estimated that the reduction in buckling resistance due to a very low value of $R_e$ should be covered by the safety factor.
Fig. 4.5.1.

Fig. 4.5.2.
more severe than in the case of buckling according to A. This proves to be quite natural, considering that formula B represents two buckling modes at a time which is much more complicated than the simple plastic bending considered in A. Formula B provides safety to 51 in a total of 62 tests.

Therefore, formulas A and B together give a safety of 74 out of 90 cases.

An investigation of the causes producing the scattering of the points in Fig. 4.5.3 and 4.5.4, show that the most important factor is the coefficient of proportionality \( \frac{1}{P} \) affecting the effective bending moment.

In this respect it should be noted that the failure load is always less than the Engesser-Shanley load for flexural buckling:

\[ P_{xy} = \frac{\pi^2 E_y I_y}{l_a} \]

Since \( P_E \) is the Euler buckling load, it is given by:

\[ P_E = \frac{\pi^2 E I_x}{l_a} \]

it results that:

\[ \frac{P_{\text{r.m.s.}}}{P_E} < \frac{E_y I_y}{E I_x} \]
Considering the above inequality, it is seen that the neglect of the coefficient of proportionality will result in a decrease of the ratio $M/M_0$ which will be:

"very sensitive" in the case of DIE 10 sections. (Due to large slenderness ratios and high values of $I_y/I_x$.)

"appreciable" for the DIE 20 sections. (Because $E_t/E$ is less than 1.)

"approximately zero" for the PN 22 S (Due to small $I_y/I_x$ values.)

The points which deviate most from the theoretical straight line in the diagram 4.5.3, are those corresponding to DIE 10 sections.

Thus it is worthwhile to recompute everything using formulas A and C.  

Figures 4.5.6 and 4.5.7 show the results of these calculations for sections DIE 10, DIE 20, and PN 22 S respectively. Therefore, Figures 4.5.3 and 4.5.4 will

\[ \text{It is important to note that Figures 4.5.3 and 4.5.4 could not be improved by simply neglecting the factor because expression C, being less severe than B, is no longer the determining factor for all cases.} \]
be replaced by 4.5.6 and 4.5.7. Figures 4.5.1 and 4.5.2 should be improved by the addition of two or three experimental points, while the points on Fig. 4.5.4 will move negligible quantities to the left.
The neglect of the coefficient of proportionality will result in:

a) an appreciable simplification regarding formula B.

b) a slight reduction in safety regarding formula B, since formula C gives safety in 45 out of 62 cases (instead of 50 cases for formula B).

c) an appreciable economy in steel in the case of wide-flange shapes, due to the reduced deviation of the test points from the theoretical straight line (1.0), (0.1).

The advantage of the new interaction formulas and their contribution to the present Belgian specifications is appreciated, by drawing similar diagrams for Figures 4.5.1 through 4.5.6 using the IBN formula:

\[
\frac{P'}{\Omega_{fB}} + \frac{M v}{I (1-0.0005 \frac{L}{h_v})} = R
\]  

(4.5.2)

Since \( \Omega_{fL} \) is defined as \( \Omega \frac{R_{fL}}{R} \), formula (4.5.2) may be written as:

\[
\frac{P'}{\Omega R_{fB}} + \frac{M'}{R \frac{h_v}{I (1-0.0005 \frac{L}{h_v})}} = 1
\]  

(4.5.3)
Let us note that eq. (4.5.3) furnishes not a failure load but a working load. Multiplication of the service load by $R_0/R$ is not sufficient for changing the working load to failure load. In view of the possibility of buckling IBN has adopted a safety factor variable with slenderness ratio. The effect of this variable factor will be well understood if formula 4.5.3 is transformed for failure load. This may be done by replacing in (4.5.3) $R$ by $R_e$, and the admissible buckling stress, $R_e$, by the critical stress, thus eliminating the effect of the variation in the factor of safety. Hence the relation (4.5.3) is transformed as:

$$\frac{p}{R_e} + \frac{M}{R_e I} = 1$$

which can be written as:

$$\frac{p}{P_e} + \frac{M}{M_e} = 1$$

where

$$P_e = R_e I$$

$$M_e = R_e I \left(1 - 0.0005 \frac{L}{b} \right)$$

The experimental points are represented on a $P/P_e$, $M/M_e$ diagram for cross sections DIE 10, DIE 20 and FN 22 S in Figures 4.5.7 and 4.5.8 respectively. Everywhere, the admissible elastic limit is $R_e = 25 \text{ kg/mm}^2$ is used.
Fig. 4.5.5

Fig. 4.5.6
These points were regrouped in Fig. 4.5.9, and the correction for the elastic limit was applied as in the case of Fig. 4.5.5.
These figures show that:

a) There is a considerable scattering of the test points in the area under the theoretical straight line. This proves that the IBN formula does not represent the actual case, but results in appreciable economy in steel.

b) On the other hand, the IBN formula is safe for 31 cases out of 90, which is better than the combinations of A and B (74/90) and A and C (63/90).

c) If the shape factor $K \geq 1.17$ in formula A is replaced by unity, the abscissa $M/M_0$ of all points in Figures 4.5.1 and 4.5.2 are multiplied by 1.17 and
so the 5 points which previously were situated on the left of the straight line move to the right. Figure 4.5.10 gives all the test points determined as above.

In other words, "the combination of formulas (A and B) with the factor \( K = 1 \) in B, gives a safety, which practically equals the one given by IBN (79/90 vs 81/90), also an appreciable economy in material.

It is proposed, therefore, to replace the IBN specifications, by those to be defined below."

In order to complete the preceding discussion, the problem ought to be examined from the service loads point of view. Therefore, the different design methods should be compared on the basis of safety factors and failure loads. The factors of safety with respect to failure loads are presented in two tables (#1 and 2) which have been constructed using:

1. the actual specifications IBN.
2. the above propositions.

The above mentioned tables have been prepared by calculating the working loads using the formulas below.
<table>
<thead>
<tr>
<th>Profil</th>
<th>Étanglement</th>
<th>Méthode de calcul</th>
<th>$e_i/e_i = +1$</th>
<th>$e_i/e_i = 0$</th>
<th>$e_i/e_i = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$m$</td>
<td>0.5</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>40</td>
<td>I.B.N.</td>
<td></td>
<td>1.85</td>
<td>1.81</td>
<td>1.88</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-1</td>
<td>1.87</td>
<td>1.83</td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td>Nouv. propos. avec $\varphi$ selon DIN 4114 gain en % I.B.N.</td>
<td>-1</td>
<td>1.86</td>
<td>1.83</td>
<td>1.89</td>
</tr>
<tr>
<td>60</td>
<td>I.B.N.</td>
<td></td>
<td>1.97</td>
<td>1.92</td>
<td>1.82</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-3</td>
<td>2.03</td>
<td>1.98</td>
<td>1.86</td>
</tr>
<tr>
<td></td>
<td>Nouv. prop. + DIN 4114 gain en % I.B.N.</td>
<td>-1</td>
<td>1.98</td>
<td>1.95</td>
<td>1.84</td>
</tr>
<tr>
<td>80</td>
<td>I.B.N.</td>
<td></td>
<td>1.90</td>
<td>1.97</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-4</td>
<td>1.98</td>
<td>2.07</td>
<td>2.02</td>
</tr>
<tr>
<td></td>
<td>Nouv. prop. + DIN 4114 gain en % I.B.N.</td>
<td>-1</td>
<td>1.92</td>
<td>2.02</td>
<td>1.99</td>
</tr>
<tr>
<td>100</td>
<td>I.B.N.</td>
<td></td>
<td>1.99</td>
<td>2.07</td>
<td>1.88</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-5</td>
<td>2.10</td>
<td>2.23</td>
<td>2.03</td>
</tr>
<tr>
<td></td>
<td>Nouv. prop. + DIN 4114 gain en % I.B.N.</td>
<td>0</td>
<td>1.99</td>
<td>2.13</td>
<td>1.97</td>
</tr>
<tr>
<td>130</td>
<td>I.B.N.</td>
<td></td>
<td>3.36</td>
<td>3.20</td>
<td>2.75</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-6</td>
<td>3.58</td>
<td>3.49</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>Nouv. prop. + DIN 4114 gain en % I.B.N.</td>
<td>+17</td>
<td>2.87</td>
<td>2.89</td>
<td>2.65</td>
</tr>
<tr>
<td>175</td>
<td>I.B.N.</td>
<td></td>
<td>4.10</td>
<td>3.91</td>
<td>3.54</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>-4</td>
<td>4.25</td>
<td>4.15</td>
<td>3.55</td>
</tr>
<tr>
<td></td>
<td>Nouv. prop. + DIN 4114 gain en % I.B.N.</td>
<td>+49</td>
<td>2.86</td>
<td>2.90</td>
<td>2.65</td>
</tr>
</tbody>
</table>

* $m = 6$
(reduced from the corresponding interaction formulas):

**IBN Specifications:**

\[
P_{\text{serv.}} = \frac{R \Omega}{m} \left( \frac{1}{\varphi} + \frac{m}{1 - 0.0005 \lambda_y} \right)
\]

\( (4.5.7) \)

**Proposed Formula:**

\[
P_{\text{serv.}} = \frac{R}{1 + \frac{m - 0.3}{\varphi} \left[ 1 + \left( \frac{\epsilon_2}{\epsilon_1} \right)^2 \right] + 0.4 \frac{\epsilon_2}{\epsilon_1}} \cdot a \left( 1 - \frac{P_{\text{ruins}}}{P_a} \right)
\]

\( (4.5.3) \)

With these as working loads the factor of safety is given by

\[ s = \frac{P_{\text{ruins}}}{P_{\text{service}}} \]

With reference to the above tables one may draw the following conclusions:

1) The proposed formulas give a much larger, and uniform safety, and they are technically superior over the IBN formulas.

2) For the case \( m = 0.5 \), the new formulas result in more limited material savings than the IBN.

This results from the fact that the proposed formulas do not change anything but the bending term in the interaction formula; the compression term remains unchanged.
<table>
<thead>
<tr>
<th>Elancement</th>
<th>Méthode de calcul</th>
<th>$\varepsilon_i/\varepsilon_i$</th>
<th>+1</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0,5</td>
<td>1</td>
<td>3</td>
<td>0,5</td>
</tr>
<tr>
<td>40</td>
<td>I.B.N.</td>
<td>2,12</td>
<td>2,16</td>
<td>2,19</td>
<td>2,44</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,11</td>
<td>0</td>
<td>2,10</td>
<td>2,16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,10</td>
<td>+1</td>
<td>0</td>
<td>2,16</td>
</tr>
<tr>
<td>60</td>
<td>I.B.N.</td>
<td>2,20</td>
<td>2,30</td>
<td>1,98</td>
<td>2,40</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,20</td>
<td>0</td>
<td>2,15</td>
<td>2,26</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,15</td>
<td>+2</td>
<td>+2</td>
<td>2,52</td>
</tr>
<tr>
<td>80</td>
<td>I.B.N.</td>
<td>2,59</td>
<td>2,58</td>
<td>2,46</td>
<td>2,92</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,60</td>
<td>0</td>
<td>2,43</td>
<td>2,52</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,43</td>
<td>+7</td>
<td>+2</td>
<td>2,52</td>
</tr>
<tr>
<td>100</td>
<td>I.B.N.</td>
<td>2,77</td>
<td>2,96</td>
<td>2,52</td>
<td>2,99</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,80</td>
<td>2,89</td>
<td>2,56</td>
<td>2,72</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,64</td>
<td>+5</td>
<td>+4</td>
<td>2,86</td>
</tr>
<tr>
<td>130</td>
<td>I.B.N.</td>
<td>3,52</td>
<td>3,79</td>
<td>3,21</td>
<td>3,82</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>3,55</td>
<td>3,86</td>
<td>3,29</td>
<td>3,47</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>-2</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,81</td>
<td>-25</td>
<td>3,14</td>
<td>2,88</td>
</tr>
<tr>
<td>175</td>
<td>I.B.N.</td>
<td>4,21</td>
<td>4,27</td>
<td>4,16</td>
<td>3,99</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.H.</td>
<td>4,25</td>
<td>4,35</td>
<td>4,32</td>
<td>3,90</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>-2</td>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Nouvelles propositions gain en % I.B.N.</td>
<td>2,85</td>
<td>+49</td>
<td>2,99</td>
<td>3,26</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>43</td>
<td>+25</td>
<td>57</td>
</tr>
</tbody>
</table>
As a result, for all cases where the compression term is larger than the bending term, the economy achieved through the bending term is of a very small order.

Comparison of the IBN failure loads vs safety factors $s$, shows an excessive safety for high slenderness ratio.

The method increasing $s$ with $\lambda$ actually is a way of taking into account the critical eccentricities and imperfections of the industrial members. However, the relation of the safety factors with the slenderness ratios, will be modified for the simple reason that the columns used in our tests were perfectly aligned.
It should be recalled that all the foreign specifications were much more conservative at the time when the first specifications on buckling were established in Belgium (1937). Since then, all the foreign specifications were improved by increasing the allowable stresses for large slenderness ratios.

At the present time, we will not try to give any precise way of modifying the relation \( s = f(\lambda) \). However, the use of increased allowable stresses will influence the safety factors of the test columns, for high slenderness ratios. For purely academic reasons, let us adopt the allowable stresses as given by the German specifications DIN 4114. In order to calculate the corresponding safety factors it is sufficient to replace in equation (4.5.3) \( \phi \) by the inverse of \( \omega \), the reduction coefficient given in DIN 4114.

Tables 1 and 2 give the safety factors obtained as above, and, percentage in material savings with respect to the IBN specifications. It is seen, that safety factors calculated by the new method present a smooth distribution. Also, for high slenderness ratios, they give sufficient safety to compensate for any reduction
in the column's load carrying capacity due to initial imperfections.

It is regrettable that the present test program does not include a large number of the critical cases, namely \( e_1 = e_2 = 0 \), as to be able to draw more general conclusions. However, additional tests will help in eliminating this gap.
DIVISION II

THEORETICAL ANALYSIS OF THE BUCKLING PHENOMENA

General

A practical design method for columns under oblique compression, has been developed and presented in Chapter IV. Next, a thorough analysis of the different instability phenomena observed during the tests will be undertaken. The calculation methods presented in this analysis are too complicated to be applied in practice. Nevertheless, this analysis gives a better understanding of the observed phenomena. Also, the validity of the developed practical design procedure could be checked for members whose form deviates excessively from the experimental ones.

The following phenomena will be studied in the present chapter:

a) Buckling by excessive plastic deformations in the plane of the web.

b) Transversal inelastic flexural-torsional buckling.

For completion we should add a study for the local buckling of the flanges and web. However, this will not be investigated in the following theoretical analysis.
CHAPTER V

PLASTIC BUCKLING IN THE PLANE OF THE WEB

5.1 Review of the Available Methods of Solution

As early as 1910, VON KARMAN had studied the buckling of eccentrically loaded columns, with a main objective to determine the influence of accidental alignment errors in his tests on the plastic buckling of axially compressed columns.

The first rational theory was proposed in 1926 by Ros and BRUNNER (Ref. 30). The authors assumed a sinusoidal initial deformation curve and elastic unloading for the stresses at the convex side of the curved member.

In 1934, CHWALLA attacked the problem with a rigorous approach (Ref. 31), and applied the method suggested by VON KARMAN to a rectangular section. After many laborious calculations, he succeeded in establishing a table of the form:

\[ \alpha = f(\lambda_\varepsilon^c) \]

which gives the ratio \( \alpha \) of the critical buckling stress
to the apparent elastic limit in compression, \( R'_e \). In these calculations, CHWALLA made use of the same stress-strain diagram as ROS and BRUNNER, namely, a diagram characterized by \( E = 2,210,000 \text{ kg/cm}^2; R'_p = 1,900 \text{ kg/cm}^2; \)
\( R'_e = 2,700 \text{ kg/cm}^2; \) and a strain of 6.5% at strain hardening.

In a subsequent study (Ref. 32) CHWALLA investigated the effect of the type of cross section.

The behavior of members with different cross sections subjected to average compressive stresses of \( \approx 1,000 \) and 1,900 kg/cm\(^2\), was also studied by the same method. The following conclusions were drawn:

The critical stresses as calculated for a rectangular section could be used to determine critical stresses for other types of cross sections. It is sufficient to correct the relative force eccentricity \( e/r \) by multiplying the core radius \( r \) by a suitable correction factor \( \phi \).

Obviously, \( \phi \) is larger than unity for all sections having most of the material concentrated around the neutral axis and less than 1 for I shaped sections where the material is placed in the flanges. It was shown by CHWALLA that the latter type of sections corresponds to a value of \( \phi = 0.7 \).
The maximum error involved in using this method of empirical correction is around 3% since the average critical stresses vary between 1,000 to 1,900 kg/cm².

Finally, CHERALLA investigated also the problem of columns of rectangular cross section subjected to oblique compression using the above method. Five cases were investigated; namely; \( \frac{e_2}{e_1} = 1.0; 0.5; 0; -0.5; -1 \); with an average critical buckling stress of 1,500 kg/cm². As a result of his studies, he proposed to use the average of the two end eccentricities \( e_m = \frac{e_1 + e_2}{2} \) for calculating the load carrying capacity of obliquely loaded columns.

The critical stresses, as determined by this procedure, compared to the rigorously obtained values are as shown below:

<table>
<thead>
<tr>
<th>Elevation</th>
<th>Stress (rigorously)</th>
<th>Stress (approached)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excellent</td>
<td>1.500</td>
<td>de 1.517 à 1.523</td>
</tr>
<tr>
<td>Bon</td>
<td>1.500</td>
<td>de 1.504 à 1.628</td>
</tr>
<tr>
<td>Passable</td>
<td>1.500</td>
<td>de 1.544 à 1.866</td>
</tr>
<tr>
<td>Mauvais</td>
<td>1.500</td>
<td>de 1.509 à 2.500</td>
</tr>
</tbody>
</table>

It should be remembered that this comparison was made only for a critical stress of 1,500 kg/cm².
The complexity of CHWALLA's computations is especially due to the use of an experimental stress-strain diagram. Such a diagram could not possibly give a simple analytical expression.

![Diagram](image)

**Fig. 5.1.1**

JAGGER (Ref. 34) reinvestigated the problem using an idealized elastic-plastic stress-strain diagram (Fig. 5.1.1). He established curves furnishing the critical stresses of eccentrically loaded columns, for different relative eccentricities e/r and many types of cross sections. The German Specifications D.I.N.4114 are based on these curves. The specified interaction formula

\[
\frac{\Omega P}{\Omega} + \frac{0.9 M_{c}}{I} = R
\]

gives a larger safety than JAGGER's charts for all values of e/r, \( \lambda \) and any type of cross section.

In connection with the experimental work carried out
at LEHIGH University on obliquely loaded WF columns (Ref. 1) WEISKOPF (Ref. 35) proposed an analytical method based along the same lines as JAGER's method.

WEISKOPF assumed:
a) A stress-strain diagram composed of three straight line segments as indicated in Fig. 5.12.
b) A simplified section of only two flanges. The web, being dimensionless, connects the two flanges in such a fashion that the cross section acts as a whole.

WEISKOPF studied the buckling of such a column under the effect of an oblique force in the plane of the web. His two conclusions are:

1) In columns where the maximum elastic stresses are developed around the mid-height, the load for which these stresses reach the apparent elastic limit, is the maximum load that the column can carry.

2) Columns, where the maximum elastic stresses are developed at one of the column ends and especially under low compressive loads (i.e. large $m = \frac{e_1}{x}$ ratios), can support higher loads than the loads for the case where $\sigma_{\text{max}} = R_o$. The author admits, however, that in
this case the failure load can be reached by premature local buckling of the flange under higher compression.

\[ \text{Fig. 5.1.2} \]

It is obvious that these conclusions do not provide us with anything new. The first one could be obtained without any calculations. An idealized section composed of two flanges, as the one assumed by WEISKOPF, does not have any plastic capacity when the material of the compressed flange is on the horizontal of the diagram. This flange can provide some resistance only in the segment BC. However, the yield strain AB is so large that by the time the compressed flange reaches point B, the column has undergone considerable plastic deformation and the equilibrium between external and internal forces is destroyed.

WEISKOPF's second conclusion is even more interesting. It involves the computation of the portion of the total
carried load corresponding to the formation of a plastic hinge at one end. But, as shown in paragraph 4, this kind of failure can be represented with sufficient accuracy by the simple interaction formula:

$$
\sigma_{\text{max}} = \frac{P}{\Omega} + \frac{M_{\text{max}}}{R_u K_v} = R_v
$$

(4.4.1)

If desired, the local buckling of the compressed flange may be taken into account by a reduced value of the shape factor $K$.

Summarizing, we can say that WEISKOPF's studies do not provide us with an accurate enough representation of the actual phenomenon. As mentioned in the paragraph 2.2, due to the residual stresses, the stress-strain diagram for such an equivalent flange has a very low proportional limit and differs completely from the diagram in Fig. 5.12. Therefore, it is believed by the author that the use of a simplified $(\varepsilon, \sigma)$ diagram introduces more serious errors than the use of an idealized cross section.

The above review of all the existing methods for I shaped columns buckling plastically in the plane of the web reveals the fact that a satisfactory solution to this problem does not exist at the present time. CHWALLA's
method could be modified to take into account the actual material properties and the effect of residual stresses, but it is very long and laborious in application. On the other hand, JAGER's and WEISKOPF's methods are based on assumptions which do not quite correspond to reality.

In the continuation of this paragraph, this problem will be analyzed by replacing the actual cross section by an ideal cross section consisting of only two flanges and which has a stress-strain diagram similar to the one obtained in paragraphs 2.2 and 2.5.

5.2. The Cross Section Consisting of Two Material Flanges and Two Vanishing Flanges

The following assumptions will be made:

The actual I section is replaced by a fictitious equivalent section as defined below.

This equivalent section is composed of two fibers of areas \( \Omega_s^* \), at a distance \( h^* \) apart, and two fibers of infinitessimal areas \( \omega_s^* \), at a distance \( H^* \) apart.

(Fig. 5.2.1)
The quantities $\Omega^*_s$, $h^*$ and $H^*$ will be chosen in such a way as to preserve the geometrical characteristics of the actual cross section. That is:

\begin{align*}
2 \Omega^*_s &= \Omega = \text{area of the actual cross section} \quad \text{(a)} \\
H^* &= H = \text{total height of the actual cross section} \quad \text{(b)} \\
I^*_x &= I_x = \text{moment of inertia of the actual cross section} \quad \text{(c)}
\end{align*}

Let us note that the introduction of the two fibers of infinitessimal area, aims only to satisfy the condition (b) above. These fibers do not complicate the numerical calculations at all since they do not really exist.

Condition (a) readily defines the area $\Omega^*_s$ of the real fibers.
Finally, noting that the fibers are very thin in the $x - x$ direction, the condition (c) can be written as:

$$I_x = I^*_x = \frac{2 \Omega^* h^{*3}}{4} = \frac{\Omega h^{*3}}{4} \tag{5.2.1}$$

from where

$$h^* = \sqrt[3]{\frac{2 I_x}{\Omega^*} = 2 \sqrt[3]{\frac{I_x}{\Omega}} \tag{5.2.2}$$

Using the two infinitesimal fibers we succeed in getting a fictitious cross section entirely equivalent to the actual one.

In fact, the two cross sections have the same radius of gyration:

$$i_x = \sqrt{\frac{I_x}{\Omega}} = \frac{h^*}{2} = i^* \tag{5.2.3}$$

and the same core radius

$$r_x = \frac{2 i_x^3}{H} = \frac{2 I_x}{H} = r^*_x \tag{5.2.4}$$

Hence, the conditions (a), (b), (c) are fulfilled.

As an example, let us determine the properties of the fictitious cross-section for the wide flange IIE 20.

From the handbook we get:
\( \mathcal{A} = 5703 \text{ mm}^2; \quad I_x^* = 3.379 \times 10^4 \text{ mm}^4; \)

\( H = 190 \text{ mm}. \)

From formula (5.2.1)

\[ \Omega^* = \frac{\mathcal{A}}{2} = 2851.5 \text{ mm}^2 \]

and from (5.2.2)

\[ h^* = 2\sqrt{\frac{3879.10^4}{5703}} = 165 \text{ mm}. \]

Finally the distance between the infinitessimal fibers is:

\( H = H = 190 \text{ mm}. \)

The above values define completely the fictitious cross section. From the handbook: \( I_x^* = I_x = 82.4 \text{ mm}. \)

and applying formula 5.2.4 we get

\[ r_x^* = r_x = \frac{2 \times 3879.10^4}{5703 \times 190} = 71.47 \text{ mm} \]

Note that the distance \( h \) between the fictitious material fibers (in this case \( h = 165 \text{ mm} \)) is much less than the total height of the actual cross section \( (h_T = 190 \text{ mm}). \)

As a matter of simplification, the astericks will be omitted in the rest of the chapter. However, it should be well understood that the letters \( h, \) etc. correspond to the equivalent fictitious cross section.
5.3 Tentative Analytical Solution for Plastic Buckling in the Plane of the Web

Let the compressive force \( P \) and the bending moment \( M \) acting on a member of a determined equivalent cross section be known.

The subscript \( i \) will be introduced for all dimensions corresponding to the interior flange, that is the flange situated near the line of action of the compressive force, and \( e \) for everything referring to the exterior flange.

The normal stresses in these flanges of zero thickness will be designated as \( \sigma_i \) and \( \sigma_e \), and assumed positive in compression.

Using the above notations, the equilibrium conditions for the internal and external forces are: (Fig. 5.3.1)

\[
P = \Omega (\alpha_i + \alpha_e)
\]

\[
M = Py = \frac{\Omega h}{2} (\sigma - \sigma_e)
\]

where \( y \) is the load eccentricity in the deformed column.
composed of the initial eccentricity plus the additional eccentricity due to the elastic-plastic deformations of the column.

Formulas (5.3.1) and (5.3.2) can be written as

\[ \sigma_i + \sigma_r = \frac{P}{\Omega} \]
\[ \sigma_i - \sigma_r = \frac{2Py}{\Omega h} \]

Introducing the average compressive stress

\[ \frac{P}{\Omega} = \sigma_m \]  \hspace{1cm} (5.3.3)

the two above relations reduce to:

\[ \sigma_i = \sigma_m \left(1 + \frac{2y}{h}\right); \sigma_r = \sigma_m \left(1 - \frac{2y}{h}\right) \]  \hspace{1cm} (5.3.4)

The corresponding strains \( \varepsilon_i \) and \( \varepsilon_e \) can be easily read off (entering with \( \sigma_i \) and \( \sigma_e \)) the stress-strain diagram for the equivalent fiber (Ref. 2.2 to 2.5).

Thus, these strains are known functions of \( y \):

\[ \varepsilon_i = \varepsilon_i (y) \]  and \[ \varepsilon_e = \varepsilon_e (y) \]  \hspace{1cm} (5.3.4)

From the classical pure bending formula \( \varepsilon = \frac{y}{P} \), it results\(^1\) that:

\(^1\) The formula \( \varepsilon = \frac{y}{P} \) interprets the law of conservation of the plane section in pure bending. It can be shown, by simple symmetry considerations, that this law holds true even in the plastic range, provided that the beam is infinitely long and the material is isotropic in terms of plastic deformations.
\[
\frac{d^2y}{dx^2} = -\frac{1}{\rho} = \frac{e_e - e_i}{h} \quad (5.3.6)
\]

Substituting the expressions for \(e_e\) and \(e_i\), as functions of \(y\) (5.2.9), into the above formula we obtain a differential equation of the following form:

\[
\frac{d^2y}{dx^2} = f(y) \quad (5.3.7)
\]

Integrating the above equation the deflected shape of the column \(y = f(x)\) can be obtained.

Equation 5.3.7 can be written as:

\[
dy' = f(y) \, dx
\]

Multiplying each by \(y' \, dx = dy\), and eliminating \(dx\) we find:

\[
y' \, dy' = f(y) \, dy
\]

from which by integration

\[
\frac{y'^2}{2} = \int f(y) \, dy + \frac{C}{2} = F(y) + \frac{C}{2} \quad (a)
\]

where

\[
F(y) = \int f(y) \, dy. \quad (5.3.8)
\]

From (a) we obtain:

\[
y' = \pm \sqrt{2F(y) + C}
\]

or

\[
\frac{\pm y'}{\sqrt{2F(y) + C}} = 1
\]

and integrating
\[ \pm \int \frac{dy}{\sqrt{2F(y) + C}} = x + C_1 \]  

(5.3.9)

We have tried to resolve the problem into the form given above. Now, a simple representative relation for the experimental stress-strain diagram of the following type should be found:

\[ \varepsilon_1 = f(\sigma) ; \varepsilon_e = f(g) \]

This relation should give simple integrals for the function \( f(y) \) encountered in 5.3.8 and 5.3.9.

Having tried many alternatives, it was proven impossible to find an analytical expression of the \((\sigma, \varepsilon)\) diagram such as to be able to conduct all integrations in formulas 5.3.8, 5.3.9 using simple elementary functions. It follows therefore, that we have to resort to a graphical integration method.

Due to all these, the analytical approach presented above will be abandoned. A new graphical method of successive approximations for obtaining the deflected shape of a column under a load \( P \), will be used.
5.4 Outline of the Graphical Method of Defining the Elastic-Plastic Deformation of a Column

**PRINCIPLE:** The adopted graphical method is based on the Mohr diagram for the elastic range. A simply supported conjugate beam, is loaded with the fictitious uniform load \( p = M \), where \( M \) is the bending moment in the actual beam. The familiar polygon of these loads \( p \), drawn with a polar distance \( EI \) gives the sought deformation \( y = f(x) \).

However, the moments \( M \) are of the form \( M = F_y \), and are not defined unless the relation \( y = f(x) \), deflection curve is known.

Therefore, the column will be given a trial deformation. Using the above graphical method, a new deflection curve will be obtained. After a number of trials the required similarity between the previously assumed and graphically obtained curves will be achieved.
The above mentioned procedure should be repeated for a series of P values. The load carrying capacity of the column P will be defined as the load for which the equilibrium becomes unstable, and there exist many possible deflection curves \( y = f(x) \) for the same value of \( P \).

For every value of \( P \) to be determined, say \( P_{K+1} \), we assume as a trial deflection curve corresponding to the preceding load \( P_K \).

The Mohr method could not be applied directly, since the elastic equation:

\[
\frac{d^2y}{dx^2} = \frac{M}{EI}
\]

does not hold true for a partly plastified section. Plastification of the material results in a reduction of the bending rigidity of the member from \( EI \) to \( kEI \), where \( k \) is a coefficient less than unity.\(^1\)

Hence, the elastic curvature as transformed for plastic deformations is equal to:

\(^1\) For example, it is shown in resistance of materials that, for a material whose \((E, \sigma)\) diagram is composed of Hooke's straight line and a horizontal part (Fig. 5.5.1), we have:

\[
k = \frac{M}{M_e} \sqrt{3 - \frac{2M}{M_e}} \quad (M > M_e \equiv R_e \frac{1}{V})
\]
\[
\frac{d^2y}{dx^2} = -\frac{M}{kEI} \tag{5.4.1}
\]

It follows, therefore, that the Mohr method can be utilized by loading the conjugate beam with an M/k diagram. The factor \( k \) is a complicated function of the degree of plastification in a section, and is dependent of the force \( P \) and bending moment \( M \) acting on this particular section. However, the function

\[
k = k(M, P) \tag{5.4.2}
\]

should be defined in order to apply Mohr's method. We are, however, not interested in the analytical expression of this function. Nevertheless, a graph defining all values of \( k \) will be constructed with \( M \) as abscissas and \( P \) as ordinates. A series of curves, of the equation \( k = \text{constant} \) (Fig. 5.4.1), drawn in this graph by interpolation will provide values of \( k \) for any given \( M \) and \( P \).

5.5 Construction of the graph \( k = f(M, P) \) - Theoretical Explanation

Let us substitute the curvature \( \frac{d^2y}{dx^2} \) in eq. (5.6.1) by its expression from (5.3.6).

\[
\frac{d^2y}{dx^2} = -\frac{M}{kEI} = \frac{\varepsilon_0 - \varepsilon_i}{h}
\]

or
Equating this expression to the one given by formula (5.3.2) we find:

\[ M = \frac{kEI(e_i - e_e)}{h} \]  

From formula (5.2.1)

\[ \frac{\Omega h}{2} = \frac{1}{h} \]

Therefore the preceding equation can be written as:

\[ \sigma_i - \sigma_e = kE(e_i - e_e) \]

or

\[ kEe_i - \sigma_i = kEe_e - \sigma_e \]  

and finally

\[ x_i = x_e, \quad x = kE - \sigma \]  

The above equation gives us enough information for the construction of the \( k = \) constant graph of the equation \( k = \xi (M, \Gamma) \). The curve \( \alpha = kE - \xi \) can be drawn for any given value of \( k \). The resulting curve has the shape shown in Fig. 5.5.1. We will give an arbitrary value of \( \xi_0 \); using the curve \( \alpha = \xi(\xi) \) we will reduce this value to the corresponding \( \alpha_1 \) (point F). The horizontal line of values \( \alpha_1 = \alpha_e \) cuts the curve \( \alpha = \xi(\xi) \)
also at $F'$ whose abscissa necessarily is $\varepsilon_e$. From $\varepsilon_i$, $\varepsilon_e$ and the $(\varepsilon, \varepsilon)$ diagram of the equivalent fiber, we can obtain the stresses $\varepsilon_i$ and $\varepsilon_e$.
Finally, applying the formulas given above, the stresses $\sigma_1$ and $\sigma_e$ can be reduced to the values of $P$ and $M$:

$$P = \Omega (\sigma_1 + \sigma_e)$$  \hspace{1cm} (5.3.1)

and

$$M = \frac{\Omega \, k}{2} (\sigma_1 - \sigma_e).$$  \hspace{1cm} (5.3.2)

which furnishes one point $(M, P)$ on the graph $k = f(M, P)$ for a given value of $k$. Then, another value of $\sigma_1$ is assumed. Repeating the same procedure followed above a second point is determined on the curve with parameter $k = \text{constant}$. This curve is completely constructed point by point. After the completion of the first curve, another round value of $k$ is assumed and a second curve corresponding to the new value of $k$ is constructed. The same procedure is repeated for every new value of $k$ introduced.

Some details on the construction of such a graph for a DIE 20 section will be given below. (Fig. 5.5.6)

It has been decided to draw the curves for parameter values of $k = 0.99$ to $0.90$ by $0.01$ increments; $0.90$ to $0.70$ by $0.02$ increments; $0.70$ to $0.10$ by $0.10$ increments, plus the curve $k = 0.65$. 

The first fundamental diagram is the \((\varepsilon, \varepsilon)\) diagram for one flange under compression; (Fig. 5.5.2) it has been constructed on the assumption that the tangent modulus \(E_t = \frac{d\varepsilon}{d\varepsilon}\) practically follows (Fig. 5.5.2) the linear relationship:

\[
E_t = 14.3 \times 10^{-4} \varepsilon + 26.667 \\
(380.10^{-8} < \varepsilon < 1880.10^{-6}).
\]

For the same interval, the \((\varepsilon, \varepsilon)\) diagram is represented with an excellent approximation by the integral of the function

\[
\sigma = -7,15.10^6 \varepsilon^2 + 26.667 \varepsilon - 1,117
\]

(5.5.4)

The calculated values from this formula do not deviate more than 0.25\% from the corrected experimental values.

From the relation \(\varepsilon_c = f(\varepsilon)\) we can express analytically the quantity \(\alpha = k E - \varepsilon_c\). Substituting \(\varepsilon_c\) by its value from (5.5.4) into the expression for \(\alpha\), we find:

\[
\alpha = k E \varepsilon - \sigma_c = 7,15.10^6 \varepsilon^2 + (21.000 k - 26.667) \varepsilon + 1,117.
\]

For each chosen value of \(k\), the above analytical expression enables us to draw with precision the curve repeated in Figure 5.5.1. The use of this method results in an appreciable saving of tedious work.
The second fundamental diagram is the \( (6, \epsilon) \) diagram of the other flange which is under tension. (Fig. 5.5.3) The derived tangent modulus is shown in this figure. However, it does not lead us to a single expression as in the previous case. Luckily though, the flange under tension remains always elastic. Therefore, its stress-strain diagram obeys to the simple relation:

\[ \sigma_e = 21,000 \epsilon_e. \]  

(5.5.5)

The figure 5.5.1 shows that any horizontal line will cut the curve \( S \) in one, two or three points. A study of the properties of this curve is therefore necessary in order to determine the useful range of the diagram.

a) It is obvious that the interior flange \((i)\) is always compressed (Fig. 5.3.1). If both of the flanges are wider compression, the most compressed flange is the flange \((i)\). Therefore, the points representing flange \((i)\) should be along the part AG of the curve \( S_i \) (Fig. 5.5.1).

b) Let us examine the form of the graph \( k = f(M, P) \) for the limiting case \( k = 1 \) which separates the elastic from the plastic region. This limiting curve represents
the beginning of plastic deformations in the interior flange, which according to Fig. 2.4.4, takes place at a strain of \( \varepsilon_i = 400 \times 10^{-6} \).

As long as the flanges are elastic, the relations \( \varepsilon_i = f(\varepsilon_i) \) and \( \varepsilon_e = f(\varepsilon_e) \) are equivalent to \( \varepsilon_i = E_i \varepsilon_i \) and \( \varepsilon_e = E_i \varepsilon_e \).

Replacing \( \varepsilon_i \) and \( \varepsilon_e \) by their values from (5.3.1) and (5.3.2) it becomes:

\[
\begin{align*}
\rho &= \Omega E (\varepsilon_i + \varepsilon_e); M = \frac{\Omega h E}{2} (\varepsilon_i - \varepsilon_e) \\
\end{align*}
\]

Elimination of \( \varepsilon_e \) between the two equations above, gives us:

\[
\begin{align*}
P + \frac{2M}{h} &= 2E \varepsilon_i \Omega, \quad \varepsilon_i = 400.10^{-6}
\end{align*}
\]

which is the equation of the limiting curve \( k = 1 \).

The above relation shows that the limiting curve is a straight line of a slope \( -\frac{h}{2} \) and has an x intercept of \( 2E \varepsilon_i \Omega \cdot h = 153 \Omega \) kg.

c) Now, let us consider the case of \( k < 1 \), i.e. the case where the column is partially inelastic. Eliminating \( \varepsilon_e \) from equations (5.3.1) and (5.3.2) we find:

\[
P + \frac{2M}{h} = 2 \Omega \varepsilon_i
\]
which is linear and is represented in the (M,P) diagram by a straight line of a slope \(-\frac{h}{2}\). We obtain the following result: On the \(k = f(M,P)\), the locus of points \(\varepsilon_i = \text{constant}\) (or \(\varepsilon_i = -\text{constant}\)) is a straight line of slope \(-\frac{h}{2}\) and having an x-intercept of \(2\Omega\varepsilon_i\).

d) The relative position of the points \(X\) and \(X'\) on the diagram \(kE\varepsilon - \varepsilon = f(\varepsilon)\) (Fig. 5.5.1) for the interior and exterior flanges respectively, can be guesses with enough accuracy, only

1) If \(X\) is at \(A\), the lowest point on the curve, \(A'\) coincides with \(A\); \(\varepsilon_i = \varepsilon_e\) and the column is under uniform compression; \(M = 0\) and \(P = 2\varepsilon_A\Omega\).

The corresponding value of \(\varepsilon_A\) for this case, and for different \(k\) values, can be obtained from the horizontal tangent condition:

\[
\frac{d (kE\varepsilon_i - \sigma_i)}{d\varepsilon_i} = 0
\]

Replacing \(\varepsilon_i\) with its expression given by (5.2.13) we find:

\[
0 = \frac{d (kE\varepsilon_i - \sigma_i)}{d\varepsilon_i} = 2 \times 7.15 \varepsilon \left(26.667 - kE\right)
\]
or

\[
kE\varepsilon_i - \sigma_i = 7.15 \varepsilon^2 - (26.667 - kE)\varepsilon + 1117
\]
and

\[
\varepsilon_A = \frac{26.667 - kE}{14.30}
\]

(5.5.7)
From where the stress \( \sigma \) and then the uniform compressive force \( P \) can be calculated as:

\[
P = 2 \Omega \sigma = 5703 \, \text{kg}.
\]

The formula that has been established, enables us to subdivide the abscissas of the diagram \( k = f(M, P) \) into \( k \) units (Fig. 5.2.10).

2) We have treated in 1) above the case of pure compression \((M = 0)\). Now, let us examine the case of pure bending \((P = 0)\). The condition \( P = 0 \) gives:

\[
\Omega \left( \varepsilon_{i} + \varepsilon_{e} \right) = 0 \quad \text{from where} \quad \varepsilon_{i} = -\varepsilon_{e}
\]

Thus, the two flanges are subjected to equal and opposite stresses. The equation:

\[
kE \varepsilon_{i} - \sigma = kE \varepsilon_{e} - \sigma_{e} \quad \text{(5.5.2)}
\]

should hold true just the same. By introducing the condition \( \varepsilon_{i} = -\varepsilon_{e} = \varepsilon \) we obtain:

\[
kE = \frac{2\sigma}{\varepsilon_{i} - \varepsilon_{e}} \quad \text{(5.5.3)}
\]

The computation of the values of \( \varepsilon_{i}, \varepsilon_{i}, \varepsilon_{e} \), satisfying the equation, is carried out in the following way:
Given a series of values for $\sigma$, and using the diagram $\sigma = f(\varepsilon)$, the corresponding values of $\varepsilon_1$ and $\varepsilon_e$ are obtained. In turn, values of $kE$, as calculated by the formula (5.5.8), are used for the construction of the $kE = f(\sigma)$ curve (Fig. 5.5.5). Values of $\sigma$ for which $k$ takes all the chosen values ($k = 0.99, 0.98, 0.97$, etc) can be read easily from this curve.

Finally, having found $\sigma$, the moment will equal

$$M = \sigma \Omega h$$

for example with $h = 165$ mm and $\Omega = 2,851.5$ mm$^2$. (See part B above)

$$M = 470.5 \sigma \text{ kgm} \quad (5.5.9)$$

This formula together with the diagram mentioned above enables us to subdivide the ordinate axis of the chart in Fig. 5.5.4 into $k$ units.

For example, for $k = 1$ we get $\sigma = E \varepsilon_1 = 16.3$ kg/mm$^2$, and $M = 3945$ kgm. For $k = 0; \sigma = 24.03$ kg/mm$^2$ and $M = 11,300$ kgm.
The corresponding points $E$ and $E'$ for the case of pure bending can be added on the curve $S$ (Fig. 5.5.1) which has been drawn for $k = 0.88$.

In order to achieve this, it is sufficient to read from the diagrams $\sigma_1 = f(\varepsilon_i)$ and $\sigma_e = f(\varepsilon_e)$ the values of the strains $\varepsilon_i$ and $\varepsilon_e$ corresponding to a compressive stress $\sigma_1$ and to a tensile stress $\sigma_e = -\sigma_i$ respectively. Then, we locate on the curve $S$, point $E$ of abscissa $\varepsilon_i$ and point $E'$ of abscissa $\varepsilon_e$.

3) Between the extreme conditions represented by points $A$, $A'$ (uniform compression) and $E$, $E'$ (pure bending), we can consider an infinite number of points corresponding to real loading conditions of the column.
Thus, points (B, B') represent a column with both flanges compressed, whereas points (D, D') correspond to a column with one flange compressed and the other under tension. Between these two conditions, there is a third one (C, C') for which the exterior flange is not stressed at all.

The graphical solution explained above, has been applied up to a value of $\varepsilon_i = 1900$ millionths. Beyond this value, it is simple to find an analytical expression for the coefficient of proportionality $k$ by assuming:

1. that the point representing the flange $i$ moves as the straight line $AB$ (Fig. 5.5.6) of equation $\varepsilon_i = \varepsilon_o + E' \varepsilon_i$ (for $\varepsilon_i > 2500 \times 10^{-5}$ where $\varepsilon_o = 23.71$ kg/mm$^2$ and $E' = 65$ kg/mm$^2$).

2. that the exterior flange remains elastic, i.e.

$$\varepsilon_e = E \varepsilon_o$$

By feeding-in the values of $\varepsilon_i$ and $\varepsilon_e$ into the
expression for \( k \) as given by equation (5.5.2), and making the necessary simplifications we obtain:

\[
\dot{k} = \frac{4M}{P} \frac{E}{E'} \left( h + 2 \frac{M}{P} - \frac{\Omega \sigma h}{P} \right) - \left( h - \frac{2M}{P} \right),
\]

the analytical expression for \( k \).

The curves \( k = \text{constant} \), calculated by the above formula, are straight lines converging to a point on the \( P \) axis. The portions of the curves corresponding to the values of \( k \) given below, have been drawn with dotted lines on the graph 5.5.4: \( k = 0.5; 0.4; 0.3; 0.2; 0.1; 0.09; 0.08; 0.07; 0.06; 0.05; 0.04; 0.03; 0.02 \).

These portions of the lines are not valid but for a small region of \((M,P)\) values bounded by the three dashed lines.

The graph for the PN 22 S section has been drawn on the same principles described above; it is shown in Figure 5.5.7.

A comparison of the two graphs 5.5.4 and 5.5.7 shows that in the case of the PN 22 sections, \( k \) decreases much more rapidly than in the case of DIE 20's.
Fig. 5.5.7

P.N.S-22

ABAQUE \( K = f(M\Phi) \)
It is assumed that this is due to the residual stresses which in DIE 20 practically provide equilibrium in one isolated flange, whereas in the case of PN 22 sections form a tensile resultant of considerable magnitude.

![Graphs showing stress-strain relationship for DIE 20 and PN 22 sections.](image)

**Fig. 5.5.8**

The result is that in the case of DIE 20 sections, the \((\varepsilon, \varepsilon)\) diagrams are almost the same for the equivalent flanges under tension and compression. (Fig. 5.5.8a) However, in the case of PN 22 they correspond to elastic limit values which are different from 8.8 kg/mm². (Fig. 5.5.8b) Besides this graph, a graph for the PN 22 section (Fig. 5.5.7) has been drawn. At first sight the curves \(k = \text{constant for this case present an interesting feature: Maxima in the region of high moments (large } M/P\). For the curve \(k = 1\) this can be explained simply as follows:
The elastic limit of the flange (e) under tension is low (-3.57 kg/mm²), whereas the compressed flange (i) has a higher elastic limit (12.85 kg/mm²). Therefore, in pure elastic bending, the maximum bending stress is 8.57 kg/mm². The application of a compressive stress of 1/2 (12.85-8.57), that is a compressive force of 2.14 kg/mm² has as a result the reduction in the existing tensile stress in one flange by 2.14 kg/mm². The application of such a compressive force, will increase the moment carrying capacity of the column, without entering the plastic range, corresponding to a pure bending stress of \( \frac{8.57 + 12.85}{2} \) kg/mm².

5.6 The Deformed Shape of the Plastically Buckled Column by the Graphical Method. Comparison with the Test Results.

The fundamentals of the graphical method are given in paragraph D above. In order to find the deflected shape of a column corresponding to a given value of \( P \), the operations listed below should be followed.

1) Assume a critical deflection curve \( y_o = f(x) \) where \( x \)'s are measured along the length of the column axis starting from one end, and \( y_o \) is the distance
between the center of gravity of the section and the line of action of the force \( P \); and is measured perpendicular to the \( x \) axis.

2) Measure the eccentricities \( y_\circ \) at 11 sections of
\[
\frac{x}{l} = 0, 0.1, 0.2, 0.3, \ldots 0.9, \text{ and } 1.0
\]
from the assumed deflection curve, and compute the corresponding bending moments \( M = P y_\circ \).

3) Knowing \( M \) and \( P \) for each section and using the graph of \( k = f(M,P) \) find the corresponding values of \( k \).

4) Using the Mohr method, of graphical construction obtain the deformed shape of the column due to the bending moment diagram \( M \). To accomplish this the conjugate beam is loaded with the \( M/k \) diagram, and using a polar distance of \( H = EI \) a furnicular polygon of the fictitious load is drawn. If a longitudinal scale of 1/10, and a deflection scale of 10 is used, the polar distance becomes
\[
H = \frac{EI}{100}
\]

5) If the deflection curve \( y_\perp = f(x) \) obtained in (4) is much different than the assumed one \( y_\circ \), the operations 1 through 4 should be repeated assuming \( y_\perp = f(x) \) as a trial curve.
This method was used in the following cases:

DIE 20, \( \lambda = 60; \ m = 1; \ e_2/e_1 = 0; \ P = 60-70-75-80 \) tons.

DIE 20, \( \lambda = 60; \ m = 1; \ e_2/e_1 = 1; \ P = 60-70-75-80 \) tons.

In both cases, the first approximation was taken as the experimental deflection curve. As an example, the construction of this curve for \( e_2/e_1 = 0 \) and \( P = 70 \) tons is given in Fig. 5.6.2. The theoretically obtained deflection curves for the case \( e_2/e_1 = 0 \) are represented by solid lines in Fig. 5.6.3.

For \( P = 80 \) tons, the method gives diverging successive deformations \( y_0, y_2, \) etc. which means that the theoretical critical load is less than 80 tons.

Fig. 5.6.3 shows the theoretical deflection curves for the case \( e_2/e_1 = -1 \). As in the previous case, the method of successive approximations gives divergent results for \( P = 80 \) tons shows that the theoretical critical load \( F_{cr} \) is smaller than this value. Figures 5.6.3 and 5.6.4 represent in dotted lines, the experimental deflection curves. It is seen that the correlation of theory and experimental results is satisfactory.
Fig. 5.6.4

Fig. 5.6.5
Figure 5.6.5 shows the experimental (dotted lines) and theoretical (solid lines) load deflection diagrams which have been drawn using Fig. 5.6.3. Similarly, load-deflection diagrams in Fig. 5.6.6 have been drawn using the deflection curves of Fig. 5.6.4.

![Diagram](image)

**Fig. 5.6.6.**

The experimental flexural torsional buckling load has been indicated on each of these two figures.

5.7 **Simplified Analysis of Eccentrically Loaded Columns and Comparison for** $e_2/e_1 = 40, 60, 80$ and $100$.

Before adopting the graphical method explained in paragraph 5.6, we have investigated the theoretical deflection curves in the plane of the web for the simplest case of eccentric loading $e_2/e_1 = 1$. The results of this study, to be presented below, are the basis for
the analysis and the behavior of such columns at the time of inelastic flexural-torsional buckling.

Let us first consider the case of \( m = 3 \). The initial eccentricity \( e_1 \) being 3 times the core radius \( r_x \),
\[ e_1 = 3 \times 71.47 = 214.41 \text{ mm} \), the additional displacements \( (y-e) \) due to the elastic-plastic deformation of the column are negligible compared with \( e_1 \). Therefore, it can be approximately taken into account, by considering that the initial deformation \( e_1 \) increases by a constant quantity \( f \) equal to the midheight deflection of the column. However, in the case of \( m = 1 \), and particularly for \( m = 0.5 \), this method does not give accurate results due to the increasing relative error involved. Nevertheless, assuming a circular deflection curve
\[
\frac{1}{\rho} = \frac{e_1 - e_2}{h},
\]
and
\[
f = \frac{l^2}{8\rho} = \frac{l^2}{8h} (e_1 - e_2) \tag{5.7.1}
\]
Thus, the equations of the problem are reduced to:
\[
P = \Omega (e_1 + e_2) \tag{5.3.1}
\]
\[
M = \frac{\Omega h}{2} (e_1 - e_2) \tag{5.3.2}
\]
\[
\frac{M}{F} = m r_x + \frac{l^2}{8h} (e_1 - e_2) \tag{5.7.2}
\]
Replacing $P$, and $M$ by their expressions into (5.7.2) we obtain:

$$\frac{\Omega h}{2} \left( \sigma_1 - \sigma_2 \right) \left( \sigma_1 + \sigma_2 \right) = mr_x + \frac{l^2}{8h} (\varepsilon_1 - \varepsilon_2).$$

Simplifying, multiplying by $2/h$ and replacing $r_x$ by its expression:

$$r_x = \frac{2 I_x}{\Omega h},$$

we get:

$$\sigma_1 - \sigma_2 = m \frac{4 I_x}{\Omega h} + \frac{l^2}{4 h^2} (\varepsilon_1 - \varepsilon_2);$$

but

$$I_x = \frac{\Omega h^3}{4},$$

from where we finally reduce:

$$\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} = m \frac{h}{h_1} + \frac{l^2}{4 h^2} (\varepsilon_1 - \varepsilon_2) \quad (5.7.3)$$

In order to establish one point on the curve $(F,f)$ first, the interior flange is given an arbitrary strain $\varepsilon_1$, and using equation (5.7.3) the corresponding $\varepsilon_2$ is computed. Since the exterior flange always remains elastic, $\varepsilon_2 = E \varepsilon_2$ and (5.7.3) becomes an equation of the second degree in $\varepsilon_2$. 
Having computed the values of $\varepsilon_1$ and $\varepsilon_e$, the values of $P$, $M$ and $f$ can be easily determined using equations (5.3.1) (5.3.2) and (5.7.1) respectively.

As an example, detailed computations for the case $\lambda = 80$, and $m = 3$ are given below.

<table>
<thead>
<tr>
<th>$\varepsilon_1 \times 10^4$</th>
<th>$\varepsilon_e \times 10^4$</th>
<th>$P$ en kg</th>
<th>$M$ en kg/m</th>
<th>$f$ en mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>-183.18</td>
<td>12.984</td>
<td>2.879</td>
<td>6.96</td>
</tr>
<tr>
<td>600</td>
<td>-272.88</td>
<td>18.871</td>
<td>4.251</td>
<td>10.41</td>
</tr>
<tr>
<td>800</td>
<td>-350.08</td>
<td>23.663</td>
<td>5.408</td>
<td>13.72</td>
</tr>
<tr>
<td>1000</td>
<td>-416.29</td>
<td>27.540</td>
<td>6.381</td>
<td>16.90</td>
</tr>
<tr>
<td>1200</td>
<td>-471.02</td>
<td>30.535</td>
<td>7.167</td>
<td>19.93</td>
</tr>
<tr>
<td>1400</td>
<td>-513.74</td>
<td>32.683</td>
<td>7.766</td>
<td>22.82</td>
</tr>
<tr>
<td>1600</td>
<td>-546.42</td>
<td>34.147</td>
<td>8.209</td>
<td>25.60</td>
</tr>
<tr>
<td>1800</td>
<td>-556.71</td>
<td>34.244</td>
<td>8.318</td>
<td>28.11</td>
</tr>
<tr>
<td>2000</td>
<td>-562.46</td>
<td>34.070</td>
<td>8.360</td>
<td>30.55</td>
</tr>
</tbody>
</table>

Figures 5.71, 5.72, and 5.73 show the theoretical load-deflection curves, as obtained by the above mentioned method, for values of relative eccentricities of $m = 0.5$, $m = 1$ and $m = 3$ respectively.

Due to the reasons presented at the beginning of the paragraph, theoretical and experimental results for the case of $m = 3$, are in closer agreement than in the cases $m = 0.5$ and $m = 1$. 


Fig. 5.7.1

Fig. 5.7.2

Fig. 5.7.3
CHAPTER VI

FLEXURAL BUCKLING THEORY OF AN ECCENTRICALLY LOADED COLUMN

Comparison with Test Results

6.1 A. General

The inelastic flexural-torsional buckling theory, which is treated in the present paragraph, is a generalization of the corresponding elastic theory.

This elastic theory, which is recent and quite complicated, is important for the review of the fundamental characteristics of the problem.

The possibility of pure torsional buckling of a compressed base was first discovered by WAGNER in 1929. Generalizing WAGNER's reasoning, KAPPUS in 1937 succeeded in establishing the differential equations for flexural-torsional buckling. He also presented the expression for the critical load for the case of a simply supported column load with a constant eccentricity whose ends were either free to warp, or perfectly fixed or completely prevented from warping. CHWALLA, in 1943, produced an explicit solution to the most general case of the problem, the case where the ends were lastically restrained against rotation as well as warping of the cross section. (Ref. 37)
In 1945, TIMOSHENKO published an excellent treatise of the entire problem (Ref. 38) which was translated into French by one of us. (Ref. 39). One of the authors, also attacked the problem of columns under oblique loading by an energy method (Ref. 28) and obtained an approximate solution. Finally, KAPPUS in 1953 investigated the problem of the influence of the elastic deformations of the eccentrically loaded column, on the critical load.

6.2 Brief review of the elastic theory. Comparison with test results.

As an introduction to the plastic flexural-torsional buckling theory to be studied later, we are going to give a brief review of the elastic theory developed by TIMOSHENKO (Ref. 39), as well as some additional comments. For the details of this demonstration refer Ref. 39.

Let us consider (Fig. 6.2.1) a prismatic column of axis Oz, having yz as its plane of symmetry. This column
is loaded with an axial force $P$, and with equal end moments $M_x$ applied in its plane of symmetry. If the elastic deformations of the column are neglected, it can be assumed that this kind of loading is equivalent to a compressive force $P$ with constant eccentricity (negative):

$$
\varepsilon_y = -\frac{M_x}{P}
$$

In order to study the stability of this type of equilibrium in the bar in the plane of its web, we shall assume that this bar, can take a deformed position which is infinitesimally near this type of equilibrium and we shall study the conditions under which it shall remain in equilibrium in this new position. This analysis is based on the assumption of an initially straight axis. Consideration of the elastic deformation of the column axis is very laborious and complicated (Ref. 41). Numerical computations executed on this subject for the case of our tests have indicated the correction factors to be negligible.

The new deformation at a section $(z)$ is composed of the translations $u$, $v$ and rotation $\phi$ all with respect to the shear center $O$. (Fig. 6.2.2)
The displacements in the $x$ and $y$ direction of any longitudinal fiber are

$$u + (y_0 - y) \varphi \quad \text{et} \quad v - (x_0 - x) \varphi.$$

The magnitude of the fictitious lateral forces and torsional moments produced by the compression load in the fibers of the slightly displaced section are given by

$$\begin{align*}
q_x &= - \int \int \sigma d \Omega [u'' + (y_0 - y) \varphi'']. \\
q_y &= - \int \int \sigma d \Omega [v'' - (x_0 - x) \varphi'']. \\
m_x &= - \int \int \sigma d \Omega [u'' + (y_0 - y) \varphi''] (y_0 - y) + \int \int \sigma d \Omega [v'' - (x_0 - x) \varphi''] (x_0 - x)
\end{align*}$$

(6.2.3)

Fig. 6.2.2
In the case of an I shaped section \( x_0 = 0 \): Moreover, by choosing a set of convenient axes:

\[
\begin{align*}
\sigma \int \Omega d\Omega &= P; \int \Omega \sigma d\Omega = \int \Omega y d\Omega = 0 \\
\int \Omega y^2 d\Omega &= I_x, \int \Omega x^2 d\Omega = I_y.
\end{align*}
\]

(6.2.4)

Using the above simplifications, formulas (6.2.3) can be written as:

\[
\begin{align*}
q_x &= -P u' - (P y_0 + M_x) v' \\
q_y &= -P v' \\
m_z &= -u' (P y_0 + M_x) + (M_x v_x - P r_{x0}) v',
\end{align*}
\]

(6.2.5)

where

\[
r^2 = x_0^2 + y_0^2 + \frac{I_x + I_y}{\Omega} \tag{6.2.6}
\]

and

\[
\beta_x = \frac{\int \Omega y^2 d\Omega + \int \Omega x^2 y d\Omega}{I_x} - 2y_0 \tag{6.2.7}
\]

(6.2.7)

Substituting the expression (6.2.5) for \( q_x \), \( q_y \), and \( m_z \) into the classical equations for bending and torsion of thin-walled members, known as:

\[
E I_x \frac{d^4 v}{ds^4} = q_x; E I_y \frac{d^4 v}{ds^4} = q_y; C_0 \phi'' - C_0' \phi' = m_z. \tag{6.2.8}
\]
we find equations 6.2.9 where \( C \) and \( C_1 \) are the torsional rigidity and the warping rigidity of the section respectively.

This system of differential equations breaks down into two independent groups. The first group, corresponding to the second equation (6.2.9), represents the Euler buckling in the plane of the web and it is not of any interest here.

From formula (6.2.1) \[ M_x = -P_\gamma, \] where \( \gamma \) is a given oriented quantity which is negative in the actual case.

Using this notation the second group of differential equations (6.2.9) is given by (6.2.10).

\[
\begin{align*}
& E_L \varphi'' + P \varphi'' + (P_{y_0} + M_{x}) \varphi'' = 0 \\
& E_x \varphi'' + P \varphi'' = 0, \\
& C_1 \varphi'' - (C + M_x \beta_x - P_{x}) \varphi'' + (P_{y_0} + M_{x}) \varphi'' = 0
\end{align*}
\]

(6.2.9)

For the case where the ends of the column are simply supported, prevented from rotation about axis \( O_z \), and free to warp, \( (u = \varphi = u'' = \varphi'' = 0 \) for \( z = 0 \) and \( z = 1 \)) the system of differential equations (6.2.10) gives a solution in the form:

\[
\varphi = A_1 \sin \frac{\pi z}{I}, \varphi = A_2 \sin \frac{\pi z}{I} \]

(6.2.11)
Replacing the values of \( u \) and \( \phi \) from (6.2.11) into (6.2.10) we get the system of differential equations given in (6.2.12).

\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( E I_y \frac{\pi^2}{2} - P \right) A_1 - P (y_0 - \epsilon_y) A_2 = 0 \\
-(P (y_0 - \epsilon_y) A_1 + \left( C_1 \frac{\pi^2}{2} + C - P \epsilon_y - \beta_x - P r_s^2 \right) A_2 = 0
\end{array} \right.
\end{aligned}
\] (6.2.12)

Equating the determinant of the coefficients of the unknown \( A_1 \) and \( A_2 \) to zero we obtain the buckling criterion.

\[
\begin{vmatrix}
E I_y \frac{\pi^2}{2} - P & -P (y_0 - \epsilon_y) \\
-P (y_0 - \epsilon_y) & C_1 \frac{\pi^2}{2} + C - P \epsilon_y - \beta_x - P r_s^2
\end{vmatrix} = 0
\] (6.2.13)

By introducing the definition of the Euler load

\[
P_1 = E I_y \frac{\pi^2}{2}
\] (6.2.14)

and the critical torsional buckling load:

\[
P_2 = \frac{1}{r_s^3} \left( C + C_1 \frac{\pi^2}{2} \right)
\] (6.2.15)

The buckling criterion (6.2.13) can be written as:

\[
(P_1 - P) \left[ P_2 - P \left( 1 + \frac{\epsilon_y}{r_s^3} \right) \right] - P^2 \left( \frac{y_0 - \epsilon_y}{r_s} \right)^2 = 0
\] (6.2.16)

From this quadratic equation, the critical elastic flexural-torsional buckling load can be easily determined.
In the case of a perfect fixed ended support condition, and complete prevention of warping at both ends, the boundary conditions are:

\[ u = u' = \phi = \phi' = 0 \text{ for } Z = 0 \text{ or } Z = l.\]

Thus, the lateral and rotational displacements of the column are given as:

\[
    u = A_1 \left(1 - \cos \frac{2\pi z}{l}\right), \quad \varphi = A_2 \left(1 - \cos \frac{2\pi z}{l}\right)
\]  

(6.2.17)

The buckling criterion remains the same as equation 6.2.16 provided that

\[
    p_1 = \frac{4\pi^2 EI}{l^2}; \quad p_2 = \frac{1}{\rho^2} \left(C + C_1 \frac{4\pi^2}{l^2}\right)
\]

(6.2.18)

In our tests, warping is almost completely prevented since the columns are connected to the bearing plates, of about 20 mm thickness (and later stiffened), which are bolted to the cast iron plate of the spherical supports. It is of importance, therefore, to provide a calculation of the critical buckling load for this particular case. An analytical expression for this critical load would be obtained using CHWALLA's analysis (Ref. 37); but a tentative analysis showed that this method leads to an extremely complicated
formula. It was decided therefore, that an approximate type of solution would be preferable, and the problem would be solved by the energy method of RAYLEIGH-RITZ-TIMOSHENKO.

The governing equation for the problem is presented in Ref. 40 page 5.

Applying the rules of variational calculus, it can be shown that the Euler equations associated to this problem (6.2.19) are exactly the same as equations (6.210). This proves the identity of the two methods.

\[
\frac{EI_y}{2} \int u'' dz + \frac{1}{2} \left[ C - P(e_y e_x + r_0^2) \right] \int \varphi'^2 dz - \frac{C_1}{2} \int \varphi'' \varphi'' dz
- \frac{P}{2} \int u'' dz - P(y_0 - e_y) \int \varphi' \varphi' dz = 0
\]  

(6.2.19)

The boundary conditions for columns prevented from warping (\( \phi = 0 \)) are:

\[
u = u'' = 0 \text{ for } z = 0 \text{ and } z = l
\]
\[
\phi = \phi' = 0 \text{ for } z = 0 \text{ and } z = l
\]  

(6.2.20)

The deflection curve of the column can be approximated by the expression:
which satisfy all boundary conditions.

Substituting $u$ and $\phi$ in equation (6.2.19) by these expressions from (6.2.21) and carrying all the necessary integrations we get:

\[
\frac{E I_y \pi^4 A_1}{2} - \pi^2 \frac{P A_1}{2l} - P (y_0 - \theta_0) \frac{8 \pi^2 A_1}{3 \pi l} = 0 \quad (6.2.22)
\]

For the real deflection of the member the variation of total energy $\Delta E$, represented by the left hand side of equation (6.2.22) should be minimum. This gives us two conditions.

\[
\begin{align*}
\frac{\delta (\Delta E)}{\delta A_1} & \equiv \frac{E I_y \pi^4 A_1}{2} - \pi^2 \frac{P A_1}{2l} - P (y_0 - \theta_0) \frac{8 \pi^2 A_1}{3 \pi l} = 0 \\
\frac{\delta (\Delta E)}{\delta A_2} & \equiv [C - P (\varepsilon + r)] \frac{2 \pi^2 A_2}{l} + C_4 \frac{8 \pi^4 A_2}{p} - P (y_0 - \theta_0) \frac{8 \pi^2 A_1}{3 \pi l} = 0
\end{align*}
\]  

(6.2.23)

Introducing the notations:

\[
\frac{\pi^2 E I_y}{l^2} = P_1 \left[ C + \frac{4 C_4 \pi^2}{l^2} \right] = P_2
\]

(6.2.24)
corresponding to transversal bending of section DIE 10 and PN 22\(^1\) decreases rapidly with strain (See diagram Fig. 6.3.3) and it is much less than the one obtained by a \((\sigma, \varepsilon)\) diagram for a compression coupon. It is concluded, therefore, that the critical load computed this way will be much lower than the ones computed by formula (6.2.26). The main reason for such a decrease in the value of \(P_{cr}\), is the reduction in the elastic modulus for the case of transversal bending, which due to the pressure of residual stresses becomes \(\frac{T_i + T_e}{2}\) instead of \(E\). Since the modulus \(T\) has not been estimated for sections DIE 10 and PN 22 S, the calculation of the above ratio is not possible. However, it should be noted that the decrease in critical load for the DIE 10 section will be quite more noticeable as compared to section PN 22 S. This is due to the residual stress distribution at the flange tips, which is a determining factor for the decrease of the modulus \(T\). These stresses are 7.9 kg/mm\(^2\) for PN 22 S.

\(^1\)For the definition of this modulus see paragraph 6.3 below.
We shall not investigate the quantitative analysis of the results any longer because the effect of the reduction in the value of modulus $T$, on the critical load will be discussed in detail in paragraph 6.5 below.

If the deformation of the column is neglected, the maximum bending and axial load stresses is related to the compressive force $P$ by

$$\sigma_{\text{max}} = \frac{P}{\Omega} (1 + m).$$

Let us replace in this formula $G_{\text{max}}$ by $R_p = 12.8$ kg/mm$^2$. The limit values of $P$ are found to be:

$$P_{\text{limite}} = \frac{12.8 \, \Omega}{1 + m},$$

and are tabulated below:

<table>
<thead>
<tr>
<th>$m$</th>
<th>DIE 10</th>
<th>PN 22 S</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>18.3</td>
<td>32.8</td>
</tr>
<tr>
<td>1</td>
<td>13.7</td>
<td>24.6</td>
</tr>
<tr>
<td>3</td>
<td>6.88</td>
<td>12.3</td>
</tr>
</tbody>
</table>

Comparing these values with the experimental critical loads it is seen that

$$P_{\text{limit}} \ll P_{\text{crit. exp.}}$$

for $\lambda = 175$, $m = 3$ and

for $\lambda = 130$, $m = 1$ and 3.
which satisfy all boundary conditions.

Substituting \( u \) and \( \phi \) in equation (6.2.19) by these expressions from (6.2.21) and carrying all the necessary integrations we get:

\[
\frac{E I_y \pi^4 A_1^2}{2} - \frac{1}{2} \left[ C - P (\phi \beta + r_0^2) \right] \frac{2 \pi^2 A_1}{l} + C_1 \frac{8 \pi^4 A_1}{\beta} - \frac{P}{2} \frac{\pi^2 A_1}{2l} - \frac{P (y_0 - e_y)}{3 \pi l} = 0 \tag{6.2.22}
\]

For the real deflection of the member the variation of total energy \( \Delta E \), represented by the left hand side of equation (6.2.22) should be minimum. This gives us two conditions.

\[
\left\{ \begin{array}{l}
\frac{\delta (\Delta E)}{\delta A_1} = \frac{E I_y \pi^4 A_1}{2} - \frac{\pi^4 P A_1}{2l} - \frac{P (y_0 - e_y) \pi^2 A_1}{3 \pi l} = 0 \\
\frac{\delta (\Delta E)}{\delta A_2} = \left[ C - P (\phi \beta + r_0^2) \right] \frac{2 \pi A_2}{l} + C_4 \frac{8 \pi^4 A_2}{\beta} - \frac{P (y_0 - e_y) \pi^2 A_2}{3 \pi l} = 0
\end{array} \right. \tag{6.2.23}
\]

Introducing the notations:

\[
\frac{\pi^4 E I_y}{\beta} = P_1; \quad \frac{\frac{1}{l} \left[ C + \frac{4 C_4}{\beta} \pi^4 \right]}{P_2}
\tag{6.2.24}
\]
and simplifying the system of equations (6.2.25) can be written as:

\[
\begin{aligned}
A_1 (P_1 - P) - P (y_0 - \epsilon_2) \frac{16}{3 \pi} A_1 &= 0 \\
A_2 \left[ P_2 - P \left( 1 + \frac{\epsilon_2 \beta \lambda}{r_0} \right) \right] - \frac{4}{3 \pi r_0} A_1 y_0 - \epsilon_2 P &= 0
\end{aligned}
\]  

(6.2.25)

In order for this system to have a non-trivial solution, the determinant of the coefficients of the unknowns \(A_1\) and \(A_2\) should equal to zero:

\[
(P_1 - P) \left[ P_2 - P \left( 1 + \frac{\epsilon_2 \beta \lambda}{r_0} \right) \right] - P_2 \left( \frac{y_0 - \epsilon_2}{r_0} \right)^2 \frac{64}{9 \pi^3} = 0
\]  

(6.2.26)

The above equation is identical to the one found for the case where the ends are free to warp, (6.2.16), except the last term which is multiplied by \(\frac{64}{9 \pi^3}\). The symbol \(P_2\) stands for different quantities in the two cases.

We will introduce the value of \(\lambda_1\), the ideal slenderness ratio, since a number of such a slenderness ratio, will fail by Euler buckling and have a critical load of:

\[
P_\sigma = \frac{\pi^2 E}{\lambda_1^3} \Omega
\]  

(6.2.27)
It was mentioned above that

\[ P_1 = \frac{\pi^2 E L_y}{I^2} = \frac{\pi^2 E}{\lambda^2 \Omega} \]  
(6.2.28)

Dividing equation (6.2.28) and (6.2.27) side by side

\[ \left( \frac{\lambda_y}{\lambda_y} \right)^2 = \frac{P_1}{P_{cr}} \]  
(6.2.29)

In order to solve equations (6.2.16) or (6.2.29) in terms of \( P_1/P \), they are first multiplied by \( \frac{r_y}{P} \) and introducing the rotation

\[ \frac{P_2 r_y^2}{P_1} = c^2 \]  
(6.2.30)

one gets:

\[ c^2 \left( \frac{P_1}{P} \right)^2 - \left( \frac{P_1}{P} \right) \left( c^2 + r_y \beta_x + r_x \beta_y \right) + r_x r_y \beta_x + r_y r_x \beta_y - M (y_0 - \epsilon_y)^2 = 0 \]  
(6.2.31)

where the coefficient of \( M \) has the values of: a) 1 for equation (6.2.16) and b) \( \frac{64}{9\pi^2} = 0.721 \) for equation (6.2.26)
New coefficients $\beta$ and $\beta_0$ will be defined below, in order to obtain a unique solution for all cases of elastic end restraints and elastic warping at the ends.

$$\beta = \frac{l^2}{l}$$

1 if column is simply supported at the ends.

0.5 if column is perfectly fixed at the ends.

1 if the column is completely free to warp at the ends.

$$\beta_0 =$$

0.5 if the column is completely prevented from warping at the ends.

With these rotations, equation (6.2.30) can be written as:

$$c^4 = \frac{1}{L_y} \left[ \left( \frac{\beta}{\beta_0} \right)^2 \frac{C_1}{E} + \frac{1}{2,6 \pi^2 G} \left( \frac{\beta}{\beta_0} \right)^3 \right], \quad (6.2.32)$$

which takes into account the 3 possible expressions of $P_1$ and $P_2$ (6.2.14) and (6.2.15); (6.2.13); (6.2.24).

Likewise

$$M = 1 - 0.093 \left[ \left( \frac{\beta}{\beta_0} \right)^2 - 1 \right], \quad (6.2.33)$$

This formula has as special cases the values of 1 and 0.721 found above.
Replacing $M$ by its expression from (6.2.33) into equation (6.2.29), and solving the resulting quadratic equation with respect to $\frac{P_1}{P}$ we finally get:

$$
\frac{P_1}{P} = \left(\frac{\lambda_i}{\lambda_y}\right)^2 = \frac{c^2 + r^2 + e_y \frac{b}{2}}{2c^2} \pm \sqrt{1 - \frac{4c^2 \left[r^2 + e_y \frac{b}{2} + e_y \left(y - y_0^2\right) + 0.093 \left[\frac{y}{y_0} - 1\right] \left(e_y - y_0^2\right)\right]}{(c^2 + r^2 + e_y \frac{b}{2})^2}}
$$

(6.2.34)

This formula is identical with different rotations, to the one given by the German Specifications DIN 4114 (Ref. Richtlinie 10.12).

**NOTE:**

It is suggested, in these specifications to take into account the possibility of flexural-torsional buckling by replacing the actual slenderness ratio of the column by the "ideal" slenderness ratio $\lambda_i > \lambda_y$ as defined by formula (6.2.34), even in a case where buckling occurs in the plastic range. The best section is then estimated on the basis of flexural buckling only.

It is obvious that this procedure is perfect for the elastic range, since it only replaces the Euler buckling
load \( P_i = \frac{\pi^2 E_l}{h^2} \) by the critical flexural-torsional buckling load.

However, in the plastic domain the above method does not have any theoretical justification.

In the following part of the present paragraph we will attempt to extend the flexural-torsional buckling theory, into the plastic range. Later (6.6.5) this theory will be used as well as the procedure recommended by the German Specifications and will both be compared in the experimental results.

As a closure to this paragraph, the elastic flexural-torsional buckling theory will be compared with the experimental results.

Before going into detailed numerical comparison, let us consider the sections which are justified by the elastic theory. The sections DIE10 and PN 22's have elastic proportional limits of 12.7 kg/mm² and 12.8 kg/mm². (See diagrams on Fig. 2.34 and 2.55) Beyond these stresses, these develop an important relaxation of the residual stresses in the flanges. Furthermore, at the ends, which are under residual compressive stresses, the Osgood tangent modulus T
corresponding to transversal bending of section DIE 10 and PN 22\(^1\) decreases rapidly with strain (See diagram Fig. 6.3.3) and it is much less than the one obtained by a \((\sigma, \varepsilon)\) diagram for a compression coupon. It is concluded, therefore, that the critical load computed this way will be much lower than the ones computed by formula (6.2.26). The main reason for such a decrease in the value of \(P_{cr}\), is the reduction in the elastic modulus for the case of transversal bending, which due to the pressure of residual stresses becomes \(\frac{T_1 + T_2}{2}\) instead of \(E\). Since the modulus \(T\) has not been estimated for sections DIE 10 and PN 22 S, the calculation of the above ratio is not possible. However, it should be noted that the decrease in critical load for the DIE 10 section will be quite more noticeable as compared to section PN 22 S. This is due to the residual stress distribution at the flange tips, which is a determining factor for the decrease of the modulus \(T\). These stresses are 7.9 kg/mm\(^2\) for PN 22 S.

\(^1\)For the definition of this modulus see paragraph 6.3 below.
We shall not investigate the quantitative analysis of the results any longer because the effect of the reduction in the value of modulus $T$, on the critical load will be discussed in detail in paragraph 6.5 below.

If the deformation of the column is neglected, the maximum bending and axial load stresses is related to the compressive force $P$ by

$$\sigma_{\text{max}} = \frac{P}{\Omega} (1 + m).$$

Let us replace in this formula $G_{\text{max}}$ by $R_p = 12.8$ kg/mm². The limit values of $P$ are found to be:

$$P_{\text{limit}} = \frac{12.8 \Omega}{1 + m},$$

and are tabulated below.

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<td>24,6</td>
</tr>
<tr>
<td>3</td>
<td>6,88</td>
<td>12,3</td>
</tr>
</tbody>
</table>

Comparing these values with the experimental critical loads it is seen that

$$P_{\text{limit}} < P_{\text{crit. exp.}}$$

for $\lambda = 175$, $m = 3$ and for $\lambda = 130$, $m = 1$ and 3.
Hence, for the above sections, we cannot obtain a good correlation between theoretical and experimental results.

The table on page 130 gives values of the experimental critical loads, together with the theoretical ones which have been calculated on the assumption of end sections being prevented from warping and for the cases $e_2/e_1 = 0$ or $-1$, using the equivalent moment given by:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$e_2/e_1$</th>
<th>$m$</th>
<th>$\text{Critique théorique}$</th>
<th>$\text{Critique expérimental}$</th>
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6.3 Theory of plastic buckling of a member under pure bending considering the influence of residual stresses and variation in material properties.

The transverse plastic buckling theory for the test columns will be presented taking into account all related
Partially plastified sections - residual stresses - variation of the elastic-plastic properties of the material for different points of the cross-section - plastic deformations by bending and torsion; it must be borne in mind that this theory will present many complications. This is the reason of first neglecting torsion and generalizing the classical d'ENGESSER-SHANLEY plastic buckling theory for pure bending. Numerical calculations based on the obtained formula, have given critical loads quite higher than the experimental failure loads which demonstrates the necessity of taking into account "torsion".

Nevertheless, it is worthwhile to present once more the generalized d'ENGESSER theory, explained above, because we will introduce for this case the tangent modulus T concept which is indispensable for a thorough understanding of the most refined theory taking into account torsion. We will assume, according to SHANLEY's theory, a lateral buckling which is produced by flexure. The critical load (for safety purposes) will be defined as the load corresponding to the beginning of such excessive bending.
Let us consider a member partially plastified to its deflection in the plane of the web under the effect of the eccentric force $P$. If the additional lever arms produced by the deflection of the column are neglected, or taken care of by a constant correction factor, the degree of plastification is the same for every section. Let $T ly$ be the flexural rigidity of the member in the plane of bending as which is perpendicular to the web. The elastic equation is

$$T ly \frac{d^4 y}{dx^4} = -M,$$  \hspace{1cm} (6.3.1)

where $T$ is the tangent modulus to be defined later.

Repeating the same Euler reasoning with $T ly$ instead of $EI_y$, we find

$$P = \frac{\pi^4 T ly}{L^2}$$  \hspace{1cm} (6.3.2)

which is the critical load for lateral buckling.

Thus, the phenomenon to be expected is the following: bending of the column in the plane of its web, followed by plastification of the web, then sudden Euler buckling normal to the web.
It only remains to define the tangent modulus $T$ in formula (6.3.2). At this stage we have been inspired by OSGOOD's study on the effect of residual stresses on the buckling load (Ref. 41).

At the verge of lateral buckling, the flanges $i$ and $e$ attain two different degrees of plastification, which are known for a given force $P$. That is, the average strains $\varepsilon_i$ and $\varepsilon_e$ can be calculated (See paragraph 5.7).

![Diagram](image)

Fig. 6.3.1

For example, a fiber of the flange $i$ at a distance $x$ from axis $y$, taking into account the known residual stress distribution, has a tangent modulus $T_i(x)$ while the corresponding fiber on the flange $e$ has a tangent modulus of $T_e(x) = T_i$. 
Let us suppose now that due to lateral buckling, cross sections also rotate around the axis of symmetry $y$. The additional strains due to such a rotation are:

$$\Delta \varepsilon = \frac{x}{\rho_x}$$

where $x$ is the distance of any fiber from the axis of rotation, and $\rho_x$ is the radius of curvature of the column in plane $x$.

The additional stress in a fiber will be:

$$\Delta \sigma = T \Delta \varepsilon \frac{T_x}{\rho_x}$$

Equilibrium of external and internal moments gives:

$$M_x = \int \int \frac{T_x x d\Omega}{\rho_x}$$

(a)

By definition:

$$\bar{T} = \frac{\int \int \Omega T_x^2 d\Omega}{\int \int \Omega x^2 d\Omega} = \frac{1}{I_y} \int \int \Omega T_x^2 d\Omega$$

(6.3.3)

where $T$ denotes "the OSGOOD average tangent modulus".

Using this notation, equation (a) can be written as:

$$M_x = \frac{\bar{T} I_y}{\rho_x} = -\bar{T} I_y \frac{d^2 v}{d x^2}$$

(5.3.1)
which is given above.

It remains to explain the practical calculation of $T$ for a given section, say the DIE 20. For a section composed of only two flanges, formula (6.3.3) becomes: (Fig.6.3.1)

$$
\bar{T} = \frac{1}{I_y} \left( \int_{x=\frac{b}{2}}^{x=\frac{b}{2}} T(x) \, dx + \int_{x=-\frac{b}{2}}^{x=-\frac{b}{2}} T_e(x) \, dx \right) = \frac{T_1 + T_2}{2}
$$

(6.3.4)

![Diagram](image)

Fig. 6.3.2

As recalled, compression tests were carried out on small coupons cut out from each flange. (Fig. 6.3.2). Therefore, both terms in formula (6.3.4) can be evaluated by replacing the integrals with equivalent sums. We find

(6.3.5)
Using formula (6.3.5) the diagram (6.3.3) which gives the variation in average tangent modulus of one flange, as a function of the strain $\varepsilon$ of the same flange can be constructed. One of the curves in this diagram corresponds to the compressed flange, whereas the other to flange under tension. The marked difference between these two curves show the important effect of the residual stresses on the column stability in lateral buckling.

In order to determine the tangent modulus $T$ to be used in formula

$$P_\sigma = \frac{\pi^2 T l_p}{I^2} \tag{6.3.2}$$

the interior and exterior flange strains $i$ and $e$ should be determined for a given load $P$, using the method explained in paragraph 5.7. Then, values of $T_i$ and $T_e$ corresponding to $\varepsilon_i$ and $\varepsilon_e$ are read off the curves given in Fig. 6.3.3. Having obtained $T_i$ and $T_e$
Some of the critical loads, obtained by the theory to be developed, will be presented in paragraph 6.5. As mentioned above, these theoretical loads exceed the experimental loads. This indicates that the torsional deformation should be taken into consideration. The analysis of the buckling phenomenon, considering torsion will be presented in paragraph 6.4.

6.4 Flexural-torsional buckling of a column considering residual stresses and varying material properties in its cross section.

A. General

We propose to extend the elastic theory (paragraph 6.2) into the elastic-plastic region. In order to simulate the actual conditions a stress-strain \((\varepsilon, \epsilon)\) diagram which differs for different points along the cross section will be chosen. A residual stress distribution, similar to the one obtained experimentally, will be assumed.

Let us consider the loading case of a compression force \(P\), and end bending moments \(M_k\) acting in the plane \(\gamma\). The
The sign convention and position of axes $x$, $y$ and $z$ are shown in Fig. 6.4.1.

The first order theory will be assumed. The effect of the force $P$ on the bending stresses will also be neglected. Having made these assumptions, we can start studying the plastic deflection of a column in the plane of its web, which will provide us with the stresses $\sigma_1$ and $\sigma_e$ and strains $\varepsilon_1$ and $\varepsilon_e$ in the fibers of the corresponding flanges.

At the present time, let it be assumed that the column axis is always straight and situated in the $yz$ plane.

Let the column be given an infinitesimal virtual deformation of combined bending and torsion and study the equilibrium of this column in the deformed state. The corresponding value of $P$ is the critical flexural-torsional buckling load.

\[ \text{Fig. 6.4.1} \]
B. Preliminary Calculations

1) Elastic-plastic state of an eccentrically loaded column.

The elastic-plastic stresses in an eccentrically loaded column can be determined by making the following simplifying assumptions:

a) "Plane sections remain plane", i.e. in any section of a member under bending and axial force we have

\[ \varepsilon = \varepsilon_0 + \frac{y}{\rho_x} \]  

(6.4.1)

where \( \varepsilon_0 \) is the average strain, and \( \frac{1}{\rho_x} \) is the curvature of the axis of the member.

b) The elastic-plastic deformation of the column will be taken care of in an approximate manner, by adding to the initial eccentricity \( e_x = n r_x \) the quantity \( f \) which is an average value of the additional lateral displacement of the point C. (Fig. 6.4.2)
If the secant modulus is taken as the slope of the straight line joining the original 0 to a point on the \((\varepsilon, \varepsilon)\) curve (Fig. 6.4.3) corresponding to a particular fiber of the member, we have simply:

\[ \sigma = S \varepsilon, \quad (6.4.2) \]

and the normal stress distribution on the cross section is given by:

\[ \sigma = S \left( \varepsilon_e + \frac{y}{\rho_e} \right). \quad (6.4.3) \]

Note that the ordinate "\(y\)" in the above formula is measured from the geometric center \(G\).

![Fig. 6.4.3](image)

For the simplified case of a section composed of only two fibers, discussed in paragraph 5.2, Fig. 6.4.3 shows that:
where $S_i$ and $S_e$ represent the secant moduli of elasticity of the interior and exterior flanges respectively.

Writing the equilibrium for internal and external forces (Fig. 6.4.4) we have:

\[ P = \frac{\Omega}{2} (a_i + a_o) \]  
(6.4.5)

\[ M_x = P (mr_x + f) = \frac{\Omega h}{4} (a_i - a_o) \]  
(6.4.6)

Using the theory explained in paragraph 5.7 and the $(E, \varepsilon)$ diagram of the equivalent fiber, we can obtain by successive trials the values of $E_i, E_o, E, \varepsilon, S_i, S_e$, 

and \( f \) for a given load \( P \).

2) **Definition of the elastic center \( B \).**

In the development of the elastic buckling theory, the axes were chosen through \( G \), where we had:

\[
\int \omega y \, d\Omega = \int \omega x \, d\Omega = \int \omega xy \, d\Omega = 0,
\]

and which had reduced our work considerably.

In the extension of this theory into the plastic range, the axes should be chosen in such a way as to benefit from the same simplifications. In other words, the position of the origin should be such that all three of the integrals

\[
\int \omega y \, d\Omega = \int \omega x \, d\Omega = \int \omega xy \, d\Omega
\]

become zero. The last two, are zero due to symmetry of the cross section and the loading with respect to the \( y \) axis. It is sufficient, therefore, to find an axis about which

\[
\int \omega y d\Omega = 0.
\]

For a section reduced to two fibers of area \( \frac{Q}{2} \), this condition is equivalent to \( \text{(Fig. 6.4.4)} \).
\[ \frac{\Omega}{2} (S_r y_r + S_s y_s) = 0, \quad \text{(a)} \]

with the provision that \( y \) will be measured from a certain point \( E \), defined as "the elastic center" and whose position is not known off-hand. Fig. 6.4.4 shows that

\[ y_i = y_e - h \quad (y_i < 0) \]

Substituting into equation (a):

\[ y_e = \frac{S_r h}{S_r + S_s}, \quad y_i = \frac{-S_r h}{S_r + S_s} \quad \text{(6.4.7)} \]

From here on, the point \( E \) will be taken as origin of the axes.

The new ordinates \((y')\) are related to the ordinates \((y)\) with respect to \( E \), by:

\[ y' - y_e = y - \frac{h}{2} \]

from where

\[ y = y' - \frac{S_r - S_s h}{S_r + S_s} \quad \text{(6.4.8)} \]

Replacing \( y \) by \( y' \) in formula (6.3.3) and omitting the \( y' \) prime:
\[ \sigma = S \left( A + \frac{V}{p_x} \right) \]  \hspace{1cm} (6.4.9)

where \( A \) is a constant equal to:

\[ A = \frac{S_t - S_s}{S_t + S_s} \frac{h}{2 p_x}, \]

which is the strain at the elastic center. Formula (6.4.9) enables us to decompose the composite axial force and bending stresses:

\[ \sigma = S \left( A + \frac{V}{p_x} \right) \]

into pure compression stresses

\[ \sigma_a = S A \]

and simple bending stresses.

\[ \sigma_b = \frac{S y}{p_x} \]

It is obvious therefore:

\[ \int \int \sigma_a \, d\Omega = \int \int \sigma \, S \, d\Omega = P \]  \hspace{1cm} (6.4.10)

and

\[ \int \int \sigma_b \, d\Omega = \int \int \frac{S y}{p_x} \, d\Omega = - M_x \]  \hspace{1cm} (6.4.11)
is the external moment $M_x$ which is balanced by the internal moment and should be measured with respect to axis $x \times x$ through the point $E$. Therefore,

$$M_x = P(mv + f + \frac{h}{2} - y_0)$$

(6.4.2)

3) **Definition of the shear center or center of elastic-plastic torsion**.

The following question arises in the study of a plastically bent column in the plane of its web: at which point $O$ on the $y$ axis should a transverse force be applied so that the column could bend without twisting in the plane $x \times z$? The point $O$ in question is defined as the **shear center** of the cross section.

![Diagram of the column and coordinate system](image)

**Fig. 6.4.5**
Let $t_e$ and $t_i$ be the shearing forces in the flanges e and i respectively. (Fig. 6.4.5). Since both of these flanges undergo the same vertical deformations $u = u(x)$, we have

$$T = -EI \frac{d^3y}{dx^3}$$

$$t_e = -\frac{T_e I_y}{2} \frac{d^2u}{dx^2} ; \quad t_i = -\frac{T_i I_y}{2} \frac{d^2u}{dx^2}$$

(6.4.13)

where $I_y$ is approximately the moment of inertia of one flange neglecting the moment of inertia of the web, $T_i$ and $T_e$ represent the average OSGOOD's tangent modulus for bending of the flanges e and i in their place as defined in paragraph 6.3. The resultant of the forces $t_e$ and $t_i$ acts through a point 0 such as (Fig. 6.4.4)

$$t_e Y_e = t_i (h - Y_e)$$

which reduces to

$$Y_e = \frac{t_i + t_e}{t_i h} = \frac{T_i h}{T_i + T_e}$$

Note that always $\frac{T_i}{T_e} < S_i < S_e$ so $Y_e < y_e$.

Therefore 0 is always situated between the exterior flange and the elastic center $E$. 
The ordinate of $O$ with respect to the coordinate axes through $E$ is:

$$y_o = y_e - Y_e = \frac{S_i h}{S_i + S_e} - \frac{T_i h}{T_i + T_e}$$

(6.4.14)

4) Definition of the warping rigidity $C_i$.

Let the member which is already in an elastic-plastic state be subjected to an infinitessimal non-uniform torsional load. This torsion will be accompanied by bending of the flanges in their plane. Let, also, as in 3) $t_i$ and $t_e$ be the shearing forces in the corresponding flanges. The cross section turns through an angle $\phi$ around a certain point $F$, and the resulting flange
displacements are equal to: (Fig. 6.4.6)

\[ u_x = -\varphi u_y, u_y = \varphi (h-y) \]

The corresponding shearing forces being:

\[ t_x = -\frac{T_y}{2} \frac{d^2 u}{dx^2} = + \frac{T_y}{2} y \frac{d^3 u}{dx^3} \]
\[ t_y = -\frac{T_y}{2} (h-y) \frac{d^3 u}{dx^3} \]

These shearing stresses should form a couple, that is, they should be equal and opposite; thus

\[ T_y y = T_y (h-y) \]

from where

\[ y = \frac{T_y h}{T_y + T} = y_s \]

It is obvious, therefore, that the center of rotation \( F \) coincides with the shear center 0, which is also evident from the MAXWELL theorem, but needed to be verified for the plastic range too.

The above computation enables us to evaluate the part of the total torsional moment \( M_t \) which is transmitted by the bending of the flanges. This part, \( M'_t \), is equal to:
-237

\[ M'_t = h t_t = - \frac{I_y}{2 \eta y_z} h \frac{d^3 \varphi}{dz^3} = - \frac{\overline{T}_r \overline{T}_i}{\overline{T}_r + \overline{T}_i} \frac{I_y h^2}{2} \frac{d^3 \varphi}{dz^3} \]

On the other hand, since the moment transmitted by uniform Saint-Venant's torsion is

\[ M'_e = C \frac{d^2 \varphi}{dz^2} \]

the total moment

\[ M_t = M'_t + M'_e = C \frac{d^2 \varphi}{dz^2} - \frac{\overline{T}_r \overline{T}_i}{\overline{T}_r + \overline{T}_i} \frac{I_y h^2}{2} \frac{d^3 \varphi}{dz^3} \]

Let

\[ \frac{\overline{T}_r \overline{T}_i}{\overline{T}_r + \overline{T}_i} \frac{I_y h^2}{2} = C_1 \quad (1) \]

and solve formula (6.4.14) with respect to \( z \). After having changed the signs on both sides of the equation we get:

\[ - \frac{dM_t}{dz} = m_z = -C \frac{d^2 \varphi}{dz^2} + C_1 \frac{d^3 \varphi}{dz^3} \quad (6.4.16) \]

which is the relation between the angle of rotation \( \varphi \) and the intensity of longitudinal torsional moment \( m_z \).

5) Computation of the torsional rigidity \( C \) of the column.

At the present time the initial transverse elastic modulus \( G_0 \) of a member which has already undergone plastic

1. For the elastic range, \( T_e = T_j = E \) and we find the
classical value
deformation is not well defined. (See Reference 42). All theories based on "Incremental Strain" imply that:

\[ G = \frac{E}{2(1 + \gamma)} \]

for both loading and unloading; whereas all theories based on "Total Strain" imply that:

\[
\begin{aligned}
\frac{G}{G_0} &= \frac{3G_0}{1 + 3G_0 \left( \frac{1}{S} - \frac{1}{E} \right)} \\
&= G_0
\end{aligned}
\]

for loading (a)

for unloading

Formula (a) shows that, if the rotation of the section occurs at a point where \( S \) is considerably smaller than \( E \), the denominator is reduced to \( \frac{3G_0}{S} \), because the term

\[ 1 - \frac{3G_0}{E} = 1 - \frac{3}{2(1 + \gamma)} = -0.15 \]

and is negligible compared with \( \frac{3G_0}{S} \).

Therefore \( G_i \) for all practical purposes, should equal to \( \frac{S}{3} \), which corresponds to a value of \( G \) calculated by formula (6.4.17) by replacing \( E \) by the secant modulus and considering the material incompressible.

\( (y = 0.5) \)
We have agreed to assure the highest value of the torsional rigidity which is:

\[
C = \frac{G_s}{3} \Sigma h b^3
\]  
(6.4.18)

C. Flexural-Torsional Buckling Theory

1) Calculation of the lateral reactions and distributions of torsional couples due to a virtual deformation of the column \((u, v, \phi)\).

Having clarified the elastic-plastic bending (lateral buckling) phenomenon of a column in the plane of its web, we can proceed now in studying the flexural-torsional buckling phenomenon.

On this subject the reassuring given in paragraph 6.2 can be repeated, with the only difference that the normal stresses have an elastic-plastic distribution.

The same formula \( (6.2.3) \) can be utilized:

\[
q_x = - \int \sigma \sigma d \Omega \left[ u'' + (y_0 - y) \varphi' \right],
\]

\[
q_y = - \int \sigma \sigma d \Omega \left[ v'' - (x_0 - x) \varphi' \right]
\]

\[
m_z = - \int \sigma \sigma d \Omega \left[ u'' + (y_0 - y) \varphi' \right] (y_0 - y) + \int \sigma \sigma d \Omega \left[ v'' - (x_0 - x) \varphi' \right] (x_0 - x).
\]  
(6.4.19)
For I shaped sections we have \( x_0 = 0 \).

Substituting \( \phi \) in (3.4.19) by its expression from (6.4.9) and recalling that due to the chosen position of \( E \):

\[
\iint_{\Omega} S y \, d\Omega = \iint_{\Omega} S x \, d\Omega = \iint_{\Omega} S xy \, d\Omega = 0
\]

we obtain,

\[
\begin{cases}
q_x = -p u'' - (p y_0 + M_x) \varphi'' \\
q_y = -p v'' \\
m_y = u'' (p y_0 + M_x) + (M_x \beta_x - p r_0^2) \varphi''
\end{cases}
\]

(6.4.20)

where

\[
\begin{align*}
\rho_0 &= y_0^2 + \frac{\iint_{\Omega} S (x^2 + y^2) \, d\Omega}{\iint_{\Omega} S \, d\Omega} \\
\rho_x &= \frac{\iint_{\Omega} S y^2 \, d\Omega + \iint_{\Omega} S x^2 y \, d\Omega}{\iint_{\Omega} S y^2 \, d\Omega} - 2 y_0
\end{align*}
\]

(6.4.21)

(6.4.22)

Also, from (6.4.10) and (6.4.11):

\[
\iint_{\Omega} A S \, d\Omega = P
\]
and \[ \frac{1}{\rho_x} \iint_S y^2 \Omega = -M_x. \]

2) **Differential equations of the buckling problem.**

The relation between the angle of rotation \( \theta \) and the distributed torsional moment \( m_z \) has been established in paragraph B (part 4) (6.4.16). Using the beam theory the relation of \( q_x \) and \( q_y \) to the lateral displacements \( u \) and \( v \) can be written as:

\[
q_x = T_y I_y \frac{du}{dx}, \quad q_y = T_x I_x \frac{dv}{dx}. \tag{6.4.23}
\]

where \( T_y \) represents the Osgood tangent modulus which was discussed in paragraph 6.3. \( T_x \) represents the tangent modulus corresponding to an additional infinitesimal bending deformation of the column in the place of its web. Formulation of \( T_x \) proves to be useless, because, as will be seen later, the corresponding term disappears from the final calculations. Replacing in equation (6.4.23) and (6.4.16) \( q_x, q_y \) and \( m_z \) by their values from (6.6.20) the following diff. equations are obtained:
The above system of equations is reduced to two independent groups; the first, which consists of the second differential equation (6.4.24), represents an Euler buckling in the plane of the web, not of any interest here.

From formula (6.4.12)

\[ M_x = P(m r_x + f + \frac{h}{2} - y_0) = -P \epsilon_y, \]

or

\[ \epsilon_y = -(m r_x + f + \frac{h}{2} + y_z) \]

(\( \epsilon_y \) being defined as negative in the actual case).

Using the above notation, the second group of differential equations from (6.4.24) is written as:

\[
\begin{align*}
\begin{cases}
T_y I_y \ u'' + P \ u'' + (P y_0 + M_x) \varphi'' = 0; \\
C_1 \varphi'' - (C + P(y_0 + M_x)) u'' + (P y_0 + M_x) u'' = 0. \\
\end{cases}
\end{align*}
\]

(6.4.24)
Introducing the critical ENGESSER-SHANLEY-OSGOOD load

\[ P_1 = \frac{T_y I_y}{2} \left( C + C_1 \frac{\pi^2}{P} \right) \] (6.4.26)

are the critical elastic-plastic flexural-torsional buckling load:

\[ P_s = \frac{1}{r_s^2} \left( C + C_1 \frac{\pi^2}{P} \right) \] (6.4.27)

one obtains the same buckling condition:

\[ (P_1 - P) \left[ P_2 - P \left( 1 + \frac{\epsilon_y}{r_s^2} \right) \right] - P_s \left( \frac{\epsilon_y}{r_s} \right)^2 = 0 \] (6.4.28)

The critical elastic-plastic flexural-torsional buckling load, \( P_{cr} \), can be found easily by solving the above quadratic equation. **Final Remark.** For I-shaped sections, for which the theory will be developed, the expressions for \( r_o^2 \) and \( \beta_x \) (6.4.21 and 6.4.22) are simplified as follows:

\[
\begin{align*}
\int \int S d \Omega &= \frac{\Omega}{2} (S_e + S_i) \\
\int \int S y^2 d \Omega &= \frac{\Omega}{2} [y_s^3 S_r + (h-y_r)^3 S_i] \\
\int \int S y^3 d \Omega &= \frac{\Omega}{2} [y_s^3 S_r - (h-y_r)^3 S_i]
\end{align*}
\]

\((y_s' s\ being\ negative\ for\ the\ flange)\)}.\)
On the other hand, by introducing the OSGood secant modulus:

\[
\bar{S}_e = \frac{2e \int_{-\frac{b}{2}}^{\frac{b}{2}} S_x x^2 dx}{I_y}
\]

we get

\[
\int \int S_x^2 dx = \frac{I_y}{2}(\bar{S}_e + \bar{S}_i)
\]

and

\[
\int \int S_x x^2 dx = \frac{I_y}{2} [y_e \bar{S}_e - (h - y_e) \bar{S}_i]
\]

Similarly, the results obtained for the elastic range in paragraph 6.2, relative to members whose warping is partially or completely prevented, are valid also in the elastic-plastic range. In particular formulas (6.2.17), (6.2.18), (6.2.24), (6.2.26), (6.2.32), (6.2.33) and (6.2.34) can be used provided that parameter \(E\) is replaced by \(T_y\) and quantities \(C\) and \(C_1\) be given their corresponding values for the elastic-plastic range.

D. Resume of Calculation Procedures for Simply Supported Columns Whose Ends are Completely Prevented from Warping.

The critical flexural-torsional buckling load \(P_{cr}\) will be calculated by the method prescribed above for a series
values of the load $P$. The curves $P = P(f)$ and $P_{cr} = P_{cr}(f)$ will be drawn. (Fig. 6.4.7).

The ordinate of the intersection point $A$ of these two curves represents the failure load of the column; the abscissa $A$ of point $A$ represents the deflection of this column at the instant of failure.

In order to obtain $P_{cr}$, the following operations should be performed:

1) The quantities defining the elastic-plastic state of the column are calculated by the method defined in paragraph 5.7; namely, $\varepsilon_i, \varepsilon_e, \varepsilon_i', \varepsilon_e', S_i, S_e$, and $f$.

2) The values of

$$y_n = \frac{S_i k}{S_i + S_e}$$
and

\[ y_\theta = \frac{S_t h}{S_t + S_\theta} - \frac{T_t h}{T_t + T_\theta} \]
\[ C_1 = \frac{T_\theta T_t}{T_\theta + T_t} - \frac{I_\theta h^2}{2} \]

\[ C = \frac{G_k}{3} \sum h b^3 \]
\[ G = \frac{E}{2,57(\eta = 0,285)} \]

are computed.

3) Also, the following quantities are calculated:

\[ e_y = - \left( m r_x + f + \frac{h}{2} \right) + y_\theta, \]
\[ r_x^2 = y_\theta^2 + \frac{I_\theta}{\Omega} \left( S_x + S_l \right) + \frac{\Omega}{2} \left[ y_\theta S_x + (h - y_\theta)^2 S_l \right] \]
\[ \beta_x = \frac{\Omega}{2} \left[ y_\theta S_x - (h - y_\theta)^2 S_l \right] + \frac{I_\theta}{\Omega} \left[ y_\theta S_x - (h - y_\theta)^2 S_l \right] - 2 y_\theta \]

\[ P_1 = T_\theta I_\theta \frac{\pi^2}{F} \]
\[ P_2 = \frac{1}{r_x^2} \left( C + 4 C_1 \frac{\pi^2}{F} \right) \]

reduced from (6.2.24)

by replacing \( E \) by \( T_\theta \).
4) Finally, the smaller root of the quadratic equation:

\[ (P_1 - P)[P_2 - P\left(1 + \frac{\delta_y \beta_x}{r_0^3}\right)] - P^2 \left(y_y - \delta_y \right)^2 = 0. \tag{6.2.26} \]

is found. This root gives the sought critical load \( P_{cr} \).

Let us suppose, for example, that \( P_1 \) is considerably smaller than \( P_2 \). The first member of the above equation is positive for \( P = 0 \) and negative for \( P = P_1 \). Therefore, the equation has one root from 0 to \( P_1 \). To find this root, let

\[
\begin{align*}
  a &= \left(1 + \frac{\delta_y \beta_x}{r_0^3}\right) - \frac{64}{9\pi^2} \left(y_y - \delta_y \right)^2 \\
  b &= -P_1 \left(1 + \frac{\delta_y \beta_x}{r_0^3}\right) - P_2 \\
  c &= P_1 P_2
\end{align*}
\]

and the equation can be written as:

\[ a P^2 + b P + C = 0. \]

with roots:

\[
P_{cr} = \frac{-b}{2a} \left[1 \pm \sqrt{1 - \frac{4ac}{b^2}}\right]
\]
6.5 Comparison of the theories developed in paragraph 6.3 and 6.4 with experimental results.

A. Theory of Torsional Buckling Only

As mentioned in paragraph 6.3, we have first tried to define the experimental critical buckling load by using the ENGESSER-SHANLEY-OSGOOD formula:

\[ p_{cr} = \frac{\pi^2 T L_y}{P} \]  

(6.3.2)

and have evaluated all calculations for 9 eccentrically loaded \( (\varepsilon_{eff} = 1) \) DIE 20 columns of \( m = 0.5 - 1 - 3 \) and \( \lambda = 60, 80, 100 \).
Table 1 below gives the assigned values of strains $\varepsilon_i$, deflections $f$ in the plane of the web, and the corresponding critical loads as calculated by formulas (5.7.1) and (6.3.2).

The exterior flange of the column remaining always elastic for all the 9 columns, $T_e$ is constant. Though, the average tangent modulus for the column

$$\overline{T} = \frac{T_1 + T_2}{2}$$

in formula (6.3.2) is dependent on $T_1$, that is of the assigned value of $\varepsilon_i$. This is the reason why all three values of $P_{cr}$ are the same for the three relative eccentricities $(m = 0.5, -1, -3)$.

If the curves $P_{cr} = P_{cr}(f)$ are drawn using the values from this table it will be noted that they always pass under the $P = P(f)$ curves or the theoretical curves of paragraph 5.7. As an example, the curve $P_{cr} = P_{cr}(f)$ corresponding to $\varepsilon_2/\varepsilon_i = 1$, $\lambda = 80$, and $m = 1$ has been represented by dotted lines in Fig. 6.5.1.

Close examination of this figure shows that the torsional
<table>
<thead>
<tr>
<th>$\varepsilon$ essuyé</th>
<th>$f$ mm</th>
<th>$P_{fr}$ Kg</th>
<th>$f$</th>
<th>$P_{fr}$ Kg</th>
<th>$f$</th>
<th>$P_{fr}$ Kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m - 0.5$</td>
<td>1</td>
<td>3</td>
<td>$m - 0.5$</td>
<td>1</td>
<td>3</td>
<td>$m - 0.5$</td>
</tr>
<tr>
<td>$\lambda = 60$</td>
<td></td>
<td></td>
<td>$\lambda = 80$</td>
<td></td>
<td></td>
<td>$\lambda = 100$</td>
</tr>
<tr>
<td>400</td>
<td>1,68</td>
<td>—</td>
<td>3,89</td>
<td>327,553</td>
<td>3,05</td>
<td>4,58</td>
</tr>
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<td>2,59</td>
<td>3,85</td>
<td>5,82</td>
<td>279,311</td>
<td>4,61</td>
<td>6,99</td>
</tr>
<tr>
<td>800</td>
<td>3,60</td>
<td>5,19</td>
<td>7,66</td>
<td>222,822</td>
<td>6,69</td>
<td>9,47</td>
</tr>
<tr>
<td>1,000</td>
<td>4,70</td>
<td>6,55</td>
<td>9,43</td>
<td>197,955</td>
<td>8,80</td>
<td>12,00</td>
</tr>
<tr>
<td>1,200</td>
<td>5,89</td>
<td>7,93</td>
<td>11,10</td>
<td>183,227</td>
<td>11,08</td>
<td>14,57</td>
</tr>
<tr>
<td>1,400</td>
<td>7,15</td>
<td>9,33</td>
<td>12,71</td>
<td>172,892</td>
<td>13,50</td>
<td>17,16</td>
</tr>
<tr>
<td>1,600</td>
<td>8,48</td>
<td>10,73</td>
<td>14,26</td>
<td>166,958</td>
<td>16,03</td>
<td>19,77</td>
</tr>
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<td>1,800</td>
<td>9,91</td>
<td>12,14</td>
<td>15,66</td>
<td>165,496</td>
<td>18,71</td>
<td>22,35</td>
</tr>
<tr>
<td>2,000</td>
<td>11,35</td>
<td>13,54</td>
<td>17,03</td>
<td>165,082</td>
<td>21,40</td>
<td>24,92</td>
</tr>
<tr>
<td>2,500</td>
<td>14,96</td>
<td>17,05</td>
<td>20,44</td>
<td>164,746</td>
<td>28,08</td>
<td>31,32</td>
</tr>
</tbody>
</table>

**TABLEAU 1**

Tableau des valeurs $P_{fr}$, $f$

On the other hand, the torsional rigidity have:

$k = 21,000 \text{ kN/mm}^2$,

$T = 20,980 \text{ kN/mm}^2$.

With the exterior flange elastic, A regime of the calculation will be found in the table 2 below.

A regime of the calculation will be found in the table 2 below.

A regime of the calculation will be found in the table 2 below.

The formulas used are given in part D of paragraph 6.6.

Complete calculations have been made for the 9 columns of D13 20 section as defined in part A above. The formulas used are given in part D of paragraph 6.6.

P. Torsional-Torsional Buckling

B. Membranous-Torsional Buckling

R. Membranous-Torsional Buckling

The fundamental role in the buckling phenomenon, previously assumed negligible, plays a very fundamental role in the buckling phenomenon.
\[
C = \frac{G k}{3} \sum h b^3 = 2059, 11.10^6 \text{ kg mm}^2
\]
and
\[
\frac{I_y k^3}{2} = 22,4768.10^{10} \text{ mm}^6.
\]

Using the values of Table 2 for each of the 9 columns a \( P_{cr} = P_{cr(f)} \) curve can be drawn, analogous to the one on Fig. 6.5.1 which meets the \( P = P(f) \) curve at one point. The ordinate of this point is the corresponding critical flexural-torsional buckling load.

Table 3 below groups theoretical and experimental values for \( P_{cr} \) for all 9 columns. It can be said, without exaggeration and remembering the complexity of the phenomenon that the accordance between theory and experiment is quite remarkable. The fact that the same correlation is valid for all 9 columns of different sizes and loadings excludes the possibility of a fortunate coincidence. We can then very well conclude that "our flexural-torsional buckling theory represents the reality very well."

It is interesting to compare the approximate theoretical values obtained by the formulas proposed in the German
### Comparaison entre les chaînes

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>m</th>
<th>$e_i \times 10^5$</th>
<th>$e_f \times 10^5$</th>
<th>$S_i$ Kg/mm²</th>
<th>$T_i$ Kg/mm²</th>
<th>$g_i$ mm</th>
<th>$g_f$ mm</th>
<th>$e_i$ mm</th>
<th>$r_i$ mm³</th>
<th>$\beta_i$ mm</th>
<th>$10^{-12} C_i$ Kg²/mm⁴</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0,5</td>
<td>1200</td>
<td>+322,11</td>
<td>17,167</td>
<td>13,285</td>
<td>2488</td>
<td>74,22</td>
<td>56,70</td>
<td>-49,97</td>
<td>12.161</td>
<td>-126,60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1400</td>
<td>+333,74</td>
<td>15,883</td>
<td>11,639</td>
<td>1166</td>
<td>71,08</td>
<td>62,38</td>
<td>-54,37</td>
<td>12.742</td>
<td>-143,62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1600</td>
<td>+336,40</td>
<td>14,656</td>
<td>10,278</td>
<td>407</td>
<td>67,82</td>
<td>64,67</td>
<td>-58,96</td>
<td>12.930</td>
<td>-154,26</td>
</tr>
<tr>
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<td>1</td>
<td>1200</td>
<td>+17,69</td>
<td>17,167</td>
<td>13,285</td>
<td>2488</td>
<td>74,22</td>
<td>56,70</td>
<td>-87,89</td>
<td>12.161</td>
<td>-126,60</td>
</tr>
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<td></td>
<td>1400</td>
<td>+9,91</td>
<td>15,883</td>
<td>11,639</td>
<td>1166</td>
<td>71,08</td>
<td>62,38</td>
<td>-92,35</td>
<td>12.742</td>
<td>-143,62</td>
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<tr>
<td></td>
<td></td>
<td>1600</td>
<td>+0,85</td>
<td>14,656</td>
<td>10,278</td>
<td>407</td>
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<td>64,67</td>
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</tr>
<tr>
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<td>12.161</td>
<td>-126,60</td>
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<td>11,639</td>
<td>1166</td>
<td>71,08</td>
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<td>10,278</td>
<td>407</td>
<td>67,82</td>
<td>64,67</td>
<td>-243,74</td>
<td>12.930</td>
<td>-154,26</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>m</th>
<th>P_Euler</th>
<th>$\gamma$</th>
<th>m</th>
<th>P_Euler</th>
</tr>
</thead>
<tbody>
<tr>
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<td>328,6</td>
<td>60</td>
<td>1</td>
<td>184,3</td>
</tr>
<tr>
<td>80</td>
<td>0,5</td>
<td>118,0</td>
<td>80</td>
<td>1</td>
<td>252</td>
</tr>
</tbody>
</table>

La valeur finale obtenue est $12.161$, soit 12,161. Rappelons que, à substituer $a$ l'élanement réel d'un élanement idéal, 

$$
\lambda_i = \gamma \sqrt{\frac{P_{\text{Euler}}}{P_{\text{cr}}}}
$$

à l'aide duquel on calcule la tension de la barre comme si elle flambait. Nous avons calculé la charge critique de flambement par flexion en utilisant l'équation (6.2.16), c'est-à-dire...
Table 3

Theoretical and Experimental Flexural-Torsional Buckling Loads in Tons.

<table>
<thead>
<tr>
<th>$m$</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>Expériment.</td>
<td>Théorique</td>
<td>Expériment.</td>
</tr>
<tr>
<td>60</td>
<td>84.8</td>
<td>85.2</td>
<td>64.8</td>
</tr>
<tr>
<td>80</td>
<td>71.0</td>
<td>75.0</td>
<td>59.0</td>
</tr>
<tr>
<td>100</td>
<td>62.5</td>
<td>62.7</td>
<td>53.5</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$m$</th>
<th>$P_{\text{Crit. Émil.}}$</th>
<th>$P_{\text{Crit. fl. m.}}$</th>
<th>$\lambda$</th>
<th>$\sigma_P$</th>
<th>$P_{\text{Crit. DIN 4114}}$</th>
<th>$P_{\text{Crit. expérim.}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.5</td>
<td>328.6</td>
<td>278.4</td>
<td>65.0</td>
<td>23.6</td>
<td>131</td>
<td>84.8</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>225.0</td>
<td>72.5</td>
<td>22.7</td>
<td>126</td>
<td>64.8</td>
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<td>3</td>
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<td>122.9</td>
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<td>19.0</td>
<td>110</td>
<td>32.5</td>
</tr>
<tr>
<td>80</td>
<td>0.5</td>
<td>184.3</td>
<td>165.2</td>
<td>84.5</td>
<td>22.4</td>
<td>124</td>
<td>71.0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>138.0</td>
<td>92.5</td>
<td>20.45</td>
<td>113</td>
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<tr>
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<td></td>
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<td>123.0</td>
<td>13.7</td>
<td>76</td>
<td>32.5</td>
</tr>
<tr>
<td>100</td>
<td>0.5</td>
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<tr>
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<td>94.5</td>
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<td>16.5</td>
<td>91.5</td>
<td>53.5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>53.6</td>
<td>148</td>
<td>9.45</td>
<td>52.5</td>
<td>29.0</td>
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</table>
Specifications, to the experimentally obtained ones. This method is explained at the end of paragraph 6.3. It consists of substituting the real slenderness ratio \( \lambda_y \) by an ideal one,

\[
\lambda_i = \lambda_y \sqrt{\frac{P_{\text{Euler}}}{P_{cr}}}
\]

with which the critical stresses are determined on the basis of lateral buckling.

The critical "elastic" flexural-torsional buckling load has been determined by equation (6.2.16), for the case of column ends free to warp. The critical buckling stresses were determined using TETMAYER's formula:

\[
\sigma_{cr} = 31 - 0.114 \lambda \text{ pour } \lambda < 105
\]

and Euler's formula

\[
\sigma_{cr} = \frac{207300}{\lambda^2} \text{ pour } \lambda > 105
\]

and an average \( \Omega \) of 5550 mm\(^2\).

All calculations are shown in Table 4 below. It can be stated that the method of calculation suggested by the German Specifications gives very bad, erroneous and unsafe results. If the ends were assumed to be
prevented from warping, which happens to be the actual case, the computed loads would have been even larger, thus more unsafe.

6.6 Extension of the Flexural-Torsional Buckling Theory for Columns Under Oblique Loading \((e_2 \neq e_1)\)

The expressions for \(q_x\) and \(q_y\) and \(m_z\) given in part C of paragraph 6.5, and formulas (6.4.23) are valid for a member with constant as well as with a variable moment of inertia.

On the other hand, formula (6.4.16) is valid only for a prismatic member. However, the error involved by using it for a member with variable elastic properties is negligible.

The resulting differential equations:

\[
\begin{align*}
\mathcal{T}_y I_y u^{iv} + P u'' + P (\gamma_0 + \gamma_y) \varphi'' &= 0 \\
\left( C_1 \varphi'' - [C + P (e_y \beta_x + r_y)] \varphi'' + P \gamma_y \right) + e_y) u'' &= 0
\end{align*}
\]  

(6.4.25)

provide a sufficiently accurate solution for the flexural-torsional buckling of obliquely loaded columns. \((e_2 \neq e_1)\)
In this case, the quantities \( T_y, y_0, e_1, C_1, C, \beta_0 \) and \( r_0 \) are variable along the length of the member. They can be computed for every section, provided that the degree of the plastification corresponding to the force \( P \) and the bending moment \( M \) are known for this particular section.

Let it be assumed that the elastic-plastic bending deformation of the column in the plane of its web, has been determined using the graphical method of paragraph 5.6.

The point in question is, as in the case \( e_2/e_1 = 1 \), to calculate the critical flexural-torsional buckling load of the column, \( P_{cr} \), for each value of \( F \).

The actual failure load \( P_{cr} \) is the one for which the load \( P \) for which the flexural elastic-plastic state of the column in its plane of symmetry has been determined.

The critical load, \( P_{cr} \), will be computed only for the case of a simply supported column whose ends are free to warp. The boundary conditions for this case are extremely simple:

\[
\begin{align*}
u = u'' = 0, & \varphi = \varphi'' = 0 \text{ pour } z = 0 \text{ et } z = l
\end{align*}
\]
For the differential equations (6.4.25) above, containing only even derivatives of \( u \) and \( \phi \), the second derivatives can be taken as unknowns; thus, for the sake of multiplication let:

\[
\begin{align*}
  u''(x) &= U(x); \quad \phi''(x) = \Phi(x) \\
\end{align*}
\]

(6.6.2)

Hence, the boundary conditions become:

\[
U = 0, \Phi = 0 \quad \text{pour} \quad z = 0 \quad \text{et} \quad z = l.
\]

Using equation (6.6.2) the system of differential equations is written as:

\[
\begin{align*}
T_y \lambda y U'' + P U + P(y_0 - \varepsilon \phi) \Phi &= 0 \\
C_1 \Phi'' - [C - P(\varepsilon \phi_0 + r \phi)] \Phi + P(y_0 - \varepsilon \phi) U &= 0.
\end{align*}
\]

(6.6.3)

This system will be solved using the classical Vianello method. \( U \) and \( \phi \) are given the expressions \( U_0(x) \) and \( \phi_0(x) \). With the end of these expressions the terms

\[
A_0 = P[\Omega_0 + (y_0 - \varepsilon \phi) \Phi_0]
\]

and

\[
B_0 = -[C + P(\varepsilon \phi_0 + r \phi)] \Phi + P(y_0 - \varepsilon \phi) U_0
\]
can be evaluated for every section. Thus, equation 6.6.3 takes the form

\[
\begin{cases}
T_IyU' + A_y = 0 \\
C_1 \Phi'' + B_0 = 0.
\end{cases}
\]

(6.6.4)

Considering equations 6.5.4 as elastic, and applying the graphical Mohr method based on the application of the furnicular polygon, improved values of \(U\) and \(\Phi\), rarely \(U_1(z)\) and \(\Phi_1(z)\) may be obtained.

By the Vianello method, it can be shown that the quotients

\[
\frac{U_{1\text{max}}}{U_{0\text{max}}} \quad \text{and} \quad \frac{\Phi_{1\text{max}}}{\Phi_{0\text{max}}}
\]

give approximate values for the critical buckling load, \(P_{cr}\).

The same procedure will be repeated with initial values \(U_1\) and \(\Phi_1\) to get even more improved values \(U_2\) and \(\Phi_2\), till the deflections \(U_n\) and \(\Phi_n\) are not affected by the preceding deformations \(U_{n-1}\) and \(\Phi_{n-1}\), and we have for every section

\[
\frac{U_n}{U_{n-1}} = \frac{\Phi_n}{\Phi_{n-1}} = \text{Constant}
\]
This constant is the sought critical load.

For support conditions other than (6.6.1) (for example in the case of prevented warping at the ends) it is impossible to reduce the system of fourth order differential equations (6.4.25) into a system of second order. (6.6.3) Nevertheless, an approximate value for the critical load may be searched for, by using an energy method and assuming for \( u \) and \( \varphi \) the expressions:

\[
\begin{align*}
    u &= A_1 \sin \frac{\pi z}{l} + A_2 \sin \frac{3\pi z}{l}; \\
    \varphi &= A_3 \left( 1 - \cos \frac{2\pi z}{l} \right)
\end{align*}
\]

It should be noted however, that the integrals resulting from the energy equation (6.2.19) contain the coefficients \( T_y, C, e_y, \beta_x, r_0^2 C_1 \), etc... all dependent of \( z \), since the degree of plastification varies along the axis of the column. It follows therefore, that the calculations to be executed will be extremely painful, and it is not worth undertaking such a numerical analysis.
CONCLUSIONS

It is hoped that the triple aim set in the preceding pages has been attained:

1) To perform with all possible care buckling tests on obliquely loaded columns, and to collect all experimental data on this problem of great practical importance.

2) To reduce from these tests a simple design method, which follows the reality closer than the existing methods, and providing an appreciable economy in steel.

3) To provide a thorough theoretical analysis of the observed phenomena, which will be in good correlation with the test results and will provide means of checking the validity of proposed design formulae of the narrow range covered by the tests.
ACKNOWLEDGEMENTS

The tests reported in the present paper, were sponsored by the "Commission pour l'Etude de la Construction Metallique" (C.E.C.M.) research organization which is composed equally from a) the steel industry and steel construction and b) the Belgian State officials represented by the "Institut pour l'Encouragement de la Recherche Scientifique de la l'Industrie et l'Agriculture (I.R.S.I.A.).

The proposal of the test program was first submitted to a committee of qualified engineers from the industry as well as administrative representatives who have made valuable suggestions.

The general program and particularly the test columns were prepared by Mr. F. HERBRANT, technical director of C.E.C.M. and Mr. H. LOULIS, scientific consultant to the C.E.C.M. and professor in the University of Liege.

The tests for the evaluation of the material characteristics were performed by the personnel of the resistance of materials laboratory in the University. The buckling tests, were executed by the personnel of the Civil Engineering and Fluid Mechanics laboratory of the Liege University,
under the direction of the chief technician Miss. M. DZULYNSKI.

The technicians of these laboratories were helped considerably by the personnel of the "Liege Technical Bureau" of C.E.C.M. in these two different types of tests. The Bureau has also spent many months in the evaluation of the test results and has performed a set of very complicated calculations of which this report gives a resume.

Mr. R. GREYSH, engineer with the C.E.C.M., deserves a special mention, as most of the above work was directed by the latter.

Finally, two students of the Civil Engineering Department, who are both graduate engineers at the time of writing, Mr. R. COLIN and A. FOSSION have largely contributed to the theoretical analysis of plastic buckling in the plane of the web. This was carried out as their diploma project.

To all those who helped in the realization of this study, the authors present their sincere appreciation.

The undersigned wish to extend their thanks to the "Column Research Council" of the U. S., one of whom is a
corresponding member for Europe, for the services that the Council has rendered in sending them progress reports relative to buckling research as conducted in the United States.
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