Welded Continuous Frames and Their Components

Progress Report No. 9

PLASTIC STRENGTH AND DEFORMATIONS OF CONTINUOUS BEAMS

by

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with an appendix prepared by Walter H. Weiskopf

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## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>2.</td>
<td>Limitations</td>
<td>8</td>
</tr>
<tr>
<td>3.</td>
<td>Methods for Computation of Deflections</td>
<td>11</td>
</tr>
<tr>
<td>(a)</td>
<td>Numerical Integration</td>
<td>13</td>
</tr>
<tr>
<td>(b)</td>
<td>Mathematical Integration</td>
<td>13</td>
</tr>
<tr>
<td>(c)</td>
<td>$\phi$ - Area Method</td>
<td>13</td>
</tr>
<tr>
<td>(d)</td>
<td>Simple Plastic Theory</td>
<td>17</td>
</tr>
<tr>
<td>(e)</td>
<td>Plastic Hinge Method</td>
<td>21</td>
</tr>
<tr>
<td>4.</td>
<td>Comparison with Experimental Results</td>
<td>21</td>
</tr>
<tr>
<td>5.</td>
<td>Influence of End Restraint</td>
<td>23</td>
</tr>
<tr>
<td>6.</td>
<td>Influence of Load Distribution</td>
<td>26</td>
</tr>
<tr>
<td>7.</td>
<td>Summary</td>
<td>28</td>
</tr>
<tr>
<td>8.</td>
<td>Acknowledgements</td>
<td>30</td>
</tr>
<tr>
<td>9.</td>
<td>References</td>
<td>31</td>
</tr>
<tr>
<td>10.</td>
<td>Nomenclature</td>
<td>34</td>
</tr>
<tr>
<td>11.</td>
<td>Tables and Figures</td>
<td>36</td>
</tr>
<tr>
<td>12.</td>
<td>Appendix by W.H. Weiskopf</td>
<td>56</td>
</tr>
</tbody>
</table>
While conventional structural design according to the theory of elasticity gives assurance against excessive deflections, the application of design methods based on the ultimate carrying capacity of a structure necessitates special consideration of critical deflections. The simplicity of the so-called plastic design methods may be overshadowed by time-consuming deflection computations unless reliable approximate approaches can be applied.

This paper demonstrates several methods for computing deflections due to bending of mild steel beams of uniform cross section. The effect of various simplifying assumptions which greatly reduce the numerical work involved is shown together with comparison with experimental results. The influence on the deflections of various degrees of end restraint and load distribution is computed.

The paper demonstrates for several loading conditions on continuous beams the possible savings by using plastic design as against conventional elastic design and suggests a specific design criteria applicable to the examples given.
The prediction of deflections of mild steel beams strained beyond the elastic limit is an important part of design methods based on the ultimate carrying capacity. This concept of structural behavior emphasized collapse (or excessive deformation) rather than a maximum unit stress as the limiting criterion on structural usefulness.

Structural design methods based on the theory of elasticity, except in problems of elastic stability, define as the limiting useful load the load causing initial yield at the most highly stressed location in the structure. The working load is taken as a certain safe fraction of this yield load, and will leave the entire structure well within the so-called elastic range. The deflections in many cases constitute no major design consideration and are readily estimated for those cases in which certain limits are imposed on their magnitude.

For statically determinate structures and in problems of stability the two concepts of structural carrying capacity yield basically the same solution. In the analysis of redundant structures, however, the differences are great.

Initial yield at some location in a redundant structure does not render the structure incapable of carrying additional load. Although the deflections will increase at a faster rate above the yield load than in the elastic range, danger of imminent collapse is obviated by the ability of mild steel to relieve the most highly strained portions of the structure. The strain-hardening characteristics of structural steel further increases the margin between initial yield and ultimate collapse. A similar influence is observed due to catenary stresses set up in beams which are not allowed to move horizontally at supports. Neglecting strain-hardening and
Catenary effects, collapse occurs when a sufficient number of cross sections have yielded to reduce the structure or a part of it to act as a mechanism under any further increase of load.

As an illustration, consider a beam over three spans carrying loads in the center span as shown in Fig. 1a. For an amount of end restraint of the center span and a load distribution which give larger moments $M_A$ at the supports than center moment $M_B$, Fig. 1b, yield will first take place at the supports (Fig. 1c). Under further increase of load the center moment $M_B$ will increase at a faster rate than in the elastic range to relieve the yielding sections at the supports. At the load $P_1$ these sections will have yield zones penetrating all the way to the neutral axis and can carry no additional moment until strain-hardening occurs. The center section, however, does not yield until the load reaches $P_2$, and loads $P_p$ are required to produce yield all through the center section. Except for strain hardening the carrying capacity is now exhausted, and the center deflection, Fig. 1d, increases rapidly under practically no increase in load.

According to this concept, the collapse load is defined as that load at which a small load increment would cause a very large increase in deflection. However, this simple definition of the ultimate load is not concerned with the magnitude of the deflections at that load but merely with their rate of change. The magnitude of the deformation at this theoretical collapse load may very well be prohibitive, and the structure in reality rendered useless at a lower load due to excessive deflections. In many cases, therefore, a design based on the ultimate carrying capacity requires the computation of the deflections at the full load.* If this deformation cannot be tolerated

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*The full load is defined as working load times the factor of safety or load factor.
as a limiting criterion, then the loading causing the maximum toler-
able deformation must be computed and substituted as ultimate load.

A constant load factor for each type of structure leads to a uni-
form safety margin and thus to economy in design. In Progress Report
No. 3 those items covered by the load factor were discussed. It was
indicated that "A structure so designed will usually be loaded within
the elastic limit in the range of its working load". However, since
the reserve strength above the yield load varies greatly with the
amount of end restraint and the type of load distribution, an econ-
omical load factor may under some conditions bring parts of the
structure above the yield point under the working load. In these
cases a check on the deflections under working load will be desirable.

The above discussion points out the need of methods for com-
puting beam deflections beyond the elastic range. The simple linear
relationship between load and deflections is lost as soon as initial
yield is reached anywhere in the beam (Fig. 1d), consequently, the
calculation of the deflections becomes more involved. It is the
purpose of this paper to show the effect of various simplifying
assumptions outlined in Ref. 19 which can be made in order to reduce
the work involved in such calculations, and also to indicate the in-
fluence on deflections of various end conditions and load distribution.

The underlying principles for predicting deflections in the
plastic range were furnished by Nadai\textsuperscript{12} and Timoshenko\textsuperscript{20}. The latter,
with references to earlier work, outlines an accurate semi-graphical
procedure based on the actual stress-strain relationship using the
conjugate beam method and the concept of reduced modulus of elas-
ticity. Van den Broek\textsuperscript{10} gives examples of the calculation of beam
deflections based on an idealized stress-strain diagram, and gives
observed deflections of simple and continuous beams of steel and
various alloys.
Roderick and Phillips in a literature survey on the carrying capacity of simply supported mild steel beams, compare the deflections according to various assumptions concerning the stress distribution through the depth of the beam, and derive the slope-deflection equations in the case of symmetrical concentrated loads.

Roderick, using Timoshenko's semi-graphical method, shows that the simple plastic theory based on an idealized stress-strain diagram determined from annealed specimens loads to a satisfactory prediction of the ultimate load. However, this theory does not predict with sufficient accuracy the deflections of as-delivered specimens. In Ref. 14 the simple plastic theory is extended to take into account strain hardening. Both these reports present results from simply-supported, rectangular beam tests.

Symonds and Neal, by means of the slope-deflection method, calculate the critical deflections according to the simple plastic theory for the load under which yielding starts at the last hinge to form ($\theta_2$ in Fig. 1d). The predictions are compared with test results of Stüssi & Kollbrunner (1935) and Maier-Leibnitz (1936) for three-span beams of varying ratio between outer and center span lengths. Strain hardening and the spread of the plastic zones is not considered. These authors call for full scale tests of frames and continuous beams made from standard commercial sections using normal fabrication procedures.

Such tests are in progress at Lehigh University as part of an investigation of welded continuous frames and their components. Results are reported for beams, columns, corner connections, and portal frames. Ref. 3 in this series shows that plastic design results in deflections which compare favorably with those resulting from elastic design, and suggests a simple method of drawing the
load-deflection diagram for beams in the plastic range. In a discussion of this paper Symonds\textsuperscript{19} suggests taking the deflection under working load as the deflection $\delta_2$, when the last hinge starts to form, divided by the load factor. This approach, however, results in deflection estimates which may be several hundred percent too large.

Although beams only are discussed in this paper, the results may be extended to portal frames with small axial loads and no side-sway. Experimental deflections from tests of miniature frames under various combinations of vertical and side loads are given by Baker and Heyman\textsuperscript{9}. Hrennikoff\textsuperscript{11} (and discussors) and Symonds\textsuperscript{17} offer analytical solutions for the plastic deflections of frames.
2. LIMITATIONS

Several factors influencing the deflections of beams in the plastic range are not included in the following treatment. The most important of these factors are briefly discussed in the following.

Local Instability

The usual rolled sections are so proportioned as to prevent local buckling of their elements in the elastic region. When part of a cross section has yielded, however, the resistance against local buckling of the compression flange at that cross section is reduced. Local instability therefore may cause greatly increased deflections and collapse before the beam is stressed very far into the plastic range.

Plastic design theories are based on the assumption that the sections used have sufficient "rotation capacity" to allow the plastic hinge moment to be maintained through the required localized angle of rotation necessary for development of hinge moments at all other required locations. Premature local buckling of cross section elements will destroy the rotation capacity and thus invalidate the above basic assumption.

Residual Stresses

As-delivered beams show large "locked-up" or residual stresses due primarily to non-uniform cooling after hot-rolling. The process of straightening or cambering after rolling and also welding has a similar although more localized effect.

When added to the stresses produced by external loads the residual stresses lower the yield load on a structure appreciably below that expected from simple tension tests. A reduction of the calculated initial yield load by one-third is commonly found. This earlier yield causes deflections somewhat larger than predicted.
Stress Concentrations

Stress concentrations in the region of application of concentrated loads are generally not considered in the analysis, and raise the actual peak stresses far above the computed nominal stresses. As a result, yield occurs at lower loads than predicted and is more extensive, thus causing a corresponding increase of actual deflections.

Shear Deformation

As is common procedure in calculating elastic deformations, the effect of shear forces is neglected. However, yielding of the web due to shear force may add appreciably to the beam deflections.

Axial Compressive Forces

Although only beams are treated in this paper the numerical examples simulate pin-ended portal frames without side-sway. If the outer spans were rotated about the interior supports to a vertical position, they would then form the legs of the portal frame. In such frames, however, all members carry compressive axial forces in addition to moments, the effect of which is two-fold:

1) For a certain moment the addition of a compressive axial force increases the curvature of the member, thus increasing the deflection. This effect is minor for thrusts smaller than about 10% of the compressive yield load.

2) As deflections grow larger the additional moment of the thrust multiplied by the deflection becomes appreciable. For one of the portal frame tests in which a uniform 8WF40 section was used, this effect added 11% to the maximum beam moment at the ultimate load. The deflections are correspondingly increased.

Catenary Effect

Large deflections tend to shorten the distance between supports. In beams with pinned supports axial tensile stresses develop and a
part of the load is carried through this catenary effect with resulting smaller deflections than if carried by bending.

In portal frames as discussed above the knees are practically free to approach one another, and the axial tension does not develop.

**Summary of Limitations**

The integrated effect of local instability, residual stresses, stress concentrations, shear deformation, and thrust is to increase the deflections and lower the ultimate carrying capacity. Ref. 5 offers a more detailed discussion of these factors on the background of experimental evidence, which is summarized as follows:

(1) The calculated deflection at the predicted initial yield load was reached at a 10 to 25% lower load.

(2) At the calculated initial yield load the deflections exceeded those predicted by 13 to 88%.

Common to all of the methods for calculating plastic deflections discussed in this paper are the following basic assumptions.

(1) The longitudinal strain varies linearly across the depth of the beam.

(2) The stress-strain relationship in bending is identical to that in simple tension tests.

(3) The stress-strain relationship is the same in tension as in compression.

(4) Under increasing load the ratios between the individual loads are held constant.
3. METHODS FOR COMPUTATION OF DEFLECTIONS

In the elastic theory the linear relation between bending moment and curvature is expressed by

\[ \phi = \frac{M}{EI} \]  

(1)

The change in slope of the deflected beam between \( x = 0 \) and \( x = x \) equals the area under the \( \phi - \) curve between these points:

\[ \alpha_o - \alpha = \int_0^x \phi \, dx \]  

(2)

In the example in Fig. 2, \( \alpha_o \) is the slope at the left support and is found by the "conjugate beam" method:

\[ \alpha_o = \frac{1}{L} \int_0^L \phi (1-x) \, dx \]  

(3)

The deflection \( \delta \) at a distance \( l \) from the support, Fig. 2, then becomes

\[ \delta = \int_0^\delta \alpha \, dx = \alpha_o \cdot l - \int_0^l \phi (l-x) \, dx \]  

(4)

which may be interpreted as the moment of the area under the \( \phi \) curve between \( x = 0 \) and \( x = l \) about the point of \( \delta \), including the effect of the slope at the support.

Above the elastic limit the linear relationship Eq. (1) is lost, but the same method as outlined above holds if the actual \( M-\phi \) relationship is used in place of Eq. (1). Assuming the same stress-strain relationship in bending as determined from simple tension tests, Fig. 3, the \( M-\phi \) diagram may be derived from the stress-strain diagram as outlined in the appendix of Ref. 1. In Fig. 3 the measured stress-strain curve is closely approximated in the elastic range by

\[ \sigma' = E \cdot \varepsilon \]  

(5)

in the plastic range for as-delivered steel having no distinct upper yield point by

\[ \sigma' = \sigma_y \]  

(6)

and in the early part of the strain-hardening range by

\[ \sigma' = B + C \varepsilon \]  

(7)

* See pg. 35 for "Nomenclature"
The latter part of the strain-hardening range will rarely be reached due to failure by local buckling, and is of little practical interest.

The $M-\phi$ diagram evaluated from this stress-strain relationship for an 8WF40 section is shown in Fig. 4. Using such a $M-\phi$ relationship in place of Eq. (1) the deflections above the elastic range may be calculated by the procedure exemplified in Eqs. (2) to (4). In statically determinate beams no difficulties are encountered, but for redundant beams the non-linear $M-\phi$ relationship necessitates a trial-and-error procedure to satisfy the boundary conditions.

In the example of Fig. 5, (a symmetric, continuous beam with third-point loads in the center span), the yield moment is exceeded at supports and the ratio between the moments at support $M_c$ and at the centerline $M_L$ is no longer given by the elastic analysis. With an assumed value of $M_b$ the centerline moment is given by

$$M_L - M_b = \frac{PL^2}{3}$$

and the $\phi$-diagram can be plotted as shown using an $M-\phi$ curve like Fig. 4. Due to symmetry, the rotation of the beam axis at the center as determined by integration of the $\phi$-diagram should be zero. This boundary condition will most likely not be satisfied by the first trial assumption for $M_b$. The zero-line of the moment diagram must then be shifted in the proper direction, the new $\phi$-diagram plotted, its areas computed and a second check on the resulting rotation at the centerline obtained. The method is theoretically satisfactory but the amount of numerical work involved is prohibitive for practical applications.

In principle the above approach is common to the various methods. The methods differ only in the extent to which simplifying assumptions are made in order to reduce the amount of work involved.
(a) **Numerical Integration of the Actual \( M-\phi \) Curve**

This method is outlined above and gives the exact deflection in-so-far as the \( M-\phi \) diagram used is correct. In as-delivered specimens, however, residual stresses and stress concentrations will cause earlier yield and delay the attainment of the predicted "plastic hinge moment" as indicated in Fig. 4. In comparison with this effect the simplifying assumptions employed in the following methods introduce relatively small errors.

The actual stress distribution across and along the beam as illustrated for a cantilever in Fig. 6 (a) forms the basis for this "exact" method.

(b) **Mathematical Integration of Idealized \( M-\phi \) Curve**

This method is due to Mr. W.H. Weiskopf and is described by him in the Appendix of this paper. Mathematical integration of the \( M-\phi \) curve is made possible by the idealization shown in Fig. 7. The resulting assumption of stress distributions in a beam is shown in Fig. 6 (b).

The effect of neglecting the short curved portion of the \( M-\phi \) diagram near the yield point (Fig. 7) is to slightly reduce the predicted deflections. This error becomes larger for sections having large relative flange thickness, especially if a considerable length of the beam is under constant moment. **Neglecting** the small triangles in stress diagram 1 Fig. 6 (b), also reduces the calculated deflections, but in significant amounts only for deep I-sections and rectangular sections.

(c) **\( \phi - \) Area Method**

C.H. Yang\(^ {21} \) suggested the further simplification of neglecting the spread of the plastic zones along the length of the beam. The equivalent assumption for the stress-strain and \( M-\phi \) relationships
are shown in Fig. 8. In a beam with a moment gradient as shown in Fig. 6 (c) the length of the perfectly plastic region will consequently be zero. The curvature in the remaining two regions is related to the moment by

\[ \psi = \frac{M}{EI} \]  

(9)

in the elastic region up to \( M_p \), and

\[ \psi = \frac{M - B Z}{C} \]  

(10)

in the strain hardening region for moments exceeding \( M_p \) (Fig. 8). In a region of constant moment \( M_p \) the curvature will have a value between the limits

\[ \frac{M_p}{EI} \leq \psi \leq \frac{M_p - B Z}{C} \]  

(11)

and is determined by the boundary conditions.

The effect of neglecting the spread of the plastic zone along a member with a moment gradient is to give somewhat smaller predicted deflections. This additional simplifying assumption thus adds to the inaccuracy of the previous method, but the deviations are small as compared to the influence of residual stress and stress concentration as indicated on the \( M-\psi \) diagram, Fig. 4. These assumptions considerably shorten the numerical work, as is shown by the following example when compared with the Appendix.

Consider an SWF40 beam over three spans 7', 14', and 7', with concentrated loads \( P \) at the third-points of the central span (Fig. 9a). The maximum deflection (which occurs at the centerline) will be calculated. The material properties and the geometrical properties of the SWF40 section as determined by tests are shown in Table I.

From the elastic analysis of the structure we have

\[ M_o = -M_b = \frac{PL}{6} \]  

(12)
where \( M_c \) and \( M_b \) are respectively the moments at centerline and support. Elastic behavior is assumed until these moments reach the full plastic hinge value \( M_p \) (Fig. 9b), corresponding to the load

\[
P = \frac{6M_c}{L} = \frac{6M_b}{L} = \frac{6M_p}{L} = 53.5 \text{ kips}
\]  

(13)

The curvature at any point is proportional to the ordinate of the moment diagram, Fig. 9b. The beam axis at the center remains horizontal due to symmetry, and the center deflection equals the moment about the support \( b \) of the shaded area of the \( \phi \)-diagram, Fig. 9b:

\[
\phi_c = \frac{1}{EI} \left( \frac{M_c L}{2} \cdot \frac{L}{2} - \frac{1}{2} \cdot \frac{2M_b L}{3} \cdot \frac{L}{3} \right) = \frac{19}{28} \frac{M_c L^2}{EI} = 0.85 \text{ in.}
\]  

(14)

A further increase in the loads \( P \) will cause yield both at center and support. Strain hardening will commence at the supports, but the center moment will not increase until the curvature is increased \( \phi_s = 11.5 \) fold (Table I). Thus for loads, say \( P = 55 \) kips (Fig. 9c),

\[
M_c = 1497 \text{ in. kip}
\]

\[
M_b = \frac{PL}{3} = M_c = 1583 \text{ in. kip}
\]

The points where the moment reaches \( M_p = 1497 \) in. kip on either side of the supports are easily determined (Fig. 9c):

\[
x_1 = \frac{1497 \cdot 84}{1583} = 79.44 \text{ in.}
\]

\[
x_2 = \frac{M_p}{P} = \frac{1497}{55} = 27.22 \text{ in.}
\]

Using Eqs. (9) to (11) or Fig. 8 the corresponding curvature diagram, Fig. 9d, can be plotted. The characteristic rotations are:

\[
\phi_p = \frac{M_p}{EI} = 344 \times 10^{-6} \text{ rad/in}
\]

\[
\phi_s = \frac{M_p - BZ}{EI} = 3944 \times 10^{-6} \text{ rad/in}
\]

\[
\phi_b = \frac{M_b - BZ}{EI} = 1583 - 1132 = 4870 \times 10^{-6} \text{ rad/in}
\]

The curvature \( \phi_c \) in the center region is of a magnitude between \( \phi_p \) and \( \phi_s \) corresponding to the horizontal part of the \( M-\phi \) diagram, Fig. 8. Its value must be determined using the condition of zero
slope of the beam axis at the centerline, $\phi_c = 0$. In this example only the area $H$ of the curvature diagram Fig. 9d depends on $\phi_c$, which then in the following computation is determined to give $\phi_c = 0$.

<table>
<thead>
<tr>
<th>AREA OF $\phi$- DIAGRAM</th>
<th>MOMENT ARM TO &quot;a&quot;</th>
<th>MOMENT ABOUT &quot;a&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = -1/2 \times 79.44 \times 0.000344 = -0.0170 \times 52.96 = -0.725$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B = -4.56 \times 0.003944 = -0.0180 \times 81.72 = -1.471$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C = -1/2 \times 4.56 \times 0.000926 = -0.0021 \times 82.48 = -0.173$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\phi_d = -2.369 = -0.0282$

<table>
<thead>
<tr>
<th>MOMENT ARM TO &quot;b&quot;</th>
<th>MOMENT ABOUT &quot;b&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = -1/2 \times 1.56 \times 0.000926 = -0.0007 \times 0.52 = -0.000$</td>
<td></td>
</tr>
<tr>
<td>$E = 1.56 \times 0.003944 = 0.0052 \times 0.78 = 0.005$</td>
<td></td>
</tr>
<tr>
<td>$F = -1/2 \times 27.22 \times 0.000344 = -0.0047 \times 10.63 = -0.050$</td>
<td></td>
</tr>
<tr>
<td>$G = +1/2 \times 27.22 \times 0.000344 = +0.0047 \times 46.93 = +0.221$</td>
<td></td>
</tr>
<tr>
<td>$H = +28.00 \times 0.001254 = +0.0351 \times 70.00 = +2.457$</td>
<td></td>
</tr>
</tbody>
</table>

$\phi_c = 0.001254$ rad/in

It is seen that the magnitude of the curvature of the yielded center portion of the beam, consistent with the condition $\phi_c = 0$, is found to be

$\phi_c = 0.001254$ rad/in

The centerline deflection for $P = 55$ kips then is $\phi_c = 2.62$ in., or $1/64$ of the span length. It follows that strain hardening in the center portion corresponding to

$\phi_c = 0.003944$ rad/in

will not take place while the deflections are tolerable and often will never be reached due to earlier collapse caused by local compression flange buckling.

Additional points on the load-deflection curve may be obtained in the same manner, but the relationship is very nearly linear above the yield point until strain hardening would commence. The curve is plotted in Fig. 10 in comparison with predictions by numerical integration and the simple plastic theory. The result from these three methods fall close together which indicate that the simplifications of the theoretical $M-\phi$ curve, Fig. 4, are of small consequence.
However, all three curves give smaller deflections than those obtained experimentally (5). As discussed earlier, the discrepancy is due to the fact that the theoretical M-Φ relationship derived from the stress-strain curve fail to give precise results for as-delivered members.

Much closer agreement with the experimental deflections are obtained when there is no portion of the beam under constant moment as shown for a cantilever in Fig. 11. In such a case a large portion of the beam is actually in the transition (elastic-plastic) range which is influenced by the assumptions made. This figure, taken from Ref. 19, shows very good agreement with tests except in the early part of the plastic region. At this load level the effect of residual stresses and stress concentration must be expected to be most pronounced.

(d) The Simple Plastic Theory

The simple plastic theory neglects the strain hardening effect, but considers the spread of the plastic zones along the member. Fig. 6(d) shows the assumed stress distribution in a cantilever according to this theory. The calculation of deflections is further simplified over the previous methods.

Using again the example of Fig. 9, the load and deflection at initial yield are obtained by Eqs. (13) and (14) with M_y instead of M_p:

\[ P_y = \frac{6M_y}{L} = 47.6 \text{ kips} \]

\[ S_y = \frac{19}{216} \frac{M_p L^2}{EI} = 0.76 \text{ in.} \]

When the full plastic hinge moment is reached,

\[ P = \frac{6M_p}{L} = 53.5 \text{ kips} \]

and deflections increase without limit since no further resistance is offered at the yielding sections.
Intermediate points on the load-deflection curve may be found by a numerical method similar to that used for the previous case except that the transfer from elastic to plastic behaviour is gradual as shown by the dotted lines in Fig. 9 (d). The resulting curve is shown in Fig. 10 and is hardly distinguishable in the early plastic region from that determined by the more exact numerical integration method. Fig. 11 shows the same result for a cantilever. The neglect of strain hardening results in a horizontal line at \( P = P_y \).

In order to demonstrate this method in greater detail the computation of the center deflection of a built-in beam with third-point loading will be shown. The resulting load-deflection curve is given in Fig. 12 in comparison with experimental values and predictions by other methods.

Initial yield takes place when the largest elastic moment \( M_y \) at the fixed ends reaches \( M_y = 1334 \text{ in.k.} \) (Table I). The loads are then

\[
P_y = \frac{9}{8} \frac{M_y}{L} = 35.8 \text{ kips}
\]

Handbook value:

\[
\delta_y = \frac{5}{144} \frac{M_y L^2}{E I} = \frac{5}{144} \cdot 8.653 = 0.30 \text{ in.}
\]

Next, determine the point on the load-deflection curve corresponding to yielded flanges at the fixed ends (Fig. 1d):

\[
M_F = M_y = 1497 \text{ in.k.} \quad \text{(Table I)}
\]

\[
M_{\phi} = \frac{1}{3} P_1 L - M_F = 55P_1 = 1497
\]

Here, the unknown value \( P_1 \) is determined by the condition of zero resultant change in slope from the fixed end to the beam center. This condition is most easily satisfied by a trial-and-error procedure, assuming a value for \( M_{\phi} \). After exceeding the yield load (\( M_{\phi} = 857 \text{ in.k.} \)) this moment is increasing at a more rapid rate than \( M_F \). Thus, \( M_{\phi} = 850 \text{ in.k.} \) is assumed in the first trial, giving the curvature-diagram below. The curvature producing yield through the flanges is

\[
\phi_p' = \frac{h}{h = 2w} \cdot \phi_y = 35.4 \cdot 10^{-3} \text{ rad/in.}
\]
Assuming the area (a) to be trapezoidal the total rotation from support to centerline is:

- \( \phi \) area (a): \( 0.5 \times (354+307) \times 3.9 \times 10^{-6} = 1290 \times 10^{-6} \) rad.
- \( \phi \) area (b): \( 0.5 \times 307 \times 31.8 \times 10^{-6} = 4880 \times 10^{-6} \) rad.
- \( \phi \) area (c): \( -0.5 \times 195 \times 20.3 \times 10^{-6} = -1980 \times 10^{-6} \) rad.
- \( \phi \) area (d): \( -195 \times 28.0 \times 10^{-6} = -5469 \times 10^{-6} \) rad.

Thus the centerline moment is smaller than assumed above, and the new value \( M_{c} = 750 \) in.k. is tried. The corresponding coordinates are shown without parentheses on the above \( \phi \) - diagram.

- \( \phi \) area (a): \( 0.5 \times (354+307) \times 4.0 \times 10^{-6} = 1320 \times 10^{-6} \) rad.
- \( \phi \) area (b): \( 0.5 \times 307 \times 33.0 \times 10^{-6} = 5110 \times 10^{-6} \) rad.
- \( \phi \) area (c): \( -0.5 \times 172 \times 18.7 \times 10^{-6} = -1610 \times 10^{-6} \) rad.
- \( \phi \) area (d): \( -172 \times 28.0 \times 10^{-6} = -4820 \times 10^{-6} \) rad.

The error is very small, and

\[ M_{c} = 750 + (850-750) \frac{30}{1370+30} = 752 \text{ in.k.}, \] by straight-line interpolation, is taken as the final value. Then

\[ P_{1} = \frac{3}{L} (M_{p} + M_{c}) = \frac{1}{56} \times (1497+752) = 40.2 \text{ kips}. \]

The corresponding center deflection \( \delta \) equals the moment of the \( \phi \)-diagram above about the centerline:
\[ \phi \text{ Area, Rad.} \quad \text{Moment Arm, In.} \quad \text{Deflection In.} \]

(a) \[ 94 \times 10^{-6} \quad 82.7 \quad 0.008 \]
(b) \[ 1228 \times 10^{-6} \quad 81.9 \quad 0.101 \]
(c) \[ 5110 \times 10^{-6} \quad 68.9 \quad 0.352 \]
(d) \[ 1610 \times 10^{-6} \quad 34.2 \quad 0.055 \]
(e) \[ 4820 \times 10^{-6} \quad 14.0 \quad -0.057 \]

\[ \delta_1 = 0.34 \text{ in.} \]

From \((P_1, \delta_1)\) the load-deflection curve closely parallels that of a simply supported beam, \(\delta_2 - \delta_1 = \frac{7}{216} \frac{(P_2 - P_1)L^5}{E}\), until the moment in the center region reaches the yield moment \(M_y = 1334 \text{ in.k.}\), which occurs for

\[ P_2 = \frac{3}{L} (M_p + M_y) = \frac{3}{L} (M_p + M_y) = \frac{1427 + 1334}{58} = 50.6 \text{ kips} \]

The deflection is then, from the equation above,

\[ \delta_2 = \delta_1 + \frac{7}{216} \frac{(P_2 - P_1)L^5}{E} = \delta_1 + \frac{7 \cdot 9 \cdot 4 \cdot P_2 - P_1 \cdot M_yL^2}{216 E} = \]

\[ 0.34 + 0.146 \cdot \frac{10.4 \cdot 8.653}{35.8} = 0.71 \text{ in.} \]

Finally, the carrying capacity is very nearly exhausted when yield has penetrated through the flanges also at the center section which occurs for a load negligibly smaller than

\[ P_p = \frac{3}{L} \cdot 2 M_p = \frac{2 \cdot 1497}{58} = 55.5 \text{ kips} \]

The curvature between load points is now \(\phi_p = 354 \times 10^{-6} \text{ rad/in.}\), while the curvature at the fixed ends is larger and can be determined by the continuity condition of zero resultant change in slope from support to beam center. From the \(\phi\) - diagram thus determined the center deflection is found following the procedure that gave \(\delta_1\) above. At this point the deflection, neglecting strain hardening and catenary effect, becomes indeterminate and the beam deflects without limit under constant loads \(P_p\).
(e) Plastic Hinge Method

Fig. 6 (e) shows a stress distribution in which yielding is assumed to be limited to the cross section which first reached initial yield. This assumption differs only from the simplified method of mathematical integration, Fig. 6 e, in that strain hardening is neglected. The beam has elastic regions and localized plastic hinges only, and the M- relationship consists of the two straight lines $M = E I \phi$ in the elastic range and $M = M_p$ in the plastic range.

The member is assumed to behave elastically until a maximum stress $f \cdot \sigma_y$ is reached ($f$ is the shape factor). This is equivalent to assuming that all of the material is concentrated in a line at the flange. In the example Fig. 9, the stress $f \cdot \sigma_y$ corresponds to a load

$$P = \frac{8N_p}{L} = 53.5 \text{ kips}$$

and a deflection at the centerline of

$$\delta = \frac{19}{216} \frac{M L^2}{E I} = 0.35 \text{ in.}$$

(Eqs. (13) and (14)). The sudden formation of a plastic hinge at that load causes a sharp break in the load-deflection curve, Fig. 10.

This simple approach was suggested in Ref. 3, and is illustrated in Appendix C of that reference by the case of a fixed-ended beam with a concentrated load off center. This method obviates the necessity of knowing the $M-\phi$ relationship above the elastic range, and makes use of deflection formulas usually available in the design office and tabulated in structural handbooks.

4. COMPARISON WITH EXPERIMENTAL RESULTS

Comparisons of the predicted and experimental deflections for a continuous beam and for a cantilever are given in Figs. 10 and 11, respectively, which were previously discussed.

Fig. 12 shows a similar comparison for an 8 WF 40 beam with fixed ends and third-point loading. As in the case of the continuous beams, Fig. 10, the differences in the predictions resulting from the
various methods outlined above are smaller in the early plastic range than the deviation from the test results for the beam with detail (a) (Fig. 12) at the supports which was also used on the tests in Figs. 10 and 13. Detail (c) gives a stiffer beam with deflections in close agreement with predictions. The deflections derived by mathematical integration are not shown, but would fall between the two upper curves. The plastic hinge method would give a curve following the simple plastic theory curve except for sharp knees at the two curved regions.

Fig. 13, finally, compares computed and experimental\(^5\) deflections for a simple beam under third-point loading. Strain hardening theoretically does not begin until the center deflection reaches about 8 inches and all methods give deflections falling on the curve shown for the simple plastic theory, the sharp knee of the plastic hinge method excepted. In the test, yielding started at about one-third of the calculated initial yield load, causing larger deflections than those predicted.

The comparison with tests, Fig. 10-13, indicates that hardly any advantage is gained by using the more refined and laborious methods outlined above for predicting deflections. The amount of work involved in a case like the continuous beam of Fig. 9 by the methods (a) to (e) is, very roughly, in the ratios 5: 4: 3: 2: 1 and these ratios rapidly grow larger with more complicated problems.

The comparisons of Figs. 10, 12, and 13 with experimental results are unfavorable due to the large influence of residual stresses over the length of the beams under constant moment. The agreement in Fig. 11 for the cantilever is more typical for most practical cases.
For engineering purposes the plastic hinge method probably would be preferred. The other methods belong in the research field where greater accuracy is often required to explain experimental behavior.

The inclusion in the calculation of deflections of strain hardening, although laborious, does not necessarily yield a truer answer since earlier failure due to local buckling is probable in many cases. In cases where strain hardening does occur but is neglected, the estimates of both the ultimate load and the deflections will be on the safe side.

5. INFLUENCE OF END RESTRAINT

As a member of a structural frame or part of a continuous beam, a single beam span may have any amount of end restraint between fully fixed and simply supported. The distribution of the live load on the span may also vary, and is usually somewhere between uniform distribution over the whole span and concentrated center loading.

In order to illustrate the effect of end restraint and load distribution, a series of examples as shown in Fig. 14 is chosen and the load-deflection curves computed by the simple plastic theory. The end restraint on the center span imposed by the outer spans varies in these examples from 100% (full fixity) through 86, 75 and 60% to the case of the simple span.

The frames simulated by the continuous beams are indicated in Fig. 14. Of these, Frame (c) has been tested as such.

Three loading conditions as shown in Fig. 14 are investigated:

1. A concentrated load 2P at center.
2. Third-point concentrated loads P.
3. Uniform load \( W = 2P \).

An 8WF40 was used throughout these examples. The section properties are listed in Table I.
The load-deflection curves for the five degrees of end restraint are shown in Fig. 15 for the case of a concentrated load at the beam center. According to the simple plastic theory, the ultimate carrying capacity is 71.5 kips for any degree of end restraint different from zero. In the latter case the ultimate load is reduced to one half of that value. Thus, there appears to be an illogical discontinuity in the value of the ultimate carrying capacity when the end restraint is increased from zero.

Fig. 15 shows, however, that the ultimate load according to the simple plastic theory should not be taken as a design criterion for beams with small end restraints. Very large deflections, which cannot be tolerated as a limit criterion for structural usefulness, will take place before this "ultimate load" is attained. Also, local flange buckling will bring about collapse before extreme deflections are reached.

Safe and tolerable limits for the deflection under the "full load" (working load multiplied by the load factor) must therefore be imposed.

The load-deflection curves for the beams of Fig. 14 with third-point loading on the center span are shown in Fig. 16, and with uniform load on the center span in Fig. 17. The discussion above applies also for these loading cases. For small end restraints, the ultimate load will not be reached and the corresponding deflections could not be tolerated as the limit of structural usefulness.

For the purpose of the following example assume these specifications for plastic design of continuous beams:

(a) Load Factor based on \( M_o = \frac{1}{2} (M_p + M_y) \) at the last hinge to form. This possibility was suggested in an earlier paper.

(b) Maximum deflection at full load arbitrarily selected at
Beams with a concentrated load at the center and end restraint varying from 0 to 100% of fixity have been analysed for these specifications. The resulting carrying capacities are given in Table II. For end restraints larger than 65%, specification (a) limits the working load. Below that value, specification (b) applies as is shown under "Deflection at full load" in Table II. The effect is to give gradual reduction in working load as the end restraint decreases to zero (the simple beam). The last two columns of Table II afford a comparison with elastic design, the elastic design working load being taken as

\[ F_W = \frac{P_W}{1.65} \]

Thus, plastic design results in an increase in working load from 5% to 33% over the elastic design. It should be noted, however, that a part of this gain is due to the choice of a different basis for the full load in the two design methods. On a strict comparative basis, therefore, \( 0.5 \cdot (f-1) \cdot 100\% \) should be subtracted from those numbers in the last column of Table II which are determined by \( F_{ult} \), giving a gain of 0 - 29% resulting from plastic design.

Fig. 18 shows the load-deflection curves for these beams according to the plastic hinge method, and in Fig. 19 are given the similar curves for the case of uniformly distributed load. The upper limits for allowable working loads according to the plastic and elastic analyses outlined above are indicated in the figures. The working loads resulting from the elastic analysis are everywhere within the elastic limit. This a point about which there has been considerable discussion. The area between the lines showing the elastic and plastic working loads indicates the economy resulting from the latter.

A possible additional specification,

(c) Maximum deflection at working load \( \frac{L}{360} \),

would decrease the working loads as indicated in Figs. 18 & 19. Since this
specification is more rigid than the limitation imposed by deflection at full load the calculation of the latter could be omitted. This simplification holds generally for continuous beam with any end restraint and load distribution since the examples in Figs. 18 and 19 cover the complete ranges of end restraint and practical loading cases. Thus, continuous beams may be designed on the basis of ultimate load and elastic deflection under working load only. However, for other types of structures, like frames subjected to side loads, it may be necessary to revert to the more basic plastic design specification (b) above.

The purpose of the above analysis is not to advocate unrestricted use of plastic design, but merely to show its possible advantage where the limitations as mentioned earlier do not prohibit its application.

The magnitude of the load factor, the definition of the ultimate load, and the magnitude of the allowable deflection at the full load may be matters of discussion. The deflection limit should be chosen so as (1) to indicate the limit of structural usefulness and (2) to eliminate danger of local flange buckling. The second requirement makes the deflection limit a function of the end restraint and the distribution of loading since the required rotation capacity at the critical sections depends on these factors.
6. **Influence of Load Distribution**

A comparison between Figs. 15, 16, and 17 reveals how the distribution of a total load \(2P\) influences the load-deflections relationship for continuous beams. In the three figures the load is concentrated at the center, divided between the third-points, and uniformly distributed, respectively.

Concentrating the load near the beam center lowers the ultimate carrying capacity and increases the deflections. As an example, some characteristic values from Figs. 15-17 are given in Table III for the beam with 75% end restraint (\(L_1/L_2 = \frac{1}{2}\)). Twice the ultimate concentrated center load can be carried as uniformly distributed over the whole span, and causes a deflection of only \(2/3\) of that of the concentrated load.

For a certain load in the elastic range, say \(2P = 50^k\), the center deflection is increased from 0.29 in. to 0.50 in. by changing the load distribution from uniformly distributed to full concentration at the center point.
(1) The plastic design of continuous beams and frames requires the calculation of critical deflections at the ultimate load, and often at the working load also.

(2) Several methods for predicting the load-deflection relationship of continuous beams are outlined. These methods are based on the moment-curvature relationship derived from the measured stress-strain relationship, and differ only in the number of assumptions made to simplify the computation.

(3) The simplest approach, the plastic hinge method, is most suitable for engineering applications. The additional work involved in the theoretically more "exact" methods is not justified by the accuracy gained except for research purposes.

(4) Tests of as-delivered beams with welded details at supports and load points give larger deflections than predicted. The discrepancy probably is due to residual stresses and stress concentrations. Beams with regions of constant moment show the largest discrepancies (Figs. 10, 12, 13), while a cantilever gives good agreement with the predicted deflections (Fig. 11).

(5) When the end restraint is small the ultimate load predicted by the simple plastic theory is accompanied with excessive deflections (Figs. 15, 16, 17). The simple plastic theory must therefore be combined with a limitation on the deflections at the ultimate load.
(6) A comparison of allowable load and resulting deflections of continuous beams analysed by elastic and plastic theory (Figs. 18 and 19) shows an increase in working load by the latter method of 0 to 29%, depending upon the end restraint and load distribution.

(7) The effect of the load distribution may be demonstrated by the fact that the ultimate load of a uniformly loaded beam is twice that which can be applied concentrated at the center. The larger capacity obtained by distributing the load is accompanied by reduced deflections (Table III).
8. ACKNOWLEDGEMENTS

This report is part of a current study of WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS being conducted at Lehigh University in the Fritz Engineering Laboratory. This research is sponsored jointly by the Department of the Navy and the Welding Research Council through the Lehigh Project Subcommittee of its Structural Steel Committee.

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10. NOMENCLATURE

A  cross section area
b  flange width
B  strain hardening off-set: \( M = BZ + CI\phi \)
C  strain hardening modulus
E  modulus of elasticity
f  shape factor \( \frac{Z}{S} \)
h  depth of section
I  moment of inertia of cross section
I_E  moment of inertia of elastic part of cross section
l  length of plastic portion of beam
L  span
M  moment
M_o  useful ultimate load
M_P  full plastic hinge moment \( \sigma_Y Z \)
M_Y  moment at initial yield \( \sigma_Y S \)
P  concentrated load or total load on the span
P_F  full load
P_W  working load
S  section modulus \( \frac{2I}{h} \)
t  web thickness
w  flange thickness
x  distance along undeformed beam axis
y_o  distance from neutral axis to nearest fibre in yield
y_s  distance from neutral axis to nearest fibre in strain hardening
Z  plastic modulus of cross section
\( Z_E \)  plastic modulus of elastic part of cross section
\( Z_p \)  plastic modulus of plastic part of cross section
\( \sigma \)  unit normal stress
\( \sigma_y \)  lower yield point stress
\( \varepsilon \)  unit normal strain
\( \phi \)  curvature of bent member
\( \alpha \)  slope of deflected beam axis
\( \delta \)  deflection
\( \delta_F \)  deflection at full load
\( \delta_w \)  deflection at working load
11. TABLES AND FIGURES
TABLE I. PROPERTIES OF 8WF40 SECTION

Material Properties (Measured)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus of Elasticity</td>
<td>$E = 29.6 \times 10^6$ psi</td>
</tr>
<tr>
<td>Yield Point (lower)</td>
<td>$\delta_y = 37760$ psi</td>
</tr>
<tr>
<td>Strain Hardening Off-set (Fig. 8)</td>
<td>$B = 28560$ psi</td>
</tr>
<tr>
<td>Strain Hardening Modulus (Fig. 8)</td>
<td>$C = 630$ psi</td>
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</table>

Geometrical Properties of 8WF40 Section (Measured)

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Flange Width</td>
<td>$b = 8.06$ in.</td>
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<tr>
<td>Flange Thickness</td>
<td>$w = 0.552$ in.</td>
</tr>
<tr>
<td>Depth</td>
<td>$h = 8.32$ in.</td>
</tr>
<tr>
<td>Web Thickness</td>
<td>$t = 0.370$ in.</td>
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<tr>
<td>Cross Section Area</td>
<td>$A = 11.66$ in$^2$</td>
</tr>
<tr>
<td>Section Modulus (strong axis)</td>
<td>$S = 35.34$ in$^3$</td>
</tr>
<tr>
<td>Plastic Modulus (strong axis)</td>
<td>$Z = 39.65$ in$^3$</td>
</tr>
<tr>
<td>Moment of Inertia (strong axis)</td>
<td>$I = 147.0$ in$^4$</td>
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<tr>
<td>Shape Factor</td>
<td>$f = 1.122$</td>
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Section Properties of 8WF40 Section (derived from above)

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<tr>
<th>Property</th>
<th>Value</th>
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<tr>
<td>Yield Moment</td>
<td>$M_y = 1334$ in. kips.</td>
</tr>
<tr>
<td>Plastic Hinge Moment</td>
<td>$M_p = 1497$ in. kips.</td>
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<tr>
<td>Curvature at Initial Yield</td>
<td>$\phi_y = \frac{M_y}{EI} = 307 \times 10^{-6}$ rad/in</td>
</tr>
<tr>
<td></td>
<td>$\phi_p = f \phi_y = 344 \times 10^{-6}$ rad/in</td>
</tr>
<tr>
<td>Curvature at Which Strain</td>
<td>$\phi_s = \frac{M_o - EZ}{EI} = 3944 \times 10^{-6}$ rad/in</td>
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<tr>
<td></td>
<td>$\frac{M_o L^2}{EI} = 9.653$ in.</td>
</tr>
<tr>
<td></td>
<td>$\frac{M_o L^2}{EI} = 9.710$ in.</td>
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## TABLE II. WORKING LOADS AND THEORETICAL DEFLECTIONS
OF CONTINUOUS BEAMS DESIGNED BY ELASTIC AND PLASTIC THEORIES

<table>
<thead>
<tr>
<th>END RESTRAINT</th>
<th>WORKING DESIGN</th>
<th>ELASTIC DESIGN</th>
<th>INCREASE IN WORKING LOAD (Plastic over elastic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{L_2}{L_1} )</td>
<td>Working load</td>
<td>Deflection at Working load</td>
<td>Deflection at full load</td>
</tr>
<tr>
<td>( % )</td>
<td>2P kips</td>
<td>( x L_2 )</td>
<td>( x L_2 )</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>40.8</td>
<td>1/370</td>
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<tr>
<td>90</td>
<td>0.17</td>
<td>42.0</td>
<td>1/340</td>
</tr>
<tr>
<td>80</td>
<td>0.38</td>
<td>42.0</td>
<td>1/320</td>
</tr>
<tr>
<td>70</td>
<td>0.64</td>
<td>42.0</td>
<td>1/300</td>
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<td>65</td>
<td>0.81</td>
<td>42.0</td>
<td>1/280</td>
</tr>
<tr>
<td>60</td>
<td>1.00</td>
<td>40.8</td>
<td>1/260</td>
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<td>50</td>
<td>1.50</td>
<td>36.1</td>
<td>1/240</td>
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<td>40</td>
<td>2.25</td>
<td>32.1</td>
<td>1/220</td>
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<td>3.50</td>
<td>29.1</td>
<td>1/200</td>
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<td>6.00</td>
<td>26.4</td>
<td>1/180</td>
</tr>
<tr>
<td>10</td>
<td>13.00</td>
<td>23.2</td>
<td>1/160</td>
</tr>
<tr>
<td>0</td>
<td>20.4</td>
<td>1/140</td>
<td>1/100</td>
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</table>
TABLE III. INFLUENCE OF THE DISTRIBUTION OF THE LOAD ON THE
STRENGTH AND DEFLECTION OF A CONTINUOUS SWF40
BEAM WITH 75% END RESTRAINT AND 14' SPAN

<table>
<thead>
<tr>
<th>LOAD DISTRIBUTION</th>
<th>ULTIMATE CAPACITY</th>
<th>DEFLECTION AT ULTIMATE LOAD</th>
<th>DEFLECTION AT 50 KIPS TOTAL LOAD</th>
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</thead>
<tbody>
<tr>
<td>Concentrated at Center</td>
<td>$71.3k$</td>
<td>1.21 in.</td>
<td>0.50 in.</td>
</tr>
<tr>
<td>Third-Point Concentrated Loads</td>
<td>$106.9k$</td>
<td>0.35 in.</td>
<td>0.40 in.</td>
</tr>
<tr>
<td>Uniformly Distributed</td>
<td>$142.6k$</td>
<td>0.31 in.</td>
<td>0.29 in.</td>
</tr>
</tbody>
</table>
FIG. 1. MOMENTS AND DEFLECTIONS OF CONTINUOUS BEAMS

(a) 

(b) 

(c) 

(d) 

CENTER DEFLECTION
**Fig. 2. Load-Deflection Relationship in the Elastic Range**

**Fig. 3. Measured and Idealized Stress-Strain Diagram**
Fig. 4. Moment-Curvature Relationship for 8 WF 40 Section

Fig. 5. Moment and Rotation Diagrams for a Plastically Bent Beam
Fig. 6. Stress Distributions in Beams above the Elastic Range Assumed in the Methods Discussed

Fig. 7. Idealized M-\(\phi\) Curve for Mathematical Integration Method
FIG. 8. STRESS-STRAIN AND L-Ø RELATIONSHIPS ASSUMED IN ø AREA METHOD
**Fig. 9. Moment and Curvature Diagrams for Continuous Beam**

*(φ-Area Method)*
Fig. 10. Load-Deflection Relationship of a Continuous Beam
**Fig. 11. Load-Deflection Relationship of a Cantilever**

- **Load, Kips**
  - 0
  - 4
  - 8
  - 12
  - 16
  - 20

- **End Deflection, Inches**
  - 0
  - 0.4
  - 0.8
  - 1.2
  - 1.6
  - 2.0
  - 2.4
  - 2.8
  - 3.2

- **Graph Lines**
  - **NUMERICAL INTEGRATION**
  - **NUMERICAL INTEGRATION, Using Control Beam M θ Curve**
  - **MATHEMATICAL INTEGRATION**
  - **ϕ-AREA METHOD**
  - **SIMPLE PLASTIC THEORY**
  - **EXPERIMENTAL RESULT**
Fig. 12. Load-Deflection Relationship, Fixed-Ended Beam
Fig. 13. Load-Deflection Relationship, Simply Supported Beam
BEAM EXAMPLE | SIMULATED FRAME | END RESTRAINT OF CENTER SPAN
(a) | ![Diagram](a) | 100 %
(b) | ![Diagram](b) | 86 %
(c) | ![Diagram](c) | 75 %
(d) | ![Diagram](d) | 60 %
(e) | ![Diagram](e) | 0 %

CENTRAL SPAN LOADINGS:
(1) ![Diagram](1)
(2) ![Diagram](2)
(3) ![Diagram](3)

FIG. 14. ILLUSTRATIVE EXAMPLES OF EFFECT ON MAXIMUM BEAM DEFLECTIONS OF END RESTRAINT AND LOAD DISTRIBUTION
Fig. 15. Influence of End Restraint on Beam Deflections
Fig. 16. Influence of End Restraint on Beam Deflections
Fig. 17. Influence of End Restraint on Beam Deflections
FIG 18. INFLUENCE OF END RESTRAINT ON BEAM DEFLECTIONS
Fig. 19. Influence of End Restraint on Beam Deflections
APPENDIX

FLEXURE OF I SECTIONS ABOVE THE ELASTIC RANGE

by

Walter H. Weiskopf
Introduction

For I-sections in flexure in the elastic range the familiar expression \( \frac{dy}{dx} = \frac{M}{EI} \) applies, and its integrations provide the slope, the deflection curve, and the solutions of innumerable problems. In this appendix corresponding expressions are developed for I-sections in the plastic and strain hardening ranges and methods of integrating them are indicated.

Fig. 1 shows the elastic, plastic and strain hardening ranges of a specimen in tension. In flexure, however, the situation is more complicated. As load on a beam is increased, some of the fibres distant from the neutral axis and where the bending moment is great enter the plastic range. There are then two portions to the beam, one fully elastic and one in which the fibres far from the neutral axis are plastic. As the load is further increased, some of the fibres enter the strain-hardening range and there are then three portions of beam, one still elastic, one in which some of the fibres are plastic, and one in which some of the extreme fibres are in the strain-hardening range. The plastic and strain-hardening sections will be analysed separately.

The Plastic Range

The stress-strain diagram is idealized as shown in Fig. 2. Applied to an I-section in flexure the strain and stress diagrams are shown in Figs. 3a and 3b. From Fig. 3a it is seen that the rotation per unit length of beam is

\[
\phi = \frac{\sigma}{E_Y} \tag{1}
\]
Taking moments about the neutral axis in Fig. 3b there is obtained

\[ M = \sigma_y \left( \frac{I_E}{y_0} + Z - Z_E \right) \]  
(2)

In this expression \( I_E \) is the moment of inertia and \( Z_E \) is the statical moment of the elastic portion of the cross section and \( Z \) is the statical moment of the entire cross section.

When the value of \( y \) is such that the elastic portion of the cross section is entirely in the web, equation (2) becomes

\[ M = \sigma_y \left( Z - \frac{y^2 t}{3} \right) \]  
(3)

In equation (3) \( t \) is the web thickness.

Eliminating \( y \) from equations (1) and (3)

\[ \phi = \frac{\sigma_y}{E} \frac{\sqrt{\sigma_t}}{\sqrt{3(M_p - M)}} \]  
(4)

In equation (4) the expression \( \sigma_y Z \) is the limiting value of the bending moment in which all of the fibres are stressed to the yield point as shown in Fig. 3c. This value, sometimes called the plastic hinge value, is designated \( M_p \).

Then equation (4) becomes

\[ \phi = \frac{\sigma_y}{E} \frac{\sqrt{\sigma_t}}{\sqrt{3(M_p - M)}} \]  
(5)

This is the fundamental relation of \( M \) to \( \phi \) for the plastic range.

In Fig. (4) this relation is plotted for an 8 WF 40. From origin to point 1 the beam is elastic. From point 1 to point 2, \( y \) is in the flange or fillet. To the right of point 2, \( y \) is in the web and equation (5) applied. The portion 1 to 2 is a small portion of the useful range. To simplify the mathematics equation (5) will
be extended to the left until it meets an extension of the elastic range at point 4. The value of the resisting moment at point 4, \( M_1 \), can be found by treating equation (5) simultaneously with the elastic expression
\[
\phi = \frac{M}{EI}
\]
\[
3M^3 - 3M_p M^2 + I^2 \sigma^* t = 0
\]

\( M_1 \) can be found by solving equation (5). It is a function of the cross section and \( \sigma^* \) only and can be tabulated for all beam sizes. The elastic range will then be taken from zero moment to \( M_1 \) and the plastic range from \( M_1 \) to \( M_p \).

**Integrating the M-\( \phi \) Curve. (Plastic Range)**

As is usually assumed in the theory of flexure \( \frac{d^2 y}{d x^2} \) is taken equal to \( \phi \) and equation (5) becomes
\[
\frac{d^2 y}{d x^2} = \frac{\sigma^*}{E \sqrt{3(M_p - M)}}
\]
This expression must be integrated along the length of the beam in the plastic range. For this integration \( M \) is a function of \( \kappa \) and the remainder of the right hand side of equation (7) is constant. The integrations have been performed for three cases: Fig. (5), the bending moment, \( M \), is constant; Fig. (6) \( M \) follows a straight line variation (as between concentrated loads); Fig. (7) \( M \) follows a parabolic diagram (as for a uniform load). In these expressions, \( \ell \) is the length of the plastic portion which is generally only a small part of the total length of the beam.

The expressions first give the slope, \( \frac{d y}{d x} \), and the deflection, \( y \), at any point in the length. The constants of integration have been determined so as to give values in terms of the bending moments at the ends and the slope, \( \kappa_L \), and deflection, \( y_L \), at the left end. Next (making \( \kappa = \ell \)) the slope, \( \kappa_R \), and deflection, \( y_F \), at
the right end are obtained. Finally expressions for \( \alpha_R \) and \( \gamma_R \) are given for the special case, \( M_c = M_p \) that is the full plastic range.

The Strain-hardening Range

Above the plastic range the stress-strain curve again rises as shown in Fig. 1. It is assumed that the diagram follows a straight line in the strain-hardening range given by the equation

\[
\sigma' = C \epsilon + B
\]

(8)

C and B are constants of the material, C being analogous to \( E \) in the elastic range. For the steel shown in Fig. 1, C equals 667 kips per square inch and B equals 28 kips per square inch. Fig. 8 represents the strain and stress diagrams for an I beam. From the figure

\[
y_s = \frac{\sigma'_s - B}{\sigma' - B} - \frac{h}{2}
\]

(9)

When the distance, \( y_s \), is entirely in the web, the resisting moment of the cross section is

\[
M = BZ + (\sigma' - B) \frac{2I}{h} + \frac{1}{3} (\sigma'_s - B) t y_s^2
\]

(10)

In the right hand member of equation (10) the first term covers the rectangle 0, 1, 4, 3; the second term the triangle 1, 5, 4 and the third term the triangle 1, 2, 6. Computations have shown this last term to be very small. If it is neglected equation (10) becomes

\[
M = BZ + (\sigma' - B) \frac{2I}{h}
\]

(11)

From Fig. 8

\[
\phi = \frac{2\epsilon}{h} = \frac{2(\sigma' - B)}{hC}
\]

(12)

Substituting \( \sigma' - B \) from equation (11) into (12)

\[
\phi = \frac{d^2 y}{d \zeta^2} = \frac{M - BZ}{IC}
\]

(13)
In Fig. 9 the relation of \( M \) to \( \phi \) for an 8 WF 40 is again shown. This is similar to Fig. 4, but extended into the strain-hardening range. The solid line is the theoretical curve and the dotted line is equation (13).

It has been found that failure takes place before the stress gets very far into the strain-hardening range. Using the approximation of equation (13) therefore appears well justified.

In integrating equation (13) \( M \) is a function of \( \lambda \) and the remaining symbols are constants. The integrations are so simple that they can be performed for each case without difficulty and will not be given here.

**Numerical Example**

To illustrate the method it will be applied to a beam with fixed ends and loads at the third points. This is Test No. B2 and is shown in Fig. 10. The following values are used:

- **8WF 40**
  - \( I = 146.3 \text{ in}^4 \)
  - \( S = 35.5 \text{ in}^2 \)
  - \( Z = 39.8 \text{ in}^3 \)
  - \( t = .365 \text{ in} \)
  - \( B = 28 \text{ ksi} \)
  - \( C = 667 \text{ ksi} \)
  - \( \sigma_y \) is taken as 37.6 ksi which is an average value of the test coupon results,
  - \( M_y = 35.5 \times 37.6 = 1335'' k \)
  - \( M_p = 39.8 \times 37.6 = 1500'' k \)

From equation (6)

\[
3M_L^2 - 3 \times 1500 M_L^2 + 146.3^2 \times 37.6 \times .365 = 0
\]

Solving this

\[
M_L = 1433'' k
\]

In the elastic range the fixed end moment is \( \frac{2PL}{9} \) and the center moment \( \frac{PL}{9} \).

Then

\[
\frac{2PL}{9} = M_L
\]

\[
P = \frac{9M_L}{2L} = \frac{9 \times 1433}{2 \times 168} = 38.4'' k
\]

This is the limiting load for the elastic range. For greater loads the ends of the beam enter the plastic range. The
bending moment diagram for this condition is shown in Fig. 10b.
The plastic section extends from the left and to the point where
the bending moment equals \( M_p \). The coordinate of this point is
\( \lambda_p \). Treating the center section of the beam and the elastic
portion of the left section (where \( \lambda \) is greater than \( \lambda_p \)) an
expression for the slope can be found by the usual elastic theory.
This is
\[
\frac{dy}{d\lambda} = \frac{1}{EI} \left[ M_F (\lambda - \frac{l}{2}) + P \left( \frac{l^2}{4} - \frac{\lambda^2}{2} \right) \right]
\]  

(14)

The fixed end moment \( M_F \) is of course not the same as when
the beam is entirely in the elastic range. In equation (14) it is
unknown.

The slope where the elastic portion meets the plastic
portion is found by putting \( \lambda_p \), for \( \lambda \) in equation (14).

\[
\frac{dy}{d\lambda} = \frac{1}{EI} \left[ M_F (\lambda_p - \frac{l}{2}) + P \left( \frac{l^2}{4} - \frac{\lambda_p^2}{2} \right) \right]
\]  

(15)

The slope at this point working from the left end of
the beam through the plastic portion can be found by putting the
proper values in equation (38). \( \lambda_p \) equals zero since the beam is
fixed at the end.

For \( M_R \) put \( M_p \) and for \( M_L \) put \( M_F \). For \( l \) put \( \lambda_p \)

Then
\[
\lambda_p = \frac{2AC_t}{EI} \frac{\lambda_p}{M_F - M_p} \left[ \sqrt{M_p - M_l} - \sqrt{M_F - M_F} \right]
\]  

(16)

Obviously \( \frac{dy}{d\lambda} \) of equation (15) must equal \( \lambda_p \) of equa-
tion (16). Equating gives an expression in which \( \lambda_p \) and \( M_F \) are
unknown. From Fig. 10

\[
\lambda_p = \frac{M_F - M_l}{P}
\]  

(17)
Substituting this in equation (15) and (16) equating the values, and simplifying there is obtained

\[
\frac{2I}{\rho} \sqrt{\frac{G_{0}}{3}} \left[ \sqrt{M_p - M_0} - \sqrt{M_p - M_0} \right] = \frac{M_0^2 - M_0^2}{2\rho} - \frac{M_0L}{2} + \frac{P}{2},
\]

(18)

For any load, \( P \), equation (18) can be solved for the fixed end moment, \( M_F \), by trial.

The largest value that \( P \) can have within the plastic range is that which makes the fixed end moment, \( M_F \), equal to \( M_P \). This can be obtained by making \( M_P \) equal to \( M_p \) in equation (18) and solving for \( P \). The resulting equation is a quadratic and its solution is

\[
P = \frac{9M_P}{4L} \pm \frac{3}{L} \frac{M_0L}{2} + 2I \sqrt{\frac{G_{0}}{3}(M_p - M_0)}
\]

(19)

For the numerical example in Fig. 10 equation (19) gives

\( P = 41.0^k \) and \( \kappa = 1.63 \). This is the greatest value \( P \) can have before the fibres near the end of the beam enter the strain-hardening range. For this value of \( P \) the moment at the center of the beam is \( 800^k \), well within the elastic range.

For values of \( P \) greater than \( 41.0^k \) the strain-hardening range is entered and the condition is that shown in Fig. 10c. The bending moment at any point in the end third is of course

\[
M = M_F - P \kappa
\]

(20)

From the figure the following relations can be obtained:

\[
\kappa = \frac{M_F - M_0}{P}
\]

(21)

\[
\kappa_p = \frac{M_F - M_p}{P}
\]

(22)

\[
\kappa = \kappa_p = \frac{M_p - M_0}{P}
\]

(23)

The slope at any point on the elastic portion of the end third of
the beam is again given by equation (14). The slope when \( x \) equals \( x_c \) can be found by putting \( x_c \) for \( x \) in this equation. Then using the value of \( x_c \) as given in equation (21):

\[
\frac{dy}{dx} = \frac{1}{\epsilon I} \left[ \frac{M_p^2 - M_c^2}{2P} - M_cX + \frac{P_x^2}{q} \right]
\]  

(24)

For the plastic section equation (40) applies. \( \alpha_p \) is given by equation (24) and \( M_p \) is \( M_c \). \( \ell \) is \( \frac{M_p - M_c}{P} \) given by equation (23).

\[
\alpha_p = -\frac{2\alpha_p}{E} \sqrt{\frac{C_{xt}}{3}} \sqrt{\frac{M_p - M_c}{P}} + \frac{1}{\epsilon I} \left[ \frac{M_p^2 - M_c^2}{2P} - M_cX + \frac{P_x^2}{q} \right]
\]  

(25)

In the strain-hardening section, from equation (20)

\[
\frac{d^2y}{dx^2} = \frac{M - BZ}{C_I} = \frac{1}{C_I} \left[ M_F - P \alpha - BZ \right]
\]  

(26)

Integrating

\[
\frac{dy}{dx} = \frac{1}{C_I} \left[ M_F X - \frac{P_x^2}{2} - BZ X + K \right]
\]  

(27)

From the condition that \( \frac{dy}{dx} = 0 \) when \( x = 0 \) it is seen that the constant of integration, \( K \), equals zero.

Putting \( \alpha_p \) for \( x \) in equation (27) gives the slope where the plastic and strain-hardening sections meet. At this point, from (27) and (22)

\[
\frac{dy}{dx} = \frac{1}{2C IP} \left[ M_F^2 - M_p^2 - 2BZ (M_F - M_p) \right]
\]  

(28)

The slopes as given by equations (28) and (25) must be the same. Equating them gives an expression that can be solved for \( M_F \). It is

\[
\left( -\frac{1}{C} + \frac{1}{E} \right) \frac{M_p^2}{2I} + \left( \frac{BZ}{C_I} - \frac{M_c}{2EI} \right) + \frac{M_F^2 - M_p^2}{2EI} + \frac{P_x^2}{qEI} - \frac{BZ M_p}{C I} = 0
\]  

(29)
This is a quadratic and values of $M_F$ can be computed without difficulty. Fig. 11 shows the end and center moments for Test Specimen B2 as measured. The plotting extends through the elastic, plastic and strain-hardening ranges. A curve of values as given by the theory here presented is shown marked "Theoretical". Values were computed up to a loading which makes the center moment enter the plastic range. Beyond this point the end moments would turn upward and the center moment would flatten out.

**Bending About the Weak Axis**

For I sections subject to flexure about the weak axis all of the equations for the plastic range (Figs. 5, 6, and 7) can be used, if for the web thickness, $t$, twice the flange thickness is used. These equations can also be used for rectangular sections if the width of the beam is used instead of the web thickness, $t$. The strain-hardening equations do not apply as accurately to the weak axis or to rectangular shapes. As a matter of fact, however, failure about the weak axis or in rectangular shapes is apt to take place before going far into the strain-hardening range if at all, so that for these cases $M_F$ could be considered the ultimate bending moment.
Fig. 1  TENSION STRESS-STRAIN CURVES
Fig. 2

\[ \frac{\sigma}{E} \]

\[ \sigma_y \]

---

Fig. 3

a. STRAIN

\[ \frac{\sigma}{E} \]

\[ y_0 \]

b. STRESS

c. STRESS
\( M = 12.72 \times 10^4 \)  
\( \phi = 3.54 \times 10^{-4} \)  
\( M_y = 11.72 \times 10^{-4} \)  
\( \phi = 2.72 \times 10^{-4} \)  
\( M_p = 1313 \)  

\( \text{WF} 40 \)  
\( \sigma_y = 33 \% \)
**Constant Moment**

\[
\begin{align*}
\frac{dy}{dx} &= \frac{\alpha_x}{E} \sqrt{\frac{\alpha_x t}{3(M_p-M)}} \\
\frac{dy}{dz} &= \frac{\alpha_x}{E} \sqrt{\frac{\alpha_x t}{3(M_p-M)}} \times x + \alpha_L \\
y &= \frac{\alpha_x}{E} \sqrt{\frac{\alpha_x t}{3(M_p-M)}} \times \frac{L^2}{2} + \alpha_L L + y_L
\end{align*}
\]

For full length \((x = L)\)

\[
\begin{align*}
\alpha_R &= \frac{\alpha_x}{E} \sqrt{\frac{\alpha_x t}{3(M_p-M)}} \times L + \alpha_L \\
y_R &= \frac{\alpha_x}{E} \sqrt{\frac{\alpha_x t}{3(M_p-M)}} \times \frac{L^2}{2} + \alpha_L L + y_L
\end{align*}
\]

---

**Fig. 5**
\textbf{Straight Line Moment Diagram}

\[ \frac{d^2 y}{d\alpha^2} = \frac{d^2 \alpha}{E} \sqrt{\frac{G_t}{3}} \cdot \frac{1}{\sqrt{M_p - M_L + (M_L - M_R) \frac{\alpha}{2}}} \]  

(35)

\[ \frac{dy}{d\alpha} = \frac{2a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{d\alpha}{M_L - M_R} \left[ \sqrt{M_p - M_L + (M_L - M_R) \frac{\alpha}{2}} - \sqrt{M_p - M_L} \right] + \alpha_L \]  

(36)

\[ y = \frac{4a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{\frac{d\alpha}{M_L - M_R}}{\frac{\alpha}{M_L - M_R}} \left[ \left( \frac{M_p - M_L + (M_L - M_R) \frac{\alpha}{2}}{M_p - M_L} \right)^{\frac{3}{2}} - \left( \frac{M_p - M_L}{M_p - M_L} \right)^{\frac{3}{2}} \right] - \frac{2a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{d\alpha}{M_L - M_R} \sqrt{M_p - M_L} + \alpha_L \cdot \frac{d\alpha}{M_L - M_R} + y_L \]  

(37)

For full length \( \alpha = \frac{l}{2} \)

\[ \alpha_R = \frac{2a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{\frac{d\alpha}{M_L - M_R}}{\frac{\alpha}{M_L - M_R}} \left[ \sqrt{M_p - M_R} - \sqrt{M_p - M_L} \right] + \alpha_L \]  

(38)

\[ y_R = \frac{4a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{\frac{d\alpha}{M_L - M_R}}{\frac{\alpha}{M_L - M_R}} \left[ \left( \frac{M_p - M_R}{M_p - M_L} \right)^{\frac{3}{2}} - \left( \frac{M_p - M_L}{M_p - M_L} \right)^{\frac{3}{2}} \right] - \frac{2a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{d\alpha}{M_L - M_R} \sqrt{M_p - M_L} + \alpha_L \cdot \frac{d\alpha}{M_L - M_R} + y_L \]  

(39)

When \( M_L = M_p \)

\[ \alpha_R = \frac{2a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{\frac{d\alpha}{M_p - M_R}}{\frac{\alpha}{M_p - M_R}} + \alpha_L \]  

(40)

\[ y_R = \frac{4a_t}{3} \sqrt{\frac{G_t}{3}} \cdot \frac{\frac{d\alpha}{M_p - M_R}}{\frac{\alpha}{M_p - M_R}} + \alpha_L \cdot \frac{d\alpha}{M_p - M_R} + y_L \]  

(41)

\textbf{Fig. 6}
PARABOLIC MOMENT DIAGRAM

\[ V_L = \frac{M_R - M_L}{l} - \frac{Wl}{2} \]
\[ V_R = \frac{M_R - M_L}{l} + \frac{Wl}{2} \]

shear \[ V_L \]
slope \[ \alpha_L \]
deflection \[ y_L \]

\[ \frac{d^2 y}{dx^2} = \frac{\sigma_0}{E} \sqrt{\frac{\sigma_0 t}{3}} \sqrt{M_p - M_L - V_L x - \frac{Wx^2}{2}} \]
\[ \frac{dy}{dx} = \frac{\sigma_0}{E} \sqrt{\frac{\sigma_0 t}{3W}} \left[ \sin^{-1} \left( \frac{Wx + V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right) - \sin^{-1} \left( \frac{V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right) \right] + \alpha_L \]
\[ y = \frac{\sigma_0}{E} \sqrt{\frac{\sigma_0 t}{3W}} \left[ \frac{Wx + V_L \sin^{-1} \left( \frac{Wx + V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right)}{W} \right. \\
\left. - \frac{Wx + V_L \sin^{-1} \left( \frac{V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right)}{W} \right] + \alpha_L x + y_L \] (44)

For full length (\( x = l \))
\[ \alpha_R = \frac{\sigma_0}{E} \sqrt{\frac{\sigma_0 t}{3W}} \left[ \sin^{-1} \left( \frac{V_R}{\sqrt{2W(M_p - M_L) + V_L^2}} \right) - \sin^{-1} \left( \frac{V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right) \right] + \alpha_L \] (45)
\[ y_R = \frac{\sigma_0}{E} \sqrt{\frac{\sigma_0 t}{3W}} \left[ \frac{V_R \sin^{-1} \left( \frac{V_R}{\sqrt{2W(M_p - M_L) + V_L^2}} \right)}{W} \right. \\
\left. - \frac{V_R \sin^{-1} \left( \frac{V_L}{\sqrt{2W(M_p - M_L) + V_L^2}} \right)}{W} \right] + \alpha_L l + y_L \] (46) 

Fig. 7
When $M_L = M_P$

\[
\alpha_R = \frac{\sigma^*}{E} \sqrt{\frac{2 \sigma^* t}{3W}} \left[ \sin^{-1} \frac{V_R}{V_L} - \frac{\pi}{2} \right] + \alpha_L \tag{47}
\]

\[
y_R = \frac{\sigma^*}{E} \sqrt{\frac{2 \sigma^* t}{3W}} \left[ \frac{V_R}{W} \sin^{-1} \frac{V_R}{V_L} - \frac{\pi V_R}{2W} + \sqrt{\frac{2(M_P - M_R)}{W}} \right] + \alpha_L s + y_L \tag{48}
\]

Fig. 7
Fig. 9  \( \phi \) in Radians \( \times 10^{-4} \)

- Plastic
- Strain-Hardening

- \[ \phi = \frac{M-B}{C_L} \]

- \( 8W = 40 \)
- \( \sigma_y = 33 \text{ ksi} \)
- \( B = 2.8 \text{ ksi} \)
- \( C = 274 \text{ ksi} \)
Fig. 10
Fig. 11 Load-Moment Curves
Test B2