PLASTIC DESIGN IN STRUCTURAL STEEL

by

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INTRODUCTION

During the past twenty to twenty-five years a considerable amount of research has been conducted on the ultimate strength of steel structures. These studies have revealed possibilities for the use of maximum (plastic) strength as a basis for structural design. While the subject is by no means new, it is only in recent years that sufficient tests of large-size structural members and frames have been performed and adequate analytical techniques developed to make the method of practical use.

Many investigators have contributed prominently to the application of plastic analysis to structural design. Some of the more recent advancements are due to the efforts of J. F. Baker, J. W. Roderick, M. R. Horne, and B. G. Neal at Cambridge University, England; and W. Prager, P. S. Symonds, and D. C. Drucker at Brown University in this country.

Since 1946 a program of research has been underway at Lehigh University under the sponsorship of the American Institute of Steel Construction, the American Iron and Steel Institute, the Welding Research Council and the Navy Department (Office of Naval Research, Bureau of Ships, Bureau of Yards and Docks). This program has included studies of the component parts of rigid frames, an examination of possible modifications to the "simple plastic theory", and development of practical design procedures -- the program being supplemented where necessary by suitable tests using as-delivered rolled structural shapes.
Whereas the traditional basis of design for construction purposes has been the "elastic limit" load, it has long been known that rigidly connected members possess a much greater load-carrying capacity. The capacity of structural steel to deform plastically allows an indeterminate structure to draw upon the reserve strength of its less heavily stressed portions.

The application of plastic design is justified, first of all, since it offers a satisfactory explanation of the observed ultimate strength of steel structures. By plastic analysis the engineer is able to determine the true load-carrying capacity of the structure. On the other hand, by conventional elastic methods the true factor of safety against ultimate strength can and does vary significantly from one structure to another.

In the second place, plastic design has an appeal on the basis of its simplicity. Most of the time-consuming analysis of equations necessary for an elastic solution is eliminated. Further, "imperfections" that seriously affect elastic limit strength of a structure (such as spreading of supports, sinking of supports, differences in flexibility of connections, residual stresses) have little or no effect upon the maximum plastic strength.

Finally, these techniques promise to produce substantial savings through the more economic and efficient use of steel and the savings in design office time.
Plastic design will not replace all other design procedures, since in some instances criteria other than maximum plastic strength (such as fatigue, instability, limiting deflection, etc.) may actually constitute the basis for design. In ordinary building construction, however, this is usually not the case. Therefore it can be expected that plastic design will find considerable application, particularly in continuous beams, industrial frames, and also in tier buildings. As a matter of fact, it has been reported that upwards of 175 industrial frames have been designed in England by the plastic method -- also a school building and a five-story office building.

* * * * * * *

In the following fourteen lectures the fundamental concepts of plastic analysis are presented. Specific plastic design techniques are described together with examples to illustrate their application. These lectures are supplemented by a series of demonstration tests of actual structures to illustrate the principles.

At the end of each lecture are given such references as are appropriate to the topic. A list of general references is also included at the end of the notes.

The authors wish to express their sincere appreciation for the helpfulness and cooperation of all members of the Fritz Laboratory staff in the preparation of these lecture notes.
Mr. George Heimberger prepared the drawings, Miss Patricia Torres typed the manuscript and Miss Lucille Fox and Mrs. R. Walther reproduced and assembled the notes.

The review of the manuscript by Mr. T. R. Higgins, Director of Engineering and Research, and Mr. E. R. Estes, Research Engineer, of the American Institute of Steel Construction was most helpful and is gratefully acknowledged.

The demonstrations which supplement these notes were prepared by members of the staff at Fritz Laboratory. Among these were George C. Driscoll (who had immediate charge of the demonstrations), S. J. Errera (Engineer of Tests), Kenneth R. Harpel (Foreman) and his staff, I. J. Taylor (Instrumentation); A. W. Huber, Y. Fujita, G. Haaijer, M. W. White, B. C. Chapman took charge of individual tests.

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Lecture No. 1

FUNDAMENTAL CONCEPTS

SCOPE: The ductility of steel (basis for plastic analysis) is illustrated. Conventional (elastic) design described and examples given to show that even these procedures are often based on implicit assumption of plastic action under overstress. Ultimate strength of several types of structures computed. Historical notes given on development of plastic design.

OUTLINE: 1. MECHANICAL PROPERTIES OF STEEL
2. CONVENTIONAL ELASTIC DESIGN
3. CARRYING CAPACITY OF STRUCTURES
4. HISTORICAL NOTES

1.1 MECHANICAL PROPERTIES OF STEELS

Fig. 1.1 shows typical tensile stress-strain curves of different types of steels which have structural applications:

1. Carbon (A-7)
2. Mayari - R
3. Otiscoloy
4. Tiva
5. Silicon
6. T-1

The Figure is self-explanatory. Note sudden change from elastic range to yield level. Extension between
Fig. 1.1
TYPICAL $\sigma$-$\varepsilon$ CURVES

Note: A yield level is very pronounced. $\varepsilon_{st} = 7\varepsilon_y$ is common!

- **T-1 Steel** ($\sigma_{ult} = 111.8$ ksi)
  - $\varepsilon_{max} = 235 \times 10^{-3}$
  - RA = 125%

- **Silicon** ($\sigma_{ult} = 92.5$ ksi)
  - $\varepsilon_{max} > 160 \times 10^{-3}$
  - RA = 46%

- **Tiva** ($\sigma_{ult} = 80.5$ ksi)
  - $\varepsilon_{max} > 160 \times 10^{-3}$
  - RA = 60%

- **Mayari-R** ($\sigma_{ult} = 77.0$ ksi)
  - $\varepsilon_{max} > 200 \times 10^{-3}$
  - RA = 62%

- **Carbon** ($\sigma_{ult} = 57.5$ ksi)
  - $\varepsilon_{max} > 260 \times 10^{-3}$
  - RA = 63%
yielding and strain hardening varies from about 6 to 16 times elastic strain at yielding. Remarkable is the ductility (varies from 16% to 26%) at fracture. Ordinary elastic design does not make a "conscious" use of this remarkable property. "Plastic Design" proposes to make use of the ductility up to the point at which strain hardening commences ($\epsilon = \epsilon_{st}$). It may be mentioned that experimental evidence is almost exclusively collected for structural carbon steel. But other types of steels, as shown in Fig. 1.1 should lead essentially to the same result.

1.2 CONVENTIONAL ELASTIC DESIGN

Example:

Elastic Analysis:

$$M_1 = \frac{8}{91} PL$$
$$M_2 = \frac{5}{27} PL$$

Maximum Stress $\leq$ Allowable (working) Stress

$$\sigma_{max} = \frac{M_2}{S} \leq \sigma_W$$

Allowable Load: Load Producing Yielding:

$$P_W = \frac{27}{5} \frac{S\sigma_W}{L}$$
$$P_Y = \frac{27}{5} \frac{S\sigma_Y}{L}$$  

(1.1)
Factor of Safety:

\[ F = \frac{\text{Yield Stress}}{\text{Allowable Stress}} = \frac{\sigma_y}{\sigma_W} = \frac{P_Y}{P_W} \]  (1.3)

What does ordinary factor of safety, \( F \), mean?

If the applied allowable load \( P_W \), acting on the idealized structure (assumed dimensions, simplified stress distribution, material with minimum prescribed yield point etc.) is increased to \( F \times P_W \), the most stressed fiber has just reached the yield stress \( \sigma_y \).

What is known about the actual carrying capacity?

Before answering this question let us examine whether or not elastic design actually adheres to the principle that no yielding should ever occur. Some examples:

1. Connections:

   ![Diagram of connections with forces](image)

   The same is true for welded connections.

   Basis of design is not maximum stress but the strength of connection as a whole!
2. Residual Stresses: (Cooling residuals)

Possibility: \( \sigma_r = \text{from } 1/3 \text{ to } 2/3 \sigma_y \)

Bending produces yielding of flange tips at loads less than working load.

Welded specimens show \( \sigma_r \) up to 80% of yield stress.

3. Cambering, Straightening, Cold Bending

\[ \sigma_r = \sigma_y (f - 1)^* \]

Residual Stresses

Loading Stresses

Loading + Residual Stress \( \geq \sigma_W \)

* \( f = \) shape factor. See Lecture No. 2.
4. "Erection" Stresses:

Forcing-in of members during erection causes stresses not accounted for in analysis.

Many more cases, as stress concentrations, secondary stresses etc. could be cited.

Conclusions:

1. Maximum stresses are very often larger than $\sigma_w$. Agreed.

2. Design of connections is actually based on failure load, not on elastic concept. Possible?

3. As justification, ductility of material is advocated. Shall we use.

4. Why not take the next step, and introduce ductility in main member design in a "conscious" manner?

These examples should be sufficient to cause reader some concern about adequacy of present elastic methods. However, no real concern is necessary. Future lectures will show that elastic design constitutes a "possible
equilibrium solution which solution is always less than (or at most equal to) the true ultimate load. (This is called a "lower bound" of the ultimate load.)

In the meanwhile it should now be evident that the actual carrying capacity of a structure is best described by considering the ductility of the material. Such is the basis for the following section.

1.3 CARRYING CAPACITY OF STATICALLY DETERMINATE AND STATICALLY INDETERMINATE STRUCTURES

Neglect purposely any possibility of instability or brittle fracture. (See Lecture No. 9)

Simple schematic examples:

1. Tension Bar: (Determinate System)

Stress: \[ \sigma = \frac{P}{A} \] (1.4)

Elongation: \[ \delta_y = \frac{P L}{E A} \] (1.5)
2. 3-Bar Truss: (Indeterminate System)

Areas: $A_1 = A_2 = A_3 = A$
Lengths: $L_1 = L_3 = \sqrt{2}L_2 = \sqrt{2}L$
Forces: $T_1, T_2, T_3$
Elongations: $\Delta L_1, \Delta L_2, \Delta L_3$

a) Elastic Solution:

Equilibrium: $2T_1 \cos 45^\circ + T_2 = P$ (1.6)

Compatibility: $\Delta L_1 \sqrt{2} = \Delta L_2$ (1.7)

Solving:

$$T_1 = \frac{P}{2 + \sqrt{2}}$$
$$T_2 = \frac{2P}{2 + \sqrt{2}}$$ (1.8)

Elastic Limit: $T_2 = A\sigma_y = \frac{2P}{2 + \sqrt{2}}$ (1.9)

Yield Load: $P_y = \frac{2 + \sqrt{2}}{2} A\sigma_y = 1.707 A\sigma_y$ (1.10)

b) Elastic-Plastic Solution:

Like statically determinate system but with constant force of $A\sigma_y$
c) **Ultimate Load:**

Forces:  
\[ T_1 = A\sigma_y \]
\[ T_2 = A\sigma_y \]

Equilibrium:  
\[ P_u = \sqrt{2} A\sigma_y + A\sigma_y = (1 + \sqrt{2})A\sigma_y = 2.414A\sigma_y \]  
(1.11)

Note: Compatibility condition dropped!!

d) **Load-deflection Curve:**

![Load-deflection Curve Diagram](image)

Yield Deflection:  
\[ (\Delta L_2)_y = \frac{\sigma_y L}{E} \]  
(1.12)

Deflection at Ultimate Load:  
\[ (\Delta L_2)_u = \frac{2\sigma_y L}{E} \]  
(1.13)

Ultimate Load = Load at which unrestricted plastic flow sets in!  
Note that deflection at ultimate load is same as yield deflection of
3. Rectangular, Simple Beam in Bending:

\[ M_{\text{max}} = \frac{PL}{4} \]
\[ \sigma_{\text{max}} = \frac{PL}{4S} \]

a) **Yield Load:**

\[ P_y = \frac{4}{L} S \sigma_y \] \hspace{1cm} (1.14)

With further load increase, yielding penetrates toward the neutral axis. If the entire cross section is yielded, the corresponding internal moment is called the "plastic moment," and for the rectangular section is defined by the equation:

\[ M_p = \frac{bh^2}{4} \sigma_y = Z\sigma_y \]

where

\[ Z = \text{Plastic Modulus} \]

b) **Ultimate Load:**

\[ P_u = \frac{4}{L} Z \sigma_y \] \hspace{1cm} (1.15)
c) Ratio:

\[
\frac{P_u}{P_y} = \frac{Z}{S} = \frac{bh^2}{4} \cdot \frac{6}{bh^2} = 1.5
\]

"Plastification" of cross section resulted in an increase in load of 50%. Note that this ratio depends only on cross-sectional dimensions.

4. Statically - Indeterminate Beam:

![Diagram of a statically indeterminate beam with sections labeled](image)

Elastic Analysis: See Eq. (1.1)

a) Yield Load, Eq. (1.2)

\[
P_y = \frac{27}{5} \frac{S\sigma_y}{L} = \frac{9}{10} \frac{bh^2}{L} \sigma_y
\]  

(1.2)

If P is increased further, section 2 starts to yield and simultaneously will rotate at a much faster rate than previously. (See Lecture No. 2) Section 1 will pick-up

* The term "plastification" means attainment of yield stress on the entire cross-section.
more moment until a final state is reached with plastic moments at Sections 1 and 2 \((M_1 = M_2 = M_p)\). 

b) Ultimate Load:

The corresponding ultimate load \(P_u\) can directly be obtained from the above Figure:

\[
\frac{2P_u L}{9} = (1 + \frac{2}{3}) M_p
\]

\[
P_u = \frac{15}{2} \frac{M_p}{L} = \frac{15}{8} \frac{bh^2}{L} \sigma_y
\]  \(1.16\)

c) Ratio:

\[
\frac{P_u}{P_y} = \frac{25}{12} = 2.08
\]

This increase is due to two effects:

1. **Plastification** of cross-section (as in statically determinate cases)

2. **Redistribution of Moments** (possible only in statically indeterminate cases)

* See "Nomenclature" for moment convention.
In the following lectures general methods for determining this ultimate load-carrying capacity will be given.

It may be of interest to indicate how "Plastic Design" developed to its present state.

1.4 HISTORICAL NOTES

1914 : Kazinczy - Tests on indeterminate beams, concept of "yield hinge".

1917 : Kist - Design procedures utilizing ultimate load capacity.

1926 : Grüning - Difficulties with general loading (Shake-down problem).


1931 : Girkmann - Discusses Portal Frames.


1936 : Cambridge University, Prof. Baker and Colleagues.

1941 : Van den Broek
1946: Brown University, Prof. Prager and Colleagues.

1946: Lehigh University, Investigations leading to design applications.

Lecture No. 2

FLEXURE OF BEAMS

SCOPE: Objective is to determine how a beam deforms beyond elastic limit under the action of bending moments, i.e., what is Moment-Curvature (M-Ø) relationship? It is shown how procedures of plastic analysis are based on the formation of plastic hinges and subsequent redistribution of moment. Since structural members and frames are usually acted upon by shear and direct forces (in addition to bending moments), the resulting stress-distributions are described. Since their effect on ordinary engineering structures is usually small, they are treated later as modifications to Simple Plastic Theory (Lecture No. 9).

OUTLINE:
1. ASSUMPTIONS AND CONDITIONS
2. BENDING OF RECTANGULAR BEAM
3. BENDING OF WF BEAM
4. PLASTIC HINGE CONCEPT
5. REDISTRIBUTION OF MOMENT
6. SHEAR AND DIRECT STRESSES

2.1 ASSUMPTIONS AND CONDITIONS

1. Strains proportional to distance from neutral axis ("plane" sections).
2. Idealized stress-strain relationship:

\[ \sigma = E\epsilon \quad (0 < \epsilon < \epsilon_y) \]

\[ \sigma = \sigma_y \quad (\epsilon_y < \epsilon < \infty) \quad (2.1) \]

Properties in compression are the same as those in tension.

Behavior of fibres in bending is the same as in tension.
3. **Equilibrium conditions:**

Normal Force: \[ P = \int_A \sigma dA \]  
Moment: \[ M = \int_A \sigma y dA \]  

![Equilibrium diagram](image)

4. Deformations sufficiently small so that \( \phi = \tan \phi \)  
\( (\phi = \text{curvature}) \).

2.2 **BENDING OF RECTANGULAR BEAM**

1. **Elastic Bending**

Deformation:

![Deformation diagram](image)

Curvature: \[ \phi = \frac{1}{R} = \frac{\varepsilon}{y} = \frac{\sigma}{E y} \]  

![Curvature diagram](image)
Moment:

\[ M = \int_{-\phi_2}^{+\phi_2} \sigma \cdot y \cdot dA \]  

(2.5)

Moment-curvature relationship (Eq. 2.4 and 2.5):

\[ M = E I \phi \]  

(0 < \phi < \phi_y)  

(2.6)

Graph of Eq. 2.6:

![Graph of Eq. 2.6](image)

Yield Moment (Eq. 2.5):

\[ M_y = \sigma_y S \]  

(2.7)

2. Plastic Bending

The following sketches show the development of strain, stress, and yield distribution as a rectangular beam is bent in successive stages beyond the elastic limit and up to plastic limit. The strain distribution is first selected
or assumed and this fixes the stress-distribution.

Strain Distributions: (Assumption No. 1)

Resulting Stress Distributions: (Assumption No. 2)

Yield Distributions:

The expressions for curvature and moment (and, thus, the resulting M- curve) follow directly from Fig. 2.4. Curvature at a given stage is obtained from particular stress-distribution. Corresponding moment-value is obtained by integration of stress-areas:

* Even though curvature is a measure of strain distribution, the stress-distribution diagram is used since, in the elastic range, the stress varies linearly with strain.
a) Curvature:

\[ \phi = \frac{\sigma_y}{E'\gamma} \]  

(Example: Stage 2)

\[ \phi = \frac{\sigma_y}{E'\gamma} \]  

\[ \text{Fig. 2.5} \]

b) Moment:

\[ (\text{Fig. 2.5}) \quad M = \int A \sigma \cdot y \cdot dA \]  

\[ M = \text{Moment of stress-areas of Fig. 2.5 at neutral axis} \]

\[ = 2 \int y^0 \sigma \cdot b y \cdot c y + 2 \int y^2 \sigma \cdot b y \cdot d y \]

\[ = \sigma_y \int \frac{2}{d/2} \int y^0 y^2 b d y + \sigma_y \int \frac{2}{y^2} y b d y \]

\[ = \sigma_y \int \frac{2}{d/2} \int y^0 y^2 b d y + \sigma_y \int \frac{2}{y^2} y b d y \]

Subscript "e" denotes elastic part of cross-section, "p" denotes plastic part of cross-section. \( S = \) section modulus, \( Z = \) plastic modulus. Thus:

\[ M = \sigma_y S_e + \sigma_y Z_p \]  

(2.9)

and the moment of resistance is made up of an elastic part and a plastic part (Fig. 2.6):

\[ M = \sigma_y S_e + \sigma_y Z - \sigma_y Z_e \]  

(2.10)
Section modulus, $S$, and Plastic modulus, $Z$:

\[
Z_e = 2by_0 \frac{y_o}{2} = by_o^2
\]

\[
S_e = \frac{b(2y_o)^2}{6} = \frac{2}{3} by_o^2 = \frac{2}{3} Z_e
\]

\[
Z_p = Z - Z_e
\]

\[
Z = \frac{bd^2}{4}
\]  

Moment in terms of $Z$: \[ M = \sigma_y (Z - \frac{Z_e}{3}) \]  

Maximum Moment: \[ M_p = \sigma_y Z \]

("Plastic Moment")

**c) Moment-curvature Relationship:**

In terms of $y_0$: \[ M = \sigma_y (Z - \frac{by_o^2}{3}) \]  

In terms of $\phi$: \[ M = \sigma_y (Z - \frac{bcy_y^2}{3E^2\phi^2}) \]  

(Eq. 2.14)

\[ (\phi_y < \phi < \infty) \]

Non-dimensional relationship is obtained by dividing both sides of Eq. 2.14 by $M_y = \sigma_y S$ and by reference to diagram below:

\[
\frac{M}{M_y} = \frac{3}{2} \left[ 1 - \frac{1}{3}(\frac{\partial y}{\phi})^2 \right]
\]  

\[ (\phi_y < \phi < \infty) \]
There is a 50% increase in strength above computed elastic limit (Stage 1) due to plastification of cross-section. (Numbers in circles in Fig. 2.7 correspond to "stages" of Fig. 2.4). Stage 4, approached as a limit, represents complete plastic yield of cross-section.

3. Shape Factor

\[ f = \frac{M_p}{M_y} = \frac{\sigma_y Z}{\sigma_y S} = \frac{Z}{S} \]  

(2.16)

Rectangle: \( f = \frac{bd^2}{4} : \frac{bd^2}{6} = 1.50 \)

2.3 BENDING OF WF BEAM

1. Elastic Bending

Same as rectangular beam. See Eqs. 2.6, 2.7.

2. Plastic Bending

Development essentially the same as for rectangle. Due to variation of width of section with depth, separate expressions are necessary when yielding is limited to the flanges (case 1) and when yielding has penetrated to the web (case 2). Also, two approaches are possible: one is to compute \( M \) and \( \phi \).
at certain discrete strain stages\(^{(2.2)}\), the other is to obtain general expressions for M in terms of \(\varphi\) for the two above-mentioned cases. The latter approach will be used in this discussion.

a) **Successive stages of plastic yield** (Assumptions 1 and 2)

\textbf{Strain Distributions:} See Fig. 2.4

\textbf{Stress Distribution:} See Fig. 2.4

\textbf{Yield Distribution:}

\begin{align*}
\text{Initial Yield} & \quad \text{Flange Yield} & \quad \text{Yield to } \frac{1}{4} \text{ depth} & \quad \text{Yield to } \frac{3}{8} \text{ depth} & \quad \text{Complete Yield} \\
1 & \quad 2 & \quad 3 & \quad 4 \quad \text{Fig. 2.8}
\end{align*}

b) **Curvature:** See Fig. 2.5 and Eq. 2.8.

\[ \varphi = \frac{\sigma_y}{E\gamma} \quad , \quad \varphi_y = \frac{\sigma_y}{E\gamma^2} \]  \hspace{1cm} (2.8)

\[ \frac{\varphi}{\varphi_y} = \frac{d/2}{\gamma_y} \]  \hspace{1cm} (2.17)

c) **Moment:** See Fig. 2.6 and Eq. 2.9.

\[ M = \sigma_y S_e + \sigma_y Z_p \]  \hspace{1cm} (2.9)

\[ M_p = \sigma_y Z \]  \hspace{1cm} (2.12)
d) **Moment-curvature relationship:**

**Case 1: Yielding in Flange**

For use in Eq. 2.29:

\[
S_e = \frac{I_e}{y_o} = \frac{I - I_p}{y_o}
\]

\[
\frac{I}{y_o} = S \frac{d\phi}{y_o}
\]

\[
I_p = \frac{bd^3}{12} - b \frac{(2y_o)^3}{12}
\]

\[
Z_p = \frac{bd^2}{4} - by_o^2
\]

In terms of \( y_o \):

\[
M = \sigma_y \left[ S \frac{d/2}{y_o} - \frac{bd^3}{12y_o} + \frac{bd^2}{4} - \frac{by_o^2}{3} \right]
\]

\[
(\frac{d}{2} - t) < y_o < \frac{d}{2}
\]

In terms of \( \phi \): (Eqs. 2.18 and 2.8)

\[
M = \frac{E\phi d}{2} (S - \frac{bd^2}{6}) + \sigma_y b \left( \frac{d^2}{4} - \frac{\sigma_y^2}{3E^2\phi^2} \right)
\]

\[
(\phi_y < \phi < \frac{d/2}{(d/2 - t)})
\]

In non-dimensional terms: (Eq. 2.18, 2.8, 2.7, 2.6)

\[
M_y = \sigma_y S = E I \phi_y
\]

\[
\frac{M}{M_y} = \frac{\phi}{\phi_y} \left( 1 - \frac{bd^2}{6S} \right) + \frac{bd^2}{4S} \left[ 1 - \frac{1}{3} \left( \frac{\phi_y}{\phi} \right)^2 \right]
\]

\[
(1 < \frac{\phi}{\phi_y} < \frac{d/2}{(d/2 - t)})
\]
Case 2: Yielding in Web

\[ M = \sigma_y S_e + \sigma_y Z_p \]

\[ Z_p = Z - Z_e \]

\[ Z_e = \frac{w y_0^2}{3} \]

\[ S_e = \frac{w(2y_0)^2}{6} = \frac{2}{3} w y_0^2 = \frac{2}{3} Z_e \]

\[ M = \sigma_y \left( Z - \frac{Z_e}{3} \right) \text{ (yield within web)} \quad (2.21) \]

Note: Due to uniform web thickness, these expressions are similar to those for rectangular section. (Eq. 2.11)

In terms of \( y_0 \): \[ M = \sigma_y \left( Z - \frac{w y_0^2}{3} \right) \quad (2.22) \]

\( 0 < y_0 < \frac{d}{2} - k \)

In terms of \( \varphi \): \[ M = \sigma_y \left( Z - \frac{w}{3} \frac{\sigma_y^2}{E^2 \varphi^2} \right) \quad (2.23) \]

\( (\varphi y \frac{d/2}{(d/2 - k)} < \varphi < \infty) \)

Non-dimensional:

\[ \frac{M}{M_y} = \frac{Z}{S} - \frac{w d^2}{12 S} \left( \frac{\sigma_y}{\varphi} \right)^2 \quad (\text{Eq. 2.22 and 2.17}) \]

\[ \frac{M}{M_y} = f - \frac{w d^2}{12 S} \left( \frac{\sigma_y}{\varphi} \right)^2 \quad (\text{Eq. 2.16}) \]

\[ \left( \frac{d/2}{d/2 - k} < \varphi < \infty \right) \]}
Plot of $M - \phi$ relationship for WF shape: (Example: 8WF13)

Note: 1. Shape factor is smaller than rectangle (Compare Fig. 2.7).

2. Average value of "f" for all WF beams = 1.14.

3. Rapid approach to $M_p$ (Compare Fig. 2.7).

3. Calculation of $Z$

The plastic modulus, $Z$, equals twice the static moment about the neutral axis of the half-sectional area. (symmetric section)

Eq. (2.5): $M_p = 2 \int_A \sigma_y \ dA \cdot \frac{y}{2}$

$$M_p = \sigma_y \cdot 2 \int_A y \ dA \left( \frac{1}{2} - \frac{1}{Z} \right)$$

(2.25)
From split-tee properties (given in AISC Handbook)

\[ Z = 2 \, A_{st} \, \bar{y} \]

An approximation that neglects fillets:

\[ Z \approx bt \, (d-t) + \frac{w}{4} \, (d-2t)^2 \]  (2.27)

An approximation that makes use of the average shape factor \( f \) for WF's:

(from Eq. 2.16): \[ Z = 1.14 \, S \] (2.27a)

2.4 PLASTIC HINGE CONCEPT

The reason a structure will support the computed ultimate load is that plastic hinges are formed at certain critical sections. What is the plastic hinge? What factors influence its formation? What is its importance?

1. Features

1) \( M-\phi \) curve is characteristic of plastic hinge (Fig. 2.11)
2) Rapid approach to \( M = M_p = \sigma_y \, Z \)
3) Indefinite increase in \( \phi \) at constant \( M \).

a) Idealized \( M-\phi \) curve

Assume material concentrated in flanges; idealized stress-
strain relationship (Fig. 2.2):

![Diagram showing Plastic Hinge and Actual Hinge with M-\phi Curve]

Unit Rotation, \( \phi \) ———

\[
\begin{align*}
M &= EI\phi \quad (0 < \phi < \phi_p) \\
M &= M_p \quad (\phi_p < \phi < \infty) \\
\phi_p &= \frac{M_p}{EI}
\end{align*}
\]

Note: The behavior shown in Fig. 2.12 is basic to plastic analysis. According to it, member remains elastic until \( M \) reaches \( M_p \). Thereafter, rotation occurs at constant moment; i.e., member acts as if it were hinged except with constant restraining moment, \( M_p \).

2. Factors Affecting Bending Strength and Stiffness (M-\( \phi \) curve)

Several factors influence the ability of members to form plastic hinges. In certain cases, some are important from the design point of view and are treated in Lecture 9.

a) Shape Factor

Ratio \( \frac{M_p}{M_y} \) depends on shape of cross-section.

Examples: Fig. 2.13
b) Material Properties  (Refs. 2.1, 2.2, 2.7)

Variation in strength → direct effect.
Variation in proportional limit → negligible effect.

c) Residual Stress  (Refs. 2.1, 2.7)

Residual stresses due to cooling, cold-bending, welding, reduce proportional limit in bending and tend to increase deflections. They have negligible effect on bending strength.

\[ \text{Fig. 2.14} \]

\[ \text{Fig. 2.15} \]

d) Stress-Concentrations  (Ref. 2.7)

Similar to residual stresses.

e) Strain-Hardening  (Ref. 2.9)

Beneficial Effect. Hardening at \( \psi_{st} \approx 15\psi_y \) (Fig. 2.2) prevents hinge from "running away".

Fig. 2.15 is an approximation.

f) Shear

g) Axial Load

Important factors, all tending to reduce carrying capacity.

h) Local Buckling

i) Lateral Buckling

Treated as "modifications". See Lectures 9 and 12.

\[ \text{Fig. 2.15} \]

j) Unsymmetrical Cross-sections

Introduces combined bending and torsion. Consider only symmetrical sections.
k) Encasement

Beneficial effect that is neglected. Consider only main frame.

1) Brittle Fracture

Specify proper material, workmanship, design details.

m) Stress-Distribution

Beneficial effect that is ignored. See Ref. 2.2 and 2.5.

3. Distribution of Plastic Hinge ("Hinge Length")

For the idealized $M-\phi$ curve of Fig. 2.12, the plastic hinges form at discrete points at which all plastic rotation occurs; hinge length $\Rightarrow 0$. In actuality "hinge" extends over a length of member that is dependent on loading and geometry.

Examples: Rectangular beam:

$$(M_y = 0.67 \, M_p)$$

Wide-Flange beam:

$$(M_y = 0.88 \, M_p)$$

Hinge Length $= \Delta L = \text{Length of beam in which } M \geq M_y$
4. Importance of $M-\phi$ Relationship

As indicated in Section 1 above (Fig. 2.12) $M-\phi$ curve is basic to plastic analysis. In addition to providing a measure of strength, it has a two-fold role:

(1) Characterizes "Rotation Capacity" of structural element -- ability of a structural member to rotate at near maximum moment. (Ref. 2.9)

(2) It is the foundation of deformation computations.

\[ \phi \] - diagram replaces the $\frac{M}{EI}$ diagram in deformation analysis.

See Lecture 8.

5. Principles

(1) Plastic hinges form at points of maximum moment

(2) A plastic hinge is characterized by large rotation at near-constant moment.

(3) The plastic moment, $M_p$, equals $\sigma_y Z$.

(4) The shape factor ($f = \frac{Z}{S}$) is one source of reserve strength beyond the elastic limit.

Application of plastic hinge to analysis is outlined in the next article (2.5).
2.5 REDISTRIBUTION OF MOMENT

A second factor contributing to reserve of strength (statically indeterminate structure) is redistribution of moment. When the plastic moment is reached at a critical section, this moment remains constant as section rotates (action of a plastic hinge). Thereafter, moment is redistributed to other portions of structure, thus allowing an increase in load.

Example:

(How does a plastic hinge allow redistribution and subsequent increase in load?)

Fig. 2.19 shows uniformly-loaded, fixed-ended beam. Deflected shape, moment diagram, load-deflection and $M-\phi$ relationship is shown at 3 stages of loading (numbers in circles):

Stage 1 (Elastic Limit)
- Yield point reached at ends
- By elastic analysis,
  
  $$M_A = \frac{wL^2}{12}$$
  
  $$M_E = \frac{wL^2}{24}$$

- A reserve of Moment at $E_L$ of 50% still exists ($M-\phi$ curve).
Stage 1 - 2

- Moment capacity at ends is exhausted. The beam "hinges".
- Deflection increases at somewhat faster rate (simply-supported beam).

Stage 2 (Ultimate Load, \( W_u \))

- Hinge just formed at \( Q \)
- Total moment capacity is exhausted.

Stage 3 (Arbitrary Deformation)

- Beam continues to deform at constant load
- Action of plastic hinges creates a "mechanism" or "Hinge system", all further rotation occurring at joints.

Note: Shaded portion of moment diagram (Fig. 2.19) represents increase in load due to redistribution of moment.

The ultimate load, \( W_u \), is reached when a mechanism forms.

Load Computations

By equilibrium (from moment diagram of Fig. 2.19) the yield and ultimate loads may be computed.

"Yield" \[ \frac{W_y L}{8} = \frac{3}{2} M_y \]

Ultimate" \[ \frac{W_u L}{8} = 2 M_p \]
Reserve strength due to redistribution

\[
\frac{W_u}{W_y} = \frac{16M_p/L}{12M_y/L} = \frac{4}{3} \frac{M_p}{M_y}
\]

(2.31)

Note: In idealization we assumed \( M_p = M_y \).

Reserve: Redistribution + shape factor

\[
\frac{W_u}{W_y} = \left(\frac{4}{3}\right)(1.14) = 1.52
\]

Principles

(1) Plastic hinges are reached first at sections subjected to greatest deformation (curvature).

(2) Formation of plastic hinges allows a subsequent redistribution of moment until \( M_p \) is reached at each critical ("maximum") section.

(3) The maximum load is reached when a mechanism forms.

2.6 SHEAR AND DIRECT STRESS

Thus far the analysis of flexure of beams has neglected shear and direct stresses. These are practically always present. Two questions are of interest:

(1) What is distribution of shear and direct stress in the inelastic range? See next pages.

(2) How do these stresses influence ability of a member to form plastic hinges? Lecture #9

Distribution of shear and direct stresses is outlined here, particularly as it affects flexural stress-distributions. (Fig. 2.4)
1. Shear Stress

Principle: In the regions made plastic due to flexural yielding.

\[ I_{xy} = 0 \]

Result: Shear stresses are carried in the elastic core.

Example:

Fig. 2.20 shows cantilever WF beam with \( M > M_y \).

Typical flexure stress and corresponding shear stress distributions shown.

The stress distributions of Fig. 2.20 point to the following possibilities:

(1) Yielding in the region of the flange-web juncture due to combined flexural and shear stresses (Distribution "A").
(2) Yielding at midheight due to shear stress (Distribution C).

(3) Combination of (1) and (2) resulting in a yield zone effectively limiting ability of member to carry further shear force.

Note: After complete strain-hardening, shear stress tends to redistribute according to distribution "A".

2. Direct Stress

The problem is simpler than that of shear distribution since only normal stresses must be considered.

Fig. 2.21 shows stress-distribution at various stages of deformation due to M and P:

Two parts of Distribution "D":

![Diagram showing stress distribution](image-url)
Conclusions

(1) Yielding on one side of section will precede that on the other, depending on the magnitude of direct stress present.

(2) In the presence of direct stress, the total bending moment capacity theoretically will not be available.

REFERENCES


2.9 Toprac, A. A. "CONNECTIONS FOR WELDED CONTINUOUS PORTAL FRAMES", Welding Journal 30(7), 30(8), and 31(11), 1951 and 1952. (Progress Report #4)
UPPER AND LOWER BOUND THEOREMS

(Theorems for Fixing Upper and Lower Limits of Ultimate Load-Carrying Capacity for Frames Structures)

SCOPE: Purpose of lecture is to establish two fundamental theorems giving an upper and lower bound for the ultimate load the structure will carry. Rather than make a general approach, a simple example is used to derive the theorems.

OUTLINE: 1. ASSUMPTIONS
   2. PRINCIPLE OF VIRTUAL DISPLACEMENT
   3. UPPER BOUND THEOREM
   4. LOWER BOUND THEOREM
   5. SUMMARY

3.1 ASSUMPTIONS

1. Moment-Curvature relationship as established in Lecture 2.

2. First order theory, i.e. deformations are small such that equilibrium conditions can be formulated for undeformed structure (same as in elastic analysis).

3. No instability of structure will occur prior to ultimate load (very often the case, however attention required to such problems as lateral buckling).

4. Connections provide full continuity such that plastic moment $M_p$ can be transmitted (does not exclude actual hinges). (See Lecture #10)
5. Influence of normal and shearing forces on plastic moment $M_p$ are neglected (see Lecture #9 for necessary modifications).

6. Loading is proportional, i.e. all loads are fixed by single parameter such that they increase in fixed proportions. However, independent increase can be allowed, provided no local failure occurs. Definitely excluded is repeated loading. (See Lecture #9)

3.2 PRINCIPLE OF VIRTUAL DISPLACEMENTS

**Principle:**

If a system of forces in equilibrium is subjected to a virtual displacement, the total work done vanishes, i.e. the work done by the external forces equals the work done by the internal forces.

**Virtual Displacement:**

1. Small; it approaches zero.
2. Geometrically possible.
3. Piecewise continuous within structure.
   (This means that a "kink" may be assumed at a hinge.)

In subsequent paragraph this principle is used to determine external load such that equilibrium is established.
The basic ideas behind the two concepts to be discussed are as follows:

**Upper Bound:**

1. Assume plastic hinges for mechanism.
2. Compute loads to establish equilibrium.

Under what condition is $|M| \leq M_p$.

**Lower Bound:**

1. Keep at any state equilibrium.
2. Always $|M| \leq M_p$.

When does structure become mechanism?

### 3.3 UPPER BOUND THEOREM

#### Example:

- **(a)** Beam, plastic Moment $M_p$
  - Plasticity Condition $|M| \leq M_p$
- **(b)** Assumed Hinges at ① and ② - Mechanism 1 - 2
- **(c)** Plasticity Condition violated between ② and ④.
- **(d)** Reinforced Beam

![Fig. 3.1](image-url)
Beam shown is 1 times statically indeterminate.

**Problem:** What is the ultimate load $P_u$ that the beam will sustain?

As the load is increased, the plastic moment $M_p$ will first be developed at one section. Further increase in load over this value will eventually produce a hinge at another section. From here on no further load increase is possible, since under this condition the structure is reduced to a mechanism ("first order" movements possible without increase in load)! Where will hinge form? Hinge 1 is the only possible negative one, hence it will form. Hinges 2 to 4?

**Arbitrary Assumption:**

Mechanism with Hinges 1 and 2 as shown in Fig. 3.1(b).
Virtual Displacement*:

\[ P\left(\frac{1}{4} \theta L x 3 + \frac{1}{6} \theta L x 1 + \frac{1}{12} \theta L x 1\right) = M_p(\theta + \frac{4}{3} \theta) \quad (3.1) \]

or:

\[ P = \frac{7}{3} \frac{M_p}{L} = 2.33 \frac{M_p}{L} \quad (3.2) \]

Moment diagram shows that \( M \) between 2 and 4 is greater than \( M_p \). Hence the assumed mechanism will not be the one that is developed. However, strengthening of beam between 2 and 4 to 1.25 \( M_p \) will produce assumed mechanism. (Fig. 3.1(d))

Second Assumption:

Mechanism with hinges 1 and 3 (Fig. 3.2(a))

Virtual Displacement:

\[ P\left(\frac{\theta L}{4} x 3 + \frac{\theta L}{2} x 1 + \frac{\theta L}{4} x 1\right) = M_p(\theta + 2\theta) \quad (3.3) \]

\[ P = 2 \frac{M_p}{L} \quad (3.4) \]

Moment diagram shows that nowhere \( |M| \leq M_p \) hence plasticity condition is fulfilled. Furthermore number of hinges is sufficient such that further deflection is possible without an increase in load.

* Use of Virtual Displacement is not necessary. Equilibrium would result in same \( P \); Virtual Displacement only an easy means to compute \( P \).
Further Assumption:
Mechanism with hinges 1 and 4.

Result: \[ P = \frac{5}{2} \frac{M_p}{L} = 2.5 \frac{M_p}{L} \] (3.5)

Conclusion:
Mechanism 1-3 will actually develop; corresponding load will be ultimate load.

Generalization is possible such that the following theorem can be formulated:

Upper Bound Theorem:

"A load computed on the basis of an assumed mechanism will always be greater or at best equal to the ultimate load."

3.4 LOWER BOUND THEOREM

Previous example:

![Diagram of a statically determinate system with force and moment distributions.]

(b) Statically determinate system.

(c) Assumption of arbitrary equilibrium configuration with \(|M| = M_p\).

Fig. 3.3
The system is made statically determinate by introducing a hinged support at the left end. External loading produces a definite moment diagram e.g. $M_p$ at 2 and 3. (Fig. 3.3(c)) Disregarding the condition of zero slope for the deflection line at the left support, an arbitrary value for the redundant moment $M_1$ is introduced, e.g. $-M_p$. Structure is still in equilibrium, however has only one plastic hinge at 1, hence is not a mechanism. Increase in load changes only statically determinate bending moment diagram. Maximum load is reached at moment when second plastic hinge forms.

**Conclusion:** Any equilibrium configuration with arbitrarily assumed values for the redundants corresponds to a loading below the ultimate loading - provided all $|M| \leq M_p$.

**Generalization:**

**Lower Bound Theorem:**

"The load corresponding to an equilibrium configuration with arbitrarily assumed values for the redundants is smaller than or at best equal to the ultimate loading - provided that all $|M| \leq M_p$."

3.5 SUMMARY

Comparison between elastic solution, plastic solution and upper and lower bound theorems.

**Elastic Solution:**

To fulfill:
1. Equilibrium
2. "Elasticity" Condition: \(|M| \leq M_{\text{yielding}}\)
3. Compatibility (continuity)

"Plastic" Solution:

To fulfill:
1. Equilibrium
2. "Plasticity" Condition: \(|M| \leq M_{\text{plastic}}\)
3. Mechanism (additional deformation are possible without load increase.)

Assumed "Mechanism Solution" (Upper Bound)

Fulfills:
1. Equilibrium (Principle of Virtual Displacement)
2. Mechanism (is presupposed)

May violate 2. \(|M| \leq M_p\). (possibly \(|M| > M_p\))

Assumed "Equilibrium Solution" (Lower Bound)

Fulfills:
1. Equilibrium
2. \(|M| \leq M_p\) (is presupposed)

May violate 3. Mechanism (insufficient \(M_p\)'s)
Each of the methods of solutions that will be discussed during the remainder of these lectures will be based on one of these two theorems. In simple cases the "Equilibrium Method" (Lower Bound Theorem) leads to straightforward solutions (Lecture #4). However, for more complicated cases the "Mechanism Method" (Upper Bound Theorem) becomes much more powerful.

References


3.2 Symonds, P. S. Neal, B. G. "RECENT PROGRESS IN THE PLASTIC METHODS OF STRUCTURAL ANALYSIS", Journal of the Franklin Institute, Vol. 252, No. 5 and 6, 1951.
Equilibrium Method of Analysis

Scope: Calculation of ultimate strength of continuous beams and single span frames on the basis of lower bound theorem.

By solving several sample problems, the general method of attack will be illustrated. In each case the following approach will be used, based on the Lower Bound Theorem (Lecture #3).

Given the structure and loading

1. Select the redundant(s) - (can be moments, forces, etc.)
2. Draw moment diagram for determinate structure (neglecting redundant)
3. Draw moment diagram for redundant - (keeping redundant moment values in general terms)
4. Sketch composite moment diagram in such a way as to insure formation of mechanism (still keeping redundant in general terms)
5. Compute value of redundant(s) by solving equilibrium equation \(|M| = M_p\) at sections of maximum moment
6. Check to see that there are sufficient hinges for mechanism and that \(|M| \leq M_p\).
4.1 EXAMPLE (Fig. 4.1)

A two span continuous beam of uniform cross-section; to find the maximum value of P in terms of the plastic moment, $M_p$.

Selecting the moment at "C" as the redundant, the determinate and redundant loadings are then as shown in Fig. 4.2(a) and (b). The corresponding moment diagrams are given in Fig. 4.2(c) and (d). If these are now combined as shown in Fig. 4.2(e), it is noted that moment values are maximum at sections B, C, and D. For these to be equal to the full plastic moment, $M_p$. 

Fig. 4.1

Fig. 4.2
(4.1) \[ \frac{PL}{4} - \frac{Mc}{2} = M_p \] 

and

(4.2) \[ M_c = M_p \]

This then gives on substitution of Eq. 4.2 in Eq. 4.1

(4.3) \[ P_u = 6 \frac{M_p}{L} \]

A consideration of Fig. 4.2(f) indicates that the formation of plastic hinges at sections "B", "C" and "D" is sufficient to produce a mechanism. Nowhere is \( M > M_p \).

### 4.2 EXAMPLE

Fig. 4.3 shows a Fixed Ended beam subjected to both concentrated and uniform loads. Here, again, the object is to determine \( P_u \) in terms of \( M_p \), the plastic moment.

Selecting \( M_A \) and \( M_D \) as the redundants, the moment diagrams will be as shown in Fig. 4.3(b), (c), and (d). Rather than attempt to draw the composite moment diagram for these conditions, it is observed
that maximum negative moment occurs at each end of the beam. Maximum positive moment will occur somewhere within the structures, say at some point (c). Moment at this point, which is located at a distance \( x \) from A, is then given by the equation,

\[
M_c = \frac{3}{2} PL \left( \frac{x}{L} - \frac{x^2}{L^2} \right) + \frac{1}{4} PL \left( 1 - \frac{x}{L} \right) - M_A \left( 1 - \frac{x}{L} \right) - M_D \frac{x}{L} \quad (4.4)
\]

(Note: This assumes \( M_A \) and \( M_D \) negative as shown)

or

\[
M_c + M_A \left( 1 - \frac{x}{L} \right) + M_D \frac{x}{L} = \frac{PL}{4} \left( 1 - \frac{x}{L} + 6 \frac{x}{L} - 6 \frac{x^2}{L^2} \right) \quad (4.5)
\]

For the left hand side to be a maximum

\[
\left( 1 + 5 \frac{x}{L} - 6 \frac{x^2}{L^2} \right) \text{ should be a maximum}
\]

or

\[
\frac{\partial}{\partial x} \left( 1 + 5 \frac{x}{L} - 6 \left( \frac{x}{L} \right)^2 \right) = 0 = \frac{5}{L} - 12 \frac{x}{L^2}
\]

(a consideration of the shear diagram will give this same result.)

Therefore

\[
x = 5/12 L \quad (4.6)
\]

Assuming now that the beam is of uniform cross-section,

\[
\begin{align*}
M_A &= M_p \\
M_D &= M_p \\
M_c &= M_p
\end{align*} \quad (4.7)
\]
Substituting these values and Eq. 4.6 in Eq. 4.5 gives
\[
\frac{PL}{4} = \frac{2M_p}{(1 + 5\frac{x}{L} - \frac{6x^2}{L^2})} = \frac{2M_p}{49/24}
\]
or
\[
P_U = (3.92) \frac{M_p}{L}
\]

From Fig. 4.3(e), it is evident that a mechanism can form and the necessary conditions for a "plastic" solution are therefore satisfied.

4.3 EXAMPLE

For this example, it is desired to determine the value "k", the ratio of the plastic moment of the end spans to that of the center span, for the case where ultimate strength is realized simultaneously in all spans. Also determine the load at which this occurs.

From symmetry the redundant moment at "C" will equal that at "E". Furthermore, since these joints can only develop the strength of the
weakest joining member,

\[ M_C = M_E = k M_p \]

Therefore from Fig. 4.4(d)

\[ M_D = \frac{PL}{4} - k M_p = M_p \] (4.9)

or

\[ M_p = \frac{PL}{4(1+k)} \]

Also from this Figure

\[ M_B + k M_p = \text{Determinate Moment at "B"} \quad \text{Redundant Moment at "B"} \] (4.10)

It should be noted that "B", the section of maximum positive moment is not at the center of the span A-C.

To determine the location of this section consider a free body diagram of that part of the span from A to C as shown in Fig. 4.5.

\[ M_C = k M_p \]

\[ R_A = \frac{P}{2} - k \frac{M_p}{L} \] (4.11)
Therefore to determine the distance to the section of zero shear and thereby maximum moment

\[ R_A - w_1(x) = 0 \quad \text{where} \quad w_1 = \frac{P}{L} \]

\[ \frac{P}{2} - k \frac{M_p}{L} - \left( \frac{P}{L} \right) (x) = 0 \]

or

\[ x = \frac{L}{2} - k \left( \frac{M_p}{P} \right) \tag{4.12} \]

The moment at "B" then equals (using this computation in place of Eq. 4.10)

\[ M_B = R_A(x) - \frac{w_1}{2} (x)^2 = -k M_p \tag{4.13} \]

where

\[ w_1 = \frac{P}{L} \]

\[ R_A \text{ is as given by Eq. (4.11) and} \]

\[ x \text{ is given by Eq. (4.12)} \]

Substitution of these values in Eq. 4.13 gives the following Eq.

\[ \left( \frac{P}{2} - k \frac{M_p}{L} \right) \left( \frac{1}{2} - k \frac{M_p}{PL} \right) L - \frac{1}{2} \left( \frac{P}{L} \right) \left( \frac{1}{2} - k \frac{M_p}{PL} \right)^2 L^2 = k M_p \tag{4.14} \]

which reduces to

\[ \frac{k^2 M_p^2}{PL^2} - 3 k \frac{M_p}{L} + \frac{P}{4} = 0 \tag{4.15} \]

Substituting in the value of \( M_p \) as given in Equation (4.9) and reducing gives

\[ k^2 + \frac{4}{7} k - \frac{4}{7} = 0 \]

or

\[ k = 0.524 \tag{4.16} \]
Then from Eq. 4.12

\[ x = 0.414 \, L \]  \hspace{1cm} (4.17)

Restating Results:

For a ratio of plastic moments of

- Span AC - 0.524
- Span CE - 1.000, and
- Span EG - 0.524,

plastic hinges will form simultaneously at sections B, C, D, E, and F, at a \( P_u \) value, as determined from Eq. 4.9, of

\[ P_u = 6.1 \frac{M_p}{L} \]  \hspace{1cm} (4.18)

Note: Since all hinges form simultaneously, this load was carried by the least weight of material. This procedure therefore suggests one possible technique of designing for minimum weight in continuous beams.

Not only beam type problems but also those of the frame type can be solved by this Equilibrium method of solution. (This method, however, may not be the easiest or quickest, as will be shown in subsequent lectures.) Consider for example, the next problem.
4.4 EXAMPLE

For this problem the horizontal reaction at 5 will be assumed as the redundant. For ease of illustration the legs have been "opened" as shown in Fig. 4.6(b). Assuming that the frame is made up of the same section throughout -

\[ M_2 = M_4 = -M_p \]

and

\[ M_3 = -M_p \]

then from Fig. 4.6(b) and (d)

\[ M_3 = \frac{PL}{4} - M_p = +M_p \]

or

\[ P_u = 8 \frac{M_p}{L} \]  \hspace{1cm} (4.14)

If the columns had different stiffnesses than the beam the result would be somewhat as shown in Fig. 4.7 for three different stiffness relationships.
4.4a EXAMPLE

Consider, now, this same structure subjected to a horizontal load at section (2) as shown in Fig. 4.8. Again, select the horizontal reaction at "5" as the redundant.

From Fig. 4.8(d) it is evident that hinges will form at sections 2 and 4 with

\[ M_4 = M_p \]

and

\[ M_2 = Ph - M_p = + M_p \]
\begin{align*}
\text{Therefore} \\
P &= \frac{2 M_p}{h} \\
\text{(4.15)}
\end{align*}

In considering the influence of a combination of both a horizontal and a vertical force -- it should be remembered that \textsc{superposition is not valid}. It is necessary to compute the composite case as if it were a new problem.

\textbf{4.4b EXAMPLE}

Again select the horizontal reaction at 5 as the redundant.

Noting from Fig. 4.9(d) that hinges will form at section 3 and 4 with

\begin{align*}
M_4 &= M_p, \text{ and} \\
M_3 &= M_p, \\
M_3 &= P\left(\frac{L}{4} + \frac{h}{2}\right) - M_p
\end{align*}

or

\begin{align*}
M_p &= P\left(\frac{L}{4} + \frac{h}{2}\right) - M_p \\
P_u &= \frac{2 M_p}{\left(\frac{L}{4} + \frac{h}{2}\right)} \quad \text{(4.16)}
\end{align*}

\textbf{Fig. 4.9}
4.5 EXAMPLE

Selecting \( B_H \), the horizontal reaction at B, as the redundant, the solution is as shown in Fig. 4.10. Since the "hinges" form at points 1, 2, 4, and 5 (see Fig. 4.10(d)), at sections 1 or 5

\[ M_p = 2L B_H \]  \hspace{1cm} (4.17)

and at sections 2 and 4

\[ M_p = P L - 2.5L B_H \]  \hspace{1cm} (4.18)

Substituting \( B_H \) as determined from Eq. 4.17 into Eq. 4.18 gives

\[ P_u = 2.25 \frac{M_p}{L} \]  \hspace{1cm} (4.19)

From Fig. 4.10 it is evident that a mechanism has formed and therefore Eq. 4.19 is the correct solution.
Summarizing:

The examples that have been presented in this section are "Equilibrium" solutions and are based on the Lower Bound Concept. This method of solution is primarily concerned with the selection of redundant moment values such that the necessary plastic hinges are formed at critical sections throughout the structure. Then from Equilibrium considerations and a knowledge that the full plastic moment values are realized at these critical sections, the unknown ultimate carrying capacity (or size of member - depending on which is given) can be computed.

References

4.1 "THE COLLAPSE METHOD OF DESIGN",
British Constructional Steelwork Association,
Publication No. 5, 1952.
Lecture No. 5

MECHANISM METHOD OF ANALYSIS

SCOPE: Assumption of plastic hinge mechanism; determination of corresponding load by method of virtual displacement. Possibilities of local and combined mechanisms. Development of systematic procedure.

Recapitulation of Upper Bound Theorem:

"A load computed on the basis of an assumed mechanism will always be greater or at best equal to the ultimate load."

EXAMPLE 5.1

A 2-span-frame as shown in Fig. 5.1 is subjected to indicated loading. The actual distributed load is replaced by equivalent concentrated loads, the reason being that the exact location of a hinge in case of distributed loads causes some minor difficulties. (See Example 4.2, for instance) The concentrated load produces a moment diagram circumscribed about the one for the actual distributed loads Fig. 5.1(b); hence the structure is subjected to a more severe loading condition and as a result the ultimate load will be somewhat smaller. More about this later. The example will be solved for the concentrated loads.

In Summary:

Concentrated Loads: Max. M under Loads, hence position of hinges is fixed.

Distributed Loads: Max. M at V = 0, hence hinge position is not known.
The plastic moment values of different members are indicated in Fig. 5.1. What is ultimate load, $P_u$, of the structure?

Assuming an arbitrary mechanism will give upper bound to $P_u$ by methods of Lecture No. 3, three local mechanisms are easily conceivable:

**Actual load distribution**

**Equiv. concentrated loads**

Mech. I

Mech. II

Mech. III

Mech. IV

 Mech. V

(Comb. II, III & IV)

Fig. 5.1
Mech. I: Failure of Beam 4-6

\[
\frac{P}{3} \theta L \left( \frac{1}{6} + \frac{1}{2} + \frac{1}{6} \right) = M_p \theta (1 + 2 + 1) \tag{5.1}
\]

\[
P_I = \frac{54}{5} \frac{M_p}{L} = 10.8 \frac{M_p}{L} \tag{5.2}
\]

Mech. II: Failure of Beam 8-10

\[
P \theta \ell (\frac{1}{3} + 1 + \frac{1}{3}) = M_p \theta (1 \cdot 3 + 2 \cdot 3 + 1 \cdot 2) \tag{5.3}
\]

\[
P_{II} = \frac{33}{5} \frac{M_p}{L} = 6.60 \frac{M_p}{L} \tag{5.4}
\]

Mech. III: Side-sway of Frame (Panel-Mechanism)

\[
\frac{1}{2} P \theta \ell = M_p \theta (1 + 2 + 2 + 1 + 2 + 2) \tag{5.5}
\]

\[
P_{III} = 20 \frac{M_p}{L} \tag{5.6}
\]

Combinations of these three local mechanisms are possible, and hence should be investigated. To reduce the internal work, i.e. the work done by the plastic hinges should be reduced; hence such combinations should be investigated which eliminate plastic hinges. In this connection it is advantageous to consider a rotation of joint 6-7-8 also as a local mechanism (Fig. 5.1(f)). In any combination the joint should be so oriented as to produce the minimum amount of internal work. Combining Mech. II and III the joint will be turned such that only one hinge at 6 forms (Fig. 5.1(g)): 
Mech. V: Combination of Mech. II, III, and IV

\[ P\theta L \left( \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{3} \right) = M_p \theta (1+2+2+1+1+2+3+2+2) \]  
(5.7)

\[ P_V = \frac{102}{13} \frac{M_p}{L} = 7.84 \frac{M_p}{L} \]  
(5.8)

Further trial-combination will not produce any load smaller than \( P_{II} \). Hence it must be concluded that Mech. II gives the smallest load or

\[ P_u = P_{II} = 6.60 \frac{M_p}{L} \]  
(5.9)

Example was purposely chosen to fail in this manner for 2 reasons:

(1) The statement often encountered that \( n + 1 \) plastic hinges are required to transform an \( n \)-times statically indeterminate structure into mechanism is wrong.

(2) Equilibrium check for Mech. V (Fig. 5.1(g)) is trivial, because system is determinate. Not so for Mech. II (Fig. 5.1(d)). Equilibrium check requires special attention. (Lecture #7)

5.2 DISTRIBUTED LOAD

As indicated, the equivalent concentrated loads lead to conservative results. Mech. II is applied to distributed load (Fig. 5.2).
Location of hinge is fixed by parameter $x$, as it is not known. Virtual displacement $\theta$ gives:

$$3P \theta \frac{x}{2} = M_p \theta \left(1.3 + \frac{2}{L} \cdot \frac{x}{L} \cdot 2 + \frac{x}{2L-x} \cdot 2\right) \quad (5.10)$$

$$P = \frac{M_p}{L} \frac{12 - \frac{x}{L}}{2 - \frac{x}{L}} \cdot \frac{2}{3x/L} \quad (5.11)$$

To find minimum value of $P$, minimize (5.11) ($x = 1.045L$); or few trials

For $x = 1.045L$ \quad $P_u = 7.33 \frac{M_p}{L} \quad (5.12)$

Comparison with Eq. 5.4 shows that equivalent concentrated load gives result which underestimates the carrying capacity by about 10%. With four equivalent concentrated loads the error is about 2%.

5.3 SYSTEMATIZED PROCEDURE

1. Location and Number of Possible Plastic Hinges

The 2 span-frame had 10 possible locations of plastic hinges, labeled 1 to 10 in Fig. 5.1(a). No hinge within columns is possible as shear is constant, hence
M_{\text{max.}} is at ends. Possibility of hinge within beams exists as shear may change through zero.

2. Indeterminacy

Frame is 6 times statically indeterminate.

3. Mechanisms

The number of local mechanisms is four (2 beams-, 1 panel-, 1 joint-mech.) corresponding to the difference between the possible number of plastic hinges and the number of redundancies. This is no accident, for to each elementary mechanism there corresponds an equation of equilibrium. (For example Equation 5.5 for Mech. III expresses the equilibrium between horizontal shear and applied horizontal load P/2; Mech. IV corresponds to equilibrium equation expressing that sum of moments connecting into a joint is zero.) In case of N possible plastic hinges and X redundancies there must be (N-X) independent equilibrium equation and hence (N-X) elementary mechanisms.

Rule: \[ N = \text{number of possible plastic hinges} \]
\[ X \quad \text{redundancies} \]
\[ (N-X) \quad \text{elementary mechanisms} \]

These plus any combination of them should be investigated to determine smallest possible load. Experience leads to many short-cuts. However in complicated cases one never is quite sure if smallest P is found. Hence equilibrium check becomes necessary. This is simple if mechanism
reduces the structure to a statically determinate system (e.g. Mech. V, Fig. 5.1(g)). A case in which structure is still statically indeterminate (as Mech. II, Fig. 5.1(d)) will be treated in Lecture #7.

The above rule will be tested in subsequent cases.

References

5.1 Symonds, P. Neal, B. G. "RECENT PROGRESS IN THE PLASTIC METHODS OF STRUCTURAL ANALYSIS", Journal of the Franklin Institute, Vol. 252, No. 5 and 6, 1951.
APPLICATION OF MECHANISM METHOD

The MECHANISM METHOD of analysis, as demonstrated in the preceding lecture, is characterized by the selection of combinations of possible plastic hinges into geometrically possible mechanisms. Each of these mechanisms will have associated with it a certain critical load and the mode in which the structure will eventually fail will correspond to the lowest critical load. However, to be sure that the correct mechanism has been investigated, it is necessary that an equilibrium check be carried out for this supposed correct case to determine if the plastic moment at any sections within the structure has been exceeded. If the plastic moment is nowhere exceeded, then the solution is the correct one for the loading condition assumed.

The procedure then is as follows:

1. Determine the locations of possible plastic hinges.
2. Select possible mechanisms (elementary mechanisms and combinations therefrom).
3. For each possible mechanism - displace the structure a virtual amount and compute the corresponding internal and external work.
4. From the Equation
   \[ \text{EXTERNAL WORK} (W_E) = \text{INTERNAL WORK} (W_I), \]
   compute the critical load.
5. Select the lowest critical load and thereby the correct mechanism.

6. Carryout an equilibrium check to ensure that

\[ M \leq M_p. \]

To illustrate this method of analysis, several examples will be considered in this section. These are:

1. Portal Frame
2. Gable Frame
3. Multispan Gable Frame
4. Industrial Frame

Also in this lecture will be introduced the concept of the instantaneous center of rotation for the determination of internal and external work expressions.

**EXAMPLE 6.1**

The problem is the determination of the maximum load, \( P_u \), that the structure shown in Fig. 6.1 can sustain. For ease of solution, the uniform load will be concentrated at the quarter points as shown in Fig. 6.2. (The resulting design will be slightly more conservative than the one that deals directly with the uniform load.)
Since shear is constant between the base of the frame and the beam connections, maximum moments in the columns will occur only at the upper ends. Therefore these are points of possible plastic hinge formation. Under each of the concentrated loads, moments again may be maximum. These are also points of possible "hinge" formation. The total number of possible plastic hinges then is 4, as numbered in Fig. 6.2. For this one time indeterminate structure, the equation on page 5.6 gives

\[
4 - \text{number of possible plastic hinges} \\
1 - \text{redundant} \\
3 - \text{elementary mechanisms}
\]

These are shown in Fig. 6.3.
Combinations of these elementary mechanisms are also possible and should be investigated. These are shown in Fig. 6.4.

\[ \begin{align*}
4 &= 1 + 2 \\
5 &= 1 + 3 \\
6 &= 2 + 3
\end{align*} \]

**COMPOSITE MECHANISMS**

The next step is the computation of the external and internal work associated with each of these 6 possible mechanisms. Then from the equation

\[ \text{WORK EXTERNAL} = \text{WORK INTERNAL} \]

determine the critical load \( P \) for each case.

**BEAM MECHANISM 1**  PLASTIC HINGES at 1, 2, and 4

**EXTERNAL WORK** = **INTERNAL WORK**

\[ \frac{3P}{2} (\theta \frac{L}{4}) + \frac{3P}{2} (\theta \frac{L}{3}) - M_P \theta + M_P (\frac{4}{3} \theta) + M_P (\frac{1}{3} \theta) \]

Load at 2  Load at 3  Hinge at 1  Hinge at 2  Hinge at 4

or \[ \frac{3P}{2} \theta \cdot 3 = M_P \theta \]

\[ (6.1) \]

\[ P = \frac{M_P}{L} \cdot 3.33 \]
BEAM MECHANISM 2  PLASTIC HINGES at 1, 3 and 4

\[ W_E = W_I \]
\[ \frac{3}{2}P\left(\frac{\theta}{3}\right)\left(\frac{L}{4}\right) + \frac{3}{2}P\left(\theta\right)\left(\frac{L}{4}\right) = M_p\left(\frac{L}{3}\right) + M_p\left(\frac{L}{3}\right) + M_p\left(\theta\right) \]

or
\[ P = \frac{M_p}{L} 5.33 \]  (6.2)

PANEL MECHANISM 3  PLASTIC HINGES at 1 and 4

\[ W_E = W_I \]
\[ P\left(\frac{L}{2}\right) = M_p\theta + M_p\theta \]

Hinge  Hinge
at 1  at 4

or
\[ P = \frac{M_p}{L} 4.000 \]  (6.3)

COMPOSITE MECHANISM 4  PLASTIC HINGES at 1, 2, 3, and 4

\[ W_E = W_I \]
\[ \frac{3}{2}P\left(\frac{\theta}{4}\right) + \frac{3}{2}P\left(\theta\right)\left(\frac{L}{4}\right) = M_p\theta \left(1+1+1+1\right) \]

or
\[ P = \frac{M_p}{L} 5.33 \]  (6.4)
From these calculations of $P$ values for each of the 6 combinations of possible hinges investigated, composite mechanism 5, which combines elementary mechanisms 1 and 3, gives the lowest allowable load. Therefore Fig. 6.5(e) is the correct failure mode PROVIDING all possible combinations have been considered and that no arithematic mistake has been made. To eliminate this possibility, the next step is to compute the moment diagram for this assumed correct case.

The structure and loading to be considered is as shown in Fig. 6.6.
Considering moment equilibrium of the right hand column

\[ M_p = B_H \left( \frac{L}{2} \right) \]

or

\[ B_H = \frac{2M_p}{L} \quad (6.7) \]

Then for the structure as a whole

\[ A_H = 0.67 \frac{M_p}{L} \]

\[ B_V = 5.33 \frac{M_p}{L} \quad (6.8) \]

\[ A_V = 2.67 \frac{M_p}{L} \]

With this information the moment diagram can be plotted as shown in Fig. 6.7. Since the plastic moment value, \( M_p \), is nowhere exceeded in the structure; Equation 6.5 is the correct solution for the structure loaded as shown in Fig. 6.2. But the loading assumed in this figures is not the actual one to which the structure is subjected. The question then is, how good is this approximation. It has been shown in previous lectures that it will be on the safe side since the structure will be subjected to a moment diagram which is greater than the actual one. But just how much is the error involved?

Rather than discuss the question of the amount of error associated with the replacement of a uniform load by... Load... concentrated loads for this one particular case (pin-based),
consider the structure shown in Fig. 6.8(a). It should be noted that if the value \( m \), which relates the plastic moment of the bottom beam to the other members of the frame, equals zero the structure is pin based as considered in the previous example. For the case where \( m = 1.0 \) the structure is fixed at the bases in-so-far as the developed mechanism is concerned.

For the quarter point, concentrated load approximation, it can be demonstrated that the correct failure mode is as shown in Fig. 6.8(b) with

\[
P_u = 2 \left[ m + 1.333 \right] \frac{M_p}{L} \quad (6.9)
\]

(Note that when \( m = 0 \), this checks Equation 6.5)

For the uniform load, the failure mode is given in Fig. 6.8(c) with

\[
P_u = \frac{M_p}{L} \cdot 4 \left[ \frac{m (1 - \frac{X}{L}) + 1}{(1 - \frac{X}{L})(1 + 3\frac{X}{L})} \right] \quad (6.10)
\]

It will be noted, however, that this equation contains the parameter \( x \) which defines the distance to the plastic hinge in
the upper beam. To determine the correct distance and thereby the "exact" load it is necessary that Equation 6.10 be minimized with respect to this distance $x$.

The following equation results:

$$\frac{x}{L} = \left[1 + \frac{1}{m}\right] \pm \sqrt{(1 + \frac{1}{m})^2 - (1 + \frac{2}{3m})} \quad (6.11)$$

A plot of these equations (6.9 and 6.10) is given in Fig. 6.9. For the pin based case the error introduced by the quarter point approximation is 10%: For the fixed base, 2 1/2%.

![Diagram](image-url)
INSTANTANEOUS CENTER - Method of Solution

In computing the internal and external work expressions associated with any given mechanism a knowledge of the instantaneous center of rotation may prove of considerable help. Consider for illustration the mechanism shown in Fig. 6.5(e). (Redrawn in Fig. 6.10.) (It should be remembered that these solutions are based on a first order theory that considers equilibrium of the undeformed structure.)

The chosen mechanism is composed of essentially three "movable parts", part A12, part 234 and part 4B. The first of these, part A12 can rotate about hinge A as shown. The last, part 4B, rotates about hinge B. For part 234 the only conditions known are that point 2 moves in a direction perpendicular to a line between A and 2; and that point 4 moves along a line perpendicular to line 4B. With this information, however, it can be shown that member 234 rotates about point C, its instantaneous center of rotation for the position considered. (The location of this point is determined from geometrical considerations.)
From Fig. 6.10, it is noted that if member 4B is given a virtual clockwise rotation $\theta$ about point B, point 4 moves to the right an amount $\theta \cdot \frac{L}{2}$. Since the distance from 4 to point C is equal to $\frac{3}{2}L$, the rotation at C of member 234 will be equal to $\theta \cdot \frac{L}{2} + \frac{3}{2}L$ or $\frac{\theta}{3}$. Furthermore since the distance from C to 2 is 3 times that of 2 to A, member A12 will have a rotation of $\theta$ about A.

Hinge 2 then rotates through an angle equal to $\theta$ (due to the rotation of A12 about A) plus $\theta/3$ (due to the rotation of 234 about C).

Hinge 4 rotates through $\theta/3$ (due to rotation of 234 about C) plus $\theta$ (due to rotation of 4B about B).

**INTERNAL WORK IS COMPUTED AS FOLLOWS:**

\[
\begin{align*}
\text{at Hinge 2} & \quad - M_p \left( \frac{4\theta}{3} \right) \\
\text{at Hinge 4} & \quad = M_p \left( \frac{4\theta}{3} \right) \\
& \quad = M_p \cdot \theta \left[ \frac{8}{3} \right] \\
& \quad = \frac{8}{3} \cdot M_p \cdot \theta \\
& \quad = (6.12)
\end{align*}
\]

For **EXTERNAL WORK**: the horizontal force $P$ acting at 1 moves through a distance equal to $\theta \cdot \frac{L}{2}$ or the work equals $PL \cdot \frac{\theta}{2}$. The $3P/2$ force acting at point 2 acts on both parts A12 and 234 so that the work computed from considering either rotation about point A or C should be the same, namely $\frac{3P}{2} \cdot \theta \cdot \frac{L}{4}$.

For the $3P/2$ force at point 3, external work equals $3P/2 \cdot \frac{L}{4} \cdot \frac{\theta}{3}$. 
Therefore

\[
\text{EXTERNAL WORK} = \frac{PL}{2} \theta + \frac{3PL}{8} \theta + \frac{PL}{8} \cdot \theta = PL\theta \tag{6.13}
\]

Using the equality

\[
W_{\text{external}} = W_{\text{internal}},
\]

\[
PL\theta = M_p \cdot \theta \left[ \frac{8}{3} \right] \tag{6.14}
\]

or

\[
P = \frac{M_p}{L} \cdot 2.667 \tag{6.15}
\]

which is the same as Equation 6.5.

In using this method of solution it should be remembered that instantaneous centers are determined from a consideration of the undeformed structure and that external work is equal to force times the distance the force moves in the direction of the force. Using the instantaneous center, this distance is equal to the virtual angle change times the perpendicular distance from the instantaneous center to the force in question.

EXAMPLE 6.2

For the single span gable frame shown in Fig. 6.11, it is desired to compute the maximum load, \( P_u \), assuming a constant section throughout. As indicated, there are 7 possible hinges.

- 7 number of possible hinges
- 3 redundants
- 4 elementary mechanisms

![Fig. 6.11](image)
ELEMENTARY MECHANISMS

BEAM MECHANISMS

PLθ = Mpθ [1 + 2 + 1]

or

P = \frac{Mp}{L} (4.00) \quad (6.16)

MECHANISM 3

P = \frac{Mp}{L} (4.000) \quad (6.17)

GABLE MECHANISM

PLθ = \frac{1}{4} + 1 = \frac{1}{4} + 2 + 2 + 1

P = \frac{Mp}{L} (3.000) \quad (6.18)

PANEL MECHANISM

Combination of Mechanisms 1 and 3

P = \frac{M_p}{L} (2.285) \quad (6.19)
Assuming that this combination 1 plus 3 (Mechanism 5) is the correct mechanism, it is now necessary to carry out an equilibrium check.

![Fig. 6.13]

Considering Fig. 6.13(b) - member 6 - 7, summing moments about 6:

\[ 2M_p = B_H (2L) \]

or

\[ B_H = \frac{M_p}{L} \quad (6.20) \]

Now considering the structure as a whole, \( \Sigma F_H = 0 \)

\[ A_H = [1.143 - 1.000] \frac{M_p}{L} = 0.143 \frac{M_p}{L} \quad (6.21) \]

\( \Sigma M_D = 0 \)

\[ B_V = 2.357 \frac{M_p}{L} \quad (6.22) \]

\( \Sigma F_V = 0 \)

\[ A_V = 2.213 \frac{M_p}{L} \quad (6.23) \]
The moment diagram using these values is shown in Fig. 6.14. The moments have been plotted on the tension side of the members.

Equation 6.19 is therefore correct, since the moment is nowhere greater than $M_p$.

**EXAMPLE 6.3**

(Constant Section) Multi-Span Gable Frame.

The problem is to determine the required plastic moment, $M_p$, in terms of the applied loads, $P$, and length parameter, $L$. It is assumed that the frame will be constant section throughout.

- 23 - Number of possible plastic hinges
- 7 - Redundants
- 16 - Elementary Mechanisms
Due to symmetry, will need to investigate only one half structure, the other half will react the same.

Note: Up to this point the Virtual work equations have been solved for the load $P_u$, the maximum load the structure will carry. However, in design it is the loads that are given and the problem is to determine the required plastic moment, $M_p$, so that a section may be selected. In this problem, the latter procedure has been used. Instead of looking for the minimum load, $P_u$, the maximum required $M_p$ is sought.
For the beam mechanisms (1 through 8):

\[
M_p \theta \left[ 1 + 2 + 1 \right] = PL\theta
\]

or

\[
M_p = \frac{PL}{4.000} = 0.250 \, PL
\]

(6.24)

Since there is no external work associated with elementary mechanism 9, this pattern of hinge combinations need not be considered. (It should be noted, however, that this does not mean that it cannot combine with other elementary mechanisms to produce failure.)

Gable mechanisms 10 and 11:

\[
M_p \theta \left[ 3 + 4 + 2 \right] = PL\theta \left[ 2 + 2 \right]
\]

\[
M_p = \frac{PL}{\frac{4}{9}} = 0.444 \, PL
\]

(6.25)

Combining mechanisms 2 and 10 (Fig. 6.17(b))

\[
M_p \theta \left[ 3 + 8 + 6 \right] = PL\theta \left[ 2 + 6 \right]
\]

\[
M_p = PL \left( \frac{8}{17} \right) = 0.470 \, PL
\]

(6.26)

Combining mechanisms 1 and 10 (Fig. 6.17(c))

\[
M_p \theta \left[ 7 + 8 + 2 \right] = PL\theta \left[ 6 + 2 \right]
\]

\[
M_p = PL \left( \frac{8}{17} \right) = 0.470 \, PL
\]

(6.27)
Note that Equations (6.26), (6.27) and (6.29) give the same size of member. Assume that the correct value is given by these equations,

\[ M_p = 0.470PL \quad \text{or} \quad P = 2.13 \frac{M_p}{L} \]  

(6.30)

It is now necessary to carry out an equilibrium check.
By considering equilibrium of various parts of the above structure the following reactions are computed

\[
\begin{align*}
A_H &= 0.500 \frac{M_p}{L} \\
A_V &= 2.125 \frac{M_p}{L} \\
B_H &= 0 \\
B_V &= 4.265 \frac{M_p}{L}
\end{align*}
\]  

\[ (6.31) \]

The moment diagram is then as shown in Fig. 6.19.

The solution therefore is the correct one: i.e. \( M_p = 0.470 \text{ PL} \)
EXAMPLE 6.4

For the structure shown in Fig. 6.20 it is desired to obtain the maximum load, $P_u$, to which it may be subjected. To simplify the solution the uniform vertical loads have been assumed as concentrated ones as shown in Fig. 6.21. For this loading there is a possibility that 17 hinges may form. Therefore,

17 - no. of possible hinges
5 - redundants
12 - elementary mechanisms

$$\begin{cases} 7 & \text{Beam Mechanisms} \\ 3 & \text{Panel Mechanisms} \\ 2 & \text{Joint Mechanisms} \end{cases}$$

These are shown in Fig. 6.22.
FOR BEAM MECHANISMS 1 and 6

(Also 2 and 7)

\[ P L \theta \left[ \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} \right] = M_p \theta \left[ 1 + \frac{4}{3} \cdot \frac{1}{4} \right] \]

\[ P = 8 \frac{M_p}{L} \]

(6.31)

FOR BEAM MECHANISMS 3 and 5

(Note: hinges 7 and 11 will have a value 2 \( M_p \).)

\[ 2P \theta \left[ \frac{1}{3} + \frac{1}{15} + \frac{1}{5} \right] = M_p \theta \left[ 2 + \frac{18}{5} + \frac{2}{5} \right] \]

\[ P = 5 \frac{M_p}{L} \]

(6.33)

FOR BEAM MECHANISM 4

\[ 2P \theta \left[ \frac{1}{3} + 1 + \frac{1}{3} \right] = M_p \theta \left[ 2 + 6 + 2 \right] \]

\[ P = 3.00 \frac{M_p}{L} \]

(6.34)
PANEL MECHANISM (8)

\[
P \frac{L}{2} \theta = M_p \theta \left[ 1 + 1 + 4 + 2 \right]
\]

\[
P = 16 \frac{M_p}{L}
\]  

(6.35)

PANEL MECHANISM (9)

\[
PL\theta = M_p \theta \left[ 2 + 2 + 2 + 2 \right]
\]

\[
P = 8 \frac{M_p}{L}
\]  

(6.36)

PANEL MECHANISM (10)

\[
P \frac{L}{2} \theta = M_p \theta \left[ 2 + 4 + 1 + 1 \right]
\]

\[
P = 16 \frac{M_p}{L}
\]  

(6.37)

COMBINATION OF MECHANISMS (8 + 9 + 10)

\[
PL\theta \left[ 1 + 2 \right] = M_p \theta \left[ 1 + 1 + 2 + 2 + 1 + 1 \right]
\]

\[
P = \frac{8}{3} \frac{M_p}{L} = 2.67 \frac{M_p}{L}
\]  

(6.38)
Using instantaneous centers

\[
\begin{align*}
\text{Horizontal Forces} & \quad \text{Vertical Forces} \\
PL \left[ 1 + \frac{1}{4} + \frac{1}{12} + \frac{2}{3} + 2 + \frac{2}{4} + \frac{1}{3} + \frac{1}{12} \right] & = M_p \theta \left[ \frac{4}{3} + \frac{4}{3} + 6 + \frac{4}{3} + \frac{4}{3} \right] \\
& \quad \text{or} \\
PL [7] & = M_p \left[ \frac{46}{3} \right] \\
& \quad \text{Assuming this is the correct solution, reactions will be determined} \\
& \quad \text{from Fig. 6.24.}
\end{align*}
\]
CONSIDERING EQUILIBRIUM OF THE VARIOUS PARTS — the following reaction values are obtained.

\[
\begin{align*}
A_V &= 0.253 \text{ M}_p/\text{L} \\
A_H &= 0.937 \text{ M}_p/\text{L} \\
B_V &= 9.346 \text{ M}_p/\text{L} \\
B_H &= 0.927 \text{ M}_p/\text{L} \\
C_V &= 8.174 \text{ M}_p/\text{L} \\
C_H &= 1.516 \text{ M}_p/\text{L} \\
D_V &= 4.127 \text{ M}_p/\text{L} \\
D_H &= 1.000 \text{ M}_p/\text{L}
\end{align*}
\]

The equilibrium moment diagram is then as shown in Fig. 6.25.

Therefore, solution is the correct one, i.e.

\[
P = 2.19 \frac{\text{M}_p}{\text{L}}
\]

In the following lecture (No. 7) cases will be treated in which the equilibrium check cannot be made by use of equations of statics alone. Such cases arise when the mechanism is a local one and portions of the structure remain statically indeterminate at ultimate load.
Note: $A = 2a\left[\frac{P}{wL}\right]$
Lecture No. 7

APPLICATION OF MECHANISM METHOD
(Equilibrium Checks on Solutions Determined by Mechanism Method)

SCOPE: Determination of equilibrium solution as a check on the analysis obtained by the Mechanism Method. Application of equilibrium check to previous examples.

PROBLEM: Is load determined by the "Mechanism Method" the actual ultimate load? Was no favorable combination overlooked? Since the Mechanism Method gives too high or at best the ultimate load, neglecting a more favorable combination could result in overestimating the true load-carrying capacity.

To make sure, an "equilibrium check" is needed, establishing a possible moment distribution throughout the structure. If no $|M| > M_p$ then a lower bound is established. Since the computed load thus becomes both an upper and a lower bound, then it can only be the correct solution.

A case will be considered which cannot be solved by the procedure used in the previous lectures (equation of static equilibrium). Referring to the example of a 2 span frame of Lecture #5, Fig. 5.1, the frame failed by beam mechanism 8-10 (Mechanism II).

\[ P_u = P_{II} = 6.60 \frac{M_p}{L} \quad (5.4) \]

It is desired to find a possible moment distribution throughout structure. Referring to the example (Fig. 5.1), out of the 10 possible plastic hinges Mechanism II (Beam 8-10)
developed 3 plastic hinges at 8, 9, and 10. Out of the 4 independent equations of equilibrium (corresponding each to a local mechanism) 3 are left over to determine the remaining 7 bending moments. Hence the structure is still 4 times statically indeterminate. As a rule it can be stated:

Rule: \( X = \) redundancies of original structure  
\[ M = \text{developed plastic hinges for given mechanism} \]  
\[ I = X - (M-1) = \text{remaining redundancies} \]  
\[ (7.1) \]

As illustration to rule,

\begin{align*}
\text{Determinate } & M = 1 \\
& I = 0 - (1-1) = 0 \\
X = 1, & M = 2; \ I = 0 \\
X = 2, & M = 3; \ I = 0 \\
X = 3, & M = 3; \ I = 1
\end{align*}

[Fig. 7.1]

For the present case:

\[
\begin{align*}
X &= 6 \\
M &= 3 \\
I &= X - (M-1) = 4
\end{align*}
\]
Using a moment-sign convention as shown in Fig. 7.2(a) the following expression can be written down:

Ultimate Load: \[ P_u = P_{II} = \frac{33}{5} \frac{M_p}{L} \] (7.2)

Developed Moments:
\[ M_8 = -3M_p \] (7.3)
\[ M_9 = 3M_p \] (7.4)
\[ M_{10} = -2M_p \] (7.5)
Equations of Equilibrium,

Beam 4-6 : \[ M_5 = \frac{1}{2} M_4 + \frac{1}{2} M_6 + \frac{5}{36} P_u L \] (7.6)

Joint 6-7-8 : \[ M_6 + M_7 - M_8 = 0 \] (7.7)

Sidesway : \[ M_1 + M_2 - M_3 - M_4 - M_7 + M_{10} + \frac{P_u}{2} L = 0 \] (7.8)

7.1 TRIAL AND ERROR METHOD

One approach is to guess "I" redundant moments (Eq. 7.1) and determine the remaining values from the equilibrium equations.

The 6 eqs. (7.3) to (7.8) relate 10 bending moments. For a solution a judicious choice of 4 moment values is made. Experience will cut down on the number of trials that must be made. The aim is to find a possible moment distribution which does
not violate the plasticity condition:

\[
\begin{align*}
M_1 &= -M_p \\
M_2 &= -M_p \\
M_3 &= +M_p \\
M_7 &= -2M_p \\
\end{align*}
\]

(7.9)

the remaining 3 moments are determined from Eqs. (7.5) to (7.7) and are

\[
\begin{align*}
M_6 &= -M_p \\
M_4 &= \frac{3}{10}M_p \\
M_5 &= \frac{17}{30}M_p \\
\end{align*}
\]

(7.10)

The corresponding bending moment diagram is shown in Fig. 7.2(b). Note that nowhere is the plasticity condition violated; hence the solution is a lower bound. Being already an upper bound it can only be the correct solution.

Very often the indeterminacy is only 1 or 2, such that a few trials will immediately lead to a result. If no distribution with \( |M| \leq M_p \) can be established, then the mechanism under consideration does not correspond to the actual ultimate load.

The presented procedure leads only in relatively simple cases to easy solutions. For more complicated cases it may be preferable to use a "method of inequalities" as applied by Neal and Symonds. (Journal Institution of Civil Engineers, Vol. 35, pp. 21-40, 1950-1951.)
7.2 MOMENT-BALANCING METHOD

For rectangular frame-works, having members intersecting at right angles (as in the present case) an equilibrium check can be made (Horne, English) that is much simpler than the use of method of inequalities. In short, the method is a moment balancing process, having some resemblance to moment distribution. Before taking up the example of the 2 span frame, the equilibrium of a single span beam is investigated, Fig. (7.3). Beam L-R is subjected to a central load P and end moments $M_L$ and $M_R$, taken positive when acting clock-wise (as shown). Moments within beam are taken positive when producing tension in lower fiber. Equilibrium for the load P is established by a simple beam moment diagram (b) and the reactions $\frac{P}{2}$. Super-imposing end moment $M_L$ does not disturb equilibrium of P, provided the reactions $\frac{M_L}{L}$ can be taken by the supports (moment diagram (c)). Similarly for $M_R$ (diagram (d)).

Result: On a given moment diagram in equilibrium, end moments can be superimposed without disturbing the equilibrium.

Following table gives influence of unit changes in moment (carry-over factors)

<table>
<thead>
<tr>
<th>$\Delta M_L$</th>
<th>$\Delta M_C$</th>
<th>$\Delta M_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Unit change of $\Delta M_L$ may change $M_C$ by $\frac{1}{2}M_R$ not at all. Or it it is desired not to change $M_C$, then $M_L = 1$ will require $M_R = 1$ etc. Change in reaction at right support is $\frac{1}{L}(\Delta M_L + \Delta M_R)$.

With this information the example of 2 span frame is investigated. Note that no consideration to continuity is given. All that is desired is a possible moment diagram in equilibrium with applied loads and fulfilling the plasticity condition $|M| \leq M_p$.

The sign convention is that positive end moments turn the end of the member clockwise, positive moments within beams produce tension in bottom fiber. In Fig. 7.4(a) a starting moment distribution is given. For load $P = 6.60 \frac{M_p}{L}$ corresponding to Mechanism II $M_8$, $M_9$, and $M_{10}$ have their respective plastic moment value $-3 M_p$, $+3 M_p$, $+2 M_p$, (Mechanism II). The loads on beam 4-6 produce a statically determinate bending moment of $\frac{5}{36} PL = \frac{11}{12} M_p$. The arbitrary assumption is made that the end moments $M_4$ and $M_6$ take half and $M_5$ the other half, hence $M_4 = -0.46 M_p$, $M_5 = +0.46 M_p$, and $M_6 = +0.46 M_p$. The column moments are determined from the horizontal (sidesway) equilibrium:

$$M_1 + M_2 + M_3 + M_4 + M_7 + M_{10} + \frac{P}{2}L = 0$$

(7.11)

Since columns 2-7 and 3-10 have twice the plastic moment value of column 1-4, it is assumed that

$$2 M_1 = 2 M_4 = M_2 = M_7 = M_3 = M_{10}$$

(7.12)
Replaced in (7.11) one gets

\[ M_1 = \frac{1}{2} M_2 = \ldots \ldots \ldots = -0.33 \frac{M_p}{L} \quad (7.13) \]
Except for the joints, the structure is in equilibrium. Joint 6-7-8 is balanced first. Considering that the plasticity condition should not be violated the only possibility to balance -3.20 is to add +2.66 to the column and +0.54 to the beam 4-6. The latter is carried over such that no change in $M_5$ occurs, hence $M_4 = +0.54$. A record is kept of the change in column moments in a separate table.

Next joint 10 is balanced by adding -1.34 to the column moment, nothing carried over. Adding +0.25 to the column moment $M_4$ balances joint 4.

As the changes in column moments affected the horizontal equilibrium, changes in $M_1$, $M_2$, and $M_3$ are required such that the sum of all changes $\sum \Delta M = 0$. Adding -0.50, -0.50, and 0.57 to $M_1$, $M_2$, and $M_3$ does not produce moments in excess of the plastic moments and results in $\sum M_i = 0$ (see table in Fig. 7.4(a)). The final moment values are obtained by adding up. As a routine check the equilibrium of the different parts is investigated.*

Beam 4-6: $M_5 = \frac{1}{2} M_4 - \frac{1}{2} M_6 + \frac{5}{36} PL = M_p (0.04-0.50+0.92) = 0.46 M_p$

Beam 8-10: $M_9 = \frac{1}{2} M_8 - \frac{1}{2} M_{10} + \frac{5}{6} PL = M_p (-1.50+1.00+5.50) = 3.00 M_p$

Joint 6-7-8: $M_6 + M_7 + M_8 = 0 = M_p (1.00+2.00-3.00) = 0$

Sidesway: $M_1 + M_2 + M_3 + M_4 + M_7 + M_{10} + \frac{PL}{2}$

$= M_p (-0.83-1.16-1.23-0.08+2.00-2.00+3.30) = 0$

* Note the difference in sign convention to Eq. 7.6 - 7.8.
The balancing process produced a bending moment distribution which is in equilibrium with the loads and which does not violate the plasticity condition $|M| \leq M_p$. Hence the load $P = 6.60 \frac{M_p}{L}$ is also a lower bound for the failure load and is therefore the correct failure load. The fact that the diagram shown in Fig. (7.4(b)) differs with Fig. (7.2(b)) is of no concern. Both are possible equilibrium solutions. Actual solution (precise moment values) is of no interest.

Remarks: The balancing process appears to be quite arbitrary, violating every rule known from the Cross Moment distribution. However it should be kept in mind that all conditions of continuity can be disregarded. A little routine is needed to apply the procedure.

Cases where no moment distribution can be found such that $|M| \leq M_p$ indicate that the assumed mechanism does not correspond to failure mechanism. The location of the new hinge becomes apparent during balancing process.
References


Lecture No. 8

CALCULATION OF DEFLECTIONS

SCOPE: Plastic design is concerned directly with load-carrying capacity. Since knowledge of deflections is sometimes desirable, this lecture presents a method of obtaining estimates. Idealization of M-Ø curve makes possible application of slope-deflection equations to estimate deflections beyond elastic limit and at ultimate load. Method applied to computation of required rotation capacity at hinges.

OUTLINE: 1. IMPORTANCE OF DEFLECTIONS
2. FUNDAMENTAL CONCEPTS
3. DEFLECTION AT ULTIMATE LOAD
4. DEFLECTION AT WORKING LOAD
5. ROTATION CAPACITY

8.1 IMPORTANCE OF DEFLECTIONS

Primary design requirement -- structure must carry the load
Secondary requirement -- it must not deform too much out of shape.

Problem of deflections is not critical to plastic design since a structure proportioned by plastic methods has restraining moments that are not present in conventional "simple beam" design. The frequent result is that "simple beam" deflection is usually greater than that of a structure designed by plastic methods.
Purpose of Deformation Analysis

1. Determine approximate magnitude of deflection at ultimate load

   Load factor of safety does not preclude the rare overload.

   What is corresponding deflection?

2. Estimate of deflection at working load

   In certain cases, design may limit deflection at $P_w$.

3. Research purposes

   Does a test structure behave as assumed in the theory?

   Preliminary to study of tolerable deflection limits.

   A check on hinge action (Rotation Capacity).

8.2 FUNDAMENTAL CONCEPTS

1. Assumptions and Conditions

   (In addition to the assumptions inherent to plastic analysis):

   (a) Idealized $M-\theta$ relationship (Fig. 2.12); consequently,

   (b) each span retains its flexural rigidity, EI, for the whole length between hinge sections,

   (c) unlimited rotation is possible at hinge sections ($M = M_p$).
An important principle:

Although "kinks" form at the other hinge sections, just as the structure attains the computed ultimate load there is continuity at that section at which the last plastic hinge forms.

Since the moments are everywhere known (Lecture #7), the slope-deflection equations may be used to solve for relative deflection of segments of the structure.

2. Limitations

None of the "Factors Affecting M-\(\phi\) Curve" (Art. 2.4) are considered in the analysis. Also ignored are:

(a) Catenary forces -- tend to decrease deflection and increase strength.

(b) Second-order effects -- tend to increase deflection and decrease strength.

Illustration:

Ignore: \(\Delta M = H \cdot \delta\)

* See, for example, Stage 2, Fig. 2.19.
3. Comment on Methods of Analysis

Deformation analysis is based on \( M-\theta \) curve \((\theta = \frac{M}{EI} \) in the elastic range. Eq. 2.6). Note, for example, the term \( \theta \) in the expressions, \( \theta_x = \int_0^x \theta \, dx \) and \( \delta_x = \int_0^x \theta \cdot x \, dx \).

The \( M-\theta \) Curve:

Effect of assumptions on \( \theta \)-diagram for simple beam:

Note: For a linear moment distribution, the \( \theta \)-diagram is a rotated plot of the \( M-\theta \) curve.

Assumption "2" simplifies calculations of \( \theta \) and \( \delta \).

Member is elastic except at hinge sections where "kinks" form.
Influence of assumptions on load-deflection curves:

![Diagram of simple beam](image)

Conclusion: The "Hinge Method" (2) gives reasonably precise approximation to load-deflection curve.

4. Slope-Deflection Equations

The following form of the slope-deflection equations will be used throughout (see Fig. 8.4 for nomenclature.

Clockwise M and θ are +)

\[
θ_A = θ_A' + \frac{Δ}{L} + \frac{L}{3EI} \left( M_{AB} - \frac{M_{BA}}{2} \right)
\]  

(8.1)
8.3 DEFLECTION AT ULTIMATE LOAD

The information needed for computing relative deflection of segments at ultimate load is now available, namely

(a) Ultimate load and moment diagram (from plastic analysis)
(b) Slope-deflection equations (Eq. 8.1)
(c) The principle of continuity at "last hinge" (see page 8.3)

But which is the last hinge to form? A somewhat complicated elastic-plastic analysis could be carried out to determine the step-by-step formation of hinges -- and thus the last hinge. However, a few examples will demonstrate that a simpler method is available; calculate the deflection on the assumption that each hinge, in turn, is the last to form. The result:

(d) The correct deflection at ultimate load is the maximum value obtained from the various trials.

1. Examples

EXAMPLE 8.1 (Fixed-ended beam, uniform vertical load.)

(a) Ultimate Load (Eq. 2.30)

\[ W_u = \frac{16M_p}{L} \]

(b) Moment Diagram and Mechanism:

Fig. 8.5 (From Fig. 2.19)
(c) Computation of Vertical and Deflection

**TRIAL AT SECTION 2** (Section 2 assumed as last hinge to form)

Free-body diagram: Fig. 8.6

Slope-deflection Equation for 2-1: \((\theta_2 = 0):\)

\[ \theta_2 = \theta_2' + \frac{\delta V_2}{l} + \frac{l}{3EI} (M_{21} - \frac{M_{12}}{2}) \]

\[ \theta_2' = \text{Simple beam end rotation}^* = -\frac{M_p L}{12EI} \]

\[ \delta V_2 = \text{Vertical deflection with continuity at Section 2 assumed at Section 2} \]

\[ 0 = -\frac{M_p L}{12EI} + \frac{\delta V_2}{L/2} + \frac{L/2}{3EI} (-M_p + \frac{M_p}{2}) \]

\[ \delta V_2 = + \frac{M_p L^2}{12EI} \]

**TRIAL AT SECTION 1** Even though it is obvious that last hinge forms at "2", what is the effect of incorrect assumption?

Free body:

Slope-deflection Equation for 1-2 \((\theta_1 = 0):\)

\[ \theta_1 = \theta_1' + \frac{\delta V_1}{l} + \frac{l}{3EI} (M_{12} - \frac{M_{21}}{2}) \]

\[ 0 = + \frac{M_p L}{12EI} + \frac{\delta V_1}{L/2} + \frac{L/2}{3EI} (-M_p + \frac{M_p}{2}) \]

\[ \delta V_1 = 0 \]

* See Ref 8.5 for tabulated values in terms of load.
Note: (1) A lesser deflection was obtained for the incorrect assumption.

(2) A "Kink" has been removed at Section 1 (compare with Fig. 8.5)

(3) A "negative" slope discontinuity has been created at Section 2.

By rigid-body rotation through the angle $\theta_2$ (Fig. 8.7), the kink is removed and correct deflection may be obtained:

Slope at Section 2 ($\theta_2$):

$$\theta_2 = \theta_2' + \frac{\delta v_1}{l} + \frac{l}{3EI} (M_{21} - \frac{M_{12}}{2})$$

$$\theta_2 = -\frac{M_p L}{12EI} + 0 + \frac{L/2}{3EI} (-M_p + \frac{M_p}{2}) = -\frac{M_p L}{6EI}$$

Rigid body rotation:

$$\delta V = \delta v_1 + \delta V'$$

$\delta v_1$ = deflection computed on basis of incorrect assumption = 0

$\delta V'$ = deflection due to rigid body rotation

(opposite to $\theta_2$)

$$= (-\theta)(l) = -\frac{L}{2} \theta$$

$$\delta V = -\frac{M_p L^2}{12EI} \uparrow k$$

This is the basis of an alternative method for computing the deflection.
EXAMPLE 8.2 (Fixed-ended beam, concentrated load off-center)

(a) Ultimate Load (By equilibrium, Fig. 8.8(b))

\[ P_u = \frac{9M_p}{L} \]

(b) Moment Diagram and Mechanism (Fig. 8.8)

(c) Computation of Vertical Deflection

**TRIAL AT SECTION 1** (Member 1-2, \( \theta_1 = 0 \))

Free-body: Fig. 8.9

Slope-deflection Eq.:

\[ \theta_1 = \theta_1' + \frac{\delta v_1}{L} + \frac{1}{3EI}(M_{12} - \frac{M_{21}}{2}) \]

\[ 0 = \frac{\delta v_1}{L} + \frac{L}{3} \left( -\frac{M_p}{3EI} + \frac{M_p}{2} \right) \]

\[ \delta v_1 = \frac{M_pL^2}{54EI} \]

**TRIAL AT SECTION 2** (Continuity assumed at Section 2: \( \theta_{21} = \theta_{23} \))

Free-body: Fig. 8.10

Slope-deflection Eqs. for 2-1 and 3-2:

\[ \theta_{21} = \frac{-\delta v_2}{L} + \frac{L}{3} \left( -\frac{M_p}{3EI} + \frac{M_p}{2} \right) = \frac{3\delta v_2}{L} - \frac{M_p L}{18EI} \]

\[ \theta_{23} = \frac{-\delta v_2 + 2L/3}{3EI} \left( M_p - \frac{M_p}{2} \right) = \frac{-3\delta v_2}{2L} + \frac{M_p L}{9EI} \]

\[ \theta_{21} = \theta_{23} \]

\[ \delta v_2 = \frac{M_pL^2}{27EI} \]
TRIAL AT SECTION 3  (Continuity at Section 3)

Free-body: Fig. 8.11

Slope-deflection Eqs. for 
3-2 (θ₃ = 0):

\[ θ₃ = \frac{δv₃}{l} + \frac{l}{3EI} \left( M_{32} - \frac{M_{23}}{2} \right) \]

\[ 0 = \frac{δv₃}{2l/3} + \frac{2l/3}{3EI} \left( M_p - \frac{M_p}{2} \right) \]

\[ δv₃ = + \frac{2}{27} \frac{M_pL^2}{EI} \]

The maximum deflection occurs when last hinge is assumed to form at Section 3. All other assumptions result in "negative kinks" (which, by comparison with correct mechanism, are impossible) and produce smaller deflections.

2. Conclusions and Summary of Procedure

Conclusions: (a) A wrong assumption as to which hinge forms last introduces a "negative kink" -- and thus a lesser deflection.

(b) The largest (and correct) value is obtained when the correct assumption is made as to the last hinge.

(c) The correct deflection may be determined from a wrong assumption through mechanism motion.
Summary of Procedure:

(1) Obtain the ultimate load, the corresponding moment diagram and the mechanism (from plastic analysis).

(2) Compute the deflection of the various frame segments assuming, in turn, that each hinge is the last to form
   (a) Draw free-body diagram of segment.
   (b) Solve slope-deflection equation for assumed condition of continuity.

(3) Correct deflection is the largest value (corresponds to last plastic hinge).

(4) A check: From a deflection calculation based on an arbitrary assumption, compute the "kinks" formed due to the incorrect assumption. Remove the "kinks" by mechanism motion and obtain correct deflection. (This is also an alternative procedure.)

3. Further Example  (Rectangular Portal Frame, Fixed Bases)

EXAMPLE 8.3 (Fig. 8.12)

(a) **Ultimate Load** (by plastic analysis)

\[ P_u = \frac{6 M_p}{L} \]

(b) **Moment Diagram and Mechanism**  Fig. 8.12(b),(c)

(c) **Free-body Diagrams**  Fig. 8.12(d)
(d) Computation of Vertical Deflection

**RATIO OF $\delta_H$ AND $\delta_V$:**

Continuity at Section 2,
\[
\theta_{23} = \theta_{21}
\]
\[
\theta_A = \theta'_A + \frac{\delta v_1}{l} + \frac{l}{3EI} (M_{AB} - \frac{M_{BA}}{2})
\]
\[
\theta_{23} = 0 + \frac{\delta v_2}{l/2} + \frac{l/2}{3EI} (0 + \frac{M_p}{2})
\]
\[
\theta_{23} = \frac{2\delta v_2}{L} + \frac{M_p L}{12EI}
\]
\[
\theta_{21} = 0 + \frac{\delta h_2}{l/2} + \frac{l/2}{3EI} (0 + \frac{M_p}{2})
\]
\[
\theta_{21} = \frac{2\delta h_2}{L} + \frac{M_p L}{12EI}
\]
\[
\frac{2\delta v_2}{L} + \frac{M_p L}{12EI} = \frac{2\delta h_2}{L} + \frac{M_p L}{12EI}
\]
\[
\delta v = \delta h
\]

**TRIAL AT SECTION 1**: Member 1-2, $\theta_1 = 0$
\[
0 = 0 + \frac{\delta h_1}{L/2} + \frac{L/2}{3EI} (-M_p + 0)
\]
\[
\delta h_1 = \frac{M_p L^2}{12EI}
\]
\[
\delta v_1 = \frac{M_p L^2}{12EI}
\]
(8.3)

\textbf{TRIAL AT SECTION 3} : \( \theta_{32} = \theta_{34} \)

\[
\theta_{32} = 0 + \frac{\delta V_3}{L/2} + \frac{L/2}{3EI} (-M_p + 0) = \frac{2\delta V_3}{L} - \frac{M_p L}{6EI}
\]

\[
\theta_{34} = 0 - \frac{\delta V_3}{L/2} + \frac{L/2}{3EI} (M_p - M_p/2) = -\frac{2\delta V_3}{L} + \frac{M_p L}{12EI}
\]

\[\delta V_3 = \frac{M_p L^2}{16EI}\]

\textbf{TRIAL AT SECTION 4} : Similar procedure using \( \theta_{43} = \theta_{45} \)

\[\delta V_4 = \frac{M_p L^2}{24EI}\]

\textbf{TRIAL AT SECTION 5} : Similar procedure using \( \theta_5 = 0 \)

\[\delta V_5 = \frac{M_p L^2}{24EI}\]

Correct Answer = \( \frac{M_p L^2}{12EI} \) (Last hinge at Section 1)

8.4 DEFLECTION AT WORKING LOAD

1. Beam Deflections at Working Load

Usually the structure will be "elastic" at working load. But it is desirable to avoid the elastic analysis, if at all possible. For certain standard cases of loading and restraint, solutions are already available (AISC handbook, for example).

The Method: (1) Divide computed ultimate load by \( F \), the load factor of safety.

(2) Solve for working load deflection from tables.
EXAMPLE 8.4 (Fixed-ended beam, uniform load)

From Fig. 8.13 and Eq. (2.30) :

\[ W_w = \frac{W_u}{F} \]

\[ W_w = \left(\frac{16M_p}{L}\right)\left(\frac{1}{1.88}\right) = 8.5 \frac{M_p}{L} \]

Note: \( W_y = 12 \frac{M_p}{L} \)

From tables,

\[ \delta = \frac{WL^3}{384EI} \quad (8.2) \]

\[ \delta_w = 0.022 \frac{M_pL^2}{EI} \]

Note: When end restraint conditions are not known, they may often be estimated and the above technique employed.

2. A Crude Approximation ("Last Hinge" Approximation)

As illustrated in Fig. 8.13 (dashed line) a crude approximation may be obtained from:

\[ \delta_w' = \delta_u/F \quad (8.3) \]

The error will be greater than 100%; but it gives upper limit to \( \delta_w \) and indicates when more refined calculations are necessary.

3. General Nature of Load-Deflection Curves

The "Hinge Method" provides a convenient means for visualizing (and calculating) load-deflection relationships.
EXAMPLE 8.5 (Fixed-ended beam of Example 8.2)

Consider the structure as load is gradually applied and just prior to formation of each hinge:

Phase 1 (0-A) - Elastic - Represents slope of deflection curve of structure (a)

Phase 2 (A-B) - Represents slope of deflection curve of structure (b)

Phase 3 (B-C) - Represents slope of deflection curve of cantilever (structure (C))

Phase 4 (C-D) - Mechanism

EXAMPLE 8.6 (Fixed-ended, uniformly-loaded beam)

From Eqs. (2.29) and (8.2), the "elastic limit" deflection, \( \delta_y' \), is

\[
\delta_y' = \frac{M_p L^2}{32EI} \tag{8.4}
\]

Above the yield load, the slope of the load-deflection curve is the same as that of a simple beam,

\[
\delta = \frac{5(\Delta W)L^3}{384EI} \tag{8.5}
\]

By comparison with Eq. 8.2 it is seen that the slope of portion AB (Fig. 8.13) is 1/5th of the portion OA.
The total deflection, from Fig. 8.13 is given by

\[ \delta_u = \delta_y' + \Delta W \delta_{AB} \]

Where

\[ \Delta W = W_u - W_y = \frac{1}{3} (W_y) \quad \text{[See Eq. 2.31]} \]

Thus

\[ \delta_u = \frac{M_p L^2}{32EI} + \frac{1}{3} \left( \frac{12M_p}{L} \right) \left( \frac{5L^3}{384EI} \right) \]

\[ \delta_u = \frac{M_p L^2}{12EI} \quad \text{[Checks with Example 8.1]} \]

Conclusion:

1. Each portion of the curve represents the P-\(\delta\) curve of a "new" structure containing one less redundant than previous one.

2. Deflection curve may be computed by determining corresponding deflection increments.

8.5 ROTATION CAPACITY

1. Definition: Rotation capacity, \( R \), is the ability of a structural element to absorb rotations at near-maximum moment after reaching the hinge condition. It is expressed as a ratio of average unit rotation, \( \varphi_A \), to \( \varphi \) at yield.

From Fig. 8.15

\[ R = \frac{\varphi_A}{\varphi_y} \quad \text{(8.6)} \]

or

\[ R = \frac{\varphi_B}{\varphi_p} \quad \text{(8.7)} \]
Closely associated is the **Hinge Rotation**: the rotation, \( \phi_h \), required at a plastic hinge in order to realize the computed ultimate load.

### 2. Importance:

(a) The plastic moment must be maintained at the first hinge to form while hinges are forming elsewhere. See Fig. 2.19, for example.

(b) Factors that may reduce the rotation capacity of a joint or section: Local and lateral buckling, general instability, fracture.

(c) Computations of "Hinge Rotation" are normally not required in design since rules of practice will assure that structural joints possess adequate "Rotation Capacity".

**Note:** Selecting as \( \phi_A \) the value \( \phi_{st} \) as the maximum probable requirement, then from Eq. 8.6.

\[
R_{\text{max}} = \frac{\phi_{st}}{\phi_y} \approx 12 \quad (8.8)
\]
What is the "Hinge Rotation" required at Section 1 of the fixed-ended beam shown. (Example 8.2)

Plastic Rotation at Joint 1:
The first hinge to form is at Section 1 and thus will rotate the most. From Eq. 8.1,

\[ \Delta \theta_1 = \theta_{12} = \theta_{12} + \frac{\delta V}{d} + \frac{d}{3EI} (M_{12} - \frac{M_{21}}{2}) \]

Example 8.2:

\[ \delta V = \frac{2MpL^2}{27EI} \]

\[ \Delta \theta_1 = \left( \frac{2MpL^2}{27EI} \right) \frac{1}{L/3} + \frac{L/3}{3EI} (-M_p + \frac{M_p}{2}) \]

\[ \Delta \theta_1 = \frac{M_pL}{6EI} \]

Hinge Length:

Above value assumes all rotation at a point. Actually it extends over a short length of the beam.

Assume: \( \Delta L = \) length of beam in which \( M > M_y \). (See Fig. 2.17)

\[ \Delta L = \frac{d}{16} = \frac{L}{48} \]

Average Unit Rotation:

\[ \phi_B = \text{Plastic deformation} + \text{unit angle change up to elastic limit} \]

\[ \phi_B = \frac{\Delta \theta_1}{\Delta L} + \frac{\Delta L \phi_p}{\Delta L} = \frac{8M_p}{EI} + \phi_p \]

\[ \phi_B = \frac{\Delta \theta_1}{\Delta L} + \phi_p \]  

(8.9)
Hinge Rotation:

\[ H = \frac{\phi_B}{\phi_p} = \frac{8M_p}{E_I} + 1 \]

\[ H = 9 \]

References

8.1 Symonds, P. S. "RECENT PROGRESS IN THE PLASTIC METHODS, ETC." Journal of Franklin Institute, 252(5), 1951.


Lecture No. 9

MODIFICATIONS TO
SIMPLE PLASTIC THEORY

SCOPE: Reduction of plastic moment due to axial load and/or shear. Possibility of progressive deformation under repeated loading. Instability problems such as local and lateral buckling.

While the simple plastic theory offers a satisfactory explanation of the observed ultimate strength behavior of proportionately loaded mild steel beams, there are several factors that it does not directly take into account. Several of these will now be considered.

1. INFLUENCE OF AXIAL THRUST
   (a) Reduction in $M_p$
   (b) Stability

2. INFLUENCE OF SHEAR

3. PROBLEM OF LOCAL BUCKLING OF FLANGES

4. PROBLEM OF LATERAL BUCKLING

5. POSSIBILITY OF PROGRESSIVE DEFORMATIONS UNDER VARYING LOADS.

9.1 INFLUENCE OF AXIAL THRUST

For generality, the section shown in Fig. 9.1 will be considered. (Note that if $A_F = 0$ the section is a rectangle.)
The following will be assumed:

(a) Plane sections remain plane
(b) Idealized $\sigma - \epsilon$ relationship (see Fig. 9.2)
(c) Small deflections, i.e. $\tan \phi = \phi$
(d) Equilibrium (from stress distribution)

\[ P = \int_A \sigma \, dA \quad (9.1) \]
\[ M = \int_A \sigma y \, dA \quad (9.2) \]

1. Reduction in $M_p$ (Influence of axial thrust on plastic moment capacity)

If a member is subjected to both an axial thrust and a bending moment, the progressive change in stress distribution across a section as these loads are increased will more than likely be of the form shown in Fig. 9.3.
Since a "plastic hinge condition" is dependent on an infinite \( \phi \) value, the MODIFIED PLASTIC MOMENT, \( M_{pc} \), (modified in that it includes the influence of axial thrust) will be determined from a consideration of stress distributions (d) of Fig. 9.3.

For this fully plastic stress distribution, two possibilities exist:

**CASE I** - Neutral axis in Web  
**CASE II** - Neutral axis in Flange

**CASE I: Neutral Axis in Web**  \((2y_o \leq d_w)\)

![Diagram](image)

The assumed stress distribution is shown in Fig. 9.4(a); however, for ease of computation this has been divided into the two parts shown in Fig. 9.4(b) and (c). These have been chosen such that the first, or (b) distribution, supplies the axial thrust resistance and the (c) distribution supplies the bending moment resistance. Therefore from Equation 9.1 (and stress distribution (b))

\[
P = \int_A \sigma \, dA = \sigma_y \, (2wy_o)
\]

Since

\[
P_Y = \sigma_y \, (A_F + A_W)
\]

\[
\frac{P}{P_Y} = \frac{2y_o}{d_w \left(1 + \frac{A_F}{A_W}\right)}
\]  

(9.3)
From Eq. (9.2) and stress distribution (c) of Fig. 9.4

\[ M_{pc} = \int_A \sigma y \, dA = \sigma_y \left( \frac{1}{2} A_F d_F + \frac{1}{4} A_w d_w - b y_o^2 \right) \]

Therefore since

\[ M_p = \sigma_y \left( \frac{1}{2} A_F d_F + \frac{1}{4} A_w d_w \right) \]

\[ \frac{M_{pc}}{M_p} = 1 - \frac{4 y_o^2}{d_w^2 (1 + 2 \frac{A_F d_F}{A_w d_w})} \] (9.4)

Solving equation (9.3) for \( y_o \) and substituting in Eq. (9.4) gives

\[ \frac{M_{pc}}{M_p} = 1 - \left[ \frac{P}{P_y} (1 + \frac{A_F}{A_w}) \right]^2 \frac{1}{(1 + 2 \frac{A_F d_F}{A_w d_w})} \] (9.5)

CASE II - Neutral Axis in Flange (i.e. \( \Delta \leq t \))

From distribution (b) Fig. 9.5 and Eq. (9.1)

\[ P = \sigma_y \left[ A_w + A_F \left( 1 - \frac{2 \Delta}{d - d_w} \right) \right], \text{ and since } P_y = \sigma_y [A_w + A_F] \]

\[ \frac{P}{P_y} = 1 + \frac{A_F}{A_w} \left( 1 - \frac{2 \Delta}{d - d_w} \right) \]

\[ \frac{P}{P_y} = 1 + \frac{A_F}{A_w} \left( 1 - \frac{2 \Delta}{d - d_w} \right) \] (9.6)
From Eq. 9.2 and distribution (c) of Fig. 9.5

\[ M_{pc} = \left( \frac{A_F}{2} \cdot \frac{2\Delta}{d-w} \right) (d - \Delta) \quad \text{and} \quad M_p = \sigma_y \left[ \frac{A_F}{2} \left( \frac{d+d_w}{2} \right) + \frac{1}{4} A_w d_w \right] \]

or

\[ \frac{M_{pc}}{M_p} = \frac{2\Delta}{d-w} \left( \frac{d - \Delta}{d} \right) \]

Neglecting \( \Delta \) in the expression \( d - \Delta \) (Resulting \( M_{pc}/M_p \) will be slightly larger than the correct value)

Eq. 9.7 gives

\[ \frac{M_{pc}}{M_p} = \frac{2\Delta}{d-w} \left[ \frac{2}{1 + d_w + A_w d_w / A_F} \right] \]  

Substituting value for \( \Delta \) from Eq. 9.6 in Eq. 9.8 gives the following interaction curve equation

\[ \frac{M_{pc}}{M_p} = \frac{2d}{1 + (1 + \frac{d}{d_w}) \frac{A_F}{A_w}} \left[ \frac{A_F}{A_w} - \left( \frac{P}{P_Y} (1 + \frac{A_F}{A_w}) - 1 \right) \right] \]

For general considerations, interaction equations 9.6 and 9.9 are best suited. For individual section computation, however, equations can be written in the following more usable form.

CASE I:

\[ \frac{M_{pc}}{M_p} = 1 - \frac{A^2}{4w} \left( \frac{P}{P_Y} \right)^2 \]

CASE II:

\[ \frac{M_{pc}}{M_p} = \frac{A^2}{bZ} \left[ t - (1 - \frac{P}{P_Y}) \right] \left[ d - t + (1 - \frac{P}{P_Y}) \right] \]

In this discussion Eqs. (9.6) and (9.9) will be used.
For most rolled WF sections the average values of $\frac{dF}{d_w}$ and $\frac{d}{d_w}$ are

$$\frac{dF}{d_w} = 1.05 \quad \text{and} \quad \frac{d}{d_w} = 1.10$$

Substituting these values in equations (9.5) and (9.9), the following curves result.

![Diagram]

Fig. 9.6
These curves are used in the following manner:

Given that \( P = 45 \) kips
\( M = 1100 \) inch kips,
determine the lightest section capable of sustaining the load.

1. Neglecting the influence of \( P \) and assuming \( \sigma_y = 33 \) ksi.
\[
M_p = (1.14) S \sigma_y = 1100 \text{ in. kips}
\]
or
\[
S = \frac{1100}{(1.14)(33)} = 29.2 \text{ in.}^3
\]

2. Tentatively select 12WF27 (lightest section in Section Modulus Table)
\[
A = 7.97 \text{ in.}^2, \quad S = 34.1 \text{ in.}^3
\]
\[
\frac{A_F}{A_w} = 1.94
\]

3. For the assumed member
\[
\frac{P}{P_y} = \frac{45}{(7.97)(33)} = \frac{45}{263} = 0.17
\]

4. From Fig. 9.6 for this value of \( P/P_y \)
\[
\frac{M_{pc}}{M_p} = 0.95
\]
which means section can only deliver 95% of its full plastic moment.

5. Therefore,
\[
S_{req.} = \frac{29.2}{0.95} = 30.8 \text{ in.}^3
\]
and tentatively selected member is okay.
2. Stability (Axial Thrust Plus Bending)

While Fig. 9.6 is a plot of the internal ability of a section to sustain combinations of thrust and moment at a fully plastic stress condition it is necessary that internal stiffness be related to external moments. In so doing, deflection (and thereby length) enters the problem, since the moment at any section will be composed of two parts: one independent of and the other dependent on deflections.

It should be pointed out that tests results have shown that when the maximum moment along a member occurs at its end, strengths can be predicted comparatively well from Fig. 9.6. For the case where maximum moment does not occur at the end of the member the possibility of instability must be considered.

For illustration consider the beam-column loaded as shown in Fig. 9.7. The moment at any section along the column is given by the equation

\[ M_x = M_e + P \delta \]  \hspace{1cm} (9.12)

The most critically deformed section is at the centerline where Eq. 9.12 becomes

\[ M_m = M_e + P \delta \]  \hspace{1cm} (9.13)
If it is postulated that the behavior of the member is governed by what happens at this one critical section (i.e. assuming a deflection curve based only on the parameters length and centerline deflection), then by assuming strain distributions across the centerline section (and thereby stress distributions) expressions can be derived which relate the quantities, P, $M_e$ and $\delta$.

For example consider Fig. 9.8, which shows three different stages of loading of the same member. Also sketched are the corresponding stress and strain distribution patterns at the centerline section. For each of these positions the centerline moment, $M_m$, and thrust, P, can be computed from the stress distribution and Eqs. 9.1 and 9.2. Curvature, $\phi_m$, can be computed from a geometric consideration of the stress distribution.

![Diagram of member with different stages of loading](image)

**Assumed strain Distributions at centerline**

**Resulting stress Distributions at centerline**

Fig. 9.8
Curvature, however, can also be related to the deflection, since
\[ \frac{d^2 y}{d x^2} = 0 \]

If, for example, it were assumed that the member deformed according to the equation
\[ y = \delta \cos \frac{\pi x}{L} \]
then
\[ \phi_m = \delta \frac{\pi^2}{L^2} \] (9.15)

Moment at the end, \( M_e \), can also be related to the center-line deflection since
\[ M_m = M_e + P\delta = M_e + P \frac{\pi^2}{L^2} \phi_m \] (9.16)

Rearranging terms,
\[ M_e = M_m - P \frac{\pi^2}{L^2} \phi_m \] (9.17)

where all quantities are known but \( M_e \). Assuming that the problem were chosen such that \( P \) was held constant at some given value and \( M_e \) was increased to collapse. Eq. 9.17 could then be plotted as shown in Fig. 9.9 (note that \( M_e \) has been plotted versus \( \delta \)). It's obvious that \( (M_e)_{cr} \) corresponds to the maximum point of this curve where
\[ \frac{d M_e}{d \delta} = 0 \] (9.18)
Jezek (see Ref. 9.4) uses this procedure and arrived at the following equations for rectangular cross-sections (these are rewritten in the notation of this section).

\[
\sigma_t \leq |\sigma_y|
\]

\[
\left[ \frac{P}{P_{cr}} \right] = \frac{\pi^2 E bh}{(L)^2} \left[ \frac{P}{P_{cr}} - 1 \right] - \frac{2M_e}{E - 1}
\]

\[
\left[ \frac{P}{P_{cr}} \right] = \frac{\pi^2 E bh}{(L)^2} \left[ \frac{P}{P_{cr}} - 1 \right] - \frac{2M_e}{E - 1}
\]

While this type of solution is readily adaptable to the rectangular section, computations for the wide-flange shape becomes rather involved and other means prove more workable. (Ref. 9.5). These are based on the same concept but involve a semi-graphical determination of certain of the inter-relationships. Solution can also be made to include the influence of residual stresses.
A slightly different means of arriving at the same end result is given in Ref. 9.6. Therein a means is developed where incipient instability is determined by equating internal stiffness to external change in moment associated with a virtual displacement of the loaded member. Using such a procedure, the following curves were obtained for the 8WF31 section bent about its strong axis. The yield point stress was assumed as 33,000 psi.

![Diagram showing the relationship between load and moment for the 8WF31 section.](Fig. 9.12)
To illustrate how these curves are used, consider the previous example, where

\[ P = 45 \text{ kips}, \quad M = 1100 \text{ in. kips}, \]

but now assume that the member is bent in single curvature and has a length

\[ L = 20 \text{ ft}. \]

1. Neglecting influence of \( P \) and \( L \), a 12WF27 is tentatively selected (\( S_{\text{req.}} = 29.2 \text{ in.}^3, S_{\text{supplied}} = 34.1 \text{ in.}^3 \)).

2. For this section the following are computed

\[ \frac{P}{P_y} = 0.17, \quad \frac{L}{r} = 47.4. \]

3. From Fig. 9.12

\[ \frac{M_e}{M_p} = 0.83 \]

4. Therefore

\[ S_{\text{req.}} = \frac{29.2}{0.83} = 35.2 \text{ in.}^3 \]

(12WF27 is therefore not strong enough.)

5. Select 14WF30. \( P/P_y = 15.5 \) and \( L/r = 41.8 \)

6. From Fig. 9.12 \( M_e/M_p = 0.86 \)

7. Therefore

\[ S_{\text{req.}} = \frac{29.2}{0.86} = 34.0 \text{ in.}^3 \quad (S_{\text{supplied}} = 41.8 \text{ in.}^3) \]

and the 14WF30 section is okay.
9.2 INFLUENCE OF SHEAR (or PLASTIC MOMENT)

When discussing the influence of axial thrust - an "ultimate strength" solution could be formulated since both bending and thrust cause normal stresses in the same direction. Moreover, from equilibrium consideration, the influence of each of these conditions could be separated. Such is not the case when considering shear versus bending moment and present theories of plasticity are not sufficiently advanced to allow a direct solution of this problem. It is possible, however, by solving certain simplified problems, to obtain a clearer insight into the behavior of members subjected to this condition of loading.

The problem to be considered is that shown in Figs. 9.13 and 9.14. The stress distribution at section A-A will be examined and it will be assumed that yielding has penetrated into the web of the section. The stress-distribution is then as shown in Fig. 9.15.

A solution will be obtained for the case where \( X = L \) and the principal stress (due to both normal
and shearing stresses) reaches a critical value.

Assumptions

1. Plane Stress Problem
2. Only Elastic Part of Section Carried Shearing Forces.*
3. Maximum Shearing Stress Criterion
   For Yielding (also Mises' Yield Condition)
4. Equilibrium
   (Note: Solution does not consider compatibility)

From assumption (1)

$$\sigma_Z = \tau_{ZX} = \tau_{ZY} = 0$$

Therefore condition (4), Equilibrium, gives

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \quad (9.19)$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad (9.20)$$

Summing moments at section A-A of Fig. 9.13

$$M_A = P \cdot X \quad (9.21)$$

From Eq. 9.2 \((M = \int_A \sigma_y dA)\), and Fig. 9.15, the moment at A will equal

$$M_A = M_p - \sigma_y \frac{w y_o^2}{3} \quad (9.22)$$

* This is proven in Refs. 9.7 and 9.8.
From Eqs. 9.21 and 9.22

\[ y_o^2 = -\frac{3}{\sigma_y w} (P X - M_p) \]  \hspace{1cm} (9.23)

Also from Fig. 9.15 it is noted that

\[ \sigma_X = \sigma_y \left( \frac{y}{y_o} \right) \]  \hspace{1cm} (Note: \( \sigma_y \) = yield point stress)  \hspace{1cm} (9.24)

Differentiating Eq. 9.24

\[ \frac{\partial}{\partial X} (\sigma_X) = \sigma_y \left[ -\frac{y_o'y}{y_o^2} \right] \]

which when substituted in Eq. (9.19) gives

\[ \frac{\partial \tau}{\partial Y} = \sigma_y \left[ -\frac{y_o'y}{y_o^2} \right] \]  \hspace{1cm} (9.25)

Integrating Eq. 9.25 between the limits \(+y\) and \(-y_o\)

\[ \tau = \frac{\sigma_y}{2} \cdot y_o' (\eta^2 - 1) \]  \hspace{1cm} (9.26)

where \( \eta = \frac{y}{y_o} \)

Equation (9.23) is then differentiated with respect to X

\[ 2y_o' y_o = -\frac{3}{w} \frac{P}{\sigma_y} \]

or

\[ y_o' = -\frac{3}{2w y_o} \frac{P}{\sigma_y} \]  \hspace{1cm} (9.27)

Substituting Eq. (9.27) into Eq. (9.26)

\[ \tau = \frac{3}{4} \frac{P}{w y_o} \left[ 1 - \eta^2 \right] \]  \hspace{1cm} (9.28)
Using a maximum shearing stress theory (and assuming that $\sigma_y = 0$) $\tau_{\text{max.}}$ will occur when $\eta = 0$. Since $\tau_{\text{max.}} = \frac{\sigma_y}{3}$

\[
\frac{3}{4} \frac{P}{WY_o} = \frac{\sigma_y}{3}
\]

or

\[
Y_o = \frac{3\sqrt{3}}{4w\sigma_y} \quad (9.29)
\]

Substituting this value back into Eq. (9.23) gives

\[
0.563 \frac{P^2}{w\sigma_y} = M_p - PX \quad (9.30)
\]

For the case where $X = L$

\[
0.563 \frac{P^2}{w\sigma_y} = M_p \left[ 1 - \frac{PL}{M_p} \right] \quad (9.31)
\]

Since $PL$ is the applied moment at the wall $PL/M_p$ is the reduction factor due to shear; i.e.

\[
\frac{PL}{M_p} = \frac{M_{ps}}{M_p} \quad (9.32)
\]

Considering the section shown in Fig. 9.1,

\[
M_p = \left[ \frac{A_F d_F}{2} + \frac{A_W d_W}{4} \right] \sigma_y \quad (9.33)
\]

Substituting this value into Eq. 9.31 and simplifying gives

\[
0.563 \left( \frac{M_{ps}}{M_p} \right)^2 \left( \frac{d_W}{2L} \right)^2 \left[ 1 + 2 \frac{A_F d_F}{A_W d_W} \right] + \left( \frac{M_{ps}}{M_p} \right) - 1 = 0 \quad (9.34)
\]

which can be solved for various values of $\frac{A_F}{A_W}$ and $d_W/L$. 
Solution will be of the form

\[
\frac{M_{ps}}{M_p} = \frac{B}{2} \left[ -1 + \sqrt{1 + \frac{4}{B}} \right]
\]

where \( B = \left[ \frac{1}{0.563(dw)^2(1 + 2AfFdF)} \right] \)

and for convenience can be expanded in a Maclaurin series of the following type

\[
\frac{M_{ps}}{M_p} = \left[ 1 - \frac{1}{B} + \frac{2}{B^2} - \frac{4}{B^3} + \frac{8}{B^4} + \cdots \right]
\]  

(9.35)

Instead of using the maximum shearing stress theory, Mises' yield condition, \( J_2 = k^2 \), could have been assumed.

This then would give

\[
\sigma_y^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau^2
\]

or, on substituting in values for the stresses,

\[
1 = \frac{\gamma^2}{4} + \frac{\gamma_o^4}{4} \eta^2 (\eta^2 - 1)^2 - \frac{\gamma_o^2}{2} \eta^2 (\eta^2 - 1) + \frac{3\gamma_o^2}{4} (\eta^2 - 1)^2
\]

Since a direct solution of this equation for assumed yield at certain points of the web will be of the form

\[
y_o' = \left( \cdots \right)
\]

it is first of all necessary to determine the minimum value of \( y_o' \) (and thereby have the first possibility of yielding). A plot of \( \eta \)-versus \( y_o' \) is shown in Fig. 9.16 and shows that the value is almost constant over a large range of \( \eta \) indicating
that once yielding starts there will not be much reserve in strength.

Using the value $y_0' = 1.325$

$$0.566 \left( \frac{M_{ps}}{M_p} \right)^{2} \left( \frac{d_{w}}{2L} \right)^{2} \left( 1 + 2 \frac{A_{pw}d_{w}}{A_{w}d_{w}} \right) + \left( \frac{M_{ps}}{M_p} \right) - 1 = 0 \quad (9.36)$$

Assuming as before that

$$\frac{d_{p}}{d_{w}} = 1.05 \text{ and } \frac{d_{l}}{d_{w}} = 1.10$$

the curves shown in Fig. (9.17) are obtained.
The limitation on Eq. 9.36 is that yielding was assumed to have penetrated through flange into web. For the case where penetration just fills this condition

\[
\left( \frac{M_{ps}}{M_p} \right) = \frac{1}{6} + \frac{A_{FdF}}{A_{wdw}}
\]

(9.37)

One other condition however, can be determined -- that where the flanges take all the bending and the web takes all the shear. For this case

\[
M_{ps} = M_p - \sigma_y \frac{1}{4} A_{wdw} = PL
\]

(9.38)

and

\[
P = V = A_w \sigma_y \sqrt{3}
\]

(9.39)

These give

\[
\frac{L}{d_w} = \frac{\sqrt{3} \frac{M_{ps}}{M_p}}{4 \left( 1 - \frac{M_{ps}}{M_p} \right)}
\]

(9.40)

where

\[
\frac{M_{ps}}{M_p} = \frac{A_{FdF}}{2 + \frac{A_{FdF}}{A_{wdw}}}
\]

(9.41)

In obtaining the curves shown in Fig. 9.17, only the magnitude of the moment and shear at the wall section were considered. Therefore this figure may be used for other conditions of loading. Consider, for example, the continuous
beam shown in Fig. 9.18(a), whose failure mechanism is shown in Fig. 9.18(b). Isolating the segment BC the free body diagram is as shown in Fig. (c). (Note that since point C is a point of maximum moment, shear will pass through zero at this section.) Drawing the moment diagram it is seen that a point of zero moment occurs midway along the segment. The equivalent cantilever is then as shown in Fig. 9.18(e) with $a$ in Fig. 9.18(e) equaling $n/2$, or

$$ n = 2a $$

Design using these curves would be carried out in a manner similar to that described earlier for the axial load cases.

Note: Since combination of high shear and high moment can only occur where the moment gradient is high, yielding will be restricted and the section may go into strain-hardening, thus increasing its ultimate moment capacity. It is therefore not expected that shear will have too
great an influence on the actual maximum bending strength of beams. The problem does become serious, however, in regions of constant shear where the web may be subjected to shearing stresses near the yield value.

9.3 INFLUENCE OF LOCAL BUCKLING OF FLANGES

It was assumed in all previous lectures that the cross-section kept its original geometrical shape regardless of the amount of deformation to which it was subjected. Obviously this is not possible. The question for discussion then is two-fold:

1. How much rotation is required to develop the necessary hinges, and

2. What geometric proportions of member segments can supply this amount of rotation without the occurrence of local flange buckling.

Considering the first of these, it is evident that this value will change for each structure and each loading condition considered. Since for most typical cases the required hinge rotation is below the ratio of $\epsilon_{st}/\epsilon_y$, if a flange can be strained to strain-hardening without local buckling, then it is satisfactory for plastic hinge action. (The advantage of using this procedure is that in the strain-hardening range properties are again linear as in the elastic range.)
Regarding the second question, the determination of the necessary geometry of the section such that this condition can be realized, the solution of this problem is dependent on the acceptance of one of the presently available theories of plasticity. Since there is at this time apparent disagreement between these various solutions, a detailed discussion will not here be attempted other than to point out the basic differences. Each has essentially been based on the following differential equation:

\[ D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = t \sigma_x \frac{\partial^2 w}{\partial x^2} \]  

(9.42)

where

\[ D_x = \frac{E_{tx} I}{1 - \nu_x \nu_y} \]

\[ D_y = \frac{E_{ty} I}{1 - \nu_x \nu_y} \]  

(9.43)

and

\[ 2H = \nu_y D_x + \nu_x D_y + 4 G_t I \]

The main differences develop at this point with regard to the assumption of the inter-relationship between the basic material properties. These have been summarized in the following chart for several of the presently available theories. (From Ref. 9.9).
Rather than discuss the merits of any of these, consider Fig. 9.19 which is a plot of test results compared to two of these theories.
From a consideration of Fig. 9.20 it is observed that if $\epsilon_{cr}$ is to equal $\epsilon_{st}$ then $b \leq 17$ will provide sufficient hinge rotation characteristics.

A similar consideration of web buckling indicates that for $\epsilon_{cr} = \epsilon_{st}$, $d = 34$ will suffice when the section is subjected to pure compression. It should be remembered, however, that seldom if ever will a member need to be compressed into strain hardening. More often will be the case where all that is desired is that the section reach the elastic limit with the flanges remaining straight. For this condition, i.e. $\epsilon_{cr} = \epsilon_{y}$, $d \leq 43$ will be the governing value. For the case of web buckling due to pure bending this value has not as yet been determined. It is expected, however, to be considerably higher than 40.
9.4 PROBLEM OF LATERAL BUCKLING

In considering the problem of lateral buckling in the inelastic range of stress, a procedure similar to that employed in the preceding discussion will be used. That is, it will be assumed that for the necessary plastic hinges to develop, a hinge rotation which corresponds to the ratio $\varepsilon_{st}/\varepsilon_y$ is required of the section under consideration. Since lateral buckling is the limiting condition under study, a solution is desired for the critical length of unsupported member that will allow the flanges to be strained to strain-hardening while still remaining in the plane of applied bending moments.

The member to be investigated is loaded as shown in Fig. 9.20. It is assumed that it is pin-ended in both the strong and weak directions and free to warp.

From Ref. 9.13 (page 160, Equation 317), the following equation defines the critical moment, $M_{cr}$, for such a member:

$$M_{cr} = \frac{\pi}{L} \sqrt{EJ_y G_k} \sqrt{1 + \frac{\pi^2}{L^2} \left( \frac{ET}{G_k} \right)} \quad (9.44)$$

where, for the symmetrical I or WF shape,

$$\Pi = \frac{d^2 I_y}{4}$$
If it is assumed that the section can be represented by two flanges as shown in Fig. 9.21 (neglects influence of web)

\[
\begin{align*}
    K &= \frac{1}{3} t^2 A \\
    M_{cr} &= \frac{1}{2} A \sigma_y d_F \\
    I_y &= A r_y^2
\end{align*}
\]

Equation 9.44 may be rewritten as follows:

\[
\sigma_y = \frac{\pi}{(r_y)} \sqrt{\frac{L}{r_y}} EG \left( \frac{t}{d_F} \right)^2 \left[ 1 + \frac{\pi^2}{\left( \frac{L}{r_y} \right)^2 (\frac{t}{d_F})^2} \right] \left( \frac{3}{4} \right) \left( \frac{E}{G} \right)
\]

Using the following for the values of E, G and \( \sigma_y \) at the onset of strain-hardening

\[
\begin{align*}
    E_{st} &= 900 \text{ ksi} \\
    G_{st} &= 2,000 \text{ ksi} \\
    \sigma_y &= 33 \text{ ksi}
\end{align*}
\]

the curve shown in Fig. 9.22 is obtained. (The value of \( E_{st} \) is an average value from a large number of tension coupon tests. \( G_{st} \) was theoretically deduced using this value of \( E_{st} \) and other measured-mechanical properties and initial imperfections. Tests to determine \( G_{st} \) directly confirm this value within reasonable limits.) Also shown for comparative purposes is the elastic limit solution.
This figure points up the seriousness of the problem of inelastic lateral buckling, since the curve marked "on-set of strain-hardening" would require that \( L/r_y \) between bracing members be not greater than about 20. It should be remembered, however, that a relatively large hinge rotation was assumed \((\epsilon_{st}/\epsilon_y)\). This implied that the hinge under investigation (total length of member shown in Fig. 9.20) was the first to form. While such may be the case for a given problem, should the hinge under investigation be the last to form it would only be required that the spacing of bracing be somewhat less than that required according to elastic limit curve of Fig. 9.22.
More often than not, the case that will be encountered in practice will be the one where maximum moment (and thereby a plastic hinge) occurs not over a large length of member but rather at a point of concentrated load application, at a connection etc., thus restricting the length of the yielding. Furthermore, these points of maximum moment are usually well braced laterally.

9.5 POSSIBILITY OF PROGRESSIVE DEFORMATIONS UNDER VARYING LOADS

In the preceding lectures it has been assumed that the particular structure under consideration was subjected to proportional loading. That is, the loads remain in fixed relation one with the other. Moreover, it was assumed that these loads are steadily increased from zero to their maximum value. Although this case may be approximated for many practical problem, often the loads may vary independently with respect to each other. Since "failure" may occur due to variable repeated (non-proportional) loading, it is important that each of the other modes of failure resulting from varying loads be considered.

First of all, when subjected to an extremely large number of load applications a structure may fail due to fatigue of the material of which the structure is made. In general this is an elastic design problem and will therefore not be covered herein.
A second type of premature failure may occur when certain sections within the structure are subjected to repeated cycles of load application that cause it to yield at each cycle: first in one direction and then in another. This condition may be thought of as plastic fatigue (or alternating plasticity).

The last condition to be examined is characterized by an increase in deflection at each cycle of loading. The problem is to specify the maximum load for which deflections become stable after a few cycles of load application. (This load is often referred to as the stabilizing or "shakedown" load.)

**ALTERNATING PLASTICITY**

To illustrate this condition, consider the cantilever beam shown in the inset sketch of Fig. 9.23.

If it is assumed that the load P is first applied in a downward manner the moment-curvature relation at the wall section follows that shown from 0 to (a). If at this point the load P is gradually released and finally applied in the opposite direction, there is first of all observed a linear range of $M-\phi$ which extends
over a value of $\Delta M_y$. At this point, however, yielding commences in the opposite direction as shown at (b). Finally a point (d) is reached corresponding to $-P_{\text{max}}$. To complete the cycle, loads are then increased with the resulting behavior as shown by the dashed line d-e-f-a.

The point of this discussion so far is to indicate that there exist ranges of $M$ values for which a section behaves elastically regardless of its previous loading history. As a first approximation this range of moments, $\Delta M$, may be taken equal to

$$\Delta M_y = 2M_y = \frac{2M_{\text{p}}}{I}$$  \hspace{1cm} (9.47)

After sufficient testing has been carried out, however, this value may be revised.

The necessary condition then to eliminate the possibility of ALTERNATING PLASTICITY is

$$(M_1)_{\text{max}} - (M_1)_{\text{min}} \leq \Delta M_y$$  \hspace{1cm} (9.48)

where $M_1$ denotes the moment values at any section "i" being investigated.

DEFLECTION STABILITY

This condition can be visualized from a consideration of the continuous beam shown in Fig. 9.24. It will be assumed that each of the load $P_B$ and $P_D$ can vary independently between
the limits of 0 and W.

Consider the case where loads are increased proportionally (Fig. (b) and (c)). In Fig. 9.25 is a plot of the load, $F$, versus the absolute value of the Moment at sections B and C. The loads are first increased to the value $P_B = P_D = W$ as shown in Fig. 9.25. When the loads are removed there remains in the structure certain residual moments that try to raise the beam.
off its center support. The residual moment diagram resulting from this condition is then as shown in Fig. 9.24 (d).

Assume that the structure is now subjected to loading condition (2) (i.e., \( P_B = 0 \), \( P_D \) increased from 0 to \( W \)). Going through the same reasoning, it will be observed that yielding now occurs under load \( P_D \) and thereby results in an increased deflection at this point. If the first loading is again applied section c yields causing a still further increase in deflection at D.

This process could be continued cycle by cycle and thereby allow the plotting of a deflection versus number of cycles curve as shown in Fig. 9.26. It would be observed that if \( W \) is equal to or less than a certain critical value, \( W_s \), (that depends only on the location and variation...
in the applied loads), a set of residual moments will be set up in the structure for which all further repetitions of load are carried elastically. This is the stabilizing or shakedown load. The problem then is to determine if such a set of residual moments is possible. Neal has stated it as follows:

"If any state of residual stress can be found for a structure that enables all further variations of the external loads between their prescribed limits to be supported in a purely elastic manner, then the structure will shakedown."

To satisfy this condition, then, the following inequalities must be satisfied:

\[ m_i + (M_i)_{\text{max}} \leq M_p \]
\[ m_i + (M_i)_{\text{min}} \geq -M_p \]  
(9.48)

where as before the \( i \) denotes the section under consideration.

For the preceding illustration (Fig. 9.24) the procedure would be as follows:
At section (C):
\[ m_R + (-M_{C1}) \geq -M_p \]
(9.50)
At section (D)
\[ \frac{1}{2} m_R + M_{D2} \leq M_p \]

Substituting in each of these equations the values of the moments determined by elastic methods in terms of \( W \) will result in two simultaneous inequalities which can be solved for \( M_p \) in terms of \( W_s \), the shakedown load.
To be sure that this is the correct answer, however, it is necessary that alternating plasticity be checked (Eq. 9.48). This gives at section B

\[ \left| \left( M_{B1} \right) \right| + \left| \left( M_{B2} \right) \right| = \frac{2M_p}{f} \] (9.51)

With regard to the seriousness of this problem of deflection stability on plastic design, recent tests have shown that it may not be as critical as theory might indicate. For example, a series of continuous beam tests of the type shown in Fig. 9.24 were carried out at Fritz Laboratory. For the particular section and loading the computed shakedown load was 13.7 kips. However, in the tests the actual value was in excess of 15.5 kips.

Another point with regard to this problem is that the ratio of live load to dead load for a given structure is important. If the ratio is small, the influence of the live load variation will be of lesser importance.

A third point to consider is that if wind stresses are responsible for the variation in load, and if it is assumed that smaller factors of safety are to be used when wind stresses are included in the analysis, then it is quite possible that the design will not be governed by the loading condition that includes wind forces even when modified to include the influence of variable repeated loading.
References

Axial Thrust - Reduction in $M_p$


Axial Thrust - Stability


Shear


Local Buckling


Lateral Buckling


Deflection Stability


SCOPE: The design of connections to transmit plastic moments, and the design of certain additional details to make sure that the ultimate load is reached, are the concern of this lecture. How do the requirements for structural connections differ from conventional design? How are connections proportioned? In view of possible adverse effects of local and lateral buckling, how will compressive elements be proportioned? Requirements for lateral support to prevent buckling.

OUTLINE: I. CONNECTIONS

1. IMPORTANCE OF CONNECTIONS
2. REQUIREMENTS FOR CONNECTIONS
3. STRAIGHT CORNER CONNECTIONS
4. HAUNCHED CONNECTIONS
5. INTERIOR CONNECTIONS

II. DESIGN DETAILS

7. PROPORTIONING COMpressive ELEMENTS
8. LATERAL SUPPORT REQUIREMENTS

10.1 IMPORTANCE OF CONNECTIONS

Points of maximum moment usually occur at connections. For instance Fig. 10.1 (from Example 5.) indicates that all plastic hinges form at connections except one in right girder. Further, at corners the connection must change the

[Fig. 10.1]

[Fig. 10.2]
direction of the forces (Fig. 10.2). Also, the connecting devices (welds, rivets, or bolts) are at points subjected to the greater moments.

**Conclusion:** Connections play a key role in assuring that the structure reaches the computed ultimate load.

1. **Types**

   (a) **Method of Fabrication**

   Recent advances in plastic design are due to fully welded continuous construction.

   Plastic design also applicable to partially welded (top plate) and to riveted or bolted connections whenever demonstrated that they will form hinges ($M_o$ possibly less than $M_p$).

   (b) **Function**

   Fig. 10.3: 1. Corner Connections (Straight, Haunched)  
   2. Beam-Column Connections  
   3. Beam-to-Girder  
   4. Splices (Beam, Column, Roof)  
   5. Column Anchorages  
   6. Miscellaneous Connections (Purlins, Girts, Bracing)

   Treat only No. 1 and 2; same principles apply to other types.
10.2 REQUIREMENTS FOR CONNECTIONS

1. General Behavior

The design requirements for connections are introduced by considering the general behavior of different corner connection types under load. Two classifications are considered. Both are documented by test.

(a) Straight Connections Without Stiffening (Inadequate)

Connection joining

Typical behavior:

Discussion of Behavior "A":

- Due to insufficient web thickness in most WF's to transmit these forces, yield due to shear force commences at low load.

- Connection rotates beyond needed hinge rotation but $M_p$ is not developed.
Elastic deformation is considerably greater than value assumed.

Behavior "B":

- Elastic stiffness and maximum strength are satisfactory
- Connection buckles prior to realizing needed hinge rotation to assure that all hinges form.

(Lecture #8)

(b) Adequately Stiffened Connections

Typical behavior of several designs:

Discussion:

- Strength is greater than computed $M_p$
- Elastic stiffness is adequate. Rotation capacity meets requirements
- All failure is by plastic instability (local and/or lateral)
Haunches proportioned by Ref. 10.2 have adequate strength. But "R" is poor.

Note: To improve R at haunched connections, design them to yield at end of haunch by increasing inner flange thickness.

Lateral support is most effective when placed at point of expected local buckling.

2. Requirements

Four principal design requirements may be formulated -- requirements that are common to all connections: these are Strength, Stiffness, Rotation Capacity, Economy. They are now discussed in the light of behavior of corner and interior connections.

(a) Strength

Connection must be adequate to develop plastic moment, $M_p$, of members joined.

Dissimilar members: Develop strength of weakest member only.

Critical section:

![Diagram of haunched connections](Fig:10.7)
For straight connections (a), critical or "hinge" section assumed at point H (Fig. 10.7). In Fig. 10.7(b), critical section assumed at $R_1$.

(b) **Elastic Stiffness**

It is desirable, but not essential, that average unit rotation of connection materials not exceed that of an equivalent length of beams joined.

Equivalent Length: length of haunch measured along frame line.

Fig. 10.7(a): $L = r_1 + r_2$  \hspace{2cm} (10.1)

\[ \theta_H = \frac{M_h}{EI} \Delta L \]  \hspace{2cm} (10.2)

(c) **Rotation Capacity**

To assure that all necessary plastic hinges will form, all connections must be proportioned to develop adequate rotation capacity, $R$.

See Lecture #8. $R = 12$ suggested as adequate for most cases.

(d) **Economy**

Obviously, extra connecting materials must be kept to a minimum. Wasteful joint details will result in loss of over-all economy.

3. Problems

Above discussion focuses attention on two problems that
require solution:

(1) How shall connections be stiffened in order that they develop the necessary strength?

(2) What proportions of haunches (tapered or curved) will assure that plastic moment will occur at end of haunch?

A design guide:

```
Provide sufficient strength in the connection materials such that the critical section is at the end(s) of the haunch. Adequate rotation capacity is more easily assured there.
```

10.3 STRAIGHT CONNECTIONS

1. Analysis of Corner Connection Strength

(a) Unstiffened Connections

Connection and Loading:

Fig. 10.8

Design Objective:

to prevent behavior "A" of Fig. 10.5 which involved shear yielding at low load.

Strength Requirement:

Moment at which yield commences due to shear force, $M_h(\tau)$, should not be less than plastic moment, $M_p$. 

![Fig. 10.8](image)

![Fig. 10.9](image)
Assumptions:

(a) Maximum shear stress yield condition
(b) Shear stress is uniformly distributed in web of knee.
(c) Web of knee carries shear stress, flange carries flexural stress.

Stress distribution: Fig. 10.9

Haunch moment at shear yield:

\[ M_h(\tau) = \frac{w d^2 \sigma_y}{2(1 - \frac{d}{L})} \]  \hspace{1cm} (10.3)

Flexural strength:

\[ M_p = \sigma_y Z \]  \hspace{1cm} (2.12)

Equating to determine required web thickness:

\[ w = \frac{2fZ}{d^2} \left(1 - \frac{d}{L}\right) \leq \frac{2S}{d^2} \]  \hspace{1cm} (10.4)

Note: Examination of rolled shapes (using Eq. 10.4) shows that all WF's and most I's require stiffening to realize design objective for straight connections.

(b) Stiffened Connections

Two methods are available:

- Doublers (Fig. 10.10)
- Diagonal Stiffener (Fig. 10.11) Recommended
Doubler:

\[ W_d = w_r - w \]

\[ W_d = \frac{2S}{d^2} - w \] \hspace{1cm} (10.5)

Diagonal Stiffener:

Assume that the stiffener material acts to increase web thickness

Find: Required thickness, \( t_s \), for adequate design characteristics.

Equating volumes of diagonal stiffener and a "simulated doubler" of thickness \( \Delta w \).

\[ \Delta w \cdot d^2 = b t_s \cdot d \sqrt{2} \]

Eq. 10.5: \[ t_s = \frac{\sqrt{2}}{b} \left( \frac{S}{d} - \frac{W_d}{2} \right) \] \hspace{1cm} (10.6)

Note: 1. An alternate method results in the same expression (Ref. 10.5).

2. A check of WF's shows that but a small amount of material is wasted if design \( t_s \) to equal flange thickness, \( t \).
10.4 HAUNCHED CONNECTIONS

1. Function

Haunched connections are the product of elastic design concept by which material is placed in conformity with the moment diagram to achieve greatest possible economy. On the other hand, in plastic design (through redistribution of moment) material is used to full capacity without necessity for use of haunches.

Other functions:  

- Architectural (esthetic)
- Allow use of rolled WF's where otherwise built-up members might be required.

Since even further economies may be obtained by considering the haunch in the plastic analysis, this should be done if architectural considerations require use of such built up knees.

Types:

(a)  
(b)  
(c)  
(d)

Fig. 10.12
2. Design Requirements

(a) Strength

1. As discussed in connection with Fig. 10.6, haunched knees may exhibit poor rotation capacity. This is due to inelastic local and/or lateral buckling.

Solution: Force formation of plastic hinge at end of haunch by requiring that the haunch proper remain elastic throughout. Accomplish this by specifying adequate thickness of inner flange.

2. Sidewise "kicking out" of inner (compression) flange is prevented by the use of lateral bracing and the requirement that inner flange remain elastic.

(b) Stiffness

Automatically provided in great majority of cases.

(c) Rotation Capacity

None required. All plastic deformation occurs in the rolled sections joined.

3. Proportioning the Haunch

Given: General shape and size of haunch (determined by architectural considerations or from economical study of various haunches).
Find: Required thickness of inner flange to assure hinge formation at extremities. Also check shear stress in knee web.*

Solution: Current study seeks to present design charts with required information based on simplification proposed by Olander.\(^{10.6}\) In the meanwhile, the following somewhat arbitrary procedure is suggested.

\[
\text{Increase inner (compression) flange thickness of haunch by 25\% over that required by present procedures.}^{10.2}\]

Basis: Present rules assure elastic action at yield moment, \(M_y\). Since \(M_p\) is greater than \(M_y\) by 14\%; further, since the moment at the end of haunch may increase due to strain-hardening, then an increase of 25\% should cover both factors. (It is assumed that flanges carry the moment.)

Tapered Haunch

Critical Sections: A, B, C

Flange Thickness: \(t_F = 1.25t\)

(assumes hinge at "A")

*Normally only required for type shown in Fig. 10.12(b).
Section B: Check flange stress on basis of moment at haunch point, h, and depth d or d', whichever is smaller.

Curved Knee

Follow procedures outlined in Ref. (10.2) except to increase inner flange thickness by 25%.

Critical sections to be taken at ends of haunch.

Shear Stress

Use Eq. 10.4 to check shear stress in the corner, using as "d" a value corresponding to that shown in Fig. 10.13.

4. Effect on Analysis

The effect of haunches is to increase number of sections at which plastic hinges may form.

In Fig. 10.15 are shown possible mechanisms for frame of Fig. 10.14. Note that the number of possible plastic hinges is increased by two. Thus two additional "elementary mechanisms" are added.
Procedure

Same as outlined in Lectures #6 and #7.

Note: Reduced effective span length is an indication that lighter sections may be used when haunched corners are specified.

10.5 INTERIOR BEAM-COLUMN CONNECTIONS

1. Function and Type

These are the connections shown as "2" in Fig. 10.3 and in further detail in 10.16. Depending on their location (top, side, or interior) their function is slightly different but generally the same as outlined in Art. 10.2.

![Diagram](Fig. 10.16)

(a) Top Connection (Fig. 10.16): Transmits moment from left to right beam (column carries unbalanced moment).

(b) Side Connection: Transmits beam moment to upper and lower columns.

(c) Interior Connection: Same as "a".
Critical Section

Select critical section at H (Fig. 10.16). Some slight economy possible if it is required that plastic moment, \( M_p \), must be developed at R.

Method of Fabrication

The two basic types are direct-welded (10.16(c)) and the top-plate beam-column connections. Both types may be used, although the latter ordinarily may be counted upon for a hinge moment somewhat less than \( M_p \).

2. Stiffeners

Two types are considered: stiffener plates to transmit moment and those to transmit shear force.

(a) "Moment" Stiffeners

Three possibilities are shown in Fig. 10.18.

\[ t_s = \text{full flange thickness, } t. \]
\[ w_s \text{ as determined from Art. 26h, AISC Spec.} \]

Connections without stiffeners not permitted.
(b) "Shear" Stiffeners

"Side" connections (Fig. 10.16(b)) or interior connection with large unbalanced moments may require "shear stiffening" if the column does not carry much direct stress. In such a case the column web at the joint is called upon to transmit forces much like those of Fig. 10.9. An examination similar to that leading to Eq. 10.4 would therefore be desirable in this infrequently encountered case.

II. DESIGN DETAILS

10.6 PROPORTIONING COMpressive ELEMENTS

Lecture No. 9 provides basis for the procedures suggested here. In order to meet the requirements of strength and deformation capacity, compressive elements must have width-thickness ratios such that they will insure against premature plastic buckling. See discussion of rotation capacity in Lecture #8.

1. Flanges and Webs

Referring to the sketch:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(beams and columns)</td>
<td>b/t ≤ 17</td>
</tr>
<tr>
<td>(columns in direct compression)</td>
<td>d/w ≤ 43</td>
</tr>
<tr>
<td>(Tentative) (Beams in bending)</td>
<td>d/w ≤ 50</td>
</tr>
</tbody>
</table>

Note: The above provisions were developed for members under uniform stress or moment. In the presence of more usual moment gradient, greater d/w ratios undoubtedly would be allowable.*

* Tests of 14 and 24 WF connections with d/w = 51 have given most satisfactory curves.
Adequacy of Shapes:

b/t : All I-beams satisfactory. All WF beams except Jr. beams 6", 8", 10", and 12" and 14WF30.*

d/w : Practically all I-beams satisfactory. Most WF's satisfactory.

2. Stiffening

On the same basis as above, the width-thickness ratio of compression or load-bearing stiffeners should not be greater than 8.

Possible methods of stiffening an otherwise inadequate shape: (Stiffen in region of expected plastic moment)

(a) Flange: Cover Type
(b) Flange: Edge Type
(c) Web: Longitudinal Type
(d) Flange & Web: Vertical Type
(e) Flange & Web: "Box" Type

Note: Such devices are expensive. They should be used only when it is reasonably certain that choice of another shape will not solve the problem.

The "Box" type has advantage of somewhat higher shape factor and greatly improved lateral buckling resistance.
3. Miscellaneous Details

For design of details not covered herein, the general philosophy to be followed is to proportion details such that yield stress is not exceeded at ultimate load. "Details" should remain elastic and assure adequate plastic deformation of main framing in attaining computed ultimate load.

10.7 LATERAL SUPPORT REQUIREMENTS

Lateral bracing must be provided to beams, columns, and (in particular) to connections to assure that they will not buckle sidewise either due to lateral buckling or lateral and local buckling.

1. Enclosed Structure

   Enclosing material provides adequate support.

   Assume that adequate lateral support is provided when structure is enclosed by walls or slabs normal to plane of the frame.

2. Interior Connections (beam-column)

   On the basis of tests carried out on connections of type shown in Fig. 10.18 without any lateral support whatsoever and in which the columns also were loaded:

   Interior columns provide adequate lateral support to beams subjected to the plastic moment.
3. Corner Connections

Positive lateral support required on two connection types as shown by solid circles in Fig. 10.20.

4. Columns

Locate bracing on compression flange near expected hinge locations (connection bracing usually covers this). Additional bracing may be required normal to plane of frame and intermediate between column ends to assure adequate strength in "weak" direction.

5. Beams and Girders

As will be recognized from treatment of lateral buckling problem in Lecture No. 9, a firm rule for spacing of lateral supports cannot yet be formulated. A number of alternatives are open, pending the completion of current research.

(a) British Practice:

Ref. 10.7 suggests tentatively that members should be braced such that \( \frac{L}{r_y} \leq 100 \). \[ (10.7) \]

This will assure that \( M_p \) will be developed, but tests on short beams indicate that it will not assure rotation capacity up to strain-hardening. However, this will be satisfactory if the girder hinge is the last to form.
Hinge Bracing:

Yielding markedly reduces the resistance to lateral and local buckling. It is therefore logical to brace at those points at which plastic hinges are expected.

Fig. 10.21 shows portal frame with possible plastic hinges at section 2 to 6. Since the load is brought into the frame by the purlins, then the maximum moment will be at one of them (assumed at Section 3).

Bracing therefore to be provided at Section 3 (point of plastic moment). Check sections 2-3 to see that they are adequately protected against elastic lateral buckling. Check moment diagram (as follows) to see if additional bracing is needed at Section 3.

Bracing at the hinge

Given: A beam loaded with a gradient in moment as shown in Fig. 10.22 (typical of purlin-loaded beam)

Problem: Is further bracing required other than that at the point of maximum moment? What criterion is to be used in determining its spacing?
Length of hinge:

\[ \Delta L = \text{length of beam in which } M > M_y. \]  
(Determined from moment diagram at ultimate load)

Criterion:

As an approximation, two parameters may be considered - \( Ld/bt \) or \( L/r_y \). (Tests indicate that the latter may be the better of the two).

Preliminary analysis shows that critical length for onset of strain-hardening bears about the same relation to "elastic limit" length as exists in a column.

\[ L_{cr} \approx \frac{1}{6} L_u \]  
(10.8)

In a perfect column at the onset of strain-hardening

\[ (L/r_y)_{cr} \approx 15 \]  
(10.9)

Note: Eq. 10.8 and 10.9 both based on premise that rotation capacity up to strain-hardening is required--a conservative assumption.
The critical length, then, could be determined from one of the following, assuming, further, that the elastic member can provide at least some restraint.

\[ L_{cr} = \frac{L_u}{5} \quad (10.10) \]

or

\[ L_{cr} = 20 \times r_y \quad (10.11) \]

Note: If the value obtained from (10.10 or (10.11) is less than the purlin spacing or the value \( \Delta L \) (Fig. 10.22), check the required rotation capacity. If less than 12, recalculate \( L_{cr} \) on the basis of this less severe requirement. In most cases last hinge to form will be in the girder, in which case \( R = 1 \).

6. Design of Bracing -- What constitutes adequate bracing?

(a) Strength and Stiffness

Force required to prevent lateral buckling has been measured on many tests and has always been small. It has never exceeded a value given by

\[ T = 0.01 \times \sigma_y \times A \]

where \( A \) is area of member being braced. If this value is doubled to account for uncertain field conditions, then the required force, normal to plane of frame becomes,

\[ T = 0.02 \sigma_y A \quad (10.12) \]
The bracing member must have adequate stiffness to prevent sidewise movement. For this reason, normal stresses in bracing members should be kept low.

(b) Position

Both tension and compression sides must be braced at changes of section (connections, taper intersections). In beams, compression flange bracing is probably adequate unless required "R" is large.

Note: Bracing members (the purlins) must themselves be braced with respect to other parts of the frame such as by roof bracing.
References

10.1 Toprac, A. A. "CONNECTIONS FOR WELDED CONTINUOUS PORTAL FRAMES", Welding Journal 30(7), 30(8), and 31(11), 1951 and 1952. (Progress Report #4).

10.2 Griffith, J.D. "SINGLE SPAN RIGID FRAMES IN STEEL", American Institute of Steel Construction, 1948.

10.3 Hendry, A. W. "AN INVESTIGATION OF THE STRENGTH OF WELDED PORTAL FRAME CONNECTIONS", Structural Engineer (Br.), 28(10), 1950, p. 265.


From the preceding lectures it is evident that the methods of analysis that we have been discussing (i.e. Plastic Methods of Analysis) are concerned with the determination of sizes of members capable of supporting given loads at their ultimate strength. Or looking at it from the opposite point of view, given a structure, the determination of its ultimate load carrying capacity. For design by these methods then, it is necessary that the given loads be "stepped-up" by the desired margin of safety against this ultimate strength condition, and that member sizes be determined from a consideration of these prorated loads. It is with regard to this question of the desired margin of safety or load factor of safety, as it is sometimes called, that this discussion will be concerned.

In discussing such a general topic as this, several positions or points of view might be taken. One could, for example, start from a basic consideration of the design problem and proceed to develop methods for evaluating structural safety based on uncertainty of the various quantities entering in the design procedure. It was considered, however, that while this general problem of structural safety is in itself important, it is not unique to plastic design and should therefore not be included in this short lecture. For the sake of completeness, however, a list of several references, which gives an indication of modern thinking on this problem, has been included at the end of this lecture.
For the remainder of this discussion, the position has been taken that design based on plastic methods should have the same load factor of safety against ultimate strength that the simple beam now has when designed according to the A.I.S.C. Specification. The corresponding factors are determined and their use is illustrated.

Consider the simple beam loaded as shown in Fig. 11.1. If it is assumed that the allowable bending stress is 20,000 psi (A.I.S.C. Spec. Section 15c-3), and that the yield stress is 33,000 psi (minimum allowable for A.S.T.M. A-7 type steel), \( Q_y \) will be 1.65 times greater than \( Q_w \), the allowable working load, since \( M_y \) is 1.65 times \( M_w \). Further, since \( M_p \) is approximately 14% greater than \( M_y \) for a wide flange section bent about its strong axis (Lecture 2), the load \( Q_p \) will be approximately 14% greater than \( Q_y \). Therefore, the true load factor of safety of the simple beam is

\[
(1.65)(1.14) = 1.88
\]

or stating in a somewhat different way, a load 1.88 times the working load will cause the structure to become a mechanism.
To illustrate the manner in which this factor should be applied, consider the rigid frame shown in Fig. 11.2. It is assumed that the vertical live load plus the estimated dead load equals 1 kip/ft. along the beam. The problem is to determine the sizes of members required to sustain this load and have a factor of safety against ultimate strength of 1.88.

Assuming that the frame is of uniform section, the moment diagram at ultimate load will be as shown in Fig. 11.2(b) with hinges forming at sections 2, 3 and 4. From equilibrium consideration of this diagram

\[ M_p = \frac{w_D l^2}{16} \]  \hspace{1cm} (11.2)

where \( w_D \) is the design load/ft. of beam. Since \( M_y = S \sigma_y \), and since \( M_p \approx 1.14 \ M_y \)

\[ M_p \approx 1.14 \ S \ \sigma_y \]  \hspace{1cm} (11.3)
From Eqs. (11.2) and (11.3)

\[ S = \frac{wDl^2}{16(1.14)\sigma_y} \quad (11.4) \]

or

\[ S = \frac{(1.00)(1.88)(30)^2(12)}{(16)(1.14)(33,000)} \quad (11.5) \]

\[ = 33.8 \text{ in.}^3 = \text{REQUIRED SECTION MODULUS} \]

The A.I.S.C. specifies another condition when structures are subjected to combinations of wind and other forces. From section 15(e):

"Members subjected to stresses produced by a combination of wind and other loads may be proportioned for a unit stress 33 1/3 percent greater than those specified for dead and live load stresses, provided the section thus required is not less than that required for the combination of dead load, live load, and impact (if any). A corresponding increase may be applied to the allowable unit stresses in their connecting rivets, bolts or welds."

Going through a procedure similar to that for Fig. 11.1, it is found that the load factor of safety for this case is

\[ \frac{1.88}{1.33} = 1.41 \quad (11.6) \]

It can be stated that if design by Plastic Methods is to have the same load factor of safety as a simple beam designed by the current A.I.S.C. Spec., the following load factors of safety should be used:

<table>
<thead>
<tr>
<th>Load Factor of Safety</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ALL FORCES (including wind)</strong></td>
</tr>
<tr>
<td><strong>ALL FORCES But wind</strong></td>
</tr>
</tbody>
</table>

(11.7)
It should be noted that this factor 1.41 is a factor to be applied to all forces when in combination with wind, not just the wind itself.

For illustration, consider example problem 6.4. (See Fig. 11.3 for sketch). Assume for the case shown that

\[ \begin{align*}
L &= 30 \text{ ft.} \\
P &= 10 \text{ kips}
\end{align*} \] (11.8)

From Lecture #6, Eq. 6.40, \( P_u = 2.19 \frac{M_p}{L} \)

or

\[ M_p = \frac{P_u L}{2.19} \] (11.9)

But since

\[ M_p = S \left(1.14\right) \sigma_y \] (From Lecture #2)

\[ S \left(1.14\right) \sigma_y = \frac{P_u L \left(1.41\right)}{2.19} \] (11.10)

Therefore

\[ S_{\text{req.}} = \frac{(10)(30)(1.41)(12)}{(2.19)(1.14)(33)} = 61.5 \text{ in.}^3 \] (11.11)

The various members then require

\[ \begin{align*}
3 \ M_p &= 184.5 \text{ in.}^3 \quad \text{REQUIRED} \quad S - (24 \text{ WF 84}) \\
2 \ M_p &= 123.0 \text{ in.}^3 \quad \text{REQUIRED} \quad S - (21 \text{ WF 62}) \\
1 \ M_p &= 61.5 \text{ in.}^3 \quad \text{REQUIRED} \quad S - (16 \text{ WF 45})
\end{align*} \] (11.12)
While for this example case it has been possible to select member sizes, the solution is not complete, since a possibility exists that the beams will fail under only the vertical loads. This condition must therefore be checked using load factor of 1.88. From Eq. 6.34

\[ M_p = \frac{PL}{3.00} = S \times (1.14) \sigma_y \]

\[ S_{req.} = \frac{(10)(30)(12)(1.88)}{(3.00)(1.14)(33)} = 60.0 \text{ in.}^3 \]  \hspace{1cm} (11.13)

Since this value of Required Section Modulus is less than that given in Eq. 11.12, the former will govern.

It is interesting to point out that the present A.I.S.C. Specification (section 15(a)3) contain two other allowable bending stress provisions. These are as follows:

"Fully continuous beams and girders may be proportioned for negative moments which are maximum at interior points of support, at a unit bending stress 20 percent higher than above stated; provided that the section modulus used over supports shall not be less than that required for the maximum positive moments in the same beam or girder, and provided that the compression flange shall be regarded as unsupported from the support to the point of contraflexure.

"For columns proportioned for combined axial and bending stresses, the maximum unit bending stress \( F_b \), Sect. 12(a) may be taken at 24,000 pounds per square inch, when this stress is induced by the gravity loading of fully or partially restrained beams framing into the columns."

Rather than discuss in detail these provisions, Fig. 11.4 is a plot of the actual load factor of safety versus the ratio of end span length to center span length for the continuous beam shown. As indicated there is a range of \( k \) values for which
the load factor of safety against ultimate strength of the continuous beam is less than that of the simple beam \(F_{\text{min}} = 1.72\). In this region, design according to the current A.I.S.C. Specifications will result in a more economical choice of members than a plastic design using 1.88. This is possible only at a decrease in factor of safety over that assumed in the plastic solution.

Fig. 11.4 is also a good example of how a limiting stress design will result in a variable factor of safety against ultimate strength.
While the structure shown in Fig. 11.4 may not seem to represent a practical problem since the end spans will at least have dead weight acting, it is observed that this condition is approximated by a portal frame subjected to gravity loading. (Consider the legs as being spread.) Consider, for example, the frame shown in Fig. 11.2. Had the frame been designed by specification provisions (assuming section 15(a) 3 holds for this case) the required $S$ would have been

$$S_{req.} = 30.9 \text{ in.}^3$$

which would show a theoretical saving of $8 \frac{1}{2}\%$ in section modulus over the plastic design. (Possible due to the lower factor of safety.)

For the continuous beam problem, it is possible to vary the ratio of the side span load to its length and thus find other cases in which the load factor of safety will be 1.72. (See Fig. 11.5)
Summarizing; it can be said that if a plastic design is to have the same load factor of safety against ultimate strength as the simple beam now has according to the A.I.S.C. spec., then the following load factors should be used:

- All forces including wind: 1.41
- All forces but wind: 1.88

When comparing "conventional" designs with plastic designs, it should be realized that the former have a variable factor of safety against ultimate strength and that these may under certain circumstances be slightly lower than the above recommended values for plastic design.
Selected references on the general topic of Factor of Safety.


11.6 --  Various Papers of the 1948 and 1952 meetings of the I.A.B.S.E.
SCOPE: On the basis of previous lectures outlining the fundamentals and methods of plastic analysis, "Rules of Design" are formulated. In some cases simplifications of more precise expressions are made for convenience. These "Rules" are based on present state of knowledge. As new information becomes available (particularly with regard to column behavior) they may be revised.

This topic is presented in chart form, the "Rules" being numbered in sequence throughout. Appropriate notes and sketches are indicated, and reference to more complete treatment given. (Unless otherwise noted, Equations, Figures, Articles refer to this set of lectures.) Where further development is desirable, this is outlined immediately following the "Rule".

These guides also constitute a summary of procedures. For examples and illustrations, reference should be made to the indicated lectures or other articles.

OUTLINE: 1. GENERAL PROVISIONS 2. ANALYSIS 3. DESIGN 4. MODIFICATIONS TO PLASTIC MOMENT 5. COLUMNS 6. CONNECTIONS 7. DESIGN DETAILS 8. MISCELLANEOUS REQUIREMENTS
### 12.1 GENERAL PROVISIONS

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<tr>
<th>RULE</th>
<th>NOTES</th>
<th>REFERENCE</th>
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</thead>
<tbody>
<tr>
<td>1. Material: ASTM-A7 steel for bridges and buildings may be used with modifications, when needed, to insure weldability and toughness at lowest service temperature.</td>
<td>1. AWS has promulgated specification to take care of weldability. 2. High-strength steels with characteristic diagram (Fig. 2.2) may be used if other requirements are met.</td>
<td>ASTM-A7 ASTM-A373 Fig. 1.1</td>
</tr>
<tr>
<td>2. Yield Stress Level (A7):</td>
<td>Current research suggests value might be raised to about 36,000 psi.</td>
<td>Ref. 12.1</td>
</tr>
<tr>
<td>( \sigma_y = 33,000 \text{ ksi} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Plastic Moment:</td>
<td>For other forms of construction,</td>
<td>Lecture #11</td>
</tr>
<tr>
<td>( M_p = \sigma_y Z )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Z = \text{plastic modulus} ) ( f = S ) ( (f \neq 1.14) )</td>
<td></td>
<td>Eq. 2.12 2.26, 2.27(a)</td>
</tr>
<tr>
<td>4. Load Factor of Safety: (Building Construction, Loadings according to AISC) Dead Load + Live Load, ( F = 1.88 ) Dead Load + LL + Wind, ( F = 1.41 )</td>
<td>Method based on Lower Bound Theorem. By requiring that a mechanism form it also satisfies Upper Bound Theorem.</td>
<td>Art. 3.5 Lecture #4</td>
</tr>
<tr>
<td>5. Equilibrium Method: (For continuous beams and simple frames): By the following procedure find an equilibrium configuration (moment diagram) in which ( M \leq M_p ) such that a mechanism is formed:</td>
<td>Method based on Lower Bound Theorem. By requiring that a mechanism form it also satisfies Upper Bound Theorem.</td>
<td>Art. 3.5 Lecture #4</td>
</tr>
<tr>
<td>(1) Select redundant(s) (2) Draw Moment-diagram for determinate structure. (3) Draw Moment-diagram for structure loaded by redundant(s) (4) Sketch composite moment diagram in such a way that a mechanism is formed (sketch mechanism) (5) Compute value of ( M_p ) by solving equilibrium equation (Rule 13) (6) Check to see that ( M \leq M_p ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Mechanism Method (General application): By the following procedure find a mechanism (elementary or combination) such that ( M \leq M_p ) (1) Determine number and location of possible plastic hinges (Rule 10)</td>
<td>1. This method is based on Upper Bound Theorem. If ( M \leq M_p ) it also satisfies lower Bound. 2. Principle of Virtual Displacement is another method of obtaining Equilibrium equations.</td>
<td>Art. 3.5 Lectures 5, 6, 7</td>
</tr>
</tbody>
</table>
Mechanism Method (Continued)

(2) Determine number of redundants (Rule 7)
(3) Find number of elementary mechanisms (Rule 9)
(4) Select possible (elementary and composite) mechanisms.
(5) Solve virtual work equations for $M_p$ (maximum).
(6) Carry out equilibrium check to see that $M \leq M_p$ (Rule 11)

Indeterminacy: To determine the number of redundants, cut sufficient supports and structural members such that all loads are carried by simple beam or cantilever action.

Number of redundants, $X = \text{Forces} + \text{Moments required to restore continuity}$

Examples (loads not shown)

Partial Redundancy: With a given mechanism, the following expression indicates whether or not the structure at failure is determinate.

$I = X - (M-1) = \text{Remaining redundancies}$

Number of Mechanisms:

$n = N - X \quad \text{Number of Elementary Mechanisms}$

Location and Number of Possible Plastic Hinges:

Plastic hinges may form at concentrated load points, at the end of each member meeting at a connection, and at the point of zero shear in a span under distributed load.
**Equilibrium Check:** When it is believed that the correct solution has been found, determine a possible equilibrium configuration (draw moment diagram) to make certain that the "plastic moment" condition is satisfied.

**Methods for establishing equilibrium check:**

1. For a determinate structure (try Rule 8) either determine the reactions and compute unknown moments or solve the moment equilibrium equations (Rule 13).

2. For partial redundancy (Rule 8) there are several possibilities:
   - (a) Assume values for "I" unknown moments (I as determined from Rule 8) and determine remaining values from equilibrium equations.
   - (b) Use Moment-Balancing process (Rule 12)

**Moment-Balancing:**

1. Compute the total simple span moments ($M_s$) and total sway moment.

2. Assume values for fixed-end moments in beams (say $\frac{1}{2}$ of $M_s$).

3. Assume sway moment in columns in proportion to $\sum M_p$ ratio.

4. Balance joints. Note limiting Plastic Moment-values in each member. (Note 3)

5. Carry-over in beams (see table)

6. Balance column sway

7. Inspect for $M > \text{Plastic Moment}$.

# Lecture #5
# Lecture #7
### REFERENCES

<table>
<thead>
<tr>
<th>Rule</th>
<th>Notes</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>13. Equilibrium Moment Equations:</td>
<td></td>
<td>Ref. 7.1 Lecture #5 Art. 14.2</td>
</tr>
<tr>
<td>Beams: ( M_a = \frac{M_1}{2} + \frac{M_2}{2} + \frac{P_1L}{4} ) (Typical)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Joint: ( M_a + M_b = M_c = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel: ( M_1 - M_2 - M_3 - M_4 + M_5 ) (Sideways) ( + M_6 + P_h = 0 )</td>
<td></td>
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</tr>
<tr>
<td>14. Distributed Load: A conservative result is obtained if distributed load is replaced by equivalent concentrated loads. (See typical example)</td>
<td></td>
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<tr>
<td>If &quot;correct&quot; mechanism involves hinge in beam, further economy is gained by working out mechanism.</td>
<td></td>
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</tr>
<tr>
<td>Note: &quot;x&quot; distance and moment ratios for various end-moment conditions have been tabulated in charts in Ref. 7.1.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Where load is brought to main frame by purlins, distributed load may be converted at the outset to actual concentrated loads applied at the assumed purlin spacing.</td>
<td></td>
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<tr>
<td>15. Portal Frames: Convenient charts may be developed for the solution of portal frame problems. A number of cases have been treated and sample charts are presented in Appendix to Lecture #6. The ultimate load (or required ( M_p )) may be determined directly from such charts.</td>
<td></td>
<td>Lecture #6</td>
</tr>
</tbody>
</table>
### 12.3 DESIGN

<table>
<thead>
<tr>
<th>RULE</th>
<th>NOTES</th>
<th>REFERENCE</th>
</tr>
</thead>
</table>

1. Determine possible loading conditions
2. Compute ultimate load by multiplying working loads by \( F \) (Rule 4)
3. Estimate Plastic Moment ratio of frame members. (Rule 18 or 19)
4. Analyze each loading condition for maximum \( M_p \) (Rule 5, 6, or 14)
5. Select section (Rule 3)
6. Check the design to see that it satisfies the remaining applicable requirements. (Rules 21 to 42)

### 17. Continuous Beams - Uniform Section:
Use equilibrium method of analysis (Rule 5) supplemented by the following:

(a) On the ultimate load moment diagram draw the appropriate fixing lines across the end span and the internal span which carry the largest bending moments.
(b) Select the greater of the required \( M_p \)'s.

### 18. Continuous Beams - Non-uniform Section:
For maximum section economy, select sections such that, where practicable, mechanism forms in each span:

(a) Express \( M_p \)-ratios as unknowns and solve by analyzing each of the local mechanisms (Example 4.3).
(b) Alternatively, select a section to suit a smaller Plastic Moment requirement and reinforce with cover plates where \( M > M_p \).

1. Formation simultaneously of local mechanisms does not necessarily give minimum section. Examination of alternate possibilities is desirable.

Ref. 12.2 Lecture #4

**Diagram:**
- On the ultimate load moment diagram, draw fixing lines to identify critical spans.
- This span controls the design.
- Express \( M_p \)-ratios as unknowns and analyze local mechanisms for each span.

(1) Determine absolute plastic moment values for separate loading conditions. (Assume all joints fixed against rotation, but frame free to sway).
   (a) Beams: Solve beam mechanism equation.
   (b) Columns: Solve panel mechanism equation.
   Actual section will be greater than or at best equal to these values.

(2) Select plastic moment ratios using the following guides
   (a) Beams: Use ratio determined in step 1.
   (b) Columns: At corner connections $M_p(col) = M_p(beam)$
   (c) Joints: Establish equilibrium.

(3) Analyze for maximum required plastic moment (Rules 5, 6, or 15)

(4) Examine frame for further economies as may be apparent from consideration of relative beam and sway moments ($M_s$).

20. Tier Buildings: (Diagonal bracing in wall panels to resist shear):
Proportion beams and girders by plastic methods. Proportion columns according to conventional ("elastic") methods.

1. See also Rule 26

2. Complete plastic analysis may be applied to design of top one or two stories if desired.
### 12.4 Modifications to Plastic Moment

<table>
<thead>
<tr>
<th>RULE</th>
<th>NOTES</th>
<th>REFERENCE</th>
</tr>
</thead>
</table>
| **21. Axial Force:** (For strong axis): Neglect the effect of direct stress unless \( P > 0.15 \, P_y \). Thus:  
\[
M = M_p, \quad (P < 0.15 \, P_y) \\
\frac{M_{pc}}{M_p} = 1.18 \left(1 - \frac{P}{P_y}\right) \quad (P > 0.15 \, P_y)
\]  
\[\text{Art. 9.1} \] | Required value of \( S \) for a member is determined by multiplying the value of \( S \) found in the initial design by ratio \( M_p/M_{pc} \) |  
| **22. Shear Force (Maximum Allowable):**  
Maximum allowable shear force in a beam at ultimate load is to be computed from  
\[V_{max} = 17,000 \, w d\]  
\[\text{Fig. 9.6} \] | \( w \) = web thickness \( d \) = depth of section \( \text{Assumes that shear is carried by web.} \) |  
| **23. Shear Force (Moment Reduction):**  
The full plastic moment \( M_p \) may be assumed unless the distance "a" between hinge and point of inflection is less than about 3\( d \) for beams and 4\( d \) for columns. Otherwise,  
\[
\begin{align*}
\text{Beams:} & \quad \frac{M_{ps}}{M_p} = 0.65 + 0.17 \frac{a}{d} \\
\text{Columns:} & \quad \frac{M_{ps}}{M_p} = 0.60 + 0.1 \frac{a}{d}
\end{align*}
\]  
\[\text{Fig. 9.17} \] | Determine new value of \( S \) as noted in Rule 21. |
### RULES

| 24. Variable Repeated Loading: | 1. Several thousand cycles of complete reversal of moment may be expected without reduction in \( M_p \) | Art. 9.5 |

These rules are intended for cases normally considered as "static" loading. Where the full magnitude of load on a member is expected to vary, the ultimate load may be modified according to analysis of deflection stability.

### 12.5 COLUMNS

| 25. Simple Columns: | L = unbraced length | Ref. 12.2 |

After frame members have been selected, any columns subjected only to axial force shall be checked according to:

\[
P/A = \sigma_y - 120 \frac{L}{r} \quad \ldots(12c)
\]

\[
P/A = \frac{290,000,000}{(L/r)^2} \quad \ldots(12d)
\]

Select axis to give maximum \( L/r \)

### 26. Framed Columns - Industrial Buildings:

| 1. Use Fig. 9.12 as a basis for design of columns bent in single curvature. | 2. Column to be adequately braced to prevent lateral-torsional buckling. |

If axial force is low (\( \frac{P}{F} < 15\% \)) and unless the column is bent in single curvature, neglect effect of direct stress (Rule 21).

Note: Column 7-2 of Fig. 7.4 (reproduced in the sketch) is an example of a case that approaches "single curvature"; it is conservative to assume it so. As an illustration of the procedure for the example of Lecture #7, assume \( w = 1.0 \text{ k/ft} \), \( F \cdot w = 1.88 \text{ k/ft} \), \( L = 20 \text{ feet} \). Then the following is obtained:

\[
\begin{align*}
\text{Shape} & = 14W734 \\
L/r & = 41.2 \\
P/P_y & = 0.152 \\
M_p & = 6M_p
\end{align*}
\]

From Fig. 9.12, \( M_e/M_p = 0.87 \). A new section is therefore selected which will supply the required "\( M_p \)" on the loaded (reduced) basis. Thus the new required section modulus is given by

\[
S' = S \frac{M_p}{M_e} \quad \text{(same procedure as Rule 21)}
\]
### RULES

<table>
<thead>
<tr>
<th>27. Framed Columns — Two and Three Story:</th>
<th>28. Framed Columns — Tier Buildings:</th>
</tr>
</thead>
<tbody>
<tr>
<td>See Rule 26.</td>
<td>Where wind-bracing in tier buildings consists of diagonals in wall panels, columns shall be proportioned by present (conventional) methods.</td>
</tr>
</tbody>
</table>

This rule presumes that moment due to side forces is carried by moment connections in columns and that a complete plastic analysis is made of the framework. Where sway bracing is used, see Rule 28.

<table>
<thead>
<tr>
<th>29. Checking the &quot;Weak Axis&quot;:</th>
<th>29. Checking the &quot;Weak Axis&quot;:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check for failure of column in plane normal to principle plane of bending. Use Rule 25, a more convenient form of Eqs. 12(c) and (d) being,</td>
<td>1. Assume columns to be pinned at the ends.</td>
</tr>
</tbody>
</table>

\[ L/r < 275 \left( 1-P/P_y \right) \quad \left( P/P_y > .6 \right) \]

\[ L/r < \sqrt{8800} \quad \left( P/P_y < .6 \right) \]

### 12.6 CONNECTIONS

<table>
<thead>
<tr>
<th>30. Required Web Thickness:</th>
<th>Eq. 10.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_r = \frac{2s}{d^2} )</td>
<td></td>
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</table>

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<tr>
<th>31. Doubler:</th>
<th>Eq. 10.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_d = \frac{2s}{d^2} - w )</td>
<td></td>
</tr>
<tr>
<td>RULES</td>
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| **32. Diagonal Stiffener:**  
  \[ t_s = \frac{\sqrt{2}}{6} \left( \frac{S}{d} - \frac{w_d}{2} \right) \]  
| Generally, \( t_s = t \) is conservative yet not uneconomic. | Eq. 10.6 |
| **33. Haunched Connections -- Inner Flange Thickness:**  
In order to force formation of hinge at end of haunch, make inner flange thickness 25% greater than required by "conventional" rules. | Follow other procedures of Ref. 10.2 |
| **34. "Moment" Stiffeners:** Moment stiffeners  
are required in beam-column connections, either of the "flange" or of the "web-stiffener" type. | Fig. 10.18 |
| **35. "Shear" Stiffeners in Beam-Column Connections:**  
In exterior columns or at interior connections with large unbalanced moment, examine adequacy of web to transmit shear force. | Art. 10.5 |
| **36. Welds:** The applicable procedures of American Welding Society Code will be followed. Continuous welds are to be used at all critical sections. Unit stresses at ultimate load should not exceed the following:  
Tension and Flexure-33,000  
Shear and Combined Stress-22,400 |  |
### Cross-Section Proportions:
Compression flanges and webs of beams and columns should comply with the following:

- \( \frac{b}{t} \leq 17 \) (Beams and columns)
- \( \frac{d}{w} \leq 43 \) (Columns in direct compression)
- \( \frac{d}{w} \leq 50 \) (Beams in bending)

### Stiffening:
The width-thickness ratio of compression or load-bearing stiffeners should not be greater than 8.

### Miscellaneous Details:
Proportion details (not otherwise covered herein) such that the yield stress is not exceeded at ultimate load.

### Bracing Requirements:
Sufficient lateral support should be provided to prevent lateral buckling. The cross-sectional area (normal component) shall not be less than 4% of area of member braced. Design bracing to provide maximum stiffness. In general, brace at expected hinge locations and check other portions according to elastic design procedures.

### Notes
- See Fig. 10.19 for possible stiffening arrangements.
- See Art. 10.7 for more detailed procedures.
### 41. Deflection at Ultimate Load:

If conditions require the computation of deflection at ultimate load, the following procedure may be used:

1. Obtain $P_u$, moment diagram, and mechanism (Rule 5, 6, 11).

2. Compute deflection of frame segments assuming, in turn, that each hinge is the last one to form.

3. Correct deflection is largest value obtained.

4. Check by "kink-removal" process.

**Slope-Deflection Eq.:**

$$\theta_A = \frac{A}{E} + \frac{A}{E} + \frac{L}{EI} \left( \frac{M_{AB} - M_{BA}}{2} \right)$$

### 42. Deflection at Working Load:

If computation of beam deflections at working load is required, this may be done by reference to handbook tables.

An upper limit of the deflection of a frame at working load is obtained by dividing the deflection at ultimate load (Rule 41) by $F$.

### References

12.1 Huber, A. W. Beedle, L. S.  
"RESIDUAL STRESS AND THE COMRESSIVE STRENGTH OF STEEL",  

12.2 British Constructional Steelwork Association  
"THE COLLAPSE METHOD OF DESIGN",  
BCSA, Publication No. 5, 1952.
Lecture No. 13

ANALYSIS AND DESIGN EXAMPLE

SCOPE: Application of derived methods of analysis and design to typical gabled frame.

Given: Gabled Portal Frame with indicated Loading (see Lincoln Arc Welding Series, No. 129 to 132).
Spacing of frames 16'.
Constant section assumed throughout frame.
Load factor of safety: (See Lecture #11)
- Dead Load + Snow: $F = 1.88$
- Dead Load + Snow + Wind: $F = 1.41$

Find: Required Section.

Three Loading Condition:

(1) Dead Load + Snow, $F = 1.88$, Fig. (13.2).
(2) Dead Load + 1/2 Snow, $F = 1.88$, Fig. (13.2).
(3) Dead Load + Snow + Wind, $F = 1.41$, Fig. (13.2).

Condition (1): Fig. (13.2) and (13.3)

Taking into account symmetry with respect to $y$

Number of possible plastic hinges $N = 3$
Number of redundancies $X = 2$

$3 - 2 = 1$ mechanism
Fig. (13.3) shows mechanism. In computing rotation of hinges it is advantageous to use the notion of instantaneous center as described in Lecture #6. Assume hinge 1 rotates through the angle $\theta$. Hinge 2 will move perpendicular to 1-2, hence the instantaneous center for 2-3 is on line through 1-2. Hinge 3 will move vertically, hence the instantaneous center is on a horizontal through 3-3'; the resulting center is at C.

The location of hinge 3 is fixed by the undetermined distance $X$. The distance (2 to C) = $\frac{1}{3}X$ by geometry. The rotation about C is $\frac{14}{X/3} \theta$. From the principle of virtual displacement:

$$M_p \theta \left[1 + \left(1 + \frac{42}{X}\right) + \frac{42}{X}\right] = 41.2 \cdot \frac{42}{X} \theta \left[\frac{X}{30} \cdot \frac{X}{2} + \frac{(30-X)}{30} X\right]$$

or

$$M_p = \frac{14.4X(60-X)}{42 + X} \text{ (ft.-k)}$$

$X$ must be determined such that $M_p$ becomes a maximum (see upper bound theorem which demands $P_{\min}$, here $P$ is given, hence $M_p$ must be a maximum). Either by differentiating or by working a few trials:

For $X = 23.45$, $M_p = 188.6 \text{ ft.-k}$
The corresponding moment diagram for condition (1) is shown in Fig. (13.4)

\[ M_3 = \frac{x}{30} M_2 + 1 - \frac{x}{30} M_{\text{top}} + \frac{41.2}{2} \frac{30}{30} \left( \frac{x}{30} - \frac{x}{30} \right) \]

\[ M_{\text{top}} = 158 \text{ ft.-k} \]

As an alternative procedure the distributed loads of condition (1), Fig. 13.2, are replaced by 3 equivalent concentrated loads as shown in Fig. 13.5.
The location of hinge 3 must be under one of the concentrated loads, since the shear must vanish at that point. Assuming it as shown in Fig. 13.5 leads to the following equation:

$$M_p \theta \left[ 1 \left(1 + \frac{14}{8.33}\right) + \frac{14}{8.33} \right] = 13.73 \frac{14}{8.33} \theta (5 + 15 + 25) \quad (13.4)$$

which yields:

$$M_p = 193.2 \text{ ft.-k} \quad (13.5)$$

Compared to Eq. (13.2) the equivalent concentrated loads give an approximately 2% higher $M_p$. Note that assuming hinge 3 under the middle load (15 ft. from left) would have resulted in

$$M_p = 176.8 \text{ ft.-k}$$

which is smaller than the above given value of Eq. (13.5). An equilibrium check however, would show that this mechanism violates the plasticity condition.

**Condition (2): Fig. (13.6)**

Number of possible hinges $N = 7$

Redundancies $X = 3$

$$7-3 = 4 \text{ local mechanisms}$$
(a) Beam Mechanism 2-3-4
(b) Beam Mechanism 4-5-6
(c) Sidesway 1-2-6-7
(d) Gable 2-4-6-7

Computation for (a) to (c) are elementary. Mechanism (d) is shown in Fig. 13.7.

Virtual work equation:

\[ M_p \theta \left( 1 + \frac{7}{10} + \frac{7}{10} + \frac{7}{10} + \frac{7}{10} \right) = \frac{7}{10} \theta \left( 13.0 \cdot 15 + 41.2 \cdot 15 \right) \quad (13.6) \]

\[ M_p = 118.6 \text{ ft.-k} \quad (13.7) \]

Combination with Beam Mechanism 2-3-4 will eliminate a hinge at 4. Hinge 3 is only fixed by parameter X as shown in Fig. 13.8.

\[ M_p \theta \left( 1 + \frac{7}{10} + \frac{7}{10} + \frac{60-X}{10} \right) = \frac{7}{10} \theta \left[ 13.0 \cdot 15 + \frac{41.2}{30} (30-X)(30+\frac{30-X}{2}) \right] + \frac{60-X}{10} \frac{7}{10} \frac{41.2}{30} X \frac{X}{2} \quad (13.8) \]

\[ \downarrow \downarrow \quad DL+S = 41.2 \text{ k} \quad \downarrow \downarrow \downarrow \quad DL = 13.0 \text{ k} \quad \downarrow \downarrow \]

![Diagram](image-url)
The maximum value occurs for

\[ X = 20.1' \quad M_p = 138.1 \text{ ft.-k} \]
An equilibrium check will immediately reveal if other combinations should be considered. Using Rule 8 (derived in Lecture #7):

Number of redundants \( X = 3 \)
Developed hinges \( M = 4 \)
Remaining redundancy \( I = 3 \cdot (4-1) = 0 \)

Hence the frame is reduced to a statically determinate system, and all bending moments can be determined by simple statics. Assuming moments as positive if producing tensile stresses on inner side of frame the moments at points of plastic hinges are:

\[
\begin{align*}
M_2 &= -M_p = -138.1 \text{ ft.-k} \\
M_3 &= M_p \\
M_6 &= -M_p \\
M_7 &= M_p \\
\end{align*}
\]

Shear Equilibrium Sidesway:

\[
\begin{align*}
M_1 - M_2 + M_6 - M_7 &= 0 \\
M_1 &= M_p (-1 + 1 + 1) = M_p = 138.1 \text{ ft.-k} \\
\end{align*}
\]

Beam Equilibrium 2-4: \((X = 20')\)

\[
\begin{align*}
M_3 &= \frac{1}{3} M_2 + \frac{2}{3} M_4 + \frac{wL^2}{2} (\frac{X}{L} - \frac{X^2}{L^2}) \\
M_4 &= +47.7 \text{ ft.-k} \\
\end{align*}
\]
Beam Equilibrium 4-6 \( (M_5 \text{ at middle}) \)

\[
M_5 = \frac{1}{2} M_4 + \frac{1}{2} M_6 + \frac{13.0}{8} \times 30 = 3.30 \text{ ft.-k} \tag{13.14}
\]

The bending moment diagram shown in Fig. 13.9 does not violate plasticity condition, hence mechanism is actual failure mechanism.

Again loading condition (2) can be handled by assuming equivalent concentrated loads as shown in Fig. 13.10.

Quite obviously the mechanism will be of the same type as the one for the distributed loads, Fig. 13.8. However, the location of hinge 3 is not known in advance, the true location being the one for which the maximum value of \( M_p \) results.
For mechanism of Fig. 13.10 one gets:

\[ M_p = \left[ 1 + (1 + \frac{7}{10}) + (\frac{7}{10} + \frac{21}{10}) + \frac{21}{10} \right] = \]
\[ 13.73 \cdot \frac{21}{10} \cdot (5 + 15) + 13.73 \cdot \frac{7}{10} \cdot 35 + 4.33 \cdot \frac{7}{10} \cdot (5 + 15 + 25) \quad (13.15) \]
or

\[ M_p = 138.1 \ \text{ft.-k} \quad (13.16) \]

Changing the hinge location from 3 to 3' leaves the value for \( M_p \) practically unchanged such that exact correspondence to the value derived for distributed loads is established. Which of the two procedures -- using distributed loads or equivalent concentrated loads -- to follow is left to the readers discretion.

**Condition (3):**

Mechanism is of same type as the one of Fig. 13.8 for loading condition (2). The only difference is that wind forces are also present. Resulting expression for \( M_p \) becomes:

\[ M_p = \frac{x(1697 - 31.5x)}{84 + 2x} \quad (\text{ft.-k}) \quad (13.17) \]

Maximum value for

\[ x = 21.7' \quad M_p = 173.0 \ \text{ft.-k} \quad (13.18) \]

Equilibrium check is not necessary, would follow pattern of previous check.
Comparison of results show that loading condition (1) requires the largest plastic moment $M_p = 188.6 \text{ ft.-k}$, hence governs.

**Design of Section:**

Required $M_p = 188.6 \text{ ft.-k}$

Plastic Moment $M_p = 1.14 \sigma_y S$

\[
1.14 = \text{shape factor (Lecture #2)} \\
\sigma_y = 33 \text{ ksi (structural steel)}
\]

Required $S = \frac{188.6 \cdot 12}{1.14 \cdot 33} = 60.1 \text{ in.}^3$

Lightest WF-Beam: $16WF40$ $S = 64.4 \text{ in.}^3$

$b/t = 13.9$

Comparison with Lincoln Design (elastic) shows that the resultant beam sections are identical. However, the Lincoln Design calls for welded reinforcements at the corners and the base of the columns.
SCOPE: This lecture is a continuation of Lecture #13 on examples of analysis and design methods. Previous lecture illustrated problems associated with gabled frame, investigating all possible loading conditions, checking equilibrium and design of section. Example will be given here of complete design procedure for an industrial building frame in which the applicable "Rules" of Lecture #12 are investigated. Of particular interest is the matter of economic choice of section.

14.1 "PRELIMINARY DESIGN"

On what basis is the first choice of relative plastic moment values made? In the various examples used to illustrate methods of analysis, the problem was to find the ultimate load for a given structure with known plastic moment values of its members. In the prior design examples, an assumption was made of the relative moment strength of the various parts of the frame. With the loading specified, the actual plastic moment values were then computed. Since "uniform section throughout" may not be the most economic solution, some guide is needed for selecting the ratio(s) of plastic moment strength.

Of course, this problem exists in elastic design, so it is not a matter that is unique to design on the basis of ultimate strength. However, a few simple techniques will occur to the designer which, coupled with his experience, will enable
him to make a preliminary economic choice of relative moment strength without too many trials. This is illustrated in Example 14.1. Some general principles are as follows:

(a) In the event the critical mechanism is an elementary one, the rest of the material in the frame is not being used to full capacity. This suggests that a more efficient choice of moment ratios may be made such that the critical mechanism is a "composite mechanism" involving plastic hinges in several different members.

(b) Adjacent spans of continuous beams will often be most economically proportioned when the elementary mechanisms for each span form simultaneously. This is illustrated in Example 4.3. Numerous examples of the design of continuous beams are given in Ref. 12.2.

(c) The formation of mechanisms simultaneously in different spans of continuous beams or the creation of composite mechanisms will not necessarily result in minimum weight. Examination of alternate possibilities is desirable. Often it will be found that the span involving the greatest determinate moment ($M_s$) should be given the greatest possible restraint (generally by supplying equivalent $Z$ of adjoining members). Example 14.1 illustrates this.
(d) The absolute minimum beam section for vertical load is obtained if the joints provide complete plastic restraint (i.e., restraining members supply a restraining plastic moment equal to that of the beam). Similarly, the minimum column sections are obtained under the action of sway forces when ends are subject to complete plastic restraint. This therefore suggests that, if the important loads are the vertical loads, the design might well be commenced on the basis that all joints are restrained as described, the ratio of beam sections be determined on this basis, and that the columns be proportioned to provide the needed joint moment balance and resistance to side load (Ref. 7.1). Example 14.1 is an illustration. Alternatively, if the important loads were side loads, the design could start, instead, with the columns.

(e) Finally, it should be kept in mind that maximum over-all economy is not necessarily associated with the most efficient choice of section for each span. It is necessary to consider fabrication conditions which may dictate uniform section where, theoretically, sections of different weight might be used.

14.2 DISTRIBUTED LOAD

Previous examples indicate that special treatment is needed when analysis is made of girders under uniform load. The position of the hinge in the beam is not known precisely.
If the load is actually distributed load, then the most economic design is obtained by analysis on this basis, including a determination of the position of maximum moment. Horne (Ref. 7.1) suggests that hinges be assumed at midspan with a correction to be made at the end of the design (facilitated by the use of charts specially devised to solve the problem).

Of course, if the distributed load is actually brought to the main frame through purlins and girts, the uniform load may be converted, at the outset, to actual purlin reactions (on the basis of assumed purlin spacing). The analysis is then made on the basis of the actual concentrated loads. The only difficulty with this procedure is that numerous additional possible plastic hinges are created -- one at each purlin. And for every possible hinge position there is another possible mechanism (Rule 9). Of course, with experience the designer will be able to tell how many of these mechanisms he should investigate.

**14.3 DESIGN OF TIER BUILDINGS**

When provision is made for wind bracing in wall panels, an approach to an economic design would be achieved through a partial application of the plastic methods. The beams and girders would be proportioned for full (plastic) continuity. The columns, on the other hand, would be proportioned according to conventional ("elastic") methods.
By this procedure, none of the plastic hinges participate in the resistance to side load. Such load is all carried by the diagonal bracing. The only mechanisms are the beam mechanisms. Of course, the top one or two stories might be designed by a "complete" plastic analysis, hinges forming both in the columns and in the beams.

14.4 EXAMPLES

EXAMPLE 14.1

An industrial frame will be designed and a complete check of the applicable "Rules" will be made. Illustrated will be a preliminary design procedure by which the plastic moment ratios will be estimated.

Frame and Loading

The frame of Fig. 14.1 is to be designed to carry a vertical load of 3 k/ft. and a side load of 0.6 k/ft. Purlin and girt spacing at 7 1/2'. The span and loading are such that it will be economic to select the most efficient section for each span. Vertical distributed load will be replaced by loads at the 1/4-point.
Loading Conditions

Case 1: Dead Load + Live Load \( F = 1.88 \)
\[ w = 3.0 \times 1.88 = 5.64 \text{ k/ft}. \]

Case 2: Dead Load + Live Load + Wind Load \( F = 1.41 \)
\[ w_v = 3.0 \times 1.41 = 4.23 \text{ k/ft}. \]
\[ w_h = 0.6 \times 1.41 = 0.85 \text{ k/ft}. \]

Case 1 will be investigated first, and case 2 will be checked using the plastic moment ratios obtained in case 1.

CASE 1 - Preliminary Design (Rule 19)

Minimum plastic moment values:

Absolute minimum plastic moment values are determined by fixing the joints against rotation, but frame free to sway.

Note that there is no side load in this case.

Fig. 14.2
Solving the beam mechanism:

\((4-6 = M_p; \ 8-10 = k M_p)\)

\#1: \(M_p \cdot \theta (1 + 1 + 1 + 1) = P_1 \cdot \theta \left(\frac{L_1}{4}\right) \) \(\text{(2)}\) \(\text{(14.1)}\)

\[ M_p = \frac{P_1 L_1}{8} \]

\#2: \(k \cdot M_p \cdot \theta (1 + 1 + 1 + 1) = P_2 \cdot \theta \left(\frac{L_2}{4}\right) \) \(\text{2}\) \(\text{(14.2)}\)

\[ k \cdot M_p = \frac{P_2 L_2}{8} \]

\[ k = \frac{P_2 L_2}{P_1 L_1} = \left(\frac{5}{3}\right) \left(\frac{5}{3}\right) = \frac{25}{9} \quad k = 2.78 \]

Note: Plastic moment values determined from above are the least that would support the loads as fixed-ended beams.

Selection of Plastic Moment ratios:

\[ M_p (2-4) = M_p \]

\[ M_p (8-10) = 2.78 M_p \]

End columns will be proportioned to give full restraining moments.

\[ M_p (1-4) = M_p \]

\[ M_p (3-10) = 2.78 M_p \]

Center column ratio is determined by considering equilibrium of joint 6-7-8:

\[ M_6 - M_7 - M_8 = 0 \]

\[ M_7 = M_6 - M_8 = -M_p + 2.78 M_p \]

\[ M_p (2-7) = 1.78 M_p \]

Plastic moment values:

\[ M_p (2-4) = \frac{P_1 L_1}{8} = \frac{127 \times 45}{8} = 715.1 k \]

\[ M_p (8-10) = k M_p = 1990.1 k \]
Equilibrium Check (Rule 11)

Although an analysis of other possible composite mechanisms might be made, an equilibrium check will be made by drawing the moment diagram (Fig. 14.3)

Horizontal Shear:

\[
H_1 = \frac{M_p}{h} = 47.6 k \\
H_2 = \frac{1.78M_p}{h} = 84.8 k \\
H_3 = \frac{-2.78M_p}{h} = 132.4 k
\]

\[
H = \frac{M_p}{h} (1 + 1.78 - 2.78) = 0 \quad \text{OK}
\]

Vertical Reactions:

\[
V_1 = P_1 = 127 k \\
V_2 = P_2 = 212 k \\
V_3 = P_1 + P_2 = 339 k
\]

Since 1/4-point loading gives the same maximum mid-span moment as does uniformly distributed load, no refinement in $M_p$ is required and

\[
M_p = 715 k
\]

CASE 2 - Analysis

Will a greater plastic moment be required for Case 2 (with wind) using the same plastic moment-ratios as determined for Case 1?
No. of possible hinges = N = 7*
Redundants = X = 3
Mechanisms n = 4

2 beam mechanisms
1 panel mechanism
1 joint mechanism

Beam Mechanism:

\[ M_p \] may be determined from

Case 1:

\[ M_p = \frac{1.41}{1.88} \times 715 \cdot M_p = 536 \cdot k \]

Panel Mechanism 4:

\[ M_p \cdot \theta (1 + 1.78 + 2.78) = W \cdot \theta \cdot \frac{h}{2} \]

\[ M_p = \frac{(12.75)(7.5)}{5.56} = 17.2 \cdot k \]

Composite Mechanism 5 (1 + 2 + 3 + 4):

\[ M_p \cdot \theta \left( \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \cdot k + \frac{4}{3} \right) = P_1 \cdot \theta \left( \frac{L_1}{4} + \frac{L_1}{12} \right) + P_2 \cdot \theta \left( \frac{L_2}{4} + \frac{L_2}{12} \right) + W \cdot \frac{h}{2} \]

\[ M_p (10.08) = \frac{P_1 L_1}{3} + \frac{P_2 L_2}{3} + \frac{Wh}{2} = \frac{P_1 L_1}{3} \left( 1 + \frac{25}{9} \right) + W \cdot \frac{h}{2} \]

\[ = 1.26 \cdot (95) \cdot (45) + (12.75)(7.5) \]

\[ M_p = 543 \cdot k \]

* Only one hinge need be computed in each beam. The other is identical insofar as the virtual work equation is concerned.
Note: Since this value is very close to value obtained for Mechanism 1(536 kN), and is an upper bound, the elementary mechanism probably controls.

CASE 2 - Equilibrium Check

The unknown moment value will be computed on the basis that Mechanisms 1 and 2 are critical.

Panel (sidesway) equilibrium (Rule 13):

\[-M_4 + M_7 + M_{10} + W \frac{h}{2} = 0\]
\[M_7 = M_4 - M_{10} - \frac{wh}{2}\]
\[= -M_p + 2.78 M_p - \frac{wh}{2}\]
\[M_7 = 1.78 M_p - W \frac{h}{2}\]

Thus, the moment diagram is the same as Fig. 14.3, except \[M_7 < 1.78 M_p\].

Result: Case 1 is critical since for Case 2 required \[M_p = 536 kN\] which is less than the value \[715 kN\] found for Case 1.

Selecting the Section (Rule 3)

Left Beam Left Column \[M_p (4.6) = 715 kN = f \sigma_y S\]
\[S_{4-6} = \frac{(715)(12)}{(1.14)(33)} = \frac{228 \text{ in.}^3}{\text{use 27WF94}}\]
\[(S = 242.8)\]
\[f = 1.15\]

Right Beam Right Column \[S_{8-10} = (2.78)(228) = 634 \text{ in.}^3\]
\[\text{use 36WF104}\]
\[(S = 663.6)\]
\[f = 1.16\]

Center Column: \[S_{2-7} = (1.78)(228) = 406 \text{ in.}^3\]
\[\text{use 33WF130}\]
\[(S = 404.8)\]
\[f = 1.15\]

Note: Since \(f > 1.14\), this section will supply adequate \(M_p\).
The frame as designed is shown in Fig. 14.6.

Now that the sections have been selected, the design will be checked according to the various applicable "Rules" of Lecture #12.

**AXIAL FORCE** (Rule 21)

Left Column (27WF94, A = 27.93):

\[
\frac{P}{P_y} = \frac{V_1}{\sigma_y A} = \frac{127k}{(33)(27.93)} = \frac{.138}{\sigma_y A} (< .15 \text{ -- OK})
\]

Center Column (33WF130, A = 38.26):

\[
\frac{P}{P_y} = \frac{V_2}{\sigma_y A} = \frac{339}{(33)(38.26)} = \frac{.269}{\sigma_y A} (> .15 \text{ -- n.g.)}
\]

Thus a larger section is required to develop the required moment. From Rule 21, the new value of S is computed

\[
S_{req} = S \times \frac{M_p}{N_{pc}} = S \left( \frac{.85}{1 - \frac{P}{P_y}} \right) = \frac{406 (.85)}{1 - .269} = 472 \text{ in.}^3
\]

Note: See "Revised Design" below.

Use 36WF150

\[
(S = 502.9, \quad A = 44.16, \quad f = 1.15)
\]

Right Column:

Okay by comparison with center column.
MAXIMUM ALLOWABLE SHEAR (Rule 22)

Left Beam: \((27WF94, \ w = .490, \ d = 26.91)\)

\[ V_{\text{max}} = 17,000 \cdot w \cdot d = 17,000 \cdot (.490) \cdot (26.91) = 224k \]

\[ V_{\text{actual}} = 127k \]

Right Beam: \((36WF194, \ w = .770, \ d = 36.48)\)

\[ V_{\text{max}} = 17,000 \cdot w \cdot d = 17,000 \cdot (.770) \cdot (36.48) = 478k \]

\[ V_{\text{actual}} = 210k \]

REDUCTION DUE TO SHEAR FORCE (Rule 23)

The two beams and the right column will be the most critical.

Values of "a", given in Rule 23

Left Beam:

\[ 4a = \frac{L_1}{2}, \quad a = \frac{L_1}{8} \]

\[ \frac{L_1}{d} = \frac{\left(\frac{45}{12}\right)}{26.91} = 20.2 \]

\[ \therefore \frac{a}{d} = \frac{20.2}{8} = 2.51 \quad (< 3.0 \quad \text{-- n.g.}) \]

Since \( \frac{a}{d} < 3.0 \), a larger value of \( S \) is required. Using the equation given in Rule 23,

\[ \frac{M_p}{M_{ps}} = \frac{1}{.654 \cdot .117 \cdot \frac{a}{d}} = \frac{1}{.944} \]

\[ S_{\text{req}} = \frac{228}{.944} = 242 \text{ in.}^3 \]

The 27WF94 is still satisfactory since its section modulus is 242.8 in.³
Right Beam: (36WF194, d = 36.48)

$$4a = \frac{L_2}{2}, \quad a = \frac{L_2}{8}$$

$$L_2/d = \frac{(75)(12)}{36.48} = 24.7$$

$$a/d = \frac{24.7}{8} = 3.1 \quad (>3.0 \quad OK)$$

Right Column:  

$$a = h$$

$$\frac{h}{d} = 4.92$$

$$\frac{a}{d} = 4.92 \quad (>3.0 \quad OK)$$

**Columns - Weak Axis** (Rule 29)

Since the end columns are braced at mid-height, the center column will be the critical one (36WF150, $$r_y = 2.38$$). Is additional bracing needed?

$$\frac{L}{r_y} = \frac{(15)(12)}{2.38} = 75.6$$

Since $$\frac{P}{F_y} < .6$$, then, from Rule 29, $$\frac{L}{r}$$ is adequate without bracing.
**CORNER CONNECTIONS** (Rules 30, 32, 38) Use straight connections.

**Connection 2** \((27\text{WF}94, b = 9.99, t = .747, w = .490, d = 26.91)\)

Required web thickness:

\[
wr = \frac{2S}{d^2} = \frac{2(242.8)}{(26.91)^2}
\]

\[
w_r = .670 \\
w = .490 \\
\]

Rule 32:

\[
t_s = \frac{\sqrt{2}}{b} \left( \frac{S}{d} - \frac{wd^2}{2} \right) = 0.354"
\]

Rule 38:

\[
\left( \frac{b}{2} \right) \max. = 8.0 \\
t_s = 0.625"
\]

Use 5/8" plate for diagonal stiffener:

**Connection 10** \((36\text{WF}194, b = 12.117, t = 1.26, d = 36.5, w = .770)\)

Required thickness of diagonal stiffener:

\[
t_s = \frac{\sqrt{2}}{b} \left( \frac{S}{d} - \frac{wd^2}{2} \right) = \frac{2}{12.117} \left( \frac{664}{36.5} - \frac{(0.770)(36.5)}{2} \right) = .48"
\]

This value is much less than flange thickness \((t = 1.26")\).

Rule 38 governs in this case:

\[
\left( \frac{b}{2} \right) \max. = 8.0 \\
t_s = \frac{12.12}{2(8)} = .757"
\]

Use 3/4" plate. 
INTERIOR CONNECTION 6-7-8 (Rules 34, 35)

This connection is designed as a corner connection except the 27WF94 is joined to the "back plate" at 6. Since 6 transmits a part of the moment brought in at 8, a diagonal stiffener is probably not required.

A check may be made using Rule 30:

\[ w_r = \frac{2S}{d^2} \]  
(Values of S and d taken for member 7 -- corresponding to net moment transmitted.)

\[
\begin{align*}
w_r &= \frac{2(502.9)}{(35.84)^2} = .783'' \\
\text{w of } 36WF194 &= .770
\end{align*}
\]

OK -- no web stiffening required

Transmit thrust of lower flange of beam 4-6 by diagonal bracket (see Fig. 14.9) or by one of the two alternates shown dotted.

CROSS-SECTION PROPORTIONS (Rule 37)

<table>
<thead>
<tr>
<th>Shape</th>
<th>b/t &lt; 17</th>
<th>d/w &lt; 50</th>
</tr>
</thead>
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<tr>
<td>27WF94</td>
<td>13.3</td>
<td>55</td>
</tr>
<tr>
<td>36WF150</td>
<td>12.7</td>
<td>57.5</td>
</tr>
<tr>
<td>36WF194</td>
<td>9.5</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All OK</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The first two shapes have rather high d/w ratios. Alternates, respectively, are 24WF100 (d/w = 51) and 33WF152 (d/w = 52). These would increase the weight of frame by about 1 1/2% of total weight. On the other hand, since only beam mechanisms are involved (load carried by "fixed-ended" beams) then the full rotation capacity will probably not be required (see Example 8.7). This, coupled with the less-than-critical b/t ratios, probably makes the original choice satisfactory.

**BRACING REQUIREMENTS (Rule 40)**

At connections: In addition to the purlins at the corner and interior connections, brace at points marked "B" in Figs. 14.7 to 9. (Use light truss normal to plane of frame).

Within beams:

The critical length is given by Eq. 10.11,

\[ L_{cr} = 20 \rho_y \]  

(10.11)

This is to be compared with the purlin spacing \((L_p)\) or to the length of hinge, \(\Delta L\). The latter, for full plastic restraint, may be shown to have a length equal to one-quarter of the span.
Left Beam (27WF94, L = 45', r_y = 2.04"):

\[ L_{cr} = 20r_y = 41" \]

If braced by the purlins

\[ L_p = (7.5)(12) = 90" \]

Actual length of hinge =

\[ = \frac{L_p}{4} = 11.25 \text{ feet} \]

Since \( L_p > L_{cr} \), additional bracing is needed if full rotation capacity is required.

Undoubtedly, Section 5 is one of the last hinges to form. So additional bracing is not required.

Right Beam (36WF194, L = 75', r_y = 2.49):

\[ \frac{L_p}{r_y} = \frac{(7.5)(12)}{2.49} = 36.1 \]

Although greater than 20, again, Section 9 is one of last hinges to form. Design OK.

REVISED DESIGN

In the design just completed, it will be noted that the required section modulus for the right beam and column was 634 in.\(^3\). However, the lightest possible WF shape available for these members had a section modulus of 663.6 in.\(^3\). If further economy is desired, the frame may be analyzed again using the actual plastic moment of the 36WF194 for hinges that form in
the right-hand beam and column. This would require a restai-
ing moment of lesser magnitude at the left end of the beam --
and it would be expected, then, that a lighter section might be
specified for the left beam or center column or both. This
procedure is explored as follows:

Plastic moment ratios:

These are indicated in

Fig. 14.11. The value

$M_p$ is the same as com-
puted in the first
design, since the 27WF94 just meets the requirements.

$M_p (8-10) = \sigma_y f S = (33)(1.16)(663.6) = 25,400 \text{ in.-kip}$

$M_p (4-6) = \quad = 8,580 \text{ in.-kip}$

Analysis of beam mechanism (8-10)-Fig. 14.12

$\left( M_p + m M_p \right) \theta + 25,400 \left[ \left(1 + \frac{1}{3}\right) + \frac{1}{3} \right] = \frac{P_2 L_2}{4} \theta + \frac{P_2 L_2}{4} \theta \left( \frac{1}{3} \right)$

$m = \frac{P_2 L_2 / 3 - (1.67)(25,400)}{M_p}$

$m = \frac{(212)(75)(12)/3 - (1.67)(25,400)}{8,580} - 1 = 2.47 - 1 = 1.47$

Selection of Section for center column:

$S_1 = \frac{m M_p}{\sigma_y f} = \frac{(1.47)(8580)}{(33)(1.14)} = 336 \text{ in.}^3$
Modification for direct stress (Rule 21)

Using the \( P/P_y \)-value computed for the first design (\( .269 \))

\[
S = S_1 \left[ \frac{M_p}{M_{pc}} \right] = S_1 \left( 1 - \frac{P}{P_y} \right) = \frac{(336)(.85)}{(.73)}
\]

\[ S = 391 \text{ in.}^3 \]

Use 33WF130
\( (S = 404.8, f = 1.15) \)

By this method, 20#/ft. was saved on the center column, the other two members being the same as before.

COMPARISON WITH WEIGHT OBTAINED IN AN ELASTIC DESIGN:

Elastic Design - 27,600#
Plastic Design - 25,050#

EXAMPLE 14.2

A single span portal frame with gabled roof will be designed to resist vertical and side load. The example is chosen to illustrate use of the chart presented in the Appendix to Lecture #6. The example is the same as Design Problem No. 1 in Ref. 10.2.
Frame and Loading

The frame of Fig. 14.12 is to be designed for the vertical load shown. Treat the vertical distributed load as such. For use in the "chart" replace the horizontal distributed load by a single concentrated load, acting at the eaves, which produces the same moment about point 1. In other words, it is a concentrated load which produces an overturning moment equal to that of the uniformly distributed load. Greatest economy should be achieved for this example through use of uniform section throughout. (The "chart" was developed on this basis.)

Loading Conditions:

Load P for case 2: \( P \times 20' = 600 \times 35 \times \frac{35}{2} \times 1.41 \)

\[ P = 25.8k \]

For use in the chart: \( a = 20/100 \quad b/a = 3/4 \)

\( b = 15/100 \quad L = 100 \)
Case 1: \( A = 0 \)

From chart: \( \frac{M_p}{wL^2} = 0.046 \)

\[
\frac{M_p}{L^2} = 0.046 \times 1.88 = 0.0866 \text{ k/ft.}
\]

Case 2: \( A = 2 \alpha \)

\[
\frac{P}{wL} = 2 \left( \frac{.2}{1.41} \right) \frac{25.8}{(100)} = 0.0734
\]

From the chart: \( \frac{M_p}{wL^2} = 0.055 \)

\[
\frac{M_p}{L^2} = 0.055 \times 1.41 = 0.0775 \text{ k/ft.}
\]

Case 1 (without wind) is critical.

Section Selection:

\[
M_p = 0.0866 \ L^2 = 866 \text{ ft.-kips}
\]

\[
S_{req} = \frac{(866)(12)}{(33)(1.14)} = 276 \text{ in.}^3
\]

Use 30WF108 (S = 299)

The remaining "rules" would then be check as for example 14.1.

Note: Elastic design (Ref. 10.2) required 30WF124.

**EXAMPLE 14.3**

Example 14.2 will be solved, except that tapered haunches will be used. (This is the same as design problem No. 3 in Ref. 10.2). The problem will be worked, first, for uniform section throughout and, in another example, different shapes will be used for beam and column. The equilibrium method of analysis will be used.
Frame and Loading

The frame is sketched in Fig. 14.14. Loading is the same as in Example 14.2, except concentrated loads applied at purlins (5' spacing) and side load replaced by equivalent concentrated load as in Example 14.2.

Loading Conditions (same as investigated in Ref. 10.2)

Case 1 (without wind) D.L. + L.L. (F = 1.88)

\[ P = 5.0 \times 1.0 \times 1.88 = 9.4 \text{ kips} \]

Case 2 (without wind) D.L. + L.L. + Wind (F = 1.41)

\[ P = 5.0 \times 1.0 \times 1.41 = 7.05 \text{ kips} \]

\[ Q = \frac{600 \times (35)^2 \times 1.41}{20} = 25.8k = 3.67P \]
Analysis for Case 1 (Rule 5)

(1) Redundant selected as force $H_1$

(2) Moment diagram for determinate structure (shown by solid line):

(3) Moment diagram due to loading by redundant:

(4) Composite moment diagram:
This has been sketched on Fig. 14.15(b) such that a mechanism is formed with hinges at Section 2 and at 2nd purlin from crown. (The position of the latter hinge is determined from the composite moment diagram.)
(5) Equilibrium Solution:

Equating the moment at Section 2 to the moment at 4,

\[ H_1 (14) = M_1 - H_1 (32) = M_p \]

\[ H_1 = \frac{M_p}{14} \]

\[ M_1 = (10P)(40) - P(5+10+15+20+25+30+35+\frac{40}{2}) \]
\[ = 240P \]

\[ M_p = \frac{240P}{(1+\frac{32}{14})} = 73.3 \text{ P'} k = 688 \text{ k} \]

Analysis for Case 2 (Rule 5)

(1) Redundant selected as \( H_9 \).

(2) Moment diagram for determinate structure (solid line):

\[ M_c = Q (20) \]
\[ = (3.67P)(20) = 73.4 \text{ P} \]

\[ M_s = 250P \]

(3) Moment diagram due to loading by redundant (similar to Fig. 14.15(c))
(4) Composite moment diagram:

This has been sketched on Fig. 14.16(b) such that a
mechanism is formed with hinges at Section 8 and at the
3rd purlin from crown.

Mechanism: Fig. 14.16(c)

(5) Equilibrium Solution:

Equating moment at Section 8 to moment at 4,

\[ H_9 (14) = M_1 + (73.4 P)(\frac{65}{100}) - H_9 (30.5) = M_p \quad (14.6) \]

\[ M_1 = (10P)(35) - P(5+10+15+20+25+30+\frac{35}{2}) \]

\[ = 227.5 \text{ P} \]

\[ H_9 = M_p/14 \]

\[ M_p = \frac{227.5P + 47.6P}{1 + \frac{30.5}{14}} = 86.5 \text{ P'}k = 610'k \]

Case 1 (without wind) is critical

Selection of Section (Rule 3)

\[ S = \frac{M_p}{\sigma_y f} = \frac{(688)(12)}{(33)(1.14)} = 219 \text{ in.}^3 \]

Use 24WF94

\[ (S = 220.9 \text{ in.}^3 \quad f = 1.15) \]

The haunch would then be designed (Rule 23) and the remaining
"Rules" would next be checked as for example 14.1.

Note: (1) Elastic design (Design Problem #3 of Ref. 10.2)

required 24WF94 girder and 30WF108 column. Plastic
design saved on the columns.

(2) We should not expect too much economy, because the
use of haunches also make possible a more balanced
elastic design.
EXAMPLE 14.4

It is evident from Fig. (14.15(b)) that the girder at Section 3 is not being used to full capacity. The moment there is about half of the plastic moment value. Addition economy may be achieved by using a lighter girder. This possibility will now be investigated, following the procedure for analysis of Case 1 in Example 14.3.

(1) Composite moment diagram:

As sketched, a mechanism forms with hinges at Sections 3 and 4. The problem is to find the required plastic moment of the girder, then to proportion the columns for the required moment at Section 2.

(2) Equilibrium Solution:

Equating the moments at Sections 3 and 4,

\[ H_1 \times 23 - M_1 = M_2 - H_1 \times 32 = M_p \]  

\[ M_1 = (10P) \times 10 - P \times \left(5 + \frac{10}{2}\right) = 90P \]

\[ M_2 = 240P \]

\[ H_1 = \frac{M_2 + M_1}{23 + 32} = \frac{330P}{55} = 6P \]

\[ M_p = 240P - 6P \times 32 = 48P = 451,1k \]
Required plastic moment at Section 2 is determined from

\[ M_p (1-2) = H_1 \ (14) \]

\[ M_p (1-2) = 6P \ (14) = 789 \frac{1}{k} \]

Selection of Section:

Girder: \[ S = \frac{451 (12)}{33(1.14)} = 144 \text{ in.}^3 \]
Use 21WF73
\[ (S = 150.7, f = 1.14) \]

Column: \[ S = 144 \frac{789}{451} = 252 \text{ in.}^3 \]
Use 27WF102
\[ (S = 266.3, f = 1.14) \]

COMPARISON OF EXAMPLES 14.2, 14.3, 14.4 WITH ELASTIC DESIGN

<table>
<thead>
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<th>EXAMPLE</th>
<th></th>
<th>ELASTIC</th>
<th>PLASTIC</th>
</tr>
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<tr>
<td>14.2-No Haunch - Uniform Section</td>
<td>Girder Column</td>
<td>30WF124 30WF108</td>
<td></td>
</tr>
<tr>
<td>14.3- Haunch - Uniform Section</td>
<td>Girder Column</td>
<td>24WF94 24WF94</td>
<td></td>
</tr>
<tr>
<td>14.4- Haunch - Different Sections</td>
<td>Girder Column</td>
<td>21WF73 27WF102</td>
<td></td>
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NOMENCLATURE AND TERMINOLOGY

A = Area of Cross-section.

A_{PF} = Area of both flanges of WF shape.

A_{St} = Area of Split-tee

A_{W} = Area of web between flanges.

B = c-axis intercept of extrapolated strain-hardening modulus.

b = Flange width

c = Distance from neutral axis to the extreme fibre.

d = Depth of section.

E = Young's modulus of elasticity

E_{st} = Strain-hardening modulus = \frac{d\sigma}{d\varepsilon}\bigg|_{st}.

E_{t} = Tangent modulus.

e = Eccentricity.

F = Load Factor of Safety.

f = Shape factor = \frac{M_{P}}{M_{y}} = \frac{Z}{S}.

G = Modulus of elasticity in shear.

G_{st} = Modulus of elasticity in shear at onset of strain-hardening.

H = Hinge rotation required at a plastic hinge.

I = Moment of inertia.

I_{e} = Moment of inertia of elastic part of cross-section.

I_{p} = Moment of inertia of plastic part of cross-section.

K_{L} = Effective (pin-end) length of column. K = Euler length factor.

L = Span length. Actual column length.

L_{cr} = Critical length for lateral buckling.

\Delta L = Length of plastic hinge.
M = Moment.

\( M_o \) = End moment; a useful maximum moment; hinge moment.

\( M_p \) = Plastic moment.

\( M_{pc} \) = Plastic hinge moment modified to include the effect of axial compression.

\( M_{ps} \) = Plastic hinge moment modified to include effect of shear force.

\( M_s \) = Maximum moment of a simply-supported beam.

\( M_y \) = Moment at which yield point is reached in flexure.

\( M_{yc} \) = Moment at which initial outer fibre yield occurs when axial thrust is present.

\( P \) = Concentrated load.

\( P_{cr} \) = Useful column load. A load used as the "maximum column load".

\( P_e \) = Euler buckling load. \( P_e = \pi^2 \frac{EI}{L^2} \).

\( P_f \) = Full load.

\( P_r \) = Reduced modulus load.

\( P_s \) = Stabilizing ("shakedown") load.

\( P_t \) = Tangent modulus load --- the load at which bending of a perfectly straight column may commence.

\[ P_t = \frac{\pi^2 E_t I}{L^2} \]

\( P_u \) = Ultimate load (theoretical).

\( P_w \) = Working load.

\( P_y \) = Axial load corresponding to yield stress level; \( P = A\sigma_y \).

\( Q \) = Side load.

\( R \) = Rotation capacity.

\( r \) = Radius of gyration.

\( S \) = Section modulus, \( I/c \).

\( S_e \) = Section modulus of elastic part of cross-section.
T = Force.
t = Flange thickness.
t_s = Stiffener thickness.
V = Shear force.

u, v, w = Displacements in x, y, and z directions.
W = Total distributed load.

W_{EXT} = External work due to virtual displacement.
W_{INT} = Internal work due to virtual displacement.

w = Distributed load per unit of length, web thickness.
x = Longitudinal coordinate.
y = Transverse coordinate.
Z = Plastic modulus, $Z = \frac{M_p}{\sigma_y}$.
Z_e = Plastic modulus of elastic portion.
Z_p = Plastic modulus of plastic portion.
z = Lateral coordinate.

δ = Deflection.
ε = Strain.

ε_{max} = Elongation at fracture (8" gage length unless otherwise noted).
ε_{st} = Strain at strain-hardening.
ε_y = Strain corresponding to first attainment of yield stress levels.
θ = Measured angle change; rotation.
ν = Poisson's ratio.
ρ = Radius of curvature.
\( \sigma = \text{Normal stress.} \)

\( \sigma_{ly} = \text{Lower yield point.} \)

\( \sigma_p = \text{Prop. Limit.} \)

\( \sigma_r = \text{Residual Stress.} \)

\( \sigma_{ult} = \text{Ultimate tensile strength of material.} \)

\( \sigma_{uy} = \text{Upper yield point.} \)

\( \sigma_w = \text{Working stress.} \)

\( \sigma_y = \text{Yield stress level.} \)

\( \tau = \text{Shear stress.} \)

\( \phi = \text{Rotation per unit length, or average unit rotation; curvature.} \)

\( \phi_y = \text{Curvature corresponding to first yield in flexure.} \)

**Mechanism (or "Hinge System")**

System of members (and/or segments of a member) that deforms at constant load. Used in the special sense that all hinges are plastic hinges (except pin ends).

**Mechanism Method**

Method of Plastic Analysis in which the principle of Virtual Displacements is applied to a mechanism created by the formation of sufficient plastic hinges.

**Moment Conventions**

Moment values are plotted on the tension side.

**Plastification**

The development of full plastic yield of the cross-section.

**Hinge Length**

Length of beam in which \( M \geq M_y \).

**Rotation Capacity**

Ability of structural member to rotate at near-maximum moment.

**Hinge Rotation**

Rotation required at a plastic hinge in order to realize computed ultimate load.
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