THE EFFECTIVE WIDTH OF CIRCULAR CYLINDRICAL SHELLS REINFORCED BY RIBS
(A Theoretical Study)

by

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Abstract

Formulas are derived for the effective width of circular cylindrical shells reinforced by ribs in the circumferential direction. In cases where the shell can be considered to extend to infinity the effective width depends on two parameters, $\sqrt{ah}$ and $\lambda = n \sqrt{\frac{h}{a}}$. The first parameter is a function of the radius "a" and the thickness h of the shell, the second contains in addition the factor n representing the influence of the stress distribution in circumferential direction.

Certain simplifications, the influence of which was investigated and found to be insignificant, are introduced in order to calculate tables and present graphs for the effective width in different cases.

For the limiting case where the radius "a" of the shell increases to infinity the correspondence to the effective width of a T-Beam with a straight axis is established.

Introduction

The problem of the effective width of T-Beams with a straight axis (Fig. 1) was extensively investigated during the past 30 years (see Ref. (1) to (5)).

*For list of references see p. 108.
actual stress distribution in the flange is replaced by an imaginary constant stress distribution over the effective width. Taking instead of the actual flange a flange of width equal to the effective width, the ordinary beam theory (cross sections remain plane) can be used to calculate the fiber stresses and the deflection of the rib. The advantages of this procedure are quite obvious.

The case of a curved T-Beam was taken up by U. Finsterwalder (6), H. Bleich (7) and Th. v. Karman (8). In Ref. (6) the general unsymmetrical case is treated with certain simplifications and the solution is not developed for practical applications. H. Bleich investigates the bending of curved knees of T- and H-section. In Ref. (8) a formula* for the effective width is given which does not coincide with the results of this dissertation.

The application of cylindrical shells stiffened by ribs in circumferential direction (Fig. 2) has entered many different fields, including shell arch roofs, airplane fuselages, pressure vessels, submarines, hot metal ladles, etc. The analysis of such structures

*v. Karman gives the formula $\frac{b}{2} = 0.54 \sqrt{ah}$ without any derivation. In the present dissertation it is shown that the numerical coefficient is not a constant.
is very involved, and there seems to be a specific need for establishing the effective width of cylindrical shells stiffened by ribs in order to simplify their analysis.

The present dissertation is a comprehensive study of this problem. Special attention was given to a simple presentation of the results in order to make them applicable for practical design purposes. An extra study was made to investigate the relation between the straight T-Beam and the effective width of the shell in case the radius of the shell increases to infinity.

The topics treated are as follows: First the effective width is defined. The case of axial symmetry is treated next. This rather simple case leads to the development of the fundamental ideas and prepares to attack the general case (unsymmetrical case). Finally the limit when the radius "a" of the shell increases to infinity is found to coincide with the results of the problem of the T-Beam with a straight axis. Some general considerations regarding the application of the effective width in practical problems are discussed.
I. Definition of the Effective Width

Consider a circular cylinder of radius "a" and thickness h around which a string under a "string force S" is stretched. The direct forces $N_\varphi = C_\varphi h$ in circumferential direction will have a distribution as shown in Fig. 3. ($N_\varphi$ is defined in Fig. 5.) Equilibrium for any line $\varphi = \text{constant}$ requires:

$$S = \int N_\varphi \cdot dx$$

(1)

where the integral is taken over the entire length of the shell. $S$ is taken positive for a compression force in the "string". The actual stress distribution can be replaced by an imaginary constant stress distribution of rectangular shape. The height of this rectangle is the $N_\varphi$ force at $x = 0$ (directly under the string). The width b is determined by the equilibrium condition:

$$S = b(N_\varphi )_{x=0}$$

(2)

$$b = \frac{S}{(N_\varphi )_{x=0}}$$

(3)

The width b is called the effective width of the cylinder under a string force S because the actual cylinder and a ring of width b whose cross section in x-direction is assumed to be rigid (constant $N_\varphi$ over the width b) are equivalent under the same string force S as far as the
direct force \((N\varphi)_{x=0}\) and the strain \((\varepsilon\varphi)_{x=0}\) are concerned.

Now consider the case of a cylinder stiffened by a rib in the circumferential direction. The loads are assumed to act on the rib only. Any general load case may be solved by assuming first the rib as rigid, then applying the corresponding reactions in the opposite direction to the rib and finally superimposing these two cases.

The connection between the rib and the shell is idealized as being along the two lines \(A\) (Fig. 4). Continuity requires that the stresses in the rib and the shell in circumferential direction are identical along these two connecting lines*. In a general case the stress distribution may be as shown in Fig. 4. By integrating with respect to \(x\) all forces \(N\varphi = \sigma\varphi h\) along a line \(\varphi = \text{a constant}\),
\[
S = \int N\varphi \, dx
\]
the action of the shell on the rib can be replaced by the single string force \(S\) acting in the middle plane of the shell. The circumferential stress along the lines \(A\)

* In this case the strains \(\varepsilon\varphi\) of the shell and the rib are discontinuous, because the rib is analysed as a beam for which the strain \(\varepsilon\varphi\) is proportional to the stress \(\sigma\varphi\). The shell on the other hand is analysed as a two dimensional structure where \(\varepsilon\varphi\) is influenced by the stress in the axial direction too (Poisson's ratio \(\nu\)). If the strains \(\varepsilon\varphi\) are assumed to be equal, the stresses will be discontinuous. The same question arises in the T-Beam problem. See e.g. Ref. (4) or (5).
is called $\sigma_A$. The direct force $N_x$ of the shell along these lines is the product of the stress $\sigma_A$ and the thickness $h$ of the shell. If the actual stress distribution is again replaced by a constant stress distribution over the effective width $b$, Eq. (3), the force $S$ becomes:

$$S = \int N_x \, dx = b(N_x)_{x=0} = bh \cdot \sigma_A \quad \text{(4)}$$

For an arbitrary cross section $y = \text{constant}$ assume the total bending moment to be $M$, the normal force $N$ to be zero. The stress distribution through the depth of the rib is assumed to vary linearly (ordinary beam theory). The action of the shell on the rib is represented by the string force $S$ as given by Eq. (4), acting in the middle plane of the shell. By taking moments around the axis $n-n$, the moment of the section is:

$$M = \int_{z_u}^{z_L} \sigma z t \, dz + Sz_A \quad \text{(5)}$$

Due to the straight line distribution of the stress over the rib section $\sigma$ can be written as function of the stress $\sigma_A$ at $A$:

$$\sigma = \frac{\sigma_A}{z_A} z \quad \text{(6)}$$
By use of Eq. (4) and (6) the moment $M$ is:

$$M = \frac{G_A}{Z_A} \left[ \int z^L \left( z^2 \right) dz + z^2 A \right]$$

(7)

Making use of the assumption that $N$ should be zero another relation can be derived:

$$N = \frac{G_A}{Z_A} \left[ \int z^L \left( z^2 \right) dz + S \right] = 0$$

(8)

The parenthesis of Eq. (8) represents the statical moment of the cross section consisting of the rib and a flange equal to the effective width of the shell around the centroid (axis n-n). In Eq. (7) the parenthesis is the moment of inertia of the same cross section. A similar derivation can be made for the case of a normal force $N$ and the bending moment $M$ equal to zero.

The following conclusions can be drawn:

1. By replacing the actual combination of rib and shell by a T-section consisting of the rib as web and the effective width $b$ of the shell as flange, the ordinary beam theory ($G = \frac{Mz}{I}$) can be used to calculate the fiber stresses $\sigma_u$ and $\sigma_L$ and the deflection of the rib. This cross section will be called the effective cross section. The string force $S$ acting on the shell is found by Eq. (4).
2. The effective width of the shell is found by stretching around the shell a string under a string force $S$, calculating the direct force $N_{\phi}$ directly under the string and applying Eq. (3).

II. Symmetrical case

1. General solution of the differential equation

The string around the cylinder is supposed to have a constant string force $S = S_0$ (Fig. 5a). The stress distribution is radially symmetric with respect to the axis of the cylinder. Three forces and 2 bending moments act on an infinitesimal shell element $dx \cdot d\varphi$ (Fig. 5c). The derivation of the differential equation and the corresponding solution can be found in various books, e.g. Ref. (9), (10), (11). The solution is expressed in terms of the radial displacement $w$:

$$w = e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta (l-x)} (C_3 \cos \beta x + C_4 \sin \beta x)$$  \hspace{1cm} (9)

$C_1$ to $C_4$ are constants of integration determined by the boundary conditions. $\beta$ is a coefficient depending on
the shell radius "a", the shell thickness h and Poisson's ratio \( \nu \) of the material:

\[
\beta^* = \frac{3(1-\nu^2)}{a^2h^2}
\]  \hspace{1cm} (10)

\( w \) has the form of two damped oscillations, one originating from the boundary \( x = 0 \) and the other one from the boundary \( x = l \). If the latter is sufficiently far removed so as to be considered at infinity (this condition is discussed on p. 25) the second part of Eq. (9) may be dropped, leaving the first two terms:

\[
w = e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x)
\]  \hspace{1cm} (11)

All forces and moments in the shell can be expressed as functions of \( w \), e.g. the direct force in circumferential direction:

\[
N_p = \frac{Eh}{a} w = e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) \frac{Eh}{a}
\]  \hspace{1cm} (12)

Of special interest is the string force \( S_0(x) \) at \( x \):

\[
S_0(x) = \int_0^\infty N_p \, dx = \frac{Eh}{2\beta a} e^{-\beta x} \left[ (C_1+C_2)\cos \beta x - (C_1-C_2)\sin \beta x \right]
\]  \hspace{1cm} (13)

Any unknown quantity such as a moment or a displacement, is of the form
\[ H = H e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) \quad (14) \]

Where: \( H \) = unknown quantity

\( H \) = coefficient depending on the quantity \( H \) under consideration

\( C_1, C_2 \) = constants, combinations of the constants of integration \( C_1 \) and \( C_2 \) depending on the quantity \( H \) under consideration.

In the following Table A the most important quantities in form of Eq. (14) are given. \( D \) is the bending stiffness of the shell:

\[ D = \frac{E h^3}{12(1-\nu^2)} \quad (15) \]

The symbols are explained in the list of notations, p. 103 and in Fig. 5. See Table A on the following page.

2. Effective width of an infinitely long cylinder

The cylinder is assumed to be infinitely long (Fig. 5a). At \( x = 0 \) a constant unit string force \( S_0 = 1 \) is applied. The value of \( S_0 \) fixes one of the constants of integration, the second one is determined from the condition of zero slope in \( x \)-direction at \( x = 0 \). By using Table A the boundary conditions are:

\( x = 0: \ E \frac{d w}{d x} = -E \beta (C_1 - C_2) = 0 \)

\( S_0(0) = \frac{E h}{2 \beta a} (C_1 + C_2) = \frac{1}{2} \quad S_0 = \frac{1}{2} \)
### TABLE A

<table>
<thead>
<tr>
<th>H</th>
<th>$H$</th>
<th>$\mathcal{C}_1$</th>
<th>$\mathcal{C}_2$</th>
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<tbody>
<tr>
<td>$Ew$</td>
<td>$E$</td>
<td>$C_1$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$E \frac{dw}{dx}$</td>
<td>$-E\beta$</td>
<td>$C_1 - C_2$</td>
<td>$C_1 + C_2$</td>
</tr>
<tr>
<td>$N_\psi$</td>
<td>$\frac{Eh}{a}$</td>
<td>$C_1$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$S_0(x)$</td>
<td>$\frac{Eh}{2\beta a}$</td>
<td>$C_1 + C_2$</td>
<td>$-(C_1 - C_2)$</td>
</tr>
<tr>
<td>$M_x$</td>
<td>$2D\beta^a$</td>
<td>$-C_2$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$M_\rho$</td>
<td>$2\nu D\beta^a$</td>
<td>$-C_2$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$Q_x$</td>
<td>$2D\beta^3$</td>
<td>$C_1 + C_2$</td>
<td>$-(C_1 - C_2)$</td>
</tr>
</tbody>
</table>

**General case:** \( H = He^{-\beta x} \left[ \mathcal{C}_1 \cos \beta x + \mathcal{C}_2 \sin \beta x \right] \)

**Special case:** \( E \frac{dw}{dx} = -E\beta e^{-\beta x} \left[ (C_1 - C_2) \cos \beta x + (C_1 + C_2) \sin \beta x \right] \)
$S_0(0)$ is the integral from $x = 0$ to infinity of all $N_\varphi$ forces on the right side of the string, therefore equal to one half of the applied string force $S_0$. The constants of integration are:

$$C_1 = C_2 = \frac{\beta a}{2Eh} \quad (16)$$

In order to calculate the effective width the direct force $N_\varphi$ at $x = 0$ is needed (Table A):

$$\left( N_\varphi \right)_{x=0} = \frac{Eh}{a} C_1 = \frac{\beta}{2} \quad (17)$$

The effective width as defined by Eq. (3) is:

$$b = \frac{S_0}{\left( N_\varphi \right)_{x=0}} = \frac{2}{\beta}$$

$$b = 1.5196 \sqrt{\frac{ah}{\sqrt{1-\nu^2}}} \quad (18)$$

The small influence of Poisson's ratio $\nu$ should be noted. In case of steel ($\nu = 0.3$) the neglecting of $\nu$ gives an effective width 2.4% too small.

In a later section, p. 25, it will be shown that a cylinder whose overhang on both sides of the rib is $l$, can be considered as infinitely long if the factor $\beta l > 2.4$.

The bending moment $M_x$ in axial direction at $x = 0$ is of special interest because it gives the highest stress.
The moment is (Table A and Eq. (16)):

\[ (M_x)_{x=0} = -2D \beta^2 c_2 = -\frac{1}{4a/\beta} \]  

(19)

A simple expression for the ratio of the cross bending stress \( \sigma_x \) and the circumferential direct stress \( \sigma_\varphi \) can be derived by making use of Eq. (17), (19). The direct stress \( \sigma_\varphi \) is equal to the direct force \( N_\varphi \) per unit width divided by the shell thickness \( h \). Similarly the maximum bending stress \( \sigma_x \) is found by dividing the moment \( M_x \) per unit width by the section modulus of the shell

\[ \left| \frac{\sigma_x}{\sigma_\varphi} \right|_{x=0} = \frac{6M_x h}{h^2 N_\varphi} = \sqrt{\frac{3}{1 - \nu^2}} \]

\[ \left| \frac{\sigma_x}{\sigma_\varphi} \right|_{x=0} = \frac{1.7321}{\sqrt{1 - \nu^2}} \]  

(20)

The use of Eq. (20) is quite obvious. It gives with a minimum of calculation the maximum cross bending stress \( \sigma_x \) if the stress \( \sigma_\varphi \) is known. It should be kept in mind that \( \sigma_x \) is the larger stress.

The values of \( S_o(x) \), \( N_\varphi \) and \( M_x \) at a distance \( x \) from the applied string force \( S_o \) are calculated by replacing the constants of integration in Table A as given by Eq. (16):
\[ S_0(x) = \frac{1}{2} e^{-\beta x} \cos \beta x \cdot S_o \]  
(21)

\[ N_x = \frac{\beta}{2} e^{-\beta x} (\cos \beta x + \sin \beta x) \cdot S_o \]  
(22)

\[ M_x = -\frac{1}{4a\beta} e^{-\beta x} (\cos \beta x - \sin \beta x) \cdot S_o \]  
(23)

By putting:

\[ i_1(\beta x) = e^{-\beta x} \cos \beta x \]  
(24)

\[ i_2(\beta x) = e^{-\beta x} (\cos \beta x + \sin \beta x) \]  
(25)

\[ i_3(\beta x) = e^{-\beta x} (\cos \beta x - \sin \beta x) \]  
(26)

These equations can be written:

\[ S_0(x) = \frac{1}{2} i_1(\beta x)S_o \]  
(21a)

\[ N_x = 0.6580 \sqrt{\frac{1-\nu^2}{ah}} i_2(\beta x)S_o \]  
(22a)

\[ M_x = -0.1900 \sqrt{\frac{h}{a}} \cdot \frac{1}{\sqrt{1-\nu^2}} i_3(\beta x)S_o \]  
(23a)

The values of the function \( i \) are tabulated in Table 4/0. (Fig. 18.)

3. Effective width of a semi-infinite cylinder

The rib is at the end \( x = 0 \) of a semi-infinite cylinder (Fig. 5b). A constant string force \( S_o = 1 \) is applied at

* The functions \( i_1, i_2 \) and \( i_3 \) are tabulated in Ref. (10), p. 394, Table 45 (symbols are changed).
this end. For this case the bending moment $M_x$ is zero at the end and the string force $S_0(0)$ is equal to unity. (Table A)

\[ x = 0: \quad M_x = -2D \beta \pi \alpha C_2 = 0 \]

\[ S_0(0) = \frac{Eh}{2\beta \alpha} (C_1 + C_2) = S_0 = 1 \]

The constants of integration are therefore:

\[
\begin{align*}
C_2 &= 0 \\
C_1 &= \frac{2\beta \alpha}{Eh}
\end{align*}
\]

(27)

$N_\varphi$ at $x = 0$ follows from Table A:

\[ (N_\varphi)_{x=0} = \frac{Eh}{a} C_1 = 2\beta \]

(28)

And the effective width $b$ is (Eq. (3)):

\[ b = \frac{S_0}{(N_\varphi)_{x=0}} = \frac{1}{2\beta} \]

\[ b = 0.3799 \sqrt{\frac{ah}{\sqrt{1-\nu^2}}} \]

(29)

Comparison with Eq. (18) shows that the effective width in this case is 4 times smaller than for an infinitely long cylinder. Poisson's ratio $\nu$ is again of minor influence.

The ratio of the maximum bending stress $\sigma_x$ in axial direction to the circumferential direct stress $\sigma_\varphi$ at $x = 0$ will now be determined. The bending moment
M\(_x\) is (Table A, Eq. (27)):

\[
M_x = 2D\beta^2 e^{-\beta x} C_1 \sin \beta x = \frac{1}{a/\beta} e^{-\beta x} \sin \beta x
\]  

(30)

\(M_x\) has its maximum for \(\beta x = \frac{\pi}{4}\) and may easily be found by differentiating \(M_x\) with respect to \(x\) and putting the result equal to zero:

\[
M_x \bigg|_{\text{max}} = \frac{1}{a/\beta} e^{-\frac{\pi}{4}} \sin \frac{\pi}{4}
\]  

(31)

\(N_\varphi\) at \(x = 0\) is given by Eq. (28). The ratio of the two stresses are:

\[
\frac{\sigma_x}{\sigma_\varphi}_{\text{max}} = \frac{6M_x}{hN_\varphi} = \sqrt{\frac{3}{1-\mu^2}} e^{-\frac{\pi}{4}} \sin \frac{\pi}{4}
\]  

\[
\frac{\sigma_x}{\sigma_\varphi}_{\text{max}} = 0.5583
\]  

(32)

It is remarkable that in both cases, Eq. (20) and (32), the ratio \(\frac{\sigma_x}{\sigma_\varphi}\) is independent of the shell dimensions. Applied to a design problem this means that by making the shell thicker the cross bending stress \(\sigma_x\) decreases only insofar as the circumferential stress decreases, the ratio of the two stresses remaining constant.

The S-Force, the direct force \(N_\varphi\) and the bending moment \(M_x\) at a distance \(x\) due to a string force \(S_0\) at the end of the semi-infinite cylinder are (Table A, Eq. (27)): 

\[ S_0(x) = e^{-\beta x}(\cos \beta x - \sin \beta x) \quad S_0 \]  

\[ N_\gamma = 2\beta e^{-\beta x} \cos \beta x \cdot S_0 \]  

\[ M_x = \frac{1}{a^2} e^{-\beta x} \sin \beta x \cdot S_0 \]  

They are functions of the parameter \( \beta x \) and can be written:

\[ S_0(x) = s_1(\beta x) S_0 \]  

\[ N_\gamma = 2.6322 \sqrt{\frac{1-\nu}{\pi}} s_2(\beta x) S_0 \]  

\[ M_x = 0.7598 \sqrt{\frac{h}{a}} \cdot \frac{1}{\sqrt{1-\nu^2}} s_3(\beta x) S_0 \]  

Where: \[ s_1(\beta x) = e^{-\beta x}(\cos \beta x - \sin \beta x) \]  

\[ s_2(\beta x) = e^{-\beta x} \cos \beta x \]  

\[ s_3(\beta x) = e^{-\beta x} \sin \beta x \]  

(See Table 40 and Fig. 18)

4. Use of superposition in general cases

In case of a circular cylinder extending a distance \( l_1 \) to the right and a distance \( l_2 \) to the left of the rib the general solution, Eq. (9), must be used. The two boundaries \( x = l_1 \) and \( x = -l_2 \) furnish 4 boundary conditions. The condition of continuity at \( x = 0 \) gives two more conditions. This system is sufficient to solve Eq. (9).
Nevertheless it may be seen that the solution becomes very complicated in the general case. For some special problems, as treated in the following section, symmetry conditions introduce essential simplifications.

In a general case a solution by superposition is much simpler. Consider as an example the case where \( l_1 = l \) is finite, \( l_2 \) on the other hand extends to infinity (Fig. 6). The actual case 1 can be thought of as a superposition of the cases 2, 3, and 4. Case 2 is an infinitely long cylinder with a unit string force \( S_0 = 1 \) at \( x = 0 \). Making a cut at \( x = l \) it may be seen that the action of the left part on the right one consists in a string force \( S_0(l) \) and a bending moment \( (M_x)_{x=l} \).

In case 3 an infinitely long cylinder is acted upon by an opposite string force \( S_0 = -1 \) at \( x = 2l \). The bending moment \( M_x \) at \( x = l \) is acting in the opposite sense compared to case 2; the string force \( S_0(l) \) on the other hand has the same direction. Case 4 is a semi-infinite cylinder at the end of which a string force \( S_0 = 2S_0(l) \) is applied. By adding up the cases 2, 3 and 4 no force or moment at the cut \( x = l \) is left over and the boundary conditions of case 1 are fulfilled. For determining the effective width the direct force \( N_y \) at \( x = 0 \) must be known. The cases 2, 3 and 4 were already solved.
before (Eq. (21) to (23) and (33) to (35)):

Case 2: Eq. (22) \( (N_\gamma)_{x=0} = \frac{\beta}{2} \)

Case 3: Eq. (22) \( (N_\gamma)_{x=2l} = -\frac{\beta}{2} e^{-2\beta l} (\cos 2\beta l + \sin 2\beta l) \)

Case 4: Eq. (21), (34) \( (N_\gamma)_{x=l} = e^{-\beta l} \cos \beta l \cdot 2\beta e^{-\beta l} \cos \beta l \)

\[ \text{Case 1} = \text{Case 2} + \text{Case 3} + \text{Case 4} \]

\[ (N_\gamma)_{x=0} = \frac{\beta}{2} \left[ 1 + e^{-2\beta l} (4 \cos \beta l - \cos 2\beta l - \sin 2\beta l) \right] \]

\[ (N_\gamma)_{x=0} = \frac{\beta}{2} \left[ 1 + e^{-2\beta l} (2 \cos 2\beta l - \sin 2\beta l) \right] \] \hspace{1cm} (39)

Replacing \( N_\gamma \) in Eq. (3), the effective width is:

\[ b = \frac{S_0}{(N_\gamma)_{x=0}} = 1.5196 \sqrt{ah} (1 - v^2) \cdot \frac{1}{1 + e^{-2\beta l} (2 \cos 2\beta l - \sin 2\beta l)} \] \hspace{1cm} (40)

or:

\[ b = c_1 (\beta l) \sqrt{\frac{ah}{1 - v^2}} \] \hspace{1cm} (40a)

Where:

\[ c_1 (\beta l) = \frac{1.5196}{1 + e^{-2\beta l} (2 \cos 2\beta l - \sin 2\beta l)} \] \hspace{1cm} (40b)

Eq. (40) checks for the limits \( l = \infty \) with Eq. (18) and \( l = 0 \) with Eq. (29).

For the bending moment \( M_x \) at \( x = 0 \) an expression can be built up in a similar way:
Case 2: Eq. (23)  \[ (M_x)_{x=0} = -\frac{1}{4a\beta} \]

Case 3: Eq. (23)  \[ (M_x)_{x=2l} = \frac{1}{4a\beta} e^{-2\beta l} (\cos2\beta l - \sin2\beta l) \]

Case 4: Eq. (21)&(35)  \[ (M_x)_{x=l} = e^{-\beta l} \cos\beta l \cdot \frac{1}{a\beta} e^{-\beta l} \sin\beta l \]

Case 1:  \[ (M_x)_{x=0} = -\frac{1}{4a\beta} \left[ 1 - e^{-2\beta l} (\cos2\beta l - \sin2\beta l + 4\cos2\beta l \sin2\beta l) \right] \]

The ratio \[ \frac{\sigma_x}{\sigma_y} \] for \( x = 0 \) is (Eq. (39) and (41)):  
\[ \left| \frac{\sigma_x}{\sigma_y} \right|_{x=0} = \frac{6M_x}{hN_p} = 1.7321 (1 - \mu^2) - \frac{1}{2} \cdot \frac{1 - e^{-2\beta l} (\cos2\beta l + \sin2\beta l)}{1 + e^{-2\beta l} (2 + \cos2\beta l - \sin2\beta l)} \]  

or:  
\[ \left| \frac{\sigma_x}{\sigma_y} \right|_{x=0} = f_1(\beta l) \cdot \sqrt{1 - \mu^2} \]  

Where:  
\[ f_1(\beta l) = \frac{1 - e^{-2\beta l} (\cos2\beta l + \sin2\beta l)}{1 + e^{-2\beta l} (2 + \cos2\beta l - \sin2\beta l)} \cdot 1.7321 \]  

The functions \( c_1(\beta l) \), Eq. (40b), and \( f_1(\beta l) \), Eq. (42b) are tabulated for different values \( \beta l \) (Table 1). They are also plotted in Fig. 14 and 15.

The use of superposition gives solutions for many other problems. Due to the considerable damping of all forces and moments sufficiently far removed from the rib, any influence at a distance \( 2\beta l > \tau \) can be neglected.
5. **Three special cases**

The general solution of axially symmetrical bending, given by Eq. (9), was used for the solution of the following three problems:

a.) **T- and H-section with circular axis:**

Curved knees of T- or H-section of frames are typical examples. The moment around the knee can be taken as constant and therefore the stress distribution has radial symmetry. Fig. 7a shows two examples. For finding the effective width a constant unit string force $S_0 = 1$ at the location of the web is applied. The action of this string force is identical to a constant radial line load $p = \frac{1}{a} S_0 = \frac{1}{a}$ (see Fig. 7b). Due to symmetry conditions it is sufficient to analyse the right leg of the flange only. At the free edge $x = \ell$ the bending moment $M_x$ and the shear force $Q_x$ are zero. At $x = 0$ the slope $\frac{dw}{dx}$ is zero and the normal shear $Q_x$ is half of the applied radial line load $p$:

\[
\begin{align*}
    x = 0: & \quad \frac{dw}{dx} = 0 \\
    Q_x &= D \frac{d^3w}{dx^3} = \frac{1}{2} \frac{1}{2a} \\
    x = \ell: & \quad M_x = D \frac{d^2w}{dx^2} = 0 \\
    Q_x &= D \frac{d^3w}{dx^3} = 0
\end{align*}
\]
The 4 conditions (43) are sufficient to solve Eq. (9) for the 4 constants of integration $C_1$ to $C_4$. Eventually the expressions for the direct force $N_x$ and the bending moment $M_x$ at $x = 0$ are found. The solution for the effective width, Eq. (3), is:

$$b = 1.5196 \sqrt{ah (1-u^2)} - \frac{1}{4} \frac{\sinh 2\beta l + \sin 2\beta l}{\cosh 2\beta l + \cos 2\beta l + 2} \quad (44)$$

or:

$$c_2(\beta l) = 1.5196 \frac{\sinh 2\beta l + \sin 2\beta l}{\cosh 2\beta l + \cos 2\beta l + 2} \quad (44b)$$

$$b = c_2(\beta l) \sqrt{\frac{ah}{1-u^2}} \quad (44a)$$

The ratio of the cross bending stress $\sigma_x$ to the direct stress $\sigma_y$ at $x = 0$ is:

$$\left| \frac{\sigma_x}{\sigma_y} \right|_{x=0} = 1.7321 (1-u^2) - \frac{1}{2} \frac{\cosh 2\beta l - \cos 2\beta l}{\cosh 2\beta l + \cos 2\beta l + 2} \quad (45)$$

or:

$$f_2(\beta l) = 1.7321 \frac{\cosh 2\beta l - \cos 2\beta l}{\cosh 2\beta l + \cos 2\beta l + 2} \quad (45b)$$

$$\left| \frac{\sigma_x}{\sigma_y} \right|_{x=0} = \frac{f_2(\beta l)}{\sqrt{1-u^2}} \quad (45a)$$

Tables and graphs for the functions $c_2$ and $f_2$ are given for different values of $\beta l$. (Table 1, Fig. 14 and 15).

b.) Infinitely long cylinder with equally spaced stiffening ribs:

On a long cylinder stiffening ribs are placed at equal distances $2\ell$ (Fig. 8). Each rib is assumed to be subjected to the same axially symmetrical forces. Examples
of this type are numerous, e.g. pressure vessels, pen-
stocks, etc. It is sufficient to consider the part
from \( x = 0 \) to \( x = l \) as shown in Fig. 8. A unit string
force \( S_0 = 1 \) at \( x = 0 \) produces to its left and its
right side the normal shear forces:

\[
Q_x = \frac{1}{2a}
\]

The boundary conditions for the above mentioned
part are:

\[
\begin{align*}
\text{at } x = 0: & \quad \frac{dw}{dx} = 0 \\
Q_x &= D \frac{d^2w}{dx^2} = \frac{1}{2a} \\
\text{at } x = l: & \quad \frac{dw}{dx} = 0 \\
Q_x &= D \frac{d^2w}{dx^2} = 0
\end{align*}
\]

Solving for the 4 constants of integration \( C_1 \) to
\( C_4 \) of Eq. (9), expressions for the effective width and
the ratio \( \frac{G_x}{G_f} \) under the rib are derived:

\[
b = 1.5196\sqrt{ab} (1-v^2) - \frac{1}{2} \frac{\cosh2\beta l - \cos2\beta l}{\sinh2\beta l + \sin2\beta l}
\]

\[
\left| \frac{G_x}{G_f} \right|_{x=0} = 1.7321(1-v^2) - \frac{1}{2} \frac{\sinh2\beta l - \sin2\beta l}{\sinh2\beta l + \sin2\beta l}
\]
The terms depending on $\beta t$ are again tabulated (Table 1 and Fig. 14 and 15).

$$b = c_3(\beta t) \sqrt{\frac{a h}{1 - \nu^2}}$$

$$\left. \frac{\sigma_x}{\sigma_y} \right|_{x=0} = \frac{f_3(\beta t)}{\sqrt{1 - \nu^2}}$$

Where $c_3(\beta t)$ and $f_3(\beta t)$ stand for:

$$c_3(\beta t) = \frac{\cosh2\beta t - \cos2\beta t}{\sinh2\beta t + \sin2\beta t}$$

$$f_3(\beta t) = \frac{\sinh2\beta t - \sin2\beta t}{\sinh2\beta t + \sin2\beta t}$$

C. Circular cylindrical ring with stiffener at one end:

A circular cylindrical ring of length $l$ is stiffened at one end by a rib (Fig. 9). Angles and channels with a circular axis belong in this category. The boundary conditions in case of a unit string force $S_0 = 1$ at $x = 0$ are:

$$x = 0: \quad M_x = D \frac{d^2w}{dx^2} = 0$$

$$Q_x = D \frac{d^3w}{dx^3} = \frac{1}{a}$$

$$x = l: \quad M_x = D \frac{d^2w}{dx^2} = 0$$

$$Q_x = D \frac{d^3w}{dx^3} = 0$$

(49)
Eventually the expression for the effective width is found:

\[ b = 0.3799 \sqrt{ah} (1-\nu^2) - \frac{1}{4} \frac{\cosh 2\beta l + \cos 2\beta l}{\sinh 2\beta l - \sin 2\beta l} - 2 \]  \hspace{1cm} (50)

By writing: \[ c_4(\beta l) = 0.3799 \frac{\cosh 2\beta l + \cos 2\beta l}{\sinh 2\beta l - \sin 2\beta l} \]  \hspace{1cm} (50b)

\[ b = c_4(\beta l) \sqrt{\frac{ah}{1-\nu^2}} \]  \hspace{1cm} (50a)

c_4(\beta l) may be found in Table 1 or Fig. 14. No simple expression for the ratio of the maximum cross bending stress \( \sigma_x \) to the circumferential direct stress \( \sigma_y \) at \( x = 0 \) can be derived. Eq. (32), derived for the case of a semi-infinite cylinder, gives an upper limit of this ratio. In any case, \( \sigma_x \) is the smaller of the two stresses.

By comparing Table 1 and Fig. 14 and 15, it becomes apparent that in all cases the effective width and the ratio \( \frac{\sigma_x}{\sigma_y} \) converge rapidly to the values of the infinite or semi-infinite cylinder. If the factor \( \beta l > 2.4 \) (\( \beta l = 2.4 \) is shown in Fig. 14 and 15 by a dotted line), the cylinder can be taken as infinite or semi-infinite. The error resulting from this simplification is smaller than 5%. 

6. Conclusions

For the case of axial symmetry (the stresses and deformations are constant in circumferential direction) simple expressions for the effective width $b$ were derived; $b$ was found to depend on the dimensions (radius $a$, thickness $h$, length $l$) and the material (Poisson's ratio $\nu$) of the shell only. The influence of $\nu$ is very small. In case of steel ($\nu = 0.3$) it is 2.4%.

The ratio of the maximum cross bending stress $\sigma_x$ to the direct stress in circumferential direction $\sigma_\phi$ at the rib revealed, that in certain cases where the shell is continuous over the rib, $\sigma_x$ is the higher of the two. In case of an infinitely long cylinder:

Eq. (20): $\sigma_x = 1.7321 \cdot (1-\nu^2)^{1/2} \cdot \sigma_\phi$.

In the following chapter III the effective width for a general case of loading will be derived. It will be found that a new parameter $\lambda$ has to be introduced depending on the variation of the stresses in circumferential direction.

III. General Case

1. Solution of the Differential Equations:

The problem consists of finding the forces, moments and displacements of a circular cylindrical shell under
an arbitrary string force (Fig. 10a).

\[ S = \sum_{n=0}^{\infty} S_n \cos n\varphi + \sum_{n=1}^{\infty} S'_n \cos n\varphi \]  

(51)

\( S \) is presented in form of an infinite Fourier Series. In the following it will be sufficient to deal with one term of this series.

\[ S = S_n \cdot \cos n\varphi \]  

(52)

\( n \) gives the number of the harmonic under consideration (\( n = \) number of complete waves of the string force \( S \) around the cylinder). \( n = 0 \) reduces to the case of axial symmetry.

The internal forces and moments acting on an infinitesimal shell element \( dx \cdot ad\varphi \) are shown in Fig. 10b. The displacement of the element is expressed in 3 components, \( u \) in the direction of the \( x \)-axis, \( v \) in circumferential direction and \( w \) in radial direction, positive if directed outwards. By assuming Hook's law and the well known principle of the plate theory, that a line perpendicular to the middle surface of the shell remains perpendicular to the deformed middle surface, all internal forces and moments can be expressed as functions of the 3 displacement components \( u \), \( v \) and \( w \) and their derivatives. Inserting these expressions in
the equations of equilibrium of an infinitesimal particle of the shell, a system of 3 partial differential equations for the 3 components of displacement is found*:

\[
\begin{align*}
g_1(u, v, w) &= 0 \\
g_2(u, v, w) &= 0 \\
g_3(u, v, w) &= 0
\end{align*}
\]

g stands for a function of u, v, w and its partial derivatives.

Solutions for the case where boundary conditions for the boundaries x = constant can be arbitrarily chosen were presented by K. Miesel, Ref. (12), U. Finsterwalder, Ref. (6) and Aas Jacobsen, Ref. (13). Finsterwalder simplifies the problem by assuming the moments \( M_{\varphi} = M_{\varphi x} = M_{x\varphi} = 0 \). Aas Jacobsen finds an approximate solution by expressing the moments and forces as functions of the radial displacement w only (as it is done in the plate theory).

Miesel gives a complete solution of the problem. A very brief summary will be given here in order to understand the simplifications which will be introduced

* For derivation see Ref. (9), (10), or (11).
later on. For a complete derivation Ref. (9) or (12)
should be consulted. By assuming the following form
for the three components of displacement**:

\[
\begin{align*}
    u &= U_n(x) \cos n\varphi \\
    v &= V_n(x) \sin n\varphi \\
    w &= W_n(x) \cos n\varphi
\end{align*}
\]  

(54)

where \(U_n(x), V_n(x)\) and \(W_n(x)\) are functions of \(x\) only,
the three partial differential equations (53) for the
three unknowns \(u, v, \) and \(w\) become total differential
equations with constant coefficients. This solution
parallels the one made by Levy for solving the plate
equation*. The three unknown functions \(U_n(x), V_n(x)\)
and \(W_n(x)\) are of the form:

\[
\begin{align*}
    U_n &= A e^{\kappa \xi} \\
    V_n &= B e^{\kappa \xi} \\
    W_n &= C e^{\kappa \xi}
\end{align*}
\]  

(55)

Where \(\xi = \frac{x}{a}\)

\(\kappa = \text{constant}\)

** Miesel actually operates with the three quantities
\(\frac{u}{a}, \xi \varphi \) and \(\frac{\partial w}{\partial x}\); see Ref. (12), p. 35.

* See Ref. (10), p. 125.
Inserting expressions (55) in the equations (53), 3 linear equations for the unknown \( A, B, \) and \( C \) are found. This system has a solution different from \( A = B = C = 0 \) only if the determinant of the coefficients of \( A, B, \) and \( C \) is zero. The latter condition furnishes a 4th order equation for the coefficient \( \kappa^a. \)

The 8 roots for \( \kappa \) are:

\[
\begin{align*}
\kappa_1 &= -\mu_2 - i\mu_1 \\
\kappa_2 &= -\mu_2 + i\mu_1 \\
\kappa_3 &= +\mu_2 - i\mu_1 \\
\kappa_4 &= +\mu_2 + i\mu_1 \\
\kappa_5 &= -\mu_4 - i\mu_3 \\
\kappa_6 &= -\mu_4 + i\mu_3 \\
\kappa_7 &= +\mu_4 - i\mu_3 \\
\kappa_8 &= +\mu_4 + i\mu_3
\end{align*}
\]

(56)

To each root \( \kappa^a \) there corresponds a particular integral of the equations (53) with the constants of integration \( A, B, \) and \( C. \) The general solution for the radial displacement \( w \) is for example:

\[
w = \left[ e^{\mu_2 \xi} \left( C_1 e^{-\mu_1 \xi} + C_2 e^{\mu_1 \xi} \right) + e^{\mu_3 \xi} \left( C_3 e^{-\mu_1 \xi} + C_4 e^{\mu_1 \xi} \right) + e^{\mu_4 \xi} \left( C_5 e^{-\mu_3 \xi} + C_6 e^{\mu_3 \xi} \right) \right. \\
\left. + e^{\mu_4 (\xi - \frac{L}{a})} \left( C_7 e^{-\mu_3 \xi} + C_8 e^{\mu_3 \xi} \right) \right] \cos n\varphi
\]

(57)
Obviously $w$ has the form of 4 damped oscillations, two originating from the boundary $\xi = 0$ and the other two from the boundary $\xi = \frac{l}{a}$. If the latter boundary is sufficiently far removed it does not influence the boundary $\xi = 0$ and the corresponding part in Eq. (57) can be neglected.

Actual calculations show that from the 2 oscillations starting from $\xi = 0$ the one with the $e^{-\mu_2 \xi}$ power is by far predominant. The one with the $e^{-\mu_4 \xi}$ power is zero for the two cases $n = 0$ and $n = 1$. For higher values of $n$ it stays very small. Miesel makes use of this property to derive an approximate solution where only the underlined part of Eq. (57) is retained. It can be shown that this approximation parallels the simplifications introduced by Geckeler (Ref. (14)) in the bending theory of spherical and conical shells.

The following derivations are based on Miesel's approximate solution, the value of which will become especially apparent by the correspondence between the T-Beam problem and the effective width of a cylinder for the limiting case where the radius "a" increases to infinity (see Chapter IV).
2. **Miesel's approximate solution:**

Only the underlined part of the general solution (57) is taken. By putting the equation in its real form any unknown quantity \( H \), where \( H \) stands for a force, moment or displacement, has the form:

\[
H = H_0 e^{-\mu_2 \xi} \left[ k_1 \sin(\mu_1 \xi + \psi) + k_2 \cos(\mu_1 \xi + \psi) \right] \cos n\phi \]  

(58)

In the following Table B the most important forces, moments and displacements are given in this form. The 2 constants of integration are \( C \) and \( \psi \). \( \mu_1 \) and \( \mu_2 \) are coefficients depending on the shell dimensions, Poisson's ratio \( \nu \) and the number \( n \) of the harmonic under consideration. \( H \), \( k_1 \) and \( k_2 \) are constants depending on the quantity \( H \). Miesel's notations were changed to conform with the ones adopted in this dissertation. The string force \( S \) as defined by Eq. (1) is:

\[
S_n(x) = \int_{-\infty}^{\infty} N \phi \, dx
\]

(59)

By replacing \( N \phi \) by its value from Table B, \( S_n(x) \) becomes:

\[
S_n(x) = BC \frac{2h}{k} \left[ e^{-\mu_2 \xi} \left( \mu_2 \left( \mu_1^2 - \frac{1}{4} \lambda^2 \left( 1 - \frac{1}{n^2} \right) (2 + \nu) \right) \sin(\mu_1 \xi + \psi) 
+ \mu_1 \left( \mu_2^2 + \frac{1}{4} \lambda^2 \left( 1 - \frac{1}{n^2} \right) (2 + \nu) \right) \cos(\mu_1 \xi + \psi) \right] \cos n\phi \, dx
\]

and performing the integration:
TABLE B

a : radius of cylinder
h : thickness of cylinder
\( \nu \) : Poisson's ratio for the material
n : the harmonic under consideration (number of complete cosine-waves of the stresses in circumferential direction)
C, \( \psi \) : constants of integration

\[ \xi = \frac{x}{a} \]
\[ \lambda = n \sqrt{\frac{h}{a}} \]
\[ k = \frac{12a^3}{h^3} \]
\[ k' = k\left(1 + \frac{1}{2} \lambda^4(1 - \frac{1}{n^2})\right) \]

<table>
<thead>
<tr>
<th>H</th>
<th>( \frac{\partial w}{\partial x} )</th>
<th>H</th>
<th>k₁</th>
<th>k₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>E</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( N_x )</td>
<td>Eh\left(\frac{1}{k} + \frac{n^2 - 1}{k'}\right)</td>
<td>( \mu_2 \left[\mu_1^2 - \frac{1}{4}\lambda^4(1 - \frac{1}{n^2})(2 + \nu)\right] )</td>
<td>( -\mu_1 )</td>
<td></td>
</tr>
<tr>
<td>( N_{\psi} )</td>
<td>E \frac{2h}{k}</td>
<td>( \mu_2 \left[\mu_1^2^2 + \frac{1}{4}\lambda^4(1 - \frac{1}{n^2})(2 + \nu)\right] )</td>
<td>( -\mu_1^2 )</td>
<td></td>
</tr>
<tr>
<td>( M_x )</td>
<td>E \frac{ah}{k}</td>
<td>( \mu_2(1 - \frac{n^2\nu}{\sqrt{k'}}) )</td>
<td>( -\mu_1(1 + \frac{n^2\nu}{\sqrt{k'}}) )</td>
<td></td>
</tr>
<tr>
<td>( M_{\psi} )</td>
<td>E \frac{ah}{k}</td>
<td>( \mu_2(\nu - \frac{n^2}{\sqrt{k'}}) )</td>
<td>( -\mu_1(\nu + \frac{n^2}{\sqrt{k'}}) )</td>
<td></td>
</tr>
<tr>
<td>( S_n(x) )</td>
<td>E \frac{2ah}{k}</td>
<td>( -\frac{1}{4}\lambda^4(1 - \frac{1}{n^2})(2 + \nu) )</td>
<td>( \mu_1 \cdot \mu_2 )</td>
<td></td>
</tr>
</tbody>
</table>

**General Case:**

\[ H = \bar{H} a^2 \mu_2^2 \left[ k_1 \sin(\mu_1 \xi + \psi) + k_2 \cos(\mu_1 \xi + \psi) \right] \cos n \psi \]

**Special Case:**

\[ M_x = E \frac{ah}{k} \bar{C} \mu_2^2 \left[ \mu_2(1 - \frac{n^2\nu}{\sqrt{k'}})\sin(\mu_1 \xi + \psi) - \mu_1(1 + \frac{n^2\nu}{\sqrt{k'}})\cos(\mu_1 \xi + \psi) \right] \cos n \psi \]
The tangential shear forces acting in the plane of the shell are of importance, especially in reinforced concrete for the calculation of the diagonal reinforcing steel. Considering the equilibrium of a strip of width $ad\varphi$, extending from $x$ to infinity (Fig. 11a), the total shear force $T$ at $x$ is determined by:

$$\frac{\partial T_{adj}}{\partial \varphi} = \int_{x}^{\infty} (N_{\varphi} + dN_{\varphi}) \, dx + \int_{x}^{\infty} N_{\varphi} \, dx = 0$$

The integral of the $N_{\varphi}$ forces is equal to the string force $S_{n}(x)$, Eq. (59):

$$\frac{\partial T_{adj}}{\partial \varphi} = \int_{x}^{\infty} dN_{\varphi} \, dx = dS_{n}(x)$$

hence:

$$T = \frac{1}{a} \frac{\partial S_{n}(x)}{\partial \varphi}$$

(61)

$T$ is the total tangential shear force. In Fig. 11b it may be seen that the direct shear force $N_{x\varphi}$ and the twisting moment $M_{x\varphi}$ produce tangential shear stresses $T_{x\varphi}$. Considering a section $x = \text{constant}$ of the shell of unit length, the $N_{x\varphi}$ force is the direct shear force acting on this element. The twisting moment is arbitrarily split in a couple of 2 forces $1 \cdot M_{x\varphi}$. The
total tangential shear force acting on this unit element is $N_x \varphi$ plus the tangential component $\frac{1}{a} M_x \varphi$ of the 2 forces of this couple*:

$$T = N_x \varphi + \frac{1}{a} M_x \varphi$$

(62)

This shear force $T$ has to be used for designing the diagonal steel in reinforced concrete shells adjacent to stiffeners. The value of $T$ is determined by Eq. (61).

3. **Effective width of an infinitely long cylinder**

(Poisson's ratio $\nu = 0$)

For the symmetrical case it was found that Poisson's ratio $\nu$ is of little influence. It can be expected that this will hold in the general case too. Therefore the assumption $\nu = 0$ is made to simplify the following derivations.

The procedure of finding the effective width is the same as used in the symmetrical case. In the middle part of an infinitely long cylinder a string force $S$ is applied (Fig. 10a). $S$ can have any variation. It is always possible to present it in form of a Fourier series (Eq. (51)). The effective width will be derived for the $n$th term of this series. Consider the unit string force:

$$S = S_n \cos n \varphi = l \cos n \varphi$$

(63)

* A similar reasoning is used in deriving the boundary shear of flat plates, e.g. Ref. (10), p. 90.
Each of the two parts on both sides of the string will carry half of this string force. The continuity for the 2 parts requires that the slope in x-direction at \( x = 0 \) is zero. By using Table B the 2 conditions take the form:

\[
\begin{align*}
  x = 0: & \quad S_n(0) = E \frac{2ah}{k} \mu_1 \mu_2 \cos \psi \cos n\varphi = \frac{1}{2} \cos n\varphi \\
  & \quad E \frac{\partial w}{\partial x} = -Ec \sin \psi \cos n\varphi = 0
\end{align*}
\]

The second of these equations requires \( \psi = 0 \) and the other constant of integration becomes:

\[
C = \frac{1}{4} \frac{k}{Eah \mu_1 \mu_2}
\]

\[
\psi = 0
\]

It may be mentioned that another boundary condition, axial displacement \( u = 0 \), was completely disregarded. But the approximate solution (58) offers only 2 constants of integration which must be used to fulfill the "essential" boundary conditions. \( N_\varphi \) is calculated by replacing in Table B the two constants \( C \) and \( \psi \) by the expressions (64):

\[
(N_\varphi)_{x=0} = \frac{1}{2a\mu_2} \left( \mu_2^a + \frac{1}{2} \lambda^a (1 - \frac{1}{n^2}) \right) \cos n\varphi
\]

The effective width is the ratio of the applied string force \( S \) to the direct force \( N_\varphi \) at \( x = 0 \) (Eq. (3)): 
If $\mu_2$ is replaced by its value given in Table B,

$$
\begin{align*}
\frac{b}{(N_f)_{x=0}} &= \frac{2a}{\mu_2 \left[ 1 + \frac{\lambda^2(1 - \frac{1}{n^2})}{2\mu_2^2} \right]} \\
&\approx \frac{1.5196 \sqrt{ah}}{\sqrt{1 + \frac{1}{2}\lambda^2(1 - \frac{1}{n^2}) + \frac{1}{2}\lambda^2}} \cdot \frac{1}{1 + \frac{\lambda^2(1 - \frac{1}{n^2})}{2\mu_2^2}} 
\end{align*}
$$

(66)

In case of axial symmetry the number $n$ of the waves of the string force $S$ around the cylinder is zero (string force $S$ is constant). $\lambda = n\sqrt{\frac{h}{a}}$ being proportional to $n$ will be zero too, and Eq. (66) checks with Eq. (18) for the case $\nu = 0$.

Eq. (66) is essentially a function of the two parameters $\sqrt{ah}$ and $\lambda$. The influence of the other terms is negligible as will be shown.

Simplifications:

1.) Let's investigate the term:

$$
1 + \frac{1}{2}\lambda^2(1 - \frac{1}{n^2}) = 1 + \frac{1}{2}\lambda^2 \cdot \frac{h^2}{a^2} (1 - \frac{1}{n^2}) \approx 1 + \frac{1}{2}\lambda^2 \frac{h^2}{a^2}
$$

The right side is an approximate expression obtained by dropping the term $\frac{1}{n^2}$. The ratio of the correct expression to the approximate expression may be called $\Delta$: 
\[ \Delta = \frac{1 + \frac{1}{2} n^a \frac{h^a}{a^a} (1 - \frac{1}{n^a})}{1 + \frac{1}{2} n^a \frac{h^a}{a^a}} \]

\( \frac{h^a}{a^a} \) is the square of the ratio thickness/radius of the shell. In any practical case it is smaller than \((\frac{1}{5})^2\).

Taking: \( \frac{h^a}{a^a} = (\frac{1}{5})^a \)

\[ \Delta = \frac{1 + \frac{1}{50} n^a (1 - \frac{1}{n^a})}{1 + \frac{1}{50} n^a} , \quad n \geq 0 \]

By differentiating \( \Delta \) with respect to \( n^a \) it is found that \( \Delta \) takes a minimum value for \( n^a = 7.07 \):

\[ \Delta \bigg|_{\text{min}} = 0.93 \]

\( \Delta \) is therefore limited between:

\[ 0.93 \leq \Delta \leq 1 \]

Considering the fact that the term \( 1 + \frac{1}{2} \lambda^a (1 - \frac{1}{n^a}) \) stands under a square root, the approximate term \( 1 + \frac{1}{2} \lambda^a \) gives in the most extreme case an error smaller than 3.5%.

For any smaller ratio \( \frac{h^a}{a^a} \) than the one considered the error will decrease. The approximation

\[ \sqrt{1 + \frac{1}{2} \lambda^a (1 - \frac{1}{n^a})} \approx \sqrt{1 + \frac{1}{2} \lambda^a} \]

(a)

may therefore be used.
2.) In the second place the magnitude of the term

\[
\lambda^*(1 - \frac{1}{na}) \quad \frac{\lambda^*(1 - \frac{1}{na})}{2 \mu_2^a} = \frac{\lambda^*(1 - \frac{1}{na})}{2 \sqrt{3} \frac{a}{h}} \left(\sqrt{1 + \frac{1}{2} \lambda^*(1 - \frac{1}{na}) + \frac{1}{\sqrt{3}} \lambda^2}\right)
\]

will be analysed. For \( \lambda \leq 1 \) it is obviously insignificant. In cases where \( \lambda > 1 \) the following inequality hold:

\[
\frac{\lambda^*(1 - \frac{1}{na})}{2 \mu_2^a} \leq \frac{\lambda^*}{2 \sqrt{3} \frac{a}{h} \lambda^*(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}})} = \frac{\lambda^* h}{4.45a} = 0.225 \left(\frac{nh}{a}\right)^2
\]

In practical cases the factor \( \frac{nh}{a} \leq \frac{1}{2} \). For \( \frac{nh}{a} = \frac{1}{2} \):

\[
\frac{\lambda^*(1 - \frac{1}{na})}{2 \mu_2^a} \leq 0.225 \left(\frac{nh}{a}\right)^2 = 0.0563
\]

The influence of this term on the effective width, Eq. (66), is \( \leq 5\% \). In consequence the simplification

\[
\frac{\lambda^*(1 - \frac{1}{na})}{2 \mu_2^a} \approx 0
\]

(b)

is made.

3.) In a later equation (Eq. (72)) the term

\[
\frac{\lambda^*(1 - \frac{1}{na})}{2 \mu_1^a} = \frac{\lambda^*(1 - \frac{1}{na})}{2 \sqrt{3} \frac{a}{h} \left(\sqrt{1 + \frac{1}{2} \lambda^*(1 - \frac{1}{na}) - \frac{1}{\sqrt{3}} \lambda^2}\right)}
\]

will occur. For \( \lambda > 1 \)
\[
\frac{\lambda^s(1-\frac{1}{n^3})}{2\mu_1^2} \leq \frac{\lambda^s}{2\sqrt{\frac{a}{h}} \lambda^s (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}})} = \frac{\lambda^sh}{0.45a} = 2.22 \left(\frac{nh}{a}\right)^2
\]

For the limit of the factor \(\frac{nh}{a}\), the inequality becomes:

\[
\frac{nh}{a} = \frac{1}{2}; \quad \frac{\lambda^s(1-\frac{1}{n^3})}{2\mu_1^2} \leq 2.22 \left(\frac{nh}{a}\right)^2 = 0.555
\]

It will be seen (Eq. (72)) that the above term is multiplied by a rather unimportant factor so that the simplification

\[
\frac{\lambda^s(1-\frac{1}{n^3})}{2\mu_1^2} \approx 0
\]

is justified.

4.) A last term (Eq. (71)) which will be dropped is

\[
\frac{\lambda^s(1-\frac{1}{n^3})}{2\mu_1 \mu_2} = \frac{\lambda^s (1-\frac{1}{n^3})}{2\sqrt{\frac{a}{h}} \sqrt{1 + \frac{1}{2} \lambda^s (1-\frac{1}{n^3}) - \frac{1}{3} \lambda^s}}
\]

Proceeding as before, the following inequality is found:

\[
\frac{\lambda^s(1-\frac{1}{n^3})}{2\mu_1 \mu_2} \leq \frac{1}{\sqrt{2}} \left(\frac{nh}{a}\right)^2 = 0.177
\]
As an approximation the term can be neglected:

\[ \lambda^4 \left(1 - \frac{1}{n^2}\right) \approx 0 \]
\[ \frac{2 \mu_1}{\mu_2} \]

Referring to the effective width \( b \) as given in Eq. (66) and introducing the 2 simplifications (a) and (b):

\[ b = 1.5196 \sqrt{ah} \frac{1}{\sqrt{1 + \frac{1}{2} \lambda^4 + \frac{1}{\sqrt{3}} \lambda^2}} \]

which is an expression in terms of the two parameters \( \sqrt{ah} \) and \( \lambda \) only. For the purpose of tabulating Eq. (67) the following form is chosen:

\[ b = K \sqrt{ah} \]

\[ K = \frac{1.5196}{\sqrt{1 + \frac{1}{2} \lambda^4 + \frac{1}{\sqrt{3}} \lambda^2}} \]

In Table 2, column 3, values of \( K \) for different values of \( \lambda \) are given. Fig. 16 is the graphical representation of this Table. Note the rapid decrease of the effective width by increasing \( \lambda \).

* For design purposes 1.5196 should be replaced by 1.52.
The cross bending stress \( \sigma_x \) under the rib will be calculated. The bending moment \( M_x \) at \( x = 0 \) is:

(\text{Table B and Eq. (64)})

\[
(M_x)_{x=0} = - \frac{1}{4 \mu_2} \cos n \varphi
\]

(68)

Computing the bending stress \( \sigma_x \) and the direct stress \( \sigma_{\varphi} \) (Eq. (65)) the ratio of the two stresses becomes:

\[
\left| \frac{\sigma_x}{\sigma_{\varphi}} \right|_{x=0} = \frac{6M_x}{hN_{\varphi}} = \frac{3a}{h\left(\mu_2^a + \frac{1}{2} \lambda^a(1 - \frac{1}{n^a})\right)}
\]

\[
\left| \frac{\sigma_x}{\sigma_{\varphi}} \right|_{x=0} = \frac{1.7321}{\sqrt{1 + \frac{1}{2} \lambda^a(1 - \frac{1}{n^a}) + \frac{1}{\sqrt{3}} \lambda^a}} \cdot \frac{1}{1 + \frac{\lambda^a(1 - \frac{1}{n^a})}{2 \mu_2^a}}
\]

(69)

The same simplifications (a) and (b) as for Eq. (66) are used:

\[
\left| \frac{\sigma_x}{\sigma_{\varphi}} \right|_{x=0} = \frac{1.7321}{\sqrt{1 + \frac{1}{2} \lambda^a + \frac{1}{\sqrt{3}} \lambda^a}}
\]

(70)

In Table 3 and Fig. 17, Eq. (70) is presented as a function of \( \lambda \).

Finally the string force \( S_n(x) \), the direct force \( N_{\varphi} \) and the bending moment \( M_x \) for a distance \( x \) from an applied
string force $S = S_n \cos n\varphi$ are calculated. The constants of integration $C$ and $\varphi$ in Table B are replaced by the expressions (64):

$$S_n(x) = \frac{1}{2} e^{-\mu_2 \xi} \left[ -\frac{\lambda^2(1 - \frac{1}{n^2})}{2\mu_1\mu_2} \sin\mu_1 \xi + \cos\mu_1 \xi \right] S_n \cos n\varphi \quad (71)$$

$$N_\varphi = \frac{1}{2a} e^{-\mu_2 \xi} \cdot \left[ \frac{\mu_1(1 - \frac{\lambda^2(1 - \frac{1}{n^2})}{2\mu_1^2})}{\sin\mu_1 \xi} \right. + \left. \frac{\mu_2(1 + \frac{\lambda^2(1 - \frac{1}{n^2})}{2\mu_2^2})}{\cos\mu_1 \xi} \right] S_n \cos n\varphi \quad (72)$$

$$M_x = \frac{1}{4} e^{-\mu_2 \xi} \left[ \frac{1}{\mu_1} \sin\mu_1 \xi - \frac{1}{\mu_2} \cos\mu_1 \xi \right] S_n \cos n\varphi \quad (73)$$

Certain terms, (a), (b), (c) and (d) which were discussed previously can be neglected. Note that (c) and (d) are multiplied by $\sin\mu_1 \xi$ which is zero for the maximum values of $S_n(x)$ and $N_\varphi$. If $\sin\mu_1 \xi$ becomes maximum these forces are already greatly damped. By introducing the simplifications and inserting the values for $\mu_1$ and $\mu_2$, Eq. (71) and (73) become:

$$S_n(x) = \frac{1}{2} e^{-\mu_2 \xi} \cos\mu_1 \xi \cdot S_n \cos n\varphi \quad (74)$$

$$N_\varphi = \frac{1}{1.5196\sqrt{a}} e^{-\mu_2 \xi} \left[ \sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2 \sin\mu_1 \xi \right. \left. + \sqrt{1 + \frac{1}{2} \lambda^2} \frac{1}{\sqrt{3}} \lambda^2 \cos\mu_1 \xi \right] S_n \cos n\varphi \quad (75)$$
\[ M_x = \frac{1}{5.2644} \frac{n}{a} e^{-\mu_2 \xi} \left[ \frac{\sin \mu_1 \xi}{\sqrt{1 + \frac{1}{2} \lambda^2}} - \frac{\cos \mu_1 \xi}{\sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^2} \right] \]

The term \( \mu_1 \xi \) has the form:

\[ \mu_1 \xi = \frac{\sqrt{3}}{\sqrt{n}} \sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2 \cdot \frac{x}{a} = \frac{\sqrt{3}}{\sqrt{ah}} \cdot x \sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2 \]

The first factor is the well known shell constant \( \beta \) defined by Eq. (10) for the case \( \nu = 0 \)

\[ \nu = 0: \quad \beta = \frac{\sqrt{3}}{\sqrt{ah}} \]

\[ \mu_1 \xi = (\beta x) \sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2 \]

Similarly \( \mu_2 \xi \) can be changed to a function of \( \beta x \) and \( \lambda \):

\[ \mu_2 \xi = (\beta x) \sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^2 \]

The 3 equations, (74) to (76), will be written as:

\[ S_n(x) = \frac{1}{2} i_1(\beta x, \lambda) S_n \cos n \varphi \quad (74a) \]

\[ N_\varphi = \frac{0.6580}{\sqrt{ah}} i_2(\beta x, \lambda) S_n \cos n \varphi \quad (75a) \]
\[ M_x = -0.1900 \frac{h}{a} i_3(\beta x, \lambda) S_n \cos n\varphi \]  

(76a)

The functions:  
\[ i_1(\beta x, \lambda) = e^{-\mu_2 \xi} \cos \mu_1 \xi \]  

(77)

\[ i_2(\beta x, \lambda) = e^{-\mu_2 \xi} \left[ \sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^a \cos \mu_1 \xi \right. \]  

\[ \left. + \sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^a \sin \mu_1 \xi \right] \]  

(78)

\[ i_3(\beta x, \lambda) = e^{-\mu_2 \xi} \left[ \sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^a \right. \]  

\[ \left. \cos \mu_1 \xi \right] \]  

\[ - \frac{\sin \mu_1 \xi}{\sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^a} \]  

(79)

are calculated in Table 4. (Fig. 18 to 23 are graphs of the functions).

In the axially symmetrical case, \( \lambda = 0 \), Eq. (77) to (79) coincide with the Eq. (24) to (26).

The total tangential shear \( T \) is found by differentiating the string force \( S_n(x) \) with respect to \( \varphi \) (Eq. (61)):  

\[ T = \frac{1}{a} \frac{\partial S_n(x)}{\partial \varphi} = \frac{1}{a} \frac{\partial}{\partial \varphi} \left[ \frac{1}{\xi} i_1(\beta x, \lambda) S_n \cos n\varphi \right] \]  

\[ T = -\frac{n}{2a} i_1(\beta x, \lambda) S_n \sin n\varphi \]  

(80)
4. **Effective width of a semi-infinite cylinder:**

A unit string force \( S = 1 \cos n\varphi \), making \( n \) complete cosine-waves around the circumference, is applied to the end of a semi-infinite cylinder. Using Table B the boundary conditions are:

\[ x = 0: \]

\[ S_n(0) = \frac{E2ah}{k} c \left[ -\frac{1}{2} \lambda^4 (1 - \frac{1}{n^2}) \sin \psi + \mu_1 \mu_2 \cos \psi \right] \cos n\varphi = 1 \cos n\varphi \]

\[ M_x = \frac{Eah}{k} c \left( \mu_2 \sin \psi - \mu_1 \cos \psi \right) = 0. \]

The condition \( M_x = 0 \) furnishes the relations:

\[ \tan \psi = \frac{\mu_1}{\mu_2} \]

\[ \sin \psi = \frac{\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}} \]

\[ \cos \psi = \frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} \]

And the constant \( C \) becomes:

\[ C = \frac{k}{2Eah} \cdot \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 \mu_2 \left[ 1 - \frac{\lambda^4 (1 - \frac{1}{n^2})}{2\mu_2} \right]} \]
Note that a further boundary condition, \( N_x = 0 \), had to be overlooked as "non-essential" due to the approximate nature of solution (see also p. 36).

Inserting the expressions (81) in Table B for the constants \( C \) and \( \psi \), \( N_x \) at \( x = 0 \) is:

\[
(N_x)_{x=0} = \frac{\mu_1^2 + \mu_2^2}{\mu_2^2 \left( 1 - \frac{1}{n} \right) \cos n\psi} \tag{82}
\]

The effective width is calculated by means of Eq. (3):

\[
b = \frac{S}{(N_x)_{x=0}} = \frac{\mu_2^2 \left( 1 - \frac{1}{n} \right)}{\mu_1^2 + \mu_2^2} \tag{83}
\]

Replacing the \( \mu_1 \) and \( \mu_2 \) by their respective values (Table B), eventually the following form for \( b \) may be derived:

\[
b = 0.3799 \sqrt{3 \lambda^2} \left\{ \lambda^2 \left[ 1 - \frac{3}{2} \lambda^2 (1 - \frac{1}{n^2}) \right] \right. + \frac{1}{\sqrt{3}} \lambda^2 \left[ 1 + \frac{1}{2} \lambda^2 (1 - \frac{1}{n^2}) \right] \tag{83'}
\]

The check with Eq. (29) for axial symmetry is established if \( \lambda \) is taken equal to zero (\( \nu = 0 \)). For practical purposes Eq. (83) will be simplified. In the case of the last term the following inequality holds:
It may be remembered that in all actual cases (p. 39):

\[
\frac{nh}{a} < \frac{1}{2}
\]

For \(\frac{nh}{a} = \frac{1}{2}\):

\[
\frac{h}{2a} \lambda^a = \frac{1}{2} \left(\frac{nh}{2}\right)^a = \frac{1}{8}
\]

By using the right side of the inequality (84) instead of the left one, the effective width will be smaller. In Chapter IV it will be shown that the simplified formula for \(b\) gives the correct limit for the case where the radius "a" increases to infinity.

Making use of the approximations (84) and (a) the simplified Eq. (83) is:

\[
b = 0.3799 \sqrt{ah} \frac{1}{\sqrt{1 + \frac{1}{2} \lambda^a + \frac{1}{\sqrt{3}} \lambda^a}} \left[1 + \frac{\lambda^a}{3\sqrt{1 + \frac{1}{2} \lambda^a}}\right]
\]

(85)

For the following form

\[
b = K \sqrt{ah}
\]

(85a)

where:

\[
K = \frac{0.3799}{\sqrt{1 + \frac{1}{2} \lambda^a + \frac{1}{\sqrt{3}} \lambda^a}} \left[1 + \frac{\lambda^a}{3\sqrt{1 + \frac{1}{2} \lambda^a}}\right]
\]

(85b)
the values of $K$ as function of $\lambda$ are given in Table 2, column $2$. In Fig. 16 the same $K$-values are plotted. It is remarkable that the effective width stays almost constant between $\lambda = 0$ and $\lambda = 0.75$, whereas the effective width of the infinitely long cylinder decreases considerably for the same interval. The physical explanation for this behavior is the following. In case of a constant string force $S$ at the end of a semi-infinite cylinder, the slope $\frac{\partial w}{\partial x}$ is constant around the cylinder. There is no end-restraint. If $S$ varies in $n$ waves around the circumference, the slope $\frac{\partial w}{\partial x}$ makes the same variation. As $\frac{\partial w}{\partial x}$ changes between positive and negative slopes, the cylinder as a whole seems to become more and more restrained against end-rotations $\frac{\partial w}{\partial x}$. This influence is responsible for an increase in the effective width. On the other hand the variation of $S$ decreases the effective width. For the interval $\lambda = 0$ to $\lambda = 0.75$ the two almost compensate each other. Only for values $\lambda > 0.75$ does the second influence become predominant. In case of the infinitely long cylinder the slope $\frac{\partial w}{\partial x}$ is always zero along the line of the applied string force $S$. The variation of $S$ is the only influence on the effective width. The latter starts therefore to decrease immediately.
Next, the ratio of the maximum bending stress \( \sigma_x \) to the circumferential direct stress \( \sigma \) at \( x = 0 \) will be derived. The bending moment \( M_x \) is (Table B):

\[
M_x = E \frac{s}{k} \frac{h}{e} \left[ \frac{1}{2} \left( \mu_2^2 - \mu_1^2 \right) \sin(\mu_1 \xi + \psi) + 2 \mu_1 \mu_2 \cos(\mu_1 \xi + \psi) \right] \cos n \varphi \quad (86)
\]

By differentiating \( M_x \) with respect to \( x \) the maximum value of \( M_x \) can be found:

\[
\frac{dM_x}{dx} = E \frac{h}{k} \frac{h}{e} \left[ \frac{1}{2} \left( \mu_1^2 - \mu_2^2 \right) \sin(\mu_1 \xi + \psi) + 2 \mu_1 \mu_2 \cos(\mu_1 \xi + \psi) \right] \cos n \varphi = 0
\]

The above expression is equal to zero if the parenthesis becomes zero:

\[
\left( \mu_1^2 - \mu_2^2 \right) \sin(\mu_1 \xi + \psi) + 2 \mu_1 \mu_2 \cos(\mu_1 \xi + \psi) = 0
\]

Or:

\[
\tan(\mu_1 \xi + \psi) = \frac{2 \mu_1 \mu_2}{\mu_2^2 - \mu_1^2}
\]

Simple trigonometry gives the two relations:

\[
\begin{align*}
sin(\mu_1 \xi + \psi) &= \frac{2 \mu_1 \mu_2}{\mu_1^2 + \mu_2^2} \\
cos(\mu_1 \xi + \psi) &= \frac{\mu_2^2 - \mu_1^2}{\mu_1^2 + \mu_2^2}
\end{align*}
\]

If the expressions (87) are inserted in Eq. (86) the maximum of \( M_x \) is:
Eventually the value of \( N_\psi \) at \( x = 0 \) is found by replacing the constant of integration \( \psi \) in Table B by its value (81). Making use of certain trigonometric relations \( N_\psi \) is:

\[
(N_\psi)_{x=0} = E \frac{2h}{k} C \mu_1 \mu_2 \sqrt{\mu_1^2 + \mu_2^2} \cos n \psi \quad (89)
\]

The ratio of the two stresses becomes:

\[
\left| \frac{\sigma_x}{\sigma_\varphi} \right|_{\text{max}} = \frac{6M_x}{hN_\psi} = \frac{3a}{h \mu_2 \sqrt{\mu_1^2 + \mu_2^2}} e^{-\mu_2 \xi} \quad (89)
\]

and by introducing the values of \( \mu_1 \) and \( \mu_2 \):

\[
\left| \frac{\sigma_x}{\sigma_\varphi} \right|_{\text{max}} = 1.2247 \frac{e^{-\mu_2 \xi}}{\sqrt{(1 + \frac{1}{2} \lambda^2)(1 - \frac{1}{n^2}) + \frac{1}{\sqrt{3}} \lambda^3 \sqrt{1 + \frac{1}{2} \lambda^2(1 - \frac{1}{n^2})}}} \quad (90)
\]

Applying simplification (a) the term \( \frac{1}{n^2} \) is dropped:

\[
\left| \frac{\sigma_x}{\sigma_\varphi} \right|_{\text{max}} = 1.2247 \frac{e^{-\mu_2 \xi}}{\sqrt{(1 + \frac{1}{2} \lambda^2)(1 + \frac{1}{3} \lambda^3) \sqrt{1 + \frac{1}{2} \lambda^2}}} \quad (91)
\]

\( \xi \) must be calculated from Eq. (87). The constant of integration \( \psi \) in (87) is given in Eq. (81) as a function of the coefficients \( \mu \) and hence of the coefficient \( \lambda \) (Table B). The exponent \( -\mu_2 \xi \) is a function of \( \lambda \) only. Eq. (91) is computed in Table 3 for different values of the parameter \( \lambda \) (see also Fig. 17). In the case of \( \lambda = 0 \), Eq. (91) becomes
identical with Eq. (32), derived under the condition of axial symmetry.

For the case of a string force \( S = S_n \cos n\varphi \) the constants of integration are given by Eq. (81). Inserting these values in the expressions for the string force \( S_n(x) \), the direct force \( N_\varphi \), and the bending moment \( M_x \) of Table B, and using certain trigonometric relations the following formulas are finally found:

\[
S_n(x) = e^{-\mu_2 \xi} \left[ \cos \mu_1 \xi - \frac{\mu_1 \left[ 1 + \frac{\lambda^4(1 - \frac{1}{n^2})}{2 \mu_2^a} \right]}{\mu_2 \left[ 1 - \frac{\lambda^4(1 - \frac{1}{n^2})}{2 \mu_2^a} \right]} \sin \mu_1 \xi \right] S_n \cos n\varphi \tag{92}
\]

\[
N_\varphi = \frac{\mu_1^2 + \mu_2^a}{\alpha \mu_2} e^{-\mu_2 \xi} \left[ \frac{1}{1 - \frac{\lambda^4(1 - \frac{1}{n^2})}{2 \mu_2^a}} \cos \mu_1 \xi - \frac{\lambda^4(1 - \frac{1}{n^2})}{2 \mu_1 \mu_2 \left[ 1 - \frac{\lambda^4(1 - \frac{1}{n^2})}{2 \mu_2^a} \right]} \right] \tag{93}
\]

\[
M_x = \frac{1 + \frac{\mu_1^2}{\mu_2^a}}{-\mu_2 \xi} e^{-\mu_2 \xi} \sin \mu_1 \xi \cdot S_n \cos n\varphi \tag{94}
\]

If the values of \( \mu_1 \) and \( \mu_2 \) are introduced and the simplifications (a) and (d) and (84) are applied the above equations take the form:
It was shown on p. 44 that the terms \( f_{1} \) and \( f_{2} \) are functions of \( \beta x \) and \( \lambda \). Therefore Eq. (95) to (97) are also functions of these two parameters and can be written:

\[
S_{n}(x) = e^{-\mu_{2} \xi} \left[ \cos \mu_{1} \xi - \left( \frac{1}{1+\frac{1}{2}\lambda^{4} - \frac{1}{\sqrt{3}}\lambda^{2}} \right) \sin \mu_{1} \xi \right] s_{n} \cos n\varphi \quad (95)
\]

\[
N_{\varphi} = \frac{2.6322}{\sqrt{ah}} \sqrt{1+\frac{1}{2}\lambda^{4} + \frac{1}{\sqrt{3}}\lambda^{2}} \cdot \frac{1}{\lambda^{2}} \cdot e^{-\mu_{2} \xi} \cos \mu_{1} \xi \cdot s_{n} \cos n\varphi \quad (96)
\]

\[
M_{x} = \frac{1}{2.6322 \sqrt{a}} \sqrt{\frac{1}{1+\frac{1}{2}\lambda^{4} + \frac{1}{\sqrt{3}}\lambda^{2}}} e^{-\mu_{2} \xi} \sin \mu_{1} \xi \cdot s_{n} \cos n\varphi \quad (97)
\]

The functions \( s(\beta x, \lambda) \):
\[ s_1(\beta x, \lambda) = e^{-\mu_2 \xi} \left[ \cos \mu_1 \xi - \frac{\sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2}{\sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^2} \cdot \sin \mu_1 \xi \right] \]  \tag{98}

\[ s_2(\beta x, \lambda) = \sqrt{1 + \frac{1}{2} \lambda^2} + \frac{1}{\sqrt{3}} \lambda^2 \cdot \frac{1}{\lambda^2} \cdot e^{-\mu_2 \xi} \cos \mu_1 \xi \]  \tag{99}

\[ s_3(\beta x, \lambda) = \frac{1+\sqrt{1 + \frac{1}{2} \lambda^2} - \frac{1}{\sqrt{3}} \lambda^2}{1 + \frac{1}{\sqrt{3}} \lambda^2} \cdot \frac{1}{\lambda^2} \cdot e^{-\mu_2 \xi} \sin \mu_1 \xi \]  \tag{100}

are listed in Table 4 (Fig. 18 to 23). If \( \lambda = 0 \), Eq. (98) to (100) reduce to the \( s \)-functions (36) to (38) for the symmetrical case and in consequence (95a) to (97a) become identical with (33a) to (35a).

5.) Use of Superposition in General Cases:

Cases where the second boundary of the cylinder is at a finite distance \( I \) from the rib are considered. To the approximate solution (58) another term with an \( e^{\mu_2(\xi - I/a)} \) power (see Eq. 57) must be added to take into account the boundary conditions at \( x = I \). However, such a solution becomes so complicated that it is of no practical value. The principle of superposition gives the only practical way for a solution of this problem.
The same example as already treated for the case of axial symmetry \((\lambda = 0)\) is taken up. (p. 17 and Fig. 6). A semi-infinite cylinder has a rib at a distance \(l\) from the free edge (see Fig. 6). This actual case \(1\) can be thought of as a superposition of 3 other cases \(2, 3\) and \(4\) already solved. Further explanations of the procedure are given on p. 18. For a unit string force \(S = 1\ \cos n\phi\), the cases \(2, 3,\) and \(4\) give the following direct force \(N_\phi\) at \(x = 0\):

\[
\text{Case 2: Eq. (75a)} \quad (N_\phi)_{x=0} = \frac{0.6580}{\sqrt{ah}} 1_2(0,\lambda) \cos n\phi
\]

\[
\text{Case 3: Eq. (75a)} \quad (N_\phi)_{x=2l} = -\frac{0.6580}{\sqrt{ah}} 1_2(2\beta l, \lambda) \cos n\phi
\]

\[
\text{Case 4: Eq. (75a)} \quad (N_\phi)_{x=l} = \frac{2.6322}{\sqrt{ah}} 1_1(\beta l, \lambda) s_2(\beta l, \lambda) \cos n\phi
\]

Case \(1 = 2 + 3 + 4\):

\[
(N_\phi)_{x=0} = \frac{0.6580}{\sqrt{ah}} \left[ 1_2(0,\lambda) - 1_2(2\beta l, \lambda) + 4i_1(\beta l, \lambda) \cdot s_2(\beta l, \lambda) \right] \cos n\phi
\]

\[
(101)
\]

The effective width is (eq. (3)):

\[
b = \frac{S}{(N_\phi)_{x=0}} = 1.5196\sqrt{ah} \frac{1}{1_2(0,\lambda) - 1_2(2\beta l, \lambda) + 4i_1(\beta l, \lambda) s_2(\beta l, \lambda)}
\]

\[
(102)
\]

Eq. (102) may be checked for different limiting cases already solved. If \(\beta l = \infty\) it reduces to the effective
width of an infinitely long cylinder, Eq. (67). For
βz = 0 Eq. (102) checks the effective width of a semi-
infinite cylinder, Eq. (85). And finally by taking
λ = 0 Eq. (102) coincides with Eq. (40).

Putting Eq. (102) in the form

\[ b = K \sqrt{ah} \]  

where:

\[ K = \frac{1.5196}{i_2(0, \beta \lambda) - i_2(2\beta \lambda, \lambda) + i_1(\beta \lambda) s_2(\beta \lambda, \lambda)} \]  

the function \( K \), depending on the two parameters \( \beta \lambda \) and \( \lambda \),
can be computed for different values of the parameters
(Table 2 and Fig. 16). Fig. 16 shows that for values
\( \beta \lambda > 1 \) the effective width approaches rapidly the one
of an infinitely long cylinder ( \( \beta \lambda = \infty \)).

Any force or moment in the shell can be found by
the same superposition of the 3 known cases 2, 3, and 4.
The procedure can be applied to many other cases. The
fact that all forces and moments are damped out very
rapidly allows one to neglect influences originating
from sources sufficiently far removed from the location
under consideration.
IV. The Effective Width of Cylindrical Shells in Case the Radius of the Shell Increases to Infinity

1.) The Problem:

The question arises as to what the effective width of a cylindrical shell becomes if the radius "a" of the shell approaches infinity. Obviously the axis of the rib and the middle plane of the shell become straight and the effective width should be identical with the effective width of a flat plate reinforced by a rib (T-Beam). No difficulty exists in proving that the differential equations of the shell reduce to the differential equation of a flat plate if the following substitutions are made:

\[ a = \infty \]
\[ a \frac{d\phi}{d\psi} = dy \]

Nevertheless this does not prove that the equations for the effective width of cylindrical shells derived in the previous two chapters will check with those of the corresponding T-Beams. For the shell equations were solved by an approximate procedure (Eq. (58) instead of Eq. (57)) and further simplifications ((a) to (d) and (84)) were introduced in order to get expressions depending on two parameters \( \lambda \) and \( \sqrt{ah} \) only. Hence the limiting process \( a \to \infty \) will be applied to the derived formulas for the effective width directly and these
results will be compared to the corresponding equations of a T-Beam with a straight axis.

2.) Effective width of a T-Beam with a straight axis:

This problem was solved by a number of authors during the past 3 decades (Ref. (1) to (5)). Fig. 1 shows a cross section of a T-Beam. The plate is assumed connected to the rib along the two lines A only. For any stress in the rib along these lines the stress in the flange-plate must be the same (continuity condition). Therefore the plate is under a condition of plane stress acted upon by boundary forces along the lines A. Airy's stress function allows a rather simple solution of this problem. For the derivation of the effective width one of the above mentioned references may be consulted.

It is well known that the effective width of T-Beams is constant only for the case where the stress along the connecting lines A varies in the form of a cosine-function. If axial forces are absent this means a similar variation for the bending moment \( G = M \frac{Z}{I} \). Two cases, corresponding to the effective width of an infinitely long and a semi-infinite cylinder are considered. The results presented are taken from Girkmann's book (Ref. (11)).

* For the question if the stresses or the strains must be identical see foot-note p. 5.
a.) T-Beam with an infinitely wide flange:

A continuous T-Beam with a flange sufficiently large to be considered as extending to infinity is supported by equidistant supports with spans \( L \) (Fig. 12a). \( y \) is the coordinate in the direction of the rib, \( x \) is taken perpendicular to it. The load acting on the rib is:

\[
p = p_0 \cos \frac{y}{L}
\]

If Poisson's ratio \( \nu \) is assumed to be zero the effective width of the beam is

\[
b = \frac{4}{3} \frac{L}{\nu} = 0.4244L
\]  

(103)

A simple beam of equal span \( L \) has the same effective width if cross-beams at the supports are provided and are adequate to carry shear forces \( \tau_{yx} \).

b.) Beam with an infinitely wide flange on one side:

Differing from the previous case the beam has an infinitely wide flange on one side only (Fig. 12b). All other conditions are equal. By neglecting the torsional stiffness of the rib the effective width is:

\[
b = \frac{1}{2} \frac{L}{\nu} = 0.1592L
\]  

(104)

The only variable in the two equations (103) and
(104) is the span $L$. It is quite obvious that the
effective width increases to infinity if the span $L \to \infty$.
This fact will be of importance in the following discussion.

3.) Effective width of cylindrical shells for the

**Limiting case $a \to \infty$:**

The string force $S = S_n \cos n\varphi$ and hence the
direct stress $G_\varphi$ varies in $n$ complete cosine-waves
around the cylinder (Fig. 13). The length of one half-
wave is $L$ and the angle corresponding to this arc
length $L$ is:

$$\alpha = \frac{L}{a}$$

The number $n$ of complete waves around the cylinder is
hence:

$$n = \frac{\pi}{\alpha} = \frac{\pi a}{L}$$

(105)

And the parameter $\lambda$ (Table B) as a function of the half-
wave length $L$ becomes:

$$\lambda = n \sqrt{\frac{h}{a}} = \frac{\pi a}{L} \sqrt{\frac{h}{a}} = \frac{\pi}{L} \sqrt{ah}$$

(106)

Let's now consider the radius "a" increasing to
infinity. During this process the half-wave length $L$
or the number $n$ of the waves can be kept constant. If
the latter is done the length $L$ becomes infinitely long.
This would then correspond to a T-Beam of infinite span,
a case which certainly does not have any practical meaning.
Therefore the length $L$ will be kept constant.
a.) Infinitely long cylinder:

The effective width for an infinitely long cylinder was given in Eq. (67):

\[ b = 1.5196 \sqrt{ah} \frac{1}{\sqrt{1 + \frac{1}{2} \lambda^2 + \frac{1}{\sqrt{3}} \lambda^3}} \]

Substituting \( \lambda = \frac{\pi}{L} \sqrt{ah} \) and rearranging

\[ b = \frac{1.5196 L}{\pi} \frac{1}{\sqrt{\frac{1}{\lambda^4} + \frac{1}{2} + \frac{1}{\sqrt{3}}}} \]

Now, in the limit as \( \lambda \to \infty \)

\[ (b)_{a \to \infty} = 0.4268 L \] (107)

This checks within 0.5% the value of the effective width of the corresponding T-Beam (Eq. (103)).

b.) Semi-infinite cylinder:

The derivation for the effective width resulted in Eq. (85):

\[ b = 0.3799 \sqrt{ah} \frac{1}{\sqrt{1 + \frac{1}{2} \lambda^2 + \frac{1}{\sqrt{3}} \lambda^3}} \left[ 1 + \frac{\lambda^2}{3 \left( 1 + \frac{1}{2} \lambda^2 \right)} \right] \]

Substituting \( \lambda = \frac{\pi}{L} \sqrt{ah} \) and rearranging

\[ b = \frac{0.3799 L}{\pi} \frac{1}{\sqrt{\frac{1}{\lambda^4} + \frac{1}{2} + \frac{1}{\sqrt{3}}} \left[ 1 + \frac{1}{3 \left( \frac{1}{\lambda^2} + \frac{1}{2} \right)} \right]} \]

the limit \( a \to \infty \) (and in consequence \( \lambda \to \infty \)) gives:

\[ (b)_{a \to \infty} = 0.1570 L \] (108)
The difference between this value and the one of the corresponding T-Beam (Eq. (104) is 1.4%.

c.) The case of axial symmetry:

The string force S and hence the stress \( \sigma_p \) are constant around the cylinder. \( n, \) being the number of waves of S, is obviously zero. The effective width for an infinitely long and a semi-infinite cylinder are given by Eq. (18) and Eq. (29) respectively. \( \sqrt{\pi n} \) is the only parameter in these two equations. If the radius \( a \) tends to infinity, the effective width becomes infinitely large. It should be kept in mind that the rib gets infinitely long and the shell reinforced by a rib transforms to a T-Beam with a span \( L = \infty \) whose effective width is equally infinitely large (see Eq. (103) and (104)). The correspondence between the two problems is established for this special case too.

4.) Conclusions

1. It was shown that the effective width of cylindrical shells reduces to the effective width of the corresponding T-Beams with a straight axis if the radius of the shell is increased to infinity. The check is complete from the practical point of view (differences of 0.5% and 1.4%), but it is not a mathematically exact one. This is to be
expected as an approximate solution (Eq. (58)) was used for the derivation of the effective width of cylindrical shells.

2. The close correspondence established between the two problems for the limiting case $a = \infty$ may be considered as a justification for the use of the approximate solution Eq. (58).

3. The derived formulas for the effective width are not limited by a certain value of the radius $a$. They are based on the general principles of the theory of elasticity. The stability* of the shell and of the combination of the shell and the rib gives an upper limit, a problem which exists equally for the effective width of the T-Beam.

V. General Remarks on the Application of the Effective Width

1.) Symmetrical case:

The effective width $b$ depends on the dimensions and the material (Poisson's ratio $\nu$) of the shell only. It can be easily determined from the formulas or tables.

* See Appendix for a résumé of this problem.
The effective section consists of the rib and the effective width of the shell as a flange. Area and moment of inertia of this section can be computed. The rib stresses and the deformation of the rib calculated for this effective section are the actual stresses and deformation. The string force $S$ acting on the shell is given by Eq. (4):

$$ S = bh \cdot \sigma_A $$

where $\sigma_A$ is the stress in the rib along the connecting lines $A$ (Fig. 4). The direct force $N_x$ and the bending moment $M_x$ are readily determined by use of the $i$- and $s$- functions given in Table 4.

2.) General case

a.) Statically determinate case:

No use of deformations need be made to find the variations of the stress $\sigma_A$ (not magnitude) along the connecting lines $A$ between rib and shell (Fig. 4). Hence the expansion of this stress in a Fourier series is possible except for a constant multiplier. The effective width for each term of the series can be calculated and their superposition gives the actual effective width of the shell for the case under consideration.

Computing the area and moment of inertia of the effective section the stresses in the rib can be determined.
Eventually the string force $S$, acting between the rib and the shell, and the $N_{\phi}$, $M_x$ and $T$ in the shell are found.

b.) Statically indeterminate case:

The stress distribution depends on the deformation of the structure. In this case a certain variation of the stress must be assumed in order to calculate the different terms of the Fourier expansion. Note that only the variation, not the magnitude of the stress distribution, must be assumed for calculating the effective width. The stresses in the ribs and the forces in the shell are found by the above explained procedure.

Probably the calculated stress $\sigma_A$ along the connecting lines of rib and shell (Fig. 4) differs from the assumed variation. Therefore the whole procedure must be repeated until a sufficiently close correspondence between the assumed and the calculated stress distribution is found. Actual applications show that it is rather easy to make a first assumption which does not require any repetition of the procedure. It should be noted that the effective width is not influenced to a great extent by a small change of the assumed stress distribution. The area and moment of inertia of the effective section change much less because the effective width of the shell makes up the flange of this cross section only.
Appendix

Limitations of the Effective Width by Stability

1.) **T-Beam with Straight Axis:**

It is conceivable that the flanges of a T-Beam, as shown in Fig. 1, may buckle under a certain stress \( \sigma_y \) in direction of the rib (y-direction). The question arises for what magnitude of the stress \( \sigma_y \) this limit will be reached and what will be the effective width of the flange in the buckled state.

Th. v. Karman solved the somewhat similar problem of the effective width of a flat panel, simply supported, under uniform compression in the buckled state (Ref. (15)). In a T-Beam the stress \( \sigma_y \) will vary in general as a function of both coordinates \( x \) and \( y \) (Fig. 24). Therefore shear stresses \( \tau_{xy} \) will be present in the flange too. Over a length \( l \) (half wave length of the buckles in y-direction, equal to the effective width \( b \) as will be shown later on) the stress \( \sigma_y \) will be assumed as constant. Furthermore the distribution of the \( \sigma_y \) in the x-direction (perpendicular to the rib) will be assumed constant over the effective width \( b \), and equal to the normal stress \( \sigma_A \) acting along the connecting line between the rib and the flange. Outside of the effective width
is considered to be zero. The buckled middle surface of the flange over the effective width \( b \) (see Fig. 24) is assumed to have a deflection:

\[
w = w_0 \sin \frac{n x}{b} \sin \frac{n y}{l}
\]  

where \( b \) is the total effective width \( (1/2 \text{ of each side of the rib}) \) and \( l \) is the half-wave length of the buckles in the direction of the rib. Eq. (109) implies the following boundary conditions for the buckled flange:

\[
\begin{align*}
 x = 0 & : w = 0 \\
 M_x &= \frac{\partial^2 w}{\partial x^2} = 0 \\
 x = \pm \frac{b}{2} & : \frac{\partial w}{\partial x} = 0 \\
 y = 0 & : w = 0 \\
 M_y &= \frac{\partial^2 w}{\partial y^2} = 0
\end{align*}
\]

The action of the entire flange outside of the effective width \( \pm \frac{b}{2} \) consists merely in providing the horizontal tangent at \( x = \pm \frac{b}{2} \). The torsional stiffness of the rib is neglected, the moment \( M_x \) at \( x = 0 \) being zero.

For the buckled state the differential equation of an initially flat sheet under a uniform compression force \( N_A = \Sigma_A h \) is (e.g. Ref. (10), p. 314)
\[
\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} = -\frac{N_A}{D} \frac{\partial^2 z}{\partial y^2} \quad (110)
\]

Where: \[D = \frac{Eh^3}{12(1-v^2)}\]

Replacing \(w\) by its assumed value (109) and dividing the result by \(w_0 \cdot \sin \frac{\pi x}{b} \cdot \sin \frac{\pi y}{l}\) Eq. (110) becomes:

\[
\frac{\pi^2}{b^4} + 2 \frac{\pi^2}{b^2 l^2} + \frac{\pi^2}{l^4} = \frac{N_A}{D} \frac{\pi^2}{l^4}.
\]

And by introducing the values for \(D\) and \(N_A\)

\[
\frac{G_A}{E} = \frac{\pi^2 h^2}{12(1-v^2)} \left( \frac{l}{b^2} + \frac{1}{l} \right)^2 \quad (111)
\]

Eq. (111) has its minimum if \(l = b\) which can be found by differentiation.

The effective width \(b\) therefore is

\[
b = \frac{\pi h}{\sqrt{3(1-v^2)} \sqrt{\frac{E}{G_A}}} \quad (112)
\]

Eq. (112) gives the effective width of a T-Beam with large flanges after local buckling of the flanges took place. \(b\) is proportional to the thickness \(h\) of the flange and depends furthermore on the modulus of Elasticity \(E\), the stress \(\sigma_A\) along the connecting line of the flange and the rib and Poisson's ratio \(v\) of the material.
In the case of a beam with a large flange on one side only (Fig. 12b) an analysis similar to the one above gives an effective width \( b \) half of the one of Eq. (112):

\[
b = \frac{\pi h}{\sqrt{12(1-\nu^2)}} \sqrt{\frac{E}{\sigma_A}} \tag{113}
\]

Consider now the case where \( \sigma_A \) approaches the yield stress \( \sigma_{\text{yield}} \) of mild steel. Due to the sharp knee of the stress-strain curve it will be a very good approximation to use the modulus of the elastic part \( E \) and the yield stress \( \sigma_{\text{yield}} \). By taking:

\[
E = 30 \cdot 10^6 \text{ lb/in}^2
\]

\[
\sigma_{\text{yield}} = 33 \cdot 10^3 \text{ lb/in}^2
\]

\( \nu = 0.3 \)

the effective width \( b \) is:

\[
(112) \quad b = \frac{\pi h}{\sqrt{3(1-0.3^2)}} \sqrt{\frac{30 \cdot 10^3}{33}} = 57.33h \tag{114}
\]

\[
(113) \quad b = \frac{\pi h}{\sqrt{12(1-0.3^2)}} \sqrt{\frac{30 \cdot 10^3}{33}} = 28.67h \tag{115}
\]

Eq. (114) holds for a T-Beam with large flanges on both sides (Fig. 12a). In case of a beam with a flange on one side only, Eq. (115) has to be used (Fig. 12b).
In a practical problem the effective width will be calculated first by means of Eq. (103) or (104) respectively, which were derived under the assumption that the equilibrium of the flange is a stable one. Then the derived stress $\sigma_A$ along the connecting line of the rib and the flange is introduced in Eq. (112) or (113) respectively. If (112) or (113) furnishes an effective width smaller than the one derived by (103) or (104) it indicates that the flanges will buckle. Therefore the effective width of Eq. (112) or (113) must be used in the calculations.

For structural steel the flanges are in a stable equilibrium, if the effective width $b < 57 \ h$ or $b < 28 \ h$ for a T-Beam or a beam with a flange on one side only, provided the yield stress is not exceeded. (Compare Eq. (114) and (115)).

It may be pointed out that the above equations are limited by the simplifying assumptions made for their derivation.

2.) Cylindrical Shell:

In deriving the effective width of a cylindrical shell the influence of the tangential shear forces $N_{\varphi x}$ and $N_{x \varphi}$ (problem of shear lag in box girders) and of the radial deflection $w$ of the shell due to the direct forces $N_{\varphi}$ was considered. Therefore by increasing the load from an initial zero load the radial deflection $w$
of the shell relative to the rib will increase proportionally, and to each load corresponds a definite state of equilibrium. On the other hand for the flat flange of a T-Beam it was shown that for a certain stress \( \sigma_A \) in the flange a point of bifurcation of the equilibrium is found.

Let's consider a cylindrical shell of thickness \( h \) and a very large radius "a" stiffened by a rib in circumferential direction. By loading up the rib a certain circumferential stress \( \sigma_A \) along the connecting line of the rib and the shell may be reached for which the equilibrium of the shell may become indifferent and more than one state of equilibrium can be possible. The shell will buckle. The solution of this stability problem is certainly more involved than the one solved previously for the flange of a T-Beam. No attempt is made to derive a solution. But certain conclusions form the T-Beam solution may be drawn to have at least a qualitative insight in this problem.

Very probably the strain energy of the shell required to reach the state of indifferent equilibrium will be higher than the strain energy of the corresponding T-Beam (shell and flange of T-Beam are assumed to have the same thickness \( h \)). A shell with a very small radius "a" will not reach buckling before yielding is developed. On the other
hand a T-Beam can buckle elastically. This seems to indicate that the stability limit for the shell (curved T-Beam) occurs for a higher stress $\sigma_A$ along the connecting line of the shell and the rib than the stress of the corresponding T-Beam with a straight axis. It is therefore believed to be safe to use the Eq. (112) or (113) for the case of the effective width of a cylindrical shell too. The problem of stability for the shell arises only for large radii $a$ where the axis of the rib can be considered as almost straight, especially considering the relative short half-wave length $l$ of the buckles.

3.) Recommendations for Further Studies:

To the author no tests on the stability of the flanges of T-Beams are known. A T-Beam differs from the usually tested flat panels with stiffeners under a constant end compression. For the stresses induced in the flange are originated by shear forces along the connecting line of the rib and the flange, the direct stresses $\sigma_y$ in direction of the rib are functions of the coordinate $x$ and $y$ and the flange will have in general stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ (Fig. 24). Tests on T-Beams with a straight axis should be made in order to have experimental results to compare with the Eq. (112) and (113).
Experiments on T-Beams with a curved axis or on cylindrical shells reinforced by ribs in circumferential direction under loads applied to the ribs are necessary to find the stability limit of such types of structures. A theoretical study paralleling the tests is required to find a rational design base.
Tables 1 to 4

(Pages 75 to 83)
TABLE 1

Functions $c(\beta t)$ and $f(\beta t)$ for different values of the parameter $\beta t$

For Graphical representation see Fig. 14 and 15.

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Eq. (40b) (44b) (47b) (50b) (42b) (45b) (48b)
TABLE 2

Coefficient K of the Effective Width for different values of the two Parameters \( \lambda \) and \((\beta l)\).

For Graphical representation see Fig. 16.

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Eq. \((85b)\) \((102b)\) \((102b)\) \((102b)\) \((102b)\) \((102b)\) \((67b)\)
TABLE 3

Ratio of the maximum bending stress $\sigma_x$ to the circumferential direct stress $\sigma_\phi$ at the rib for an infinitely long and a semi-infinite cylinder.

For graphical representation see Fig. 17.

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<th>$\lambda$</th>
<th>$\frac{\sigma_x}{\sigma_\phi}_{x=0}$</th>
<th>$\frac{\sigma_x}{\sigma_\phi}_{\text{max}}$</th>
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Eq. (70) (91)
TABLES 4

Functions $i$ and $s$ for different values of the parameter $\lambda$ and the variable ($/beta{x}$).

For Graphical representation see Fig. 18.

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Eq. (24) (25) (26) (36) (37) (38) or (77) (78) (79) (98) (99) (100)
### Table 4/0.25

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<td>(79)</td>
<td>(98)</td>
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For graphical representation see Fig. 19.

### Table 4/0.50

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Eq. (77) (78) (79) (98) (99) (100)

For graphical representation see Fig. 20.

### Table 4/1.00

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Eq. (77) (78) (79) (98) (99) (100)
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Eq. (77) (78) (79) (98) (99) (100)

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For graphical representation see Fig. 21
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For graphical representation see Fig. 22

### Table 4/2.00

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For graphical representation see Fig. 22
### Table 4/2.25
\( \lambda = 2.25 \)

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**Eq.** (77) (78) (79) (98) (99) (100)

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### Table 4/2.50
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**Eq.** (77) (78) (79) (98) (99) (100)

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For graphical representation see Fig. 23
Figures 1 to 24

(Pages 85 to 101)
Imaginary stress distribution

Actual stress distribution

\[ \frac{1}{2} b \]

Flange (flat plate)

Rib

T-Beam

Fig. 1

Circular cylindrical shell reinforced by ribs.

Fig. 2
Effective width of cylindrical shells

Fig. 3

Fig. 4
a) Infinitely long cylinder

b) Semi-infinite cylinder

String Force $S_o = 1$

\[ Q_x + dQ_x \]

\[ N_x + dN_x \]

\[ N_\varphi + dN_\varphi \]

Forces

Moments

Displacements

Fig. 5

Case 1 = 2 + 3 + 4

Case 2

Case 3

Case 4

Fig. 6
Variation of String Force $S$

$S = S_2 \cos 2\varphi$

$S = S_3 \cos 3\varphi$

Fig. 10

Forces

Moments

Displacements

Fig. 11
Section A - A:

a) \[-x \quad 0 \quad 0 \quad x\]

b) \[0 \quad x\]

Fig. 12

Middle surface of the shell

String force \( S = S_n \cos n\phi \)

Fig. 13
Fig. 15

$\sqrt{1.7321}$

$\alpha_1(\beta l)$

$\alpha_2(\beta l)$

$\alpha_3(\beta l)$

$\beta l$
\[ \frac{F_0}{h} = 1.316 \]

\[ V = \frac{h}{h} \]

\[ S = 5.40s \mu g \]

Effective width \( b = f \sqrt{n} \)

- \( E_4 (850) \)
- Semi-infinite cylinder

- \( E_4 (875) \)
- Infinity long cylinder
Fig. 17

Infinitely long cylinder
-Eq. (70)-

Semi-infinite cylinder
-Eq. (91)-

Ratio \[ \frac{G_x}{G_y} \]

\( \lambda \)
Case: $\lambda = 0$
Case: $\lambda = 0.50$

Fig. 19
Case: $\lambda = 2.00$

Fig. 22
Section A-A

Buckled Middle Surface of the Flange

Section B-B

Fig. 24
Notations

List of References

Vita

(Pages 103 to 110)
NOTATIONS

Roman Alphabet

a radius of the shell
A constant of integration
b effective width
B constant of integration
c\(1(\beta l)\) function of \(\beta l\) defined by Eq. (40b)
c\(2(\beta l)\) " " \(\beta l\) " " Eq. (44b)
c\(3(\beta l)\) " " \(\beta l\) " " Eq. (47b)
c\(4(\beta l)\) " " \(\beta l\) " " Eq. (50b)
C constant of integration
C\(_1\) to C\(_8\) constants of integration
C\(_1\), C\(_2\) coefficients defined in Table A, p. 11.
D bending stiffness of the shell, Eq. (15)
e base of natural logarithm
E modulus of elasticity
f\(_1(\beta l)\) function of \(\beta l\) defined by Eq. (42b)
f\(_2(\beta l)\) " " \(\beta l\) " " Eq. (45b)
f\(_3(\beta l)\) " " \(\beta l\) " " Eq. (48b)
e\(_1\), e\(_2\), e\(_3\) functions, Eq. (53)
h thickness of the shell
H symbol for any force, moment etc., of the shell
H coefficient depending on the quantity \(H\) under consideration, Table A and Table B
function, Eq. (77); in case of $\lambda = 0$, Eq. (24)

$\alpha_2(\beta z, \lambda)$ function, Eq. (78); in case of $\lambda = 0$, Eq. (25)

$\alpha_3(\beta z, \lambda)$ function, Eq. (79); in case of $\lambda = 0$, Eq. (26)

$k, k'$ coefficients defined in Table B

$k_1, k_2$ coefficients defined in Table B

$K(\beta z, \lambda)$ function of the two parameters $\beta z$ and $\lambda$, used for the effective width.

$L$ length of the shell in axial direction, or half-wave length of buckles (Eq. 109)

$L$ span of T-Beam or half-wave length of the string force $S$

$M$ total moment of the effective section

$M_x$ bending moment per unit width of the shell in axial direction

$M_\varphi$ bending moment per unit width of the shell in circumferential direction

$M_x \varphi, M_\varphi x$ twisting moments per unit width of the shell

$n$ th term of a Fourier series (number of complete cosine-waves of the string force $S$ around the cylinder)

$N$ total normal force of the effective section

$N_x$ direct force per unit width of the shell in axial direction
$N_\phi$ direct force per unit width of the shell in circumferential direction

$N_{x\phi}, N_{\phi x}$ direct shear forces per unit width of the shell

$p$ line load due to a constant string force $S$.

(See Fig. 7b)

$Q_x$ normal shear force per unit width of the shell acting on a face $x = \text{constant}$

$Q_\phi$ normal shear force per unit width of the shell acting on a face $\phi = \text{constant}$

$r$ height of rib

$s_1(\beta x, \lambda)$ functions, Eq. (98); in case of $\lambda = 0$, Eq. (36)

$s_2(\beta x, \lambda)$ " , Eq. (99); " " $\lambda = 0$, Eq. (37)

$s_3(\beta x, \lambda)$ " , Eq. (100);" " $\lambda = 0$, Eq. (38)

$S$ string force $S$ defined by Eq. (1)

$S_0(x)$ string force at a distance $x$ in the symmetrical case, defined by Eq. (13)

$S_n(x)$ string force at a distance $x$ in the case of the $n$th harmonic, defined by Eq. (60)

$t$ width of rib

$T$ total shear force per unit width of shell in circumferential direction, Eq. (62)

$u$ displacement in axial direction

$U_n(x)$ function of $x$, Eq. (54)

$v$ displacement in circumferential direction
\( V_n(x) \) function of \( x \), Eq. (54)

\( w \) displacement in radial direction, positive outward, or deflection of the buckled middle surface of the flange of a T-Beam, Eq. (109)

\( w_0 \) maximum deflection of the buckled middle surface of the flange of a T-Beam, Eq. (109)

\( W_n(x) \) function of \( x \), Eq. (54)

\( x \) coordinate of the shell in axial direction

\( y \) coordinate of the T-Beam in direction of its axis

\( z_A \) distance between the centroid of the effective section and the connecting line of the rib and the shell (Fig. 4)

\( z_L \) distance between the centroid of the effective section and the lower fiber of the rib

\( z_U \) distance between the centroid of the effective section and the upper fiber of the rib.

Greek Alphabet

\( \alpha \) angle corresponding to a half-wave of the string force (Fig. 13)

\( \beta \) shell coefficient, depending on the thickness \( h \), the radius "a" and Poisson's ratio \( \nu \) of the shell, Eq. (10)

\( \Delta \) ratio of two expressions, p. 37

\( \varepsilon_\varphi \) strain of the middle surface of the shell in circumferential direction
\( \kappa \)  
indeterminate coefficient in Eq. (55)

\( \kappa_1 \text{ to } \kappa_8 \)  
roots of an 8th order equation, Eq. (56)

\( \lambda \)  
parameter of the effective width, see Table B

\( \mu_1 \text{ to } \mu_4 \)  
numerical coefficients of the roots \( \kappa \), Eq. (56)

\( \mu_1 \text{ to } \mu_2 \)  
numerical coefficients, defined in Table B

\( \nu \)  
Poisson's ratio

\( \sigma \)  
stress

\( \sigma_A \)  
stress along the connecting line A of the rib and the shell (Fig. 4)

\( \sigma_L \)  
stress of the lower fiber of the rib (Fig. 4)

\( \sigma_u \)  
stress of the upper fiber of the rib (Fig. 4)

\( \sigma_x \)  
bending stress in axial direction

\( \sigma_\varphi \)  
direct stress in circumferential direction

\( \tau_{x\varphi} \)  
shear stress of the shell on a cut \( x = \text{constant} \)
in circumferential direction

\( \tau_{yx} \)  
shear stress in the flange of a T-Beam

\( \varphi \)  
angular coordinate of the shell in circumferential direction

\( \xi \)  
dimensionless coordinate in \( x \)-direction \( (\xi = \frac{x}{a}) \)

\( \psi \)  
angle, constant of integration, Eq. (58)
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<td>4</td>
<td>Chwalla, E.</td>
<td>&quot;Die Formeln zur Berechnung der voll mittragenden Breite dünner Gurt- und Rippenplatten&quot;.</td>
<td>Der Stahlbau, vol. 9, 1936, p. 73.</td>
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<td>&quot;Die Spannungsverteilung in den Gurtungen gekrümmter Stäbe mit T- und I-förmigem Querschnitt&quot;.</td>
<td>Der Stahlbau, vol. 6, 1933, p. 3, or Navy Department, Translation 228, 1950.</td>
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(15) v. Karman, Th.  
VI T A

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