NON-UNIFORM TORSION
OF
PLATE GIRDER

By
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Accepted, (Date)

Special committee directing the doctoral work of Mr. Gerald Giro Kubo

Chairman
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(Date)

Gerald G. Kubo
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"NON-UNIFORM TORSION OF PLATE GIRDERS"

SYNOPSIS

As an extension of the non-uniform torsion tests of built-up structural members recently completed at the Fritz Engineering Laboratory of Lehigh University, a series of six full-size plate girders, with a rolled beam as a control, was selected for testing under non-uniform torsion conditions in the elastic range. This group included one bolted, three riveted, and two welded specimens fabricated under representative shop conditions.

A custom-built yoke with a pair of cantilevered loading arms was attached to each specimen at the center. The twisting couple, formed by a pair of equal and opposite forces, was applied by means of weights connected to the ends of the loading arms. The specimens were simply-supported at the ends but restrained against twisting.

By virtue of symmetry, the warping tendency of the section was fully restrained at the center. This condition of non-uniform torsion without bending produced transverse shears and bending moments in the flanges superimposed on the pure torsional shears.

The transverse and longitudinal variations of the bending and shear stresses, as well as the angular distortions of web and flange, were determined by means of a system of SR-4 gages and level bars.

The experimental results were compared with computed values based on available theories, such as the Timoshenko and Goodier-Barton solutions. Certain modifications were required to account for the deviations encountered; namely, the difference between web and flange distortions and variations near the free end. Special features of built-up sections in torsion were investigated. An analytical study of the effect of a variable torsion constant was also made.
CHAPTER I. INTRODUCTION

A. Program and Sponsorship

1. Overall Program

The behavior of solid sections under uniform and non-uniform torsion has been developed analytically and checked experimentally by many investigators. However, the built-up section has as yet received scant attention. Previous tests have been limited to model specimens of reduced size.

It was decided that only full-size specimens, fabricated under actual shop conditions, would provide the desired verification of the analytical modifications necessitated by the variable characteristics of fabricated sections. Therefore, a comprehensive study of the behavior of built-up structural members was undertaken at the Fritz Engineering Laboratory of Lehigh University, initially under the general supervision of Dr. Bruce G. Johnston and subsequently under Professor William J. Eney. A specially-designed torsion testing machine of record size and capacity had just been built and was available in the laboratory.

The complete program, as eventually evolved, can be summarized as follows:

<table>
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A supplementary series of tests on built-up column sections under uniform torsion was also carried out as an independent investigation.

The first phase of this program has been finished, and the results in abridged form have been published. This report covers the non-uniform torsion tests conducted at Lehigh University and completes the second phase. The third report, based on combined bending and torsion tests performed at Swarthmore College with the cooperation of Professor Samuel T. Carpenter, is in progress.

2. Sponsorship

All three phases of this program have been supported financially by the Department of Highways of the Commonwealth of Pennsylvania in cooperation with the Federal Bureau of Public Roads.

The specimens used in this project were initially contributed to the first project by Lehigh Structural Steel Company, of Allentown, Pennsylvania.
## CLASSIFICATION OF TORSION LOAD ArrANGEMENT

<table>
<thead>
<tr>
<th>Type</th>
<th>Case</th>
<th>Sketch</th>
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<tbody>
<tr>
<td><strong>UNIFORM TORSION</strong></td>
<td>(a) Circular</td>
<td><img src="image1" alt="Sketch" /></td>
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<tr>
<td></td>
<td>Unrestrained Ends</td>
<td></td>
</tr>
<tr>
<td>Shaft Loading</td>
<td>(b) Circular</td>
<td><img src="image2" alt="Sketch" /></td>
</tr>
<tr>
<td></td>
<td>Restrained End</td>
<td></td>
</tr>
<tr>
<td>(c) Non-Circular</td>
<td>Unrestrained Ends</td>
<td><img src="image3" alt="Sketch" /></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d) Non-Circular</td>
<td>Restrained End</td>
<td><img src="image4" alt="Sketch" /></td>
</tr>
<tr>
<td></td>
<td>Shaft Loading</td>
<td></td>
</tr>
<tr>
<td>(e) Non-Circular</td>
<td>Restrained End</td>
<td><img src="image5" alt="Sketch" /></td>
</tr>
<tr>
<td></td>
<td>Cantilever Loading</td>
<td></td>
</tr>
<tr>
<td>(f) Non-Circular</td>
<td>Simply-Supported</td>
<td><img src="image6" alt="Sketch" /></td>
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<tr>
<td></td>
<td>Beam</td>
<td></td>
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<tr>
<td></td>
<td>Combined Bending</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and Torsion</td>
<td></td>
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</tbody>
</table>
E. Classification of Torsion

1. Definitions

For purposes of this report, the term uniform torsion will be used to designate the St. Venant or pure torsion condition which assumes no restraint of warping. Non-uniform torsion develops in non-circular sections when the warping tendency is restrained.

2. Uniform Torsion Cases

When a member having constant torsional rigidity is subjected to uniform torsion, the unit angle of twist will be constant and only torsional shear stresses will exist. The secondary effects are assumed to be negligible in the small angle of twist range.

The common cases of uniform torsion are illustrated in Fig. 1:1.

(a) The cylindrical shaft, either solid or tubular, is the classic case of a section without a significant tendency to warp.

(b) Even when one or both ends are restrained, no change in the torsion conditions will result, except locally.

(c) A non-circular section under shaft torque without end restraints can also be treated by St. Venant's theory. The built-up specimens used in the first phase of the torsion program were tested in this manner.

3. Non-uniform Torsion Cases

When non-circular sections of constant torsional rigidity are twisted, the unit angle of twist varies along the span. Normal and shear stresses due to flange bending are produced in addition to the torsional shear stresses. The angular distortions are greatly reduced.

A few cases of non-uniform torsion are shown in Fig. 1:1.

(d) An open section fixed at one end and subjected to a shaft torque
is the basic loading adopted for investigation in this project.

(c) A cantilevered beam, when acted upon by a concentrated or distributed load eccentrically applied, will develop combined bending and torsion stresses.

(f) A simply supported beam when loaded eccentrically will be subjected to a system of bending and torsion stresses. The second part of the non-uniform torsion program was based on this load arrangement, using a single concentrated load applied eccentrically at the centerline of the span.

C. Uniform Torsion - Review

1. Analytical Development

Since many excellent treatises on this subject are available, no attempt will be made to summarize this material. The classical theory was developed by St. Venant (1) and modified by Timoshenko.(11) The membrane analogy, proposed by Prandtl, (12) was used by Griffith and Taylor, (13) Thayer and March, (14) as well as by Lyse and Johnston, (2) to check the torsion constants of a variety of solid sections. The built-up section under torsion was analyzed by Chang and Johnston. (4)

2. Experimental Programs

Several test programs have been conducted in which the angular distortion and stress distribution of specimens under uniform torsion have been measured. Some representative programs are listed below.

I-beams have been tested by Tuckerman (16) at the University of Toronto, by Campbell (9) at Northwestern University, and by Lyse and Johnston (2) at Lehigh University. Built-up sections were tested by Madsen, (15) Chang and Johnston, (4) and Jentoft and Mayo, (5) all at Lehigh University.

Note: The numbers in parentheses refer to items listed in the Bibliography, Appendix C.
CHAPTER II. OBJECTIVES AND SCOPE

A. Objectives

This project was designed to investigate the over-all behavior of plate girders under non-uniform torsion caused by shaft loading. The specific objectives can be summarized as follows:

1. To measure the actual angular distortions and to compare the results with computed values obtained from the available theories.

2. To determine the existing distribution and variation of both the normal and shear stresses and to check against theoretical values.

3. To examine the effect of section properties on the non-uniform torsion behavior both analytically and experimentally.

4. To study the characteristic behavior of built-up girders under torsional loads.

5. To investigate the differences in the actual and anticipated results and to establish the explanation for same.

6. To explore the possibility of extending the available theories to cover variations in section properties.

B. Scope

Typical, full-size, plate girder sections of bolted, riveted, or welded construction, as well as a rolled beam, were tested. The seven specimens were relatively shallow with the section properties kept constant over the span length.

Although the over-all length of all the beams was 14'9", only half of the span was actually used. The effective length, being a function
of the section properties, varied over a representative yet practical range. However, the extremely short or the very long specimen was not considered.

Shaft torque acting on a beam with one end restrained was the only loading condition considered. The combined bending and torsion case was reserved for a later study.

The magnitude of the applied torque was restricted so that the normal and shear stresses were well under the elastic limit. The specimens were eventually loaded into the plastic range in the uniform torsion project.
CHAPTER III. NON-UNIFORM TORSION THEORY - SUMMARY

A. Analytical Development

1. General - Notation

The warping tendency is present in all non-circular sections subject to torsion. The restraint of warping becomes particularly significant in the case of open sections, such as WF beams.

The original solutions for the shaft loading case were presented by Timoshenko. A more general solution was proposed by Goodier and Barton. These theories, developed for solid sections, form the basis for the present study of the behavior of built-up sections.

One of the apparently unavoidable sources of confusion in technical literature is the lack of a standard yet descriptive system of notation. Each author and/or editor usually adopts a special code for his paper. When two different articles are juxtaposed, some inconsistencies and contradictions in notations are inevitable.

It was decided, therefore, to present each of the two solutions mentioned above in summary form, modifying each to conform to the same system of notation. Modifications and proposed extensions also followed this same system.

2. Timoshenko's Solution

a. Assumptions

The basic assumptions made, besides those common to the elastic and small deflection theories, are as follows:

(1) The section is twisted as a unit; i.e., the angle of twist of flange and web are the same.

(2) The applied torque is held in equilibrium by a resisting torque
made up of two parts; namely, the torsional shear torque and
the restraint of warping torque.

(3) Bending of the web and of each flange about its weaker axis can
be neglected.

(4) Lateral deflection of the flange due to shear is negligible.

b. Derivation — Summary

A fundamental non-uniform torsion problem of a WF beam subjected
to a shaft loading is shown in Fig. 3:2a as Case I. The notation
used is summarized in Fig. 3:1.

The coordinate axes are defined in (a) of Fig. 3:2. The top
flange after twist, together with a typical flange element, is
depicted in (b). A typical cross section after twist, defining
the applied torque, the torsional shear, and restraint of warping
torques, appears in (c), (d), and (e), respectively.

The various steps in the Timoshenko solution accompanied by self-
explanatory sketches are outlined as follows:

(1) The equilibrium of a flange element leads to equation (2)\textsubscript{1} for
flange shear.

(2) The geometry of distortion relates the flange shear to the angle
of twist by equation (4)\textsubscript{1}.

(3) The equilibrium equation of torsion (6)\textsubscript{1}, when rearranged, be-
comes equation (7)\textsubscript{1}, the differential equation for the angle of
twist.

(4) The general solution of this differential equation is given as
equation (8)\textsubscript{1}.

(5) The three boundary conditions based on the assumed physical con-
considerations are stated as equations (9a)\textsubscript{1}, (9b)\textsubscript{1}, and (9c)\textsubscript{1}. 
(6) The three arbitrary constants, \( A \), \( B \), and \( C \), are evaluated by the proper combination of the general solution and the three boundary conditions. The results are listed as \((10a)_1\), \((10b)_1\), and \((10c)_1\).

(7) The angle of twist is defined by equation \((11a)_1\) which can now be evaluated for any value of \( z \).

(8) The unit angle of twist, equation \((12a)_1\), is obtained from the angle of twist by differentiation.

(9) The torsional shear or St. Venant's torque is expressed in terms of the unit angle of twist by definition as equation \((13a)_1\).

(10) The restraint of warping or flange bending torque can be written as equation \((14a)_1\).

(11) The transverse flange shear \( (V^f) \) is directly related to the restraint of warping torque as expressed by equation \((15a)_1\).

(12) The bending moment in the flange is obtained by integrating the shear equation, which results in equation \((16a)_1\).

(13) Equation \((17a)_1\) for the bending stress in the flange is based on the flexure formula.

(14) The longitudinal variation of the torsional, transverse, and combined shear stress at the centerline of the flange are summarized in equations \((18)_1\), \((19)_1\), and \((20)_1\).
### Summary of Derivation

**Case I:** Constant Torque, Constant KG, Constant $E I_y$

**Not Considering Web Deformation**

**Timoshenko Theory**

#### Notation:

- **Torsional Rigidity** - $S$
  
  \[ S = KG \]

- **Torsion Constant of Section** - $K$

- **Shear Modulus of Elasticity** - $G$

- **Angular Distortion**
  
  \[ \psi = \text{Angle of Twist of Section} \]

  \[ \theta = \text{Unit Angle of Twist} \]

  \[ \frac{d\psi}{dz} = \psi' \]

- **Warping Rigidity** - $W$

  \[ W = \frac{EI h^2}{4} \quad \text{(WF & I-Beams)} \]

  \[ I_y = \text{Moment of Inertia of Section} \]

  \[ = 2 I_F \]

  \[ h = \text{Distance between Flange Centroids} \]

  \[ E = \text{Modulus of Elasticity} \]

- **Torque** - $T$

  \[ T_S = \text{Torsional Shear Torque} \]

  \[ = KG \theta = KG \psi' = S \psi' \quad \text{(A)} \]

  \[ T_W = \text{Restraint of Warping Torque} \]

  \[ = Vh = - \frac{EI h^2}{4} \psi'' = -W \psi'' \quad \text{(B)} \]

---

**Fig. III - I:** Notation - Timoshenko Theory - Non-Uniform Torsion
Fig. 3:2  NON-UNIFORM TORSION OF WF BEAM - SHAFT LOADING

Case II:  TIMOSHENKO THEORY - WITHOUT WEB DEFORMATION
NON-UNIFORM TORSION - WF SECTION - SHAFT LOADING

CASE II: TIMOSHENKO THEORY

SUMMARY OF DERIVATION

1. Equilibrium of Flange Element

\[ \Sigma M = 0 \]
\[ V^F = -\frac{dM^F}{dz} \]  
(1)

By Flexure Formula:
\[ V^F = -EI^F \frac{d^3x}{dz^3} \]  
(2)

2. Geometry of Distortion

\[ X = \frac{h}{2} \sin \psi \]  
(3)

\[ \frac{d^3x}{dz^3} = \frac{h}{2} \frac{d^3\psi}{dz^3} = \frac{h}{2} \psi'''' \]

\[ V^F = -\frac{EI^F h}{2} \psi'''' \]

\[ = -\frac{EI^F h}{4} \psi'''' \]  
(4)

\[ M^F = \int V dz = -\frac{EI^F h}{4} \psi'''' \]  
(5)

3. Equilibrium Equation of Torsion

\[ T = T_R = T_s + T_w = S \psi' - W \psi'''' \]  
(6)

\[ \psi'''' - \frac{S}{W} \psi' = -\frac{T}{W} \]

Let \[ r^2 = \frac{S}{W} \]

\[ \psi'''' - r^2 \psi' = -\frac{T}{W} \]  
(7)

(D,E; for angle of twist)

4. General Solution of D.E.

\[ \psi = A + B \sinh(rz) + D \cosh(rz) + \frac{Tz}{S} \]  
(8)
Case II: Non-uniform Torsion

5. Boundary Conditions

At $z = 0$;
\[ \psi = 0 \]  \hspace{1cm} (9a)

$z = 0$;
\[ \psi' = \theta = 0 \]  \hspace{1cm} (9b)

$z = L$;
\[ M = \frac{EJy}{4} \psi'' = 0 ; \]
\[ \psi'' = 0 \]  \hspace{1cm} (9c)

6. Arbitrary Constants

\[ A = -\frac{T}{S} \tanh (rL) \]  \hspace{1cm} (10a)

\[ B = -\frac{T}{S} \]  \hspace{1cm} (10b)

\[ D = \frac{T}{S} \tanh (rL) \]  \hspace{1cm} (10c)

7. Angle of Twist

\[ \psi = \frac{T}{Sr} \left[ rz - \sinh (rz) + \tan (rL) \right] \left( \cosh (rz) - 1 \right) \]  \hspace{1cm} (11a)

At $z = 0$;
\[ (\psi)_R = 0 \]  \hspace{1cm} (11b)

At $z = L$;
\[ (\psi)_F = \frac{T}{GK} \left[ 1 - \frac{1}{rL} \tanh (rL) \right] \]  \hspace{1cm} (11c)

8. Unit Angle of Twist

\[ \theta = \frac{d\psi}{dZ} = \frac{T}{S} \left[ 1 - \frac{\cosh (L-z)}{\cosh (rL)} \right] \]  \hspace{1cm} (12a)

At $z = 0$;
\[ (\theta)_R = 0 \]  \hspace{1cm} (12b)

At $z = L$;
\[ (\theta)_F = \frac{T}{GK} \left[ 1 - \frac{1}{\cosh (rL)} \right] \]  \hspace{1cm} (12c)
### 9. Torsional Shear Torque

\[ T_s = KG\theta = T \left[ 1 - \frac{\cosh r(L-z)}{\cosh (rL)} \right] \]  
\[ T_s \bigg|_{z=0} = 0 \]  
\[ T_s \bigg|_{z=L} = T \left[ 1 - \frac{1}{\cosh (rL)} \right] \]

### 10. Restraint of Warping Torque

\[ T_w = V^f h = -\frac{EI_y h^2}{4} \frac{\psi'''}{T} \left[ \frac{\cosh r(L-z)}{\cosh (rL)} \right] \]
\[ T_w \bigg|_{z=0} = T \]  
\[ T_w \bigg|_{z=L} = \frac{T}{\cosh (rL)} \]

### 11. Transverse Shear in Flange

\[ V^f = \frac{T_w}{h} = \frac{T}{h} \left[ \frac{\cosh r(L-z)}{\cosh (rL)} \right] \]
\[ V^f \bigg|_{z=0} = \frac{T}{h} \]  
\[ V^f \bigg|_{z=L} = \frac{T}{h \cosh (rL)} \]

### 12. Bending Moment in Flange

\[ M^f = \int V^f d\xi = \frac{T}{rh} \left[ \frac{\sinh r(L-z)}{\cosh (rL)} \right] \]
\[ M^f \bigg|_{z=0} = \frac{T}{rh} \left[ \tanh (rL) \right] \]  
\[ M^f \bigg|_{z=L} = 0 \]
13. Bending Stress in Flange

\[ \sigma = \frac{M^f b}{I_{y/2}} = \frac{M^f b}{I_y} \]  

At \( z = 0 \); \( \sigma_R = \frac{T b \tanh (rL)}{r h I_y} \)  

At \( z = L \); \( \sigma_F = 0 \)

14. Shear Stress - Longitudinal Variation - Flange at \( z \)

(a) Torsional Shear Stress

\[ (\tau_s)_z = \left[ \frac{t_F + D}{2K} \right] T \left[ 1 - \frac{\cosh (r(L-z))}{\cosh (rL)} \right] \]  

(b) Transverse Shear Stress

\[ (\tau_v)_z = \left[ \frac{1.5}{b t_F} \right] T \left[ \frac{\cosh (r(L-z))}{\cosh (rL)} \right] \]  

(c) Combined Shear Stress

(Top Side)

\[ (\tau')_z = (\tau_s)_z + (\tau_v)_z \]
**FIG. 3:5**

**ANGULAR DISTORTION**

**NON-UNIFORM TORSION WF BEAMS**

**Fritz Engineering Laboratory Project No. 215A**
3. The Goodier-Barton Theory

a. Comparison with Timoshenko Solution

As pointed out by the authors, the Timoshenko solution does not permit the assignment of the division of torsional shear and restraint of warping torques at each end. This limitation indicates that some aspects of the problem have been neglected. It is shown that consideration of web deformation is one step towards the desired generality. However, the scope of the solution is still restricted and can only be applied to a limited number of cases.

The twisting of the section is geometrically related to the lateral deflection of the flange in its own plane. Such deflection implies a non-zero fourth derivative corresponding to some distributed load on the flange which is supplied by means of the web. The resulting lateral forces and twisting moments cause the web to form an S-curve. This web deformation, shown in exaggerated detail in Fig. 3:5B, results in a reduction in the flange rotation by an angle equal to the web chord rotation. The corresponding angular distortion assumed in the Timoshenko solution is sketched in Fig. 3:5A.

In the limiting case of a web of zero thickness, the flanges would deflect independently as beams without twisting. With relatively thick webs the flange twist will approach that of the web. The web thickness, therefore, influences the quantitative division of torque.

b. Derivation - Summary

The notation used in this condensation of the Goodier-Barton solution is assembled in Fig. 3:3. The basic problem, designated
as Case II, with the applicable geometric relationships is pictured in Fig. 3:4.

The derivation of the desired equations is condensed into the following sequence:

(1) The application of basic mechanics leads to equations \( (1)_2 \) and \( (2)_2 \).

(2) The geometry of distortion combined with the previous equation results in an expression, \( (4)_2 \), for flange shear in terms of the angle of twist.

(3) The distributed reactive force \( q \) per unit length from the web is the load on the flange, as expressed by equation \( (5)_2 \).

(4) Equilibrium of the flange connects the distributed twisting moment \( m \) per unit length with the angular distortion as equation \( (6)_2 \).

(5) Torsional equilibrium of the web strip yields equation \( (7)_2 \).

(6) Consideration of the plate action of the web strip leads to equation \( (8)_2 \).

(7) Consolidation of the foregoing equations results in two equations, \( (9)_2 \) and \( (10)_2 \), in terms of the angular distortion.

(8) The restraint of warping torque or flange bending couple is given by equation \( (11)_2 \).

(9) In contrast to the Timoshenko solution, the torsional shear torque is divided into the component parts for the web and each flange in equation \( (14)_2 \).

(10) The equilibrium equation of torsion in terms of the angles of twist is rearranged to form equation \( (15)_2 \).
(11) For convenience, the section properties are combined into dimensionless constants, as equations (16)$^2$.

(12) Equations (9)$^2$ and (10)$^2$ can be written as a system of homogeneous linear differential equations, (17)$^2$ and (18)$^2$.

(13) One obvious solution which corresponds to the uniform torsion case is given by equations (19a)$^2$ and (19b)$^2$.

(14) If the exponential forms of the solution are substituted, a set of algebraic equations (20)$^2$ results.

(15) A trivial solution (21)$^2$ will satisfy equation (20)$^2$.

(16) By setting the determinant of the coefficients equal to zero, the values of $\lambda$ for which a non-trivial solution can exist are defined by equation (22)$^2$.

(17) If the root corresponding to the uniform or simple torsion solution is eliminated, the roots of the resulting quadratic equation can be written as equation (24)$^2$.

(18) Based on a number of typical computations, the roots become complex when the web is sufficiently thin.

(19) For the case of real roots, there are four possible values of $\lambda$.

(20) The complete solution for the angle of twist is given by equation (28)$^2$.

(21) The arbitrary constants for ($\zeta$) are related to those for ($\psi$).

(22) The complete solution for the web chord rotation is expressed by equation (31)$^2$ in terms of six arbitrary constants yet to be evaluated.
NON–UNIFORM TORSION – WF SECTION – SHAFT LOADING

CASE II: Constant Torque, Constant KG, Constant E ly
Considering Web Deformation

GOODIER–BARTON THEORY

SUMMARY OF DERIVATION

NOTATION:

TORSIONAL RIGIDITY – S

\[ S = KG \]
\[ K^W = \text{Torsion Constant of Web} \]
\[ K^F = \text{Torsion Constant of Flange} \]
\[ K = \text{Torsion Constant of Section} \]
\[ = K^W + 2 K^F \]
\[ G = \text{Shear Modulus of Elasticity} \]

TORQUE T

\[ T = \text{Applied Torque} \]
\[ T_R = \text{Resisting Torque} \]
\[ T_S = \text{Torsional Shear Torque} \]
\[ (\text{St. Venant’s Couple}) \]
\[ T_W = \text{Restraint of Warping Torque} \]
\[ (\text{Flange Bending Torque}) \]

WARPING RIGIDITY – W

\[ W = \frac{EI_y h^2}{4} \]
\[ (\text{WF & I–Beams}) \]
\[ I_y = \text{Moment of Inertia of Section} \]
\[ = 2 I^F_y \]
\[ h = \text{Distance between Flange Centroids} \]
\[ E = \text{Modulus of Elasticity} \]

ANGULAR DISTORTION

\[ \psi = \text{Angle of Twist of Web} \]
\[ \alpha = \text{Chord Rotation of Web} \]
\[ \psi^F = \text{Angle of Twist of Flange} \]
\[ = (\psi - \alpha) \]
\[ \theta = \text{Unit Angle of Twist} \]
\[ = \frac{d\psi}{d\xi} = \psi' \]

\[ q = \text{Distributed Lateral Load on Web and Flange per unit length} \]
\[ m = \text{Distributed Twisting Moment on Web and Flange per unit length} \]

FIG. III – 3: NOTATION – GOODIER–BARTON THEORY
**Fig. 3.4**  NON-UNIFORM TORSION OF WF BEAM - SHAFT LOADING

GOODIER-BARTON THEORY - CONSIDERING WEB DEFORMATION
### GOODIER-BARTON THEORY

**SUMMARY OF DERIVATION**

1. **Equilibrium of Flange Element**
   
   $\Sigma M = 0$;
   
   $V_F = -\frac{dM^F}{dz}$ (1)

   By Flexure Formula:
   
   $V_F = -EI_y \frac{d^3x}{dz^3}$ (2)

2. **Geometry of Distortion**
   
   - Before distortion
   - After distortion - Timoshenko Theory
   - After distortion - Goodier-Barton Theory

   Lateral displacement of flange:
   
   $x = \frac{h}{2} \sin \gamma \approx \frac{h}{2} \gamma$ (3)
   
   $\frac{d^3x}{dz^3} = \frac{h}{2} \left( \frac{d^3\gamma}{dz^3} \right) = \frac{h}{2} \gamma^{'''}$
   
   $V_F = -EI_y h \frac{\gamma^{'''}(\psi')}{4}$ (4)

3. **Equation of Bending - Flange**

   Distributed lateral load per unit length:
   
   $q = -\frac{dV_F}{dz} = \frac{EI_y h}{4} (\psi^{IV})$ (5)

4. **Equation of Torsion - Flange**

   $\Sigma T = 0$;
   
   $mdz + GK^F(\gamma' - \alpha) = 0$
   
   $m = -GK^F(\gamma'' - \alpha)$ (6)
**Goodier-Barton Theory (continued)**

5. **Equation of Torsion**

- **Web Strip**
  (Pure Torsion)

\[ \Sigma T = 0 \]

\[ 2(m \, dq \, dz) + q \, dz(h) - K^w \, G \, \psi' = 0 \]

\[ 2m + q \, h = K^w \, G \, \psi'' \quad (7)_2 \]

6. **Bending of Web Strip**

(length \( dz \))

\[ \text{Difference of torsional shear couple in web} = \text{Distributed bending couple over height} \, h \]

\[ = K^w \, G \, \psi'' \, dz \]

\[ \text{Distributed bending couple per unit length} = \frac{K^w \, G \, \psi'' \, dz}{h} \]

**Solution of Plate Problem:**

\[ m = \frac{6 \, D_w \, \alpha}{h} \quad (8)_2 \]

\[ D_w = \frac{E \, t_w^3}{12(1-\mu^2)} \]

\[ \mu = \text{Poisson's Ratio} \]

\[ t_w = \text{web thickness} \]
Fig. 3:4

7. Combined Equations for $\psi$ and $\alpha$

Eliminate (m) and (q) from Equations (5), (6), (7), and (8)

\[
K^F G(\psi'' - \alpha'') + \frac{6D_w \alpha}{h} = 0 \quad (9)_2
\]
\[
\frac{E I_y h^2}{4} \psi''' - K^W G \psi'' + \frac{12D_w \alpha}{h} = 0 \quad (10)_2
\]

8. Restraint of Warping Torque - $T_w$

\[
T_w = V^F h = -\frac{E I_y h^2}{4} (\psi'') = -W \psi'''
\quad (11)_2
\]

9. Torsional Shear Torque - $T_s$

Web:

\[
(T_s)^W = K^W G \theta'' = K^W G \psi' \quad (12)_2
\]

Flange:

\[
2(T_s)^F = 2K^F G \theta^F = 2K^F G (\psi' - \alpha') \quad (13)_2
\]

10. Equilibrium Equation of Torsion

\[
T = T_R = (T_s)^W + 2(T_s)^F + T_w
\quad (14)_2
\]
\[
= K^W G \psi' + 2K^F G (\psi' - \alpha') - \frac{E I_y h^2}{4} \psi'''
\]
\[
= K G \psi' - 2K^F G \alpha' - \frac{E I_y h^2}{4} \psi'''
\quad (15)_2
\]

11. Dimensionless Constants

\[
k_1 = \frac{E I_y}{4 KG} ; \quad k_2 = \frac{K^W}{K} ; \quad \frac{K^F}{K} ; \quad \frac{6D_w h}{K G} .
\quad (16)_2
12. Simplified Differential Equations for $\psi$ and $\alpha$
(System of homogeneous, linear D.E.'s with constant coefficients)

\[ k_3 h^2 (\psi'' - \alpha'') + k_4 \alpha = 0 \]  
\[ k_1 h^4 \Psi^{IV} - k_2 h \psi'' + 2k_4 \alpha = 0 \]

13. One Solution of Differential Equations

Corresponding to Uniform Torsion:

\[ \psi = A_1 + A_2 z \]
\[ \alpha = 0 \]

14. Set of Algebraic Equations

Exponential forms of the solution:

\[ \psi = C_1 e^{\lambda z} \quad \xi = C_2 e^{\lambda z} \]

Substitute in Equations (17) and (18):

\[ \left\{ \begin{array}{l}
(k_3 \lambda^2 h^2 - k_4) C_2 - k_3 \lambda^2 h^2 C_1 = 0 \\
2k_4 C_2 + (k_1 \lambda^4 h^4 - k_2 \lambda^2 h^2) C_1 = 0
\end{array} \right. \]

15. Trivial Solution of Algebraic Equations

\[ C_1 = C_2 = 0 \]
16. Characteristic Equation

Theorem: For non-trivial solutions to exist, the determinant of the coefficients must equal zero.

\[ \lambda^2 h^2 (k_3 \lambda^2 h^2 - k_4) (k_1 \lambda^2 h^2 - k_2) - 2k_3 k_4 \lambda^2 h^2 = 0 \]  

Equation (22) defines all values of \( \lambda \) for which a non-trivial solution can exist.

17. Roots of Characteristic Equation

Uniform Torsion solution:

\[ (\lambda h)^2 = 0 \]  

(23)

Roots of resulting quadratic:

\[ (\lambda h)^2 = \frac{K_1}{2} \left[ 1 \pm \sqrt{1 - K_2} \right] \]  

(24)

\[ \begin{align*}
K_1 &= \frac{k_2}{k_1} + \frac{k_4}{k_3} \quad (25) \\
K_2 &= \frac{4k_4}{K_1} \left( 2 + \frac{k_2}{k_3} \right) \quad (26)
\end{align*} \]

18. Roots

Roots are real - For thick webs

Roots are complex - For sufficiently thin webs
19. **Real Roots**

Two real values of $\lambda^2$ denoted by $\lambda_1^2$ and $\lambda_2^2$.

Four possible values of $\lambda$ denoted by $\pm \lambda_1$ and $\pm \lambda_2$ (27)

20. **Complete Solution - Angle of Twist - $\psi$**

$$\psi = A_1 + A_2 x + A_3 e^{\lambda_1 x} + A_4 e^{-\lambda_1 x} + A_5 e^{\lambda_2 x} + A_6 e^{-\lambda_2 x}$$ (28)

$A_1$, $A_2$, etc. are arbitrary constants.

21. **Arbitrary Constants for $\lambda$** (Based on those for $\psi$)

From (12b):

$$\frac{C_2}{C_1} = - \left[ \frac{k_1 \lambda^4 h^4}{2k_4} - \frac{k_2 \lambda^2 h^2}{2k_4} \right]$$

$$= k_5(\lambda h)^2 - k_6(\lambda h)^4$$ (29)

Constants:

$$k_5 = \frac{k_2}{2k_4}, \quad k_6 = \frac{k_1}{2k_4}$$ (30)

22. **Complete Solution - Web Chord Rotation - $\alpha$**

$$\alpha = \left[ k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4 \right] \left[ A_3 e^{\lambda_1 x} + A_4 e^{-\lambda_1 x} \right]$$

$$+ \left[ k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4 \right] \left[ A_5 e^{\lambda_2 x} + A_6 e^{-\lambda_2 x} \right]$$ (31)
c. "Long" Beam

When the beam is sufficiently long, the effect of flange shear becomes negligible, and a condition approaching uniform torsion prevails at the free end. The "long" beam solution has been derived from the Goodier-Barton theory (8) as follows:

(23) The conditions at the fixed end are given as equations (32)₂.
(24) The angular distortions are defined by equations (33)₂ and (34)₂ when the positive exponentials are discarded, leaving the damping functions.
(25) At the free end, the uniform torsion condition is approached which leads to equation (35)₂.
(26) The boundary conditions at the restrained end are used to set up equations (36)₂, (37)₂, and (38)₂.
(27) The four arbitrary constants are evaluated by solving the four related equations, leading to equations (39)₂, (40)₂, (41)₂, and (42)₂.

The angular distortions and the other related functions can now be evaluated for any given specimen. A numerical example has been worked out for one specimen. The computations are summarized in Appendix B. The results are plotted in Fig. 5:14.
### 23. End Conditions

At $z = 0$:
- Flange is fixed
- $\psi = 0$ \hspace{1cm} (32a)
- $\psi' = 0$ \hspace{1cm} (32b)
- $\alpha = 0$ \hspace{1cm} (32c)

As $z \to L$, angle of twist approaches a linear function of $z$.

Positive exponentials must be discarded to satisfy this condition.

### 24. Angular Distortions - Modified Equations
(For complex as well as real roots)

\[
\psi = A_1 + A_2 z + A_4 e^{-\lambda_2 z} + A_6 e^{-\lambda_2 z} \tag{33}
\]

\[
\alpha = \left[ k_5 (\lambda_1 h)^2 - k_6 (\lambda_1 h)^4 \right] A_4 e^{-\lambda_1 z} + \left[ k_5 (\lambda_2 h)^2 - k_6 (\lambda_2 h)^4 \right] A_6 e^{-\lambda_2 z} \tag{34}
\]

### 25. Condition at Free End - Uniform Torsion

Torsional Shear Torque $T_6 \to T$ at end

$Z \to \infty$; \hspace{1cm} $\psi \to A_1 + A_2 z$

$\psi' \to A_2$

$T \to T_6 = KG \psi' = KG A_2 \tag{35}$
26. **Conditions at Restrained End**

\[
\begin{align*}
\psi &= 0; \quad A_1 + A_4 + A_6 = 0 \quad (36)_2 \\
\psi' &= 0; \quad A_2 - \lambda_1 A_4 - \lambda_2 A_6 = 0 \quad (37)_2 \\
\alpha &= 0; \quad \left[ k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4 \right] A_4 + \\
&\qquad \left[ k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4 \right] A_6 = 0 \quad (38)_2
\end{align*}
\]

27. **Evaluate Arbitrary Constants**

From (35)

\[
A_2 = \frac{T}{GK} \quad (39)_2
\]

From (36)

\[
A_1 = -(A_4 + A_6) \quad (40)_2
\]

Solve (37) and (38) simultaneously

\[
A_4 = \frac{A_2}{\lambda_2} \left[ \frac{\lambda_1}{\lambda_2} - \frac{k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4}{k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4} \right]^{-1} \quad (41)_2
\]

\[
A_6 = \frac{A_2}{\lambda_1} \left[ \frac{\lambda_2}{\lambda_1} - \frac{k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4}{k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4} \right]^{-1} \quad (42)_2
\]

28. **Effect of Flange Bending and Web Deformation**

Actual "Long" beam:

\[
\psi \rightarrow A_1 + A_2 Z \quad (43)_2
\]

Uniform torsion solution:

\[
\psi = A_2 Z = \frac{T}{GK} Z \quad (44)_2
\]
1. General

A review of the literature reveals a limited number of experimental investigations devoted to the study of structural shapes subjected to non-uniform torsion.

Combined bending and torsion tests will not be included in this review since the second extension of this project will be devoted to that particular phase of the problem.

An extensive test program of rolled beams under shaft loading with restraint of warping was conducted at Lehigh University in 1935. However, no report of a comprehensive investigation of the behavior of built-up structural members under similar conditions has been uncovered.

2. Lyse and Johnston - Lehigh University

In addition to an extensive investigation of the torsional properties of structural shapes based on the membrane analogy, Lyse and Johnston studied the effect of end fixity on the torsional behavior.

Twenty-two specimens ranging in size from 3" to 12" deep I-beams with both ends fixed were subjected to a shaft loading by means of a special cable torsion rig. The length of the beams varied from 3" to 72", while the factor \( rL \left( = \frac{l}{a} \right) \) ranged from 0.168 to 4.753.

Both ends of the specimen were "fixed" by means of a pair of side stiffener plates welded between the flanges, which in turn were welded all around to a heavy end plate. Since the ends were twisted in opposite directions, there was a point of inflection at the centerline.
The test results checked the theoretically-computed values quite well, as shown by the curves comparing the flange shears. The check at the center of the specimen appears to be closer than at the fixed ends. This may be due to the fact that actual boundary conditions were more closely duplicated at the centerline by virtue of point symmetry.
Fig. 4:1  BOLTED AND RIVETED SPECIMENS

L. to R.  T 1-R, T 5-R; T 3-B, T 2-R
CHAPTER IV. EXPERIMENTAL PROGRAM

A. General

An extensive investigation of the behavior of full-size built-up structural members under uniform torsion was underway at Fritz Engineering Laboratory in the summer of 1949. Since the initial tests were kept in the elastic range, it was felt that the same specimens could be used in a second series of tests designed to study the behavior of plate girders under non-uniform torsion, also in the elastic range. A wide variety of specimens was made available for this purpose.

This proved to be a very economical arrangement from the standpoint of both time and material. Many of the same SR-4 gages and level-bar supports could be used in both investigations. The specimens were all supported in the same 2,000,000 in-lb. torsion testing machine. Depending upon the particular circumstances existing at the time that the specimen was ready, the uniform or the non-uniform test would be conducted first, followed by the other.

B. Test Specimens

1. Sections

Details of all seven sections which were included in the non-uniform torsion study are assembled in Table IV:1. The make-up, the manner of fabrication, and the torsional properties are summarized in the same table. Fig. 4:1 shows an over-all view of one bolted and three riveted specimens.

The selections were made with the view of providing as wide a range of torsional constant as possible. The test values of \( K \) varied from 1.86 in\(^4\) for T1-R to 17.40 in\(^4\) for T5-R, a ninefold increase.
<table>
<thead>
<tr>
<th>TABLE IV-1</th>
<th>SHALLOW GIRDERS</th>
<th>SUMMARY OF TEST SPECIMENS - SERIES A</th>
<th>SHAFT LOADING</th>
</tr>
</thead>
<tbody>
<tr>
<td>SKETCH</td>
<td><img src="image" alt="Sketch" /></td>
<td><img src="image" alt="Sketch" /></td>
<td><img src="image" alt="Sketch" /></td>
</tr>
<tr>
<td>MARK</td>
<td>T1 - R</td>
<td>T2 - R</td>
<td>T3 - B</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>WEB R. 18&quot; X 3/8&quot;</td>
<td>WEB R. 18&quot; X 3/8&quot;</td>
<td>WEB R. 18&quot; X 3/8&quot;</td>
</tr>
<tr>
<td></td>
<td>4-L 4 X 3 1/2 X 1/2&quot;</td>
<td>4-L 4 X 3 1/2 X 1/2&quot;</td>
<td>4-L 4 X 3 1/2 X 1/2&quot;</td>
</tr>
<tr>
<td></td>
<td>2 PL. 9 1/4 X 3/8&quot;</td>
<td>2 PL. 9 1/4 X 3/8&quot;</td>
<td>2 PL. 9 1/4 X 3/8&quot;</td>
</tr>
<tr>
<td>FABRICATION</td>
<td>RIVETED 7/8&quot; Ø AT 2 3/4&quot;</td>
<td>RIVETED 7/8&quot; Ø AT 2 3/4&quot;</td>
<td>RIVETED 7/8&quot; Ø AT 2 3/4&quot;</td>
</tr>
<tr>
<td></td>
<td>(BOTH LEGS)</td>
<td>(BOTH LEGS)</td>
<td>(BOTH LEGS)</td>
</tr>
<tr>
<td>TORSIONAL</td>
<td>(K)A = 1.86</td>
<td>(K)B = 5.75</td>
<td>(K)C = 7.24</td>
</tr>
<tr>
<td>CONSTANT</td>
<td></td>
<td>5.75</td>
<td>7.24</td>
</tr>
<tr>
<td>(IN.⁴)</td>
<td></td>
<td></td>
<td>1.56</td>
</tr>
<tr>
<td></td>
<td>(K)D = 1.87</td>
<td></td>
<td>1.87</td>
</tr>
<tr>
<td>NOTES</td>
<td>(K)A - PURE TORSION TEST - CHANG</td>
<td>(K)B - INTEGRAL ACTION - NEGLECTING END &amp; HUMP EFFECT - CHANG</td>
<td>(K)C - INTEGRAL ACTION - KUBO</td>
</tr>
<tr>
<td></td>
<td>(K)D - SEPARATE ACTION - KUBO</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ALL SPECIMENS 14'-9&quot; LONG AND APPROXIMATELY 18&quot; DEEP</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FRITZ ENGINEERING LABORATORY PROJECT NO. 215 A "NON-UNIFORM TORSION OF PLATE GIRDERS" TABLE IX-1
### TABLE IV - 2

#### SUMMARY OF SPECIMEN CONSTANTS

<table>
<thead>
<tr>
<th>Specimen</th>
<th>K (Test)</th>
<th>I_y</th>
<th>a = \frac{L}{r}</th>
<th>\frac{L}{a} = rL</th>
<th>h</th>
<th>\frac{L}{h}</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 1-R</td>
<td>1.86</td>
<td>50.5</td>
<td>74.0</td>
<td>1.19</td>
<td>17.8</td>
<td>5.0</td>
</tr>
<tr>
<td>T 2-R</td>
<td>6.61</td>
<td>91.3</td>
<td>54.1</td>
<td>1.63</td>
<td>18.2</td>
<td>4.9</td>
</tr>
<tr>
<td>T 3-E</td>
<td>7.42</td>
<td>91.3</td>
<td>51.2</td>
<td>1.72</td>
<td>18.2</td>
<td>4.9</td>
</tr>
<tr>
<td>T 5-R</td>
<td>17.40</td>
<td>198.5</td>
<td>51.1</td>
<td>1.72</td>
<td>18.9</td>
<td>4.7</td>
</tr>
<tr>
<td>T 7a-W</td>
<td>2.35</td>
<td>69.4</td>
<td>88.0</td>
<td>1.00</td>
<td>17.9</td>
<td>5.0</td>
</tr>
<tr>
<td>T 7b-W</td>
<td>6.35</td>
<td>140.6</td>
<td>69.0</td>
<td>1.28</td>
<td>18.3</td>
<td>4.8</td>
</tr>
<tr>
<td>T 9-WF</td>
<td>3.23</td>
<td>88.6</td>
<td>66.7</td>
<td>1.32</td>
<td>17.3</td>
<td>5.1</td>
</tr>
</tbody>
</table>

L = 36"  \quad a = \frac{h}{2} \quad \frac{E I_y}{G K}
2. Mark

The designations of the specimens used in this series follow generally the pattern set in the uniform torsion program. The only mark which does not coincide is T3-B, which was T2-B in the original series. Specimen T7a-W does not appear in the published report of the first tests, but the test results, being available, were incorporated in this investigation.

For ease of reference, the specimen number was followed by a symbol, R, B, or W, representing riveted, bolted, or welded construction respectively.

3. Fabrication

The effect of mode of fabrication was investigated by testing 3 riveted, 1 bolted, and 2 welded specimens, as well as a rolled beam.

In order to reduce the possible variations to a minimum, shop fabrication procedures were carefully specified and supervised. Details of the shop equipment used and the technique followed are given in the Chang and Johnston report. (4)

4. Section Properties

As shown in Table IV:1, there is more than one value of the torsional constant available for each specimen.

(a) The experimental value \((K)_A\) obtained in the uniform torsion tests was used without change in Computation "A".

(b) The computed value \((K_I)_B\) proposed by Chang and Johnston for design purposes, based on the devries assumption of integral action, corrected for longitudinal continuity but neglecting end and hump effects, was utilized in Computation "B".
(c) Another set of computed values \((K_1)_0\) was obtained by including the end and hump effects to form the basis of Computation "C".

(d) The fourth set \((K_5)_0\) was made up of computed values based on the assumption of separate action behavior.

The devries assumption, as applied to riveted, bolted, and welded plate girders, is explained in detail in the uniform torsion report. Illustrative examples are also included.
FIG. 4:2
TEST SETUP
APPLICATION OF TORQUE
SCHEMATIC DRAWING

FRITZ ENGINEERING LABORATORY PROJECT NO. 215A.
"NON-UNIFORM TORSION OF PLATE GIRDERS"
C. Test Set-up

1. General Arrangement

a. Loading

In order to conform to the boundary conditions generally assumed in the analytical solutions, it was necessary to provide complete restraint at one end. Due to the difficulty of obtaining the fixed end condition, even by means of an elaborate system of welded plates, it was decided to take advantage of symmetry.

With the specimen simply supported at both ends by web framing angles and supported against twisting by flange lugs, a twisting moment was introduced at the center using a yoke loaded by means of suspended weights. A schematic sketch of the test set-up is shown in Fig. 4:2.

b. End Supports

In effect, the end torque was applied by the adapters attached to the end support plates to two opposite-hand specimens held rigidly at the centerline. It was possible to apply the end torque by means of the web angles, by the flange lugs, or by a combination of the two.

c. Instrumentation

Since each specimen was made up of two equal halves, due to the symmetry of loading, it was decided to measure strains in one half and angular distortions in the other. SR-4 gages were placed on the so-called "west" half and the level-bar supports on the "east" half. To see that the behavior was as assumed, check gages and supports were spotted at selected points in the other half.
Fig. 4:4  LOADING YOKE, FILLERS, AND WEDGES
2. **Loading Apparatus**

a. **Specifications**

Based on preliminary calculations for the contemplated tests, the following specifications for the loading jig were formulated:

1. Maximum end torque = 90,000 in-lb.
   
   Maximum applied torque = 160,000 in-lb.


3. Torque to be applied smoothly.

4. Torque to be applied in a manner conforming as nearly as possible to the assumed conditions.

5. No additional vertical bending stress to be introduced.

b. **Design of Loading Rig**

Several alternate arrangements were investigated. The use of a circular ring girder, as well as a system of pivoted supports at the centerline, was considered. The possibility of utilizing some source of load other than dead weights was thoroughly explored.

The loading rig finally adopted was made up of two loading beams welded to a central yoke, as shown in Fig. 4:3. The yoke was made up of four shallow I-beam sections arranged as a rectangular frame. The assembly was rigidly held together by a pair of adjustable "collars" placed at the top and bottom. When the "collars" were removed, the yoke could be dismantled into two units. A close-up of the yoke is pictured in Fig. 4:4.

In order to maintain the section at the point of application of the torque load, a series of shaped fillers, each backed by a pair of machined wedges, was inserted between the specimen and the yoke.
FIG. 4:3
FRITZ ENGINEERING LABORATORY PROJECT NO.25A "NON-UNIFORM TORSION OF PLATE GIRDER"
These fillers, kept tight at all times, forced the section and yoke to act together. The loading arms consisted of steel I-beams welded to the yoke and designed as cantilevers.

c. Layout and Fabrication

In anticipation of a possible ten degrees of rotation, the loading arms were laid out with an initial bevel of five degrees in the opposite direction. The working lines of the beams, when extended, intersected the centroidal axes of the section.

The points of application of the weights were so located that a practically constant couple arm of ten feet was maintained throughout the angular movement.

The entire frame, as well as the various detail pieces required, was fabricated by the Fritz Laboratory staff. The units were then assembled and welded together by a team of experienced shop men.

d. Pulley Support Frame

A nine-inch diameter pulley wheel, with a concentric ball-bearing race and axle, was mounted on a pair of angle supports. The pulley unit was tack welded to the top of a welded portal frame fabricated from I-beam sections. The frame, in turn, was anchored down to a heavy steel base plate.

e. Hangers

Two different types of hangers were used. Section "A" of Fig. 4:3 shows a unit formed by two long rods welded to the ends of a crossbar. The contact edge of the bar was machined to fit in a grooved plate which was tacked to the top of the loading arm. A flat plate, welded to the lower end of the vertical rods, served as a retainer for the disk weights.
At the other end, a Tee-shaped bracket was welded to the top of the loading beam. A stranded cable with standard fittings connected the hanger to the loading rig over the pulley wheel, as shown in Section "B".

f. Disk Weights

Over two tons of weights in the form of circular lead disks were borrowed from the Testing Laboratory of the Bethlehem Steel Company. These calibrated disks weighed a little more or a little less than the nominal 100 pounds.

Careful matching of the disks made possible the application of virtually equal weights at each end of the loading rig. These disks were always applied in the same order for each specimen. Therefore, the increment of torque was always known. At the same time, no additional vertical bending stress could develop due to an unbalance of weights.

3. End Support

a. Torsion Testing Machine

The Lehigh University torsion testing machine of 2,000,000 in-lb. capacity, one of the largest in the world, was used to support the specimens in this investigation. The various details of this specially-designed and custom-built machine have been described elsewhere.\(^{(17)}\)

Since it was not necessary to utilise its loading wheel in the non-uniform torsion tests, it was locked in the neutral position and used as a stationary support at the "west" end.

The "east" end, where the end support plate is attached to a box frame, free to move longitudinally but restrained against twisting, was used without modification.
b. End Fixture Plates

(1) Connections

In order to keep the ends of the specimen from twisting under torsional load, web framing angles and/or flange lugs, welded securely to the end support plates, were used. These connections were designed with sufficient clearance so that any and all of the shallow girders could be inserted and attached without difficulty. The resulting gaps were effectively bridged by suitable plates and shims.

(2) Evaluation

The fact that the torsional shear stresses per se could not be introduced at the ends, as specified by the Timoshenko theory, would not be serious in a long beam by St. Venant's principle. Since the specimens were relatively short, the actual end conditions would constitute a significant factor in the interpretation of the results of this program.

One further deviation from the theoretically assumed conditions would result from the introduction of a longitudinal friction force at each lug which is proportional to the flange reaction. Here again the effect, being local in extent, should wash out in a relatively short distance, depending on the length of the beam.

c. Torquemeter

The applied torque based on known weights and a virtually constant distance was a known quantity. However, an additional check of the end torque was kept by recording the strain gage readings of the specially-built and calibrated torquemeter. (17)
A comparison of end torques obtained from the applied weights and the torquemeter showed good correlation when based on a slightly different calibration constant. One possible explanation might be that the torque applied during the calibration test of the torquemeter was opposite in direction to that acting in the non-uniform torsion test.

4. Instrumentation

a. Strain Measurement

(1) SR-4 Gages – Number and Type

The normal and shear stress distribution varying along the span as well as around the section required an extensive network of strain gages. An average of approximately 70 separate gage readings was taken per load increment for each specimen.

Three different types of SR-4 gages were utilized to define the stress picture. For normal stresses in a given direction, a uniaxial gage, A-5, was selected. To evaluate the transverse shear stress combined with one longitudinal normal stress at a given point, a biaxial gage, AX-5, was oriented at 45° to the coordinate axes. To determine the principal stresses and their orientation at some special point, a triaxial gage, AR-1, forming a 45° strain rosette was used in a few instances.

(2) Gage Location

A typical layout of SR-4 gages for a built-up specimen is shown in Fig. 4:5. The gage pattern for the solid and welded specimens was similar. Individual differences in spacing and type of gage were taken care of in the computations.
FIG. 4:5 TYPICAL ARRANGEMENT OF SR-4 STRAIN GAGES

SPECIMEN T2-R (WEST END)

FRITZ ENGINEERING LABORATORY PROJECT O.215A - "NON-UNIFORM TORSION OF PLATE GIRDERS"

RIVET PITCH IN FLANGE AND WEB = 2 3/4"

L = 7' - 0"
The major stations, I, II, III, and V, were placed 1'-10" apart along the central portion of the span. An additional station, VI, was placed as close to the centerline of the specimen as practicable. Station IV, midway between III and V, was used only in a limited number of cases.

Transversely on the flange a combination of gages was placed at selected positions so as to obtain a measure of both the normal bending stress and shear stress variation.

In the web of built-up specimens the gage stations were displaced a distance equal to half the longitudinal rivet pitch relative to the corresponding flange station, since the holes in flange and web were generally staggered. At least two AX-6 gages were used at each station to establish the shear stress in the web.

To minimize the stress concentration factor in the riveted and bolted specimens, the gages on the flange were located as far from the holes as possible while, at the same time, maintaining the desired top and underside relationship.

In order to obtain a comprehensive yet related strain record in one flange, it was decided to concentrate all available gages on the top flange. The bottom flange, if used, would have produced similar results since no vertical bending stresses were introduced.
Fig. 4:6 SET-UP FOR STRAIN MEASUREMENT
(3) **Wiring**

Wire, made up of a single strand of #18-gage, tinned, copper wire with thermoplastic insulation, was used to connect the gages to the strain indicator. The static nature of the strains and the limited length of leads did not justify the use of a shielded cable.

The handling of gages, the soldering of wires, the checking of electrical resistances, and the numbering of leads followed the recommended laboratory practice.

After additional shop work was performed on a given specimen for purposes of conversion, it was not always possible to re-use all the gages since some would be damaged either in transit or in fabrication. Such losses were experienced, for example, when a bolted girder, after being tested, was sent back for riveting or when cover plates were welded to the flange of a specimen.

(4) **Switching Unit and Indicator**

Since only static strains were being measured, a balancing unit was not required. A simple 96-terminal switching box was used to connect the gage loads to the strain indicator. An over-all view of the arrangement is shown in Fig. 4:6.

A portable, battery-operated, electronic-type strain indicator was utilized to measure the unit strains directly in micro-inches per inch. The same Young Testing Machine Company unit was connected to the specimen gages in all tests. A Baldwin Type K strain indicator measured the torquemeter strains.
b. Angular Distortion

(1) Level Bar

A level-bubble unit with a 3-inch gage length has been used in a number of previous torsion tests. The points were seated on an adjustable angle section attached to the support. This combination provided a wide range which was especially suitable for the large angles of twist used in the uniform torsion tests. The same unit, shown schematically in Fig. 4:8b, was utilized in the non-uniform torsion series for reasons of expediency. To simplify the sketch, the adjustable angle seat is not shown. For the limited angular distortions introduced, a modified 20-inch level bar would have increased the degree of precision of the readings. However, the readings obtained with the smaller unit were reproducible, and the results were satisfactory.

(2) Level-bar Supports

(a) Flange

The L-shaped support was tack welded to the top of the flange along the longitudinal centerline in all cases, as shown in Fig. 4:8a.

In specimen T1-R, however, it was deemed necessary to avoid tacking the heel of the flange angles together since separate action behavior was desired. This was accomplished by tack welding a small block to the top of the web plates, and then tacking the level-bar support in turn to this block.
FIG. 4:8A

TEST SETUP
MEASUREMENT - ANGLE OF TWIST
SCHEMATIC DRAWING

FIG. 4:8B

TYPICAL SETUP
MEASUREMENT - ANGLE OF TWIST
SCHEMATIC DRAWING
Fig. 4:7 MEASUREMENT OF ANGULAR DISTORTION
(b) Web

If under non-uniform torsion the girder section is maintained, then the angle of twist of web and flange would be the same. Web deformation will, however, cause the flange angle of twist to be smaller than that of the web. Attachment of a level-bar support directly to the web would be misleading, as shown schematically in Fig. 3:5. It was decided that a more representative value would be obtained by basing the angular measurement on the movement of the trace of the edge of top and bottom flanges. An auxiliary vertical bar, equipped with a stop piece at the top end, was placed and held against the side of the flanges by means of elastic bands. The level-bar support was tacked to this vertical bar.

(3) Location

The level-bar supports for both flange and web were uniformly spaced at 11 inches along the longitudinal axis of each specimen, as shown in Fig. 4:8a. The first station was set 7 inches from the centerline. This placed the supports approximately in line with a pair of flange connectors throughout the length in the case of built-up plate girders.

Fig. 4:7 shows the author in the process of adjusting the micrometer screw, preparatory to taking a level-bar reading on the top flange. The loading rig is in place with disk weights at each end of the loading arms. The strain gage leads have been disconnected in this photograph.
D. Experimental Tests

1. Shaft Loading

   a. Preliminary

      After the specimen was properly placed in the torsion testing machine, the loading yoke was fitted into position. The strain gage leads were connected to the switching unit. The level-bar supports were attached to the specimen. Dial gages were clamped to their respective supports. An initial load, usually made up of the hangers and small weights to take up the slack in the set-up, was applied. Initial readings of a few selected gages and level bars were recorded.

   b. Loading

      (1) Dry Runs

         A trial test run was made by applying the design load in two stages, taking readings after each increment. Critical points were kept under careful observation to detect any weaknesses and other irregularities. Then the total increment of load was removed. The last readings were checked against the original ones to pick up any erratic behavior. This dry run was repeated at least once more to "shake down" the specimen as much as possible.

      (2) Test Runs

         After all foreseeable difficulties had been ironed out, a full set of readings was taken with the initial load acting. The design load was applied in about six increments, each followed by a full set of readings.

         The unloading run was usually carried out in two increments.
A check of selected readings was made to decide whether the complete set of data would be acceptable or whether the test would have to be repeated.

2. Lateral Bending Test

a. General

One of the properties of the section which plays an important part in the non-uniform torsion problem is the lateral bending constant \( I_y^F \), the moment of inertia of the flange about the Y-axis. In the analysis of wide flange beams, it is assumed that the \( I_y \) of the web is negligibly small and, therefore, \( I_y^F \) can be taken equal to one half of \( I_y \) for the section.

The moment of inertia of the flange can be computed from the given dimension, but an experimental verification for a built-up section was desired. Specimen T2-R was selected as a representative section for a lateral bending test.

b. Arrangement of Loading

Specimen T2-R, turned on its side, was subjected to a symmetrical two-point loading while supported at the ends. A view of the actual test set-up is shown in Fig. 4:9.

A 300,000 lb. hydraulic testing machine was used to apply a central load which was transferred to the specimen by a distribution beam.

A base beam, made up of two wide flange beams clamped together resting on the platform of the machine, supported a pair of cast iron block piers or a WF beam at each end. Suitable plates and fillers were utilized to level up the specimen on these piers.
c. **Instrumentation - Dial Gages**

In the center portion of the specimen, where a condition of pure bending moment existed, 3 pairs of Ames dials were spaced symmetrically about the center, a set against each flange. The dial supports were clamped to a pair of light but rigid angle trusses suspended from outriggers tacked to the centerline of the flanges of the specimen at the ends.

d. **Test Procedure**

A couple of trial runs were made to check the adequacy of the set-up and the operation of the gages. An initial load was applied, and a complete set of data was recorded. The design load was applied in 7 increments with a set of data for each. After unloading, the corresponding set of initial readings was compared and found to be acceptable.

e. **Computations**

The dial readings were plotted against the load and the graphical results combined for both flanges to obtain the average deflection at each station. Based on the well-known elastic relationships of the deflected curve, and the average modulus of elasticity for the material, a value of \((I_y)\) based on this test was computed.

f. **Evaluation**

A comparison between test and computed \((I_y)\), based on the gross section, showed a difference of 7% on the low side. It was felt that this justified the use of the computed \((I_y)\) in the case of the other specimens, although the combination of a test \((K)\) with a test \((I_y)\) would have been desirable.
E. Preliminary Tests

1. General - Tension Tests

Since the specimens, used in both phases of the torsion problem, were the same, it was convenient to use the physical constants determined in the uniform torsion program.\(^{(4)}\)

Standard tension tests were performed on representative coupons. An average value of \(29.5 \times 10^6\) psi for the modulus of elasticity and \(11.5 \times 10^6\) psi for the shear modulus was determined and used in the subsequent calculations.

2. Tests of Connections

A study of bolted and riveted connections was also conducted as part of the first program.\(^{(4)}\) On the basis of the recommendations made therein, a torque of 300 ft.-lbs. was applied to each high-strength bolt in specimen T3-B by means of a calibrated torque wrench. Standard shop practice was followed in the case of the riveted and welded specimens.
A. General Procedure

1. Load Increment

Each specimen was subjected to a torsional load which would insure elastic behavior at all times. In each case the magnitude of the applied torque was limited by the maximum lateral bending stress in the flange at the restrained end. The resulting shear stresses in both the web and the flange were relatively low and did not control.

Since the properties of each specimen varied over a wide range by design, the maximum applied end torque differed in magnitude from a minimum of 36,900 in-lb. for T1-R to a maximum of 87,500 in-lb. for T5-R.

In order to have a basis for comparison between the various specimens, an arbitrary end torque of 40,000 in-lb. was selected and used in all the subsequent data reduction and computations.

2. Test Data Reduction

The over-all behavior of the specimens being elastic, the SR-4 gage, level bar, and Ames dial readings should follow a straight line variation with load. It was finally decided to plot the increment of reading \((R-R_0)\) against the corresponding increment of end torque and then to draw a straight line through the points.

The corresponding \((R-R_0)\) reading, due to a 40,000 in-lb. increment of torque, was determined graphically and used in the subsequent conversions.

With the exception of an explainable shift of some readings in the built-up specimens and some irregularity in lightly stressed SR-4 gages, the test data showed good linearity.
FIG. 5:1

TYPICAL ARRANGEMENT OF SR-4 GAGES

MEASUREMENT OF NORMAL STRESSES

FLANGE AND WEB

LATERAL BENDING
STRESS VARIATION

NORTH SIDE
A-5 GAGE

AX-5 GAGE

TYPICAL BUILT-UP SECTION
B. Conversion and Combination

1. Strain Measurement

a. Conversion of SR-4 Readings

The conversion procedure, including the transverse correction recommended by the manufacturer, was used without modification. The constants, determined by the conventional methods in the first series of tests by Chang and Johnston, were adopted in this extension.

b. Combination of Test Results

1) Flange Gages

Wherever possible, SR-4 gages were placed on the topside and underside of the flange opposite one another as shown in Fig. 5:1. The lateral bending stress at the point corresponding to the center of the gages was taken to be the average of the two measured normal stresses.

In certain instances two different types of gages were used at a given station. For instance, a type AX-5 gage would be used on the top on the "south" side to measure both normal and shear stresses with a type A-5 gage on the "north" side at the corresponding point to indicate only the bending stress. In some cases A-5 gages were located along the sides of the flange to measure the maximum bending stress.

For comparison with the computed values, the average of the measured bending stresses on the tension and compression sides was determined and plotted. The resultant of the torsional and transverse shear stresses in the flange was measured and plotted separately at each AX-5 gage location.
(2) Web Gages

For small angles of twist, the web is theoretically subjected only to torsional shear stresses which would be constant in magnitude since the web thickness was uniform. Therefore, the measured shear stresses from gages at and above the centerline in the web at a given station were combined, and the average shear stress plotted against the computed theoretical values. Where two gages were positioned at the same point but on opposite sides of the web, the measured values were averaged.

c. Layout of Curves

For convenience, the strains were measured on the western half while the angular distortions were determined for the eastern half of each specimen. Taking advantage of symmetry, the results in terms of normal and shear stress, angle and unit angle of twist are all plotted as if for the same eastern half, with the restrained end at the left as the origin and the free end at the right.

2. Angular Distortion

a. Test Data

In taking the level-bar readings, the micrometer-actuated device was carefully placed in prepared seats so that the experimental error in the readings for a given station would be kept to a minimum. The change in micrometer reading \((R-R_0)\) for a given increment of torque loading was plotted for the flange and web at each station.

For the solid sections, such as T9-\#F and T7a-W, a continuous, straight line could be drawn through the plotted points for each station. In the case of the built-up sections, such as T2-R, T3-R, and T5-R, the most reasonable curve appeared to be made up
of a series of straight, disconnected but parallel segments. The apparent translation of the points was not always consistent. Sometimes the shift would be on the high side and sometimes on the low side. The shift, quantitatively small, would take place under different loads and extend over different ranges.

The rivets or bolts in the flange tend to keep the section acting as a unit. However, it is quite obvious that local adjustments of a highly irregular nature are taking place -- a build-up due to restraint here, a relaxation due to slip there. The flange level-bar readings, in effect, reflect the actual movement of the outer cover plate at the point of attachment of the level-bar support. It appears reasonable to expect some shift in the readings which in turn would not affect the over-all behavior of the section. It was found that the increment of level-bar readings, based on the slope of the straight-line portions for an end torque of 40 in-kip, gave results which were in good agreement with the theoretical values.
FIG. 5:2

ANGULAR DISTORTION

RELATIONSHIP OF ANGLE OF TWIST $\psi$ AND UNIT ANGLE OF TWIST $\theta$

FRITZ ENGINEERING LABORATORY PROJECT NO. 215A
"NON-UNIFORM TORSION OF PLATE GIRDER"
b. Conversion of Level-Bar Readings

The geometric relationships utilized to compute the angle of twist and the unit angle of twist from the level-bar readings are shown in Fig. 5.2.

(1) Angle of Twist

In the test as conducted, the free ends were supported against twist, and the torque applied at the centerline of the specimen. Due to symmetry, full restraint was developed at the centerline. Since the loading yoke was placed at that point, no angle of twist readings could be taken at the centerline. It was necessary to compute the angle of twist for the half span as follows:

(a) The angle of twist at station (1) relative to station (0) was assumed to be equal to the computed theoretical value based on test constants.

(b) The change in micrometer reading for adjacent stations such as (1) and (2) for an end torque change of 40 in-kip was obtained as $\Delta R_1$ and $\Delta R_2$.

(c) The angle of twist in radians at station (2), assuming tentatively that station (1) is stationary, would be equal to $\frac{\Delta R_2 - \Delta R_1}{F}$ where $F = 3^\circ$, the distance between supports points on the level bar.

(d) The actual angle of twist at station (2) would be equal to the sum of the angles found in steps (a) and (c).

(e) This procedure was repeated for each of the other six stations.
(2) Unit Angle of Twist - Test Data

The unit angle of twist for both the web and the flange was computed from the test data as follows:

(a) The change in the micrometer reading corresponding to the end torque increment of 40 in-kip for both stations (1) and (2) was divided by the gage length (F) of 317.

(b) The difference between these values gave the relative change in angle of twist ($\psi_{1-2}$) between the two stations.

(c) This difference, when divided by the distance (C) between the stations, was considered the unit angle of twist, ($\theta_{1-2}$) for a point midway between the two stations.

(d) A similar procedure was followed for each of the other stations.

(e) The initial point on the unit angle of twist curve was assumed to be equal to the theoretical value of zero since no reading of the angle of twist at station (0) was available.

(f) The final value on the measured curve was at the point midway between stations (7) and (8). It was noticed that contrary to the assumed end condition, there was a slight rotation at the end. This was attributed to the elastic and inelastic rotation of the end support plates which, in turn, were attached to a torque dynamometer at the east end and to the main shaft at the west end.
(3) Unit Angle of Twist - Curve Data

An alternate method of computing the unit angle of twist was available. Using the angle of twist curve drawn from the plotted level-bar reading, an adjusted value of the change in micrometer reading was obtainable for each station. The difference in these readings at adjacent stations could be used to compute the unit angle of twist at a point midway between the two stations.

Since the original test data would, in effect, be ironed out in this procedure, the resulting unit angle of twist values would plot as a relatively smooth curve. This curve would be an improved version of the graphical plot based on the test data.

In the summary curves, the unit angle of twist was reproduced instead of the angle of twist, since the variations in the unit angle at the end are more pronounced. It was decided to indicate the spread in the measured data by plotting the computed unit angle of twist values based on the test data as outlined in Part (2).
C. Properties of Sections

1. Torsion Constant \((K)\)

As indicated in Table IV:1, it is possible to have several computed values of the torsion constant, depending on the assumptions made and the number of secondary effects taken into account. The computed values \((K_1)_B\), \((K_1)_0\), and \((K_S)_D\) mentioned in this report are explained in Chapter IV, Section B.

Several sets of calculations were initially made, using different combinations of constants, to determine the over-all effect on the stress distribution and angular distortions. In general, these computations confirmed the theory as to the effect of changing the value of \((K)\). The lateral bending stress, the torsional shear stress, and the unit angle of twist, all increase as the torsion constant decreases, other factors being equal.

In the final analysis and summary curves comparing experimental and computed values, Computations "B" and "C" based on \((K_1)_B\) and \((K_1)_0\) respectively, were superseded by Computation "A" using test constants. This would provide a logical, consistent basis for evaluating the validity of the various theories.

The upper limit of stress and distortion would be set by the separate action constant \((K_S)_D\) which was used later in the study of the effect of variable \((K)\) along the span.

2. Lateral Bending Constant \((I_y)\)

The constant \((I_y)\) can be computed for a built-up section in a number of ways, depending on the assumptions made and the factors considered. The lateral bend test of specimen T2-R checked the computed value, based on gross section, reasonably well. It was decided to assign the calculated value based on the same assumptions to the other specimens.
D. Computations - Timoshenko Theory

1. General

The assumptions, the set-up of the differential equation and the solution for the angle of twist, are summarized in Chapter III, Section A, Article 2.

For comparison with experimental values, it was decided to use Computation "A" based on test constants as the logical standard.

2. Discussion of Results

a. Grouping

For purposes of comparison and analysis, the specimens are divided into two groups. Group (I) is made up of specimens T1-R, T7a-W, and T9-WF which are essentially "solid" sections. Group (II) includes the remaining specimens which are "built-up" sections.

No question exists as to the integral action of the rolled section, T9-WF, which was used as a criterion by which the experimental technique was evaluated. Specimen T7a-W is also a solid section since the continuous double fillet weld connecting the flange plate to the web eliminated the possibility of relative slip. Specimen T1-R was designed and analyzed assuming separate action behavior which meant that each flange angle could slip relative to the web. In other words, the specimens in Group (I) could have but one value of (K) throughout its length. Furthermore, each of the specimens in Group (I) did not contain stress raisers such as holes, discontinuities, and other irregularities.

On the other hand, each of the specimens in Group (II) could conceivably have more than one value of effective (K) due to the possibility of relative slip. In addition, each of the four specimens had stress raisers of one type or another.
b. Normal Stresses

The results are shown graphically in the "Experimental vs. Computed" curves drawn for each specimen and assembled in Appendix A.

The following results were observed:

(1) In general, the measured normal stresses compared favorably with the curve of computed values over the whole span.

(2) The measured values in Group (I) specimens were consistently lower than the Computation "A" values, while those in Group (II) specimens were slightly higher than the corresponding computed values. This difference in measured stresses might be attributed to a combination of the following factors:

(a) In the event that relative slip between angles and plates occurred, the measured stresses would tend to increase, since such adjustments would be equivalent to a reduction in effective (K).

(b) The flange holes in specimens T2-R, T3-R, and T5-R would produce stress concentrations in the immediate vicinity of the holes. The SR-4 gages were located as far from the adjacent holes as practicable, the center-to-center distance being 1-3/4 inches. The hole effect would be constant and quite small at the gages. However, the clamping action of bolts and rivets heads would be difficult to predict.

(c) The location of the neutral axis of the flange at a point in line with a pair of holes would be affected by the loss of section on the tension side. This shift would affect the pattern of stress variation across the section.
(3) The most noticeable difference between measured and computed stresses developed near the free end of Group (II) specimens, where the experimental values were somewhat higher than the Timoshenko theory would indicate. This "hump" was found to be caused principally by web deformation. Modification No. 1 and the "finite" beam solution, both based on the Goodier-Barton theory, provided a closer check of the measured values. The former method, designated Computation "E", is explained in detail in Section E, while the latter, called Computation "J", is covered in Section F.
FIG. 5:3

SHEAR STRESS DISTRIBUTION
(Assumed Variation)

NON-UNIFORM TORSION - SHAFT LOADING
Built-Up Girder

Torsional Shear Stress
Flange:
\[(\tau_s)_1^F = \frac{T_s t_1}{K}\]
\[(\tau_s)_2^F = \frac{T_s t_2}{K}\]
\[(\tau_s)_3^F = \frac{T_s t_3}{K}\]
Web:
\[(\tau_s)_w^F = \frac{T_s t_w}{K}\]

Transverse Shear Stress
Flange:
\[(\tau_v)_x^F = \frac{1.5V}{bt}\]
\[(\tau_v)_x^F = \left[1 - \frac{x^2}{(b_2)^2}\right](\tau_v)_x^F\]
Web:
\[(\tau_v)_w^F = 0\]

Combined Shear Stress
Top of Flange:
\[(\tau)_x^F = (\tau_s)_x^F + (\tau_v)_x^F\]
Underside of Flange:
\[(\tau)_x^F = (\tau_s)_x^F - (\tau_v)_x^F\]
Web:
\[\tau = (\tau_s)_w^F\]
c. Shear Stresses

(1) Transverse Variation

The assumed variation of shear stress over the cross-section of a built-up section subject to non-uniform torsion is shown graphically in Fig. 5.3.

The torsional shear stress pattern is based on the DeVries assumption: integral action between the outside rivet gage lines in the flange angles combined with separate action behavior over the rest of the section. Fig. (a) depicts the assumed torsional shear stress distribution across the top and underside of the flange, as well as the variation across the vertical side of the flange angle and the web. As the accompanying formulae indicate, the magnitude of the computed shear stress varies directly as the effective thickness for a given torsional shear torque.

In specimens T2-R, T3-B, and T5-R, this led to the situation in which the shear stress outside the rivet gage lines would be larger on the underside than on the topside of the flange.

As developed by St. Venant, the torsional shear torque can be pictured as being made up of a series of couples formed by resultant shear forces per unit length which follow a tip-to-tail procession around the section.

The flange shear (V) which makes up the restraint of warping torque will be assumed to act on a modified rectangular section formed by the cover plate and the out-standing leg of the flange angles. The transverse shear stress variation along the top and underside of the flange would then follow a parabolic pattern.
as illustrated in Fig. (b). The direction of these shear stresses would be the same at every point.

The resultant shear stress, obtained by combining the torsional shear and transverse shear stresses vectorially, will vary over the section, as shown schematically in (c). Due to the difference in the thicknesses of the component parts and the variable direction of the stresses, there would be no comparable combined shear stress values except at points of symmetry in the flange. The web shear stresses would not be changed from the torsional shear stress value.

(2) Longitudinal Variation

Since the ratio of the torsional shear torque to the flange bending torque varies along the span, the resultant shear stress along any given longitudinal line would also vary. In order to show this variation, the experimental vs. computed shear stress curves were drawn for the longitudinal line which had the largest number and, therefore, the best distribution of AX-5 gages. In the case of the flange, this line was usually near the edge where the combined shear stress was relatively small. In the web, the gage reading at and above the centerline was averaged at each station.

(3) Shear Stress - Computation #A#

The resultant transverse and longitudinal shear stress variations in the flange and web were computed for each specimen according to the Timoshenko theory, using test constants. The measured values were plotted and the results compared. The
measured and computed longitudinal variations along the selected line at top and underside of flange are shown graphically for each specimen. Similar curves were drawn for the web. These curves are assembled in Appendix A.

Inspection of the curves revealed the following:

(a) The measured shear stresses checked the computed values reasonably well, considering the relatively low level of stress.

(b) A limited number of gages behaved erratically, producing a scatter which could have been caused by a number of irregularities.

(c) The measured shear stresses showed a noticeable tendency to drop in value near the free end. This behavior could not be predicted by the Timoshenko theory. However, Computation 8H based on Modification No. 1 showed a marked improvement in this respect.
Fig. 5:4  Experimental vs. Computed $\psi$  T9-WF
d. Angle of Twist

(1) Web vs. Flange Behavior

According to the Timoshenko theory, the angle of twist of the flange and web should be the same at any point, since the section deforms as a unit. The measured values, when plotted, indicate that the flange does not twist as much as the web. A typical "Experimental vs. Computed Angle of Twist" curve, drawn for specimen T9, is shown in Fig. 5:4.

The difference in the behavior of the web and flange, predicted qualitatively by the Goodier-Barton theory, varied along the span, being more noticeable near the free end. The make-up of the section also influenced the magnitude of the divergence. In general, the Group (II) specimens showed slightly larger differences between flange and web angle of twist than those of Group (I).

(2) Computation "A"

Due to the limited effective length of the actual specimens, it was not possible to apply the available theories as a direct check on the measured values. It was noticed that an average of the web and flange twists compared favorably with the Computation "A" values, obtained from the Timoshenko theory using test constants, except near the free end.

(3) Shear Correction

In the derivation of the Timoshenko theory it was assumed that the lateral deflection of the flange was due to flexure only. As the span length decreases, however, the effect of shear becomes increasingly important.
A shear correction formula was proposed by Lyse and Johnston\(^{(2)}\) for steel beams under non-uniform torsion. For the case of a beam fixed at one end and free at the other, this factor is expressed as follows:

\[
\text{Shear correction} = 1 + 0.74 \frac{b^2}{L^2}
\]  

(5.1)

where \(b\) = width of flange  
\(L\) = length of span

Applied to specimen T9-WF, as tested, the angle of twist would be increased by about 1\(\frac{1}{2}\). Obviously, the specimens in this program were of such proportions that shear deflection could be neglected.

3. Unit Angle of Twist

(1) Web vs. Flange Behavior

The measured values for both web and flange were plotted against the Computation "A" values for each specimen and are assembled in Appendix A. These results are summarized in Fig. 5:5 and 5:6 for Groups (I) and (II) respectively.

The remarks made in the preceding section about the angle of twist can be applied to the unit angle of twist since the two are directly related. The web values were slightly larger than those for the flange in each specimen. The longitudinal variation of the average unit angle for web and flange compared reasonably well with the computed values, except near the free end where the test results showed a decided downward trend.

(2) End Condition

The actual end support could not be arrange to duplicate exactly the conditions that are assumed by the theory; namely,
Fig. 5:5

Summary Experimental vs. Computed $\theta$ Group I
FIG. 5:6 SUMMARY EXPERIMENTAL VS. COMPUTED $\theta$ GROUP II
that torsional shear stresses exist and are distributed over the section at the end. In these experiments the resisting torque at the free end was made up almost wholly of the flange shear torque \( T_W \) since the flange was maintained in position against twisting by lugs. This would mean that the torsional shear torque \( T_S \) was quite small. Since the unit angle of twist is proportional to \( T_S \), it would approach zero at the free end. By St. Venant's principle, there would be a re-adjustment of this condition within a short distance at the end which would result in a division of torque more in line with the theory in the remainder of the span.

This deviation from textbook behavior was further aggravated by the fact that the specimen was actually quite short since only half of the 14'-8" over-all length was used. The length to depth ratio was approximately 5. The \( rL \) or \( L/a \) ratio varied from 1.0 to 1.7 for the specimens as tested.

It was decided to utilize the measured angular distortions as a basis for Computation "H" which will be explained in Section E.
E. Computation "H^2 - Modification No. 1

1. General
   a. Preliminary

   An inspection of the measured normal stresses compared to computed values "A" based on the Timoshenko theory showed that there was a decided deviation of measured values on the high side near the free end. This "hump" effect was quite consistent, and several attempts were made to explain this deviation. One such theory, herein designated as Modification No. 1, was found to give satisfactory results.

b. Assumptions

   The torsional shear torque \( T_S \) was divided into its two component parts, \( (T_S)^W \) for the web and \( (T_S)^F \) for the two flanges.

   Since \( (T_S) \) was equal to \( K_0 \theta \), the revised expression was written as follows:

   \[
   T_S = (T_S)^W + 2(T_S)^F = K_W^W + 2 K_F^F
   \]  (5.2)

   where \( K_W \) = torsional constant of web

   and \( K_F \) = torsional constant of one flange

   In order to utilize the test \( (K) \), the available computed \( (K) \) was modified as follows:

   \[
   \text{Web: } (K_W)^{\text{Test}} = \frac{(K_W)^{\text{comp.}}}{(K)^{\text{comp.}}} (K)^{\text{Test}}
   \]  (5.3)

   \[
   \text{Flange: } 2(K_F)^{\text{Test}} = \frac{2(K_F)^{\text{comp.}}}{(K)^{\text{comp.}}} (K)^{\text{Test}}
   \]  (5.4)
FIG. 5-7
NON-UNIFORM TORSION - SHAFT LOADING

Computation "H" Modification No. 1

(a) Sketch

(b) Unit Angle of Twist
Web & Flange (Measured)

(c) Division of Torque

(d) Flange Shear

(e) Flange Bending Moment
c. Procedure - Normal Stress

The sequence of steps taken to obtain the flange bending stresses is outlined below and shown diagrammatically in Fig. 5:7.

The experimental values of $\theta^W$ and $\theta^F$ at each substation were recorded and used to compute the actual ($T_S$) for web and both flanges at the point.

Noting that the unit angle of twist takes a gradual but decided drop near the free end, it was assumed that the curve of $\theta$ for both web and flange converges to zero at the end, as shown in (b). This assumption is in line with the actual end conditions, since the end torque is introduced by means of flange lugs and not by surface shears.

The restraint of warping torque ($T_W$) was then computed for each substation as the difference between the applied end torque ($T$) and the sum of the torsional shear torques ($T_S$). At both the restrained and free ends, ($T_W$) was set equal to ($T$) since the unit angle of twist was theoretically zero at both points, as in (c).

The flange shear ($V$) at each substation was computed by dividing the corresponding ($T_W$) by the distance between flange centroids (h). A smooth curve connecting these computed points resulted in a curve of flange shear ($V$), sketched in (d).

Recalling the fundamental relationship that the change in bending moment between any two points is equal to the area under the shear curve between the same two points, the area under the ($V$) curve between substations was computed assuming trapezoidal areas. Starting at the free end where the bending moment was assumed to be zero, the flange bending moment at the successive stations was computed as the cumulative sum of the shear curve areas and plotted in (e).
The flange bending stress curve was found from the flexure formula, using the proper lateral distance for comparison with measured stresses from SR-4 gage readings.

d. Procedure - Shear Stress

The combined shear stress was computed from the measured unit angle of twist values. At each substation the torsional shear torques for the web and each flange was based on the experimental unit angle of twist and the proportioned torsion constant from equations (5.3) and (5.4). The restraint of warping torque and the related flange shear were determined for an applied end torque of 40 in-kip.

The torsional shear stresses in the web and at a selected line in the flange were computed, based on the applicable thickness. The transverse shear stress at the same point in the flange was derived from the parabolic variation based on the flange shear.

The combined flange shear stress and the web shear stress were plotted at the respective substations, through which points a curve designated Computation "H" was drawn.

e. Typical Computations

A set of typical computations for both normal and shear stresses, based on Modification No. 1, is included in Appendix B.
2. Discussion of Results

a. Normal Stresses

The longitudinal variation of the lateral bending stress in the flange, obtained from Computation "H" based on test 6, was plotted on the same "Experimental vs. Computed σ" sheet with Computation "A" for each specimen, assembled in Appendix A. The measured stresses are plotted against the corresponding Computation "H" curve for Group (I) and Group (II) specimens in Fig. 5:8 and Fig. 5:9 respectively.

In the case of the "solid" section specimens of Group (I), the over-all check was good with the measured values slightly on the low side. For the built-up sections of Group (II), the use of Computation "H" in place of Computation "A" resulted in a marked improvement over the whole span except for the welded specimen T7-b. Unfortunately, some of the flange gages on this specimen were apparently damaged, and only a limited number of measured values were available.

The troublesome hump in the measured values near the free end was neatly matched by the Computation "H" curve. It seems logical and reasonable to conclude that one of the major factors which affects the actual stress distribution throughout the span is the readjustment in the division of torque near the free end.
FIG. 5:8 SUMMARY EXPERIMENTAL VS. COMPUTED $\sigma$ GROUP I

FLANGE BENDING STRESS (K.S.I.)

GROUP I COMP "H"

$\bigcirc$ T1-R
$\times$ T7A-W
$\triangle$ T9-WF
FRITZ 215 A

L = 88"
FIG. 5:9 SUMMARY EXPERIMENTAL VS. COMPUTED $\sigma$ GROUP II
b. Shear Stresses

Computation "H" values, defining the longitudinal variation of the combined shear stress in both the web and the flange, were plotted with the corresponding Computation "A" curves for each specimen. Representative curves of experimental vs. computed shear stresses in web and flange are summarized in Fig. 5:10 and 5:11 for Group (I) and Fig. 5:12 and 5:13 for Group (II) specimens.

Due to the low range of stress involved, the measured values are sensitive to irregularities, the influence of which is reflected in some of the readings. In general, the check over the whole span was good. Computation "H" produced the characteristic drop in shear stress near the end which was an improvement over the results from Computation "A". This development reinforces the concept that the actual stress distribution is influenced by the readjustment in the division of torque near the free end. However, in the usual plate girder, the resultant maximum shear stress will not be affected adversely.
FIG. 5:10

SUMMARY

EXPERIMENTAL VS. COMPUTED $\tau$ (WEB) GROUP I
SUMMARY EXPERIMENTAL VS. COMPUTED $\tau$ (FLANGE) GROUP I.

FIG. 5:11

T1-R
T7A-W
T9-WF

SHEAR STRESS (FLANGE) (K.S.I.)

EXP

GROUP I

FRITZ 215A

$\tau$
FIG. 5:12

SUMMARY EXPERIMENTAL VS. COMPUTED $\tau$(WEB) GROUP II
FIG. 5: SUMMARY EXPERIMENTAL VS. COMPUTED \( \tau \) (FLANGE) GROUP II

SHEAR STRESS (FLANGE) (K.S.I.)
F. Computation "I" – Goodier-Barton Theory – ("Long" Beam)

1. General

a. Assumptions

As stated by Goodier and Barton, the Timoshenko solution is not sufficiently general to permit the assignment of the proportion of St. Venant's torsional shear torque \( T_g \) and torque of the flange shearing forces \( T_w \) at each end.

Consideration of web deformation extends the theory to thinner webs and permits the satisfaction of more general end conditions. However, it is further stated that all end conditions cannot be satisfied by this theory.

b. Derivation and Summary

The development of this extension is outlined in Chapter III, Article A, Section 3. The general expressions for the angle of twist \( \psi \) of the web and the reduction \( \alpha \) in the angle of twist of the flange in terms of the section properties are summarized.

The case of the so-called "long" beam, in which the angle of twist approaches a linear function of the longitudinal distance (i.e., the torsional shear torque approaches the uniform torsion value), is also included as Part c.

c. Numerical Example: T 9-WF

To investigate the effect of web deformation on a typical specimen, the 18 WF 77 rolled beam was selected for analysis. Following the procedure outlined, the angle of twist of both web and flange was computed assuming a "long" beam. The computations are included in Appendix B. The results are shown graphically in Fig. 5:14.
Fig. 5:15  Comparison of Computed $\Theta$

**T9-WF**

13 WF 77
T = 40 IN-KIP
FRITZ 215A

Unit Angle of Twist

(RAD/IN. $\times 10^{-5}$)

$\Theta$ - "I" (Long Beam)

$\Theta$ - "A" (Test K)
For comparison, the corresponding values based on uniform torsion and on the Timoshenko theory, as well as the actual experimental values, are plotted.

The unit angle of twist curves, computed by three different methods, are collected in Fig. 5:15.

d. Evaluation

The unit angle of twist curves indicate that the value based on the "long" beam theory approaches the uniform torsion constant and is appreciably higher than the results based on Timoshenko's theory of the measured values. It is quite apparent that the conditions assumed in Computation "I" are far from realized. The limited length of the specimen tested does not permit the full development of the theoretical torsional shear stress at the ends.

The difference between computed angles of twist for flange and web is appreciable, especially near the quarter point. The "long" beam theory assumes the gradual merging of the twist curves for web and flange at the free end, as is clearly shown by the curves. These curves are not intended for direct comparison with the experimental results. However, they do indicate the relative magnitude of the actual differences to be expected. On this basis, the test values appear to be reasonable.
2. Computation "J" - "Finite" Beam

a. Assumptions

In this case the beam flanges are rigidly attached at both ends to plates which are assumed to remain normal to the axis. The beam is twisted by torques applied to the end plates, which rotate by equal amounts in opposite directions relative to an origin at the center of the span. Since the deformation will be antisymmetrical about the plane of the center section, the angle of twist can be taken as an odd function of $z$.

The resulting equations $(45)_2$ and $(46)_2$ defining the angular distortions are summarized in Appendix B. The assumed boundary conditions are given as equations $(47)_2$.

b. Procedure

The three physical conditions lead to three equations in which the three arbitrary constants are expressed in terms of the end angle of twist. The relationship between this end angle of twist and the applied torque can be found by use of the equilibrium equation of torsion. The arbitrary constants can now be evaluated.

Substitution of these constants in the appropriate general expressions will lead to the theoretical angular distortions and stress condition.

c. Numerical Example: T9-WF

A typical set of computations for specimen T9 is included in Appendix B. The section constants are based on available test values used in Computation "A". The results are plotted in Fig. 5:14 and Fig. A9:1.
d. Evaluation

Similar computations have been made for specimens T1-R and T2-R. The computed flange bending stress has been plotted on Figures A1:1 and A2:1 respectively.

Comparison with the measured values indicates that consideration of web deformation yields a better check than the Timoshenko solution. The improvement is most noticeable in the case of specimens with appreciable differences of flange and web twist angles. The choice between the two available solutions should be based on practical considerations. Since the maximum flange bending stresses affect only a limited portion of the beam, one would be justified in using the allowable values for secondary stresses in design. The importance of the member and its deviation from normal dimensions should be weighed against the relative complexity of the finite beam solution compared to the Timoshenko solution.
G. Variable (K) - Modification No. 2

1. General

a. Preliminary

A comparison of experimental results for the specimens in Group (I) and Group (II) indicated that there was a distinct difference in behavior between "solid" sections and built-up sections.

Plate girders made up of angles and/or plates bolted, riveted, or welded together can be of variable sections if the cover plates are of different lengths. Where the section changes, then both the lateral bending constant \( I_y \) and the torsional constant \( K \) would have an abrupt change.

In the case of full-length cover plates, it is possible to have a variation in \( K \) without a corresponding change in \( I_y \). The torsion constant \( K_t \) assumes integral action insured by intrasurface friction between the outside gage lines, modified for longitudinal continuity if necessary.

If for some reason, such as a decrease in bolt tension, the elements no longer act integrally, then the effective torsion constant would approach the separate action constant \( K_b \). Another possible basis for a change in effective \( K \) would be introduced by a change in the longitudinal rivet pitch. The lateral bending constant would not be affected in either case.

b. Conditions

In the specimens tested in this program, all controllable variables were eliminated or kept to a minimum. The make-up of the section was constant. The spacing of the connectors was the same
Fig. 5:16

UNIFORM TORSION - SHAFT LOADING - VARIABLE K

(a) Plate Girder
Partial Elevation of Flange

(b) Flange Torsion
Constant $K_F$
Assumed Variation

(c) Flange Torsion
Constant $K_F$
Simplified Variation

(d) Sketch of Beam
Uniform Torsion

(e) Applied Torque
(Constant)

(f) Torsion Constant
of Section
Assumed Variation

(g) Unit Angle of Twist
Resultant Variation

(h) Angle of Twist
Resultant Variation
throughout. Riveting was done under controlled shop conditions with special supervision. All bolts were tightened manually by means of a calibrated torque wrench to the same value of 300 ft.-lb. torque.

Some variation in (K) was unavoidable, however, at the free ends since the elements are not forced to act together outside the first pair of connectors. This build-up was retarded by the relatively long end distance of 4-5/8 inches in specimens T2-R, T3-B, and T5-R.

It seems reasonable to assume that full integral action will not be developed at the first rivet line and that the transition may require several pairs of rivets, as indicated in Fig. 5:16b. A simplifying approximation might be made by assuming a straight line variation of (K) over the end portion as shown in Fig. (c).

c. Uniform Torsion Case

Assume that a built-up member with such an actual variation of (K) is subjected to a constant torque (T) without restraint of warping as in (d). The unit angle of twist \( \theta = \frac{T}{KG} \) would be constant and inversely proportional to the value of (K), as shown in Fig. (g). It is theoretically possible to have sharp transitions in (\( \theta \)) under these conditions.

Since the unit angle of twist is equal to the rate of change of the angle of twist, the slope of the (\( \psi \)) curve will be proportional to the ordinate of the (\( \theta \)) curve at the same point. The (\( \psi \)) curve will be made up of three straight-line segments, as shown in Fig. (h). No discontinuities in the (\( \psi \)) curve are physically possible, but cusps reflecting changes in slope at points of changing (K) may exist.
The total angle of twist of one end relative to the other end would include the over-all effect of the variation in \((E)\). Comparison with the theoretical total \((\Psi)\) based on constant \((K_I)\) would give a quantitative measure of the effect of the reduction in \((K)\) at the ends.

In order to obtain a more precise check, it would be necessary to measure the angle of twist at a large number of carefully selected stations along the span from end to end. A large number of SR-4 strain gages spaced along the centerline of the web would also help to define the transition zone between \((K_I)\) and \((K_S)\).

Once the experimental technique had been checked on this specially designed specimen, then the process could be applied to specimens as actually fabricated to evaluate the effect of end \((K)\) variation.

No experimental data that could be used for this purpose was available. It is hoped that further tests can be arranged to investigate this factor in greater detail.
2. Non-uniform Torsion - Variable (K) - Modification No. 2

a. Assumptions

In the case of a plate girder with a definite variation of (K), subject to a shaft loading torque with restraint of warping, an extension of the Timoshenko theory will be required. No analytical treatment of this condition was found in the references consulted.

Assume that the variation in effective (K) is as shown in Fig. 5:17b: constant (K₁) over the portion AB and constant (K₂) over the remainder BC. Although the effect of a variable (I₂) could be readily included, this factor will be assumed constant over the whole span in order to focus attention on the effect of variable (K).

b. Boundary Conditions

The general expression for the angle of twist in the Timoshenko theory neglecting web deformation contains three arbitrary constants; A, B, and D. It will be necessary to set up three conditions, usually at the boundaries, in order to evaluate these constants. Since there are two segments in this illustration, six conditions in all will be required.

1. At the restrained end (Z = 0), the angle of twist ($\psi$) is obviously zero.

2. Also at the origin (Z = 0), the rate of change of the angle of twist will be zero.

3. At the free end (Z = L), the lateral bending moment ($M_2^F$) in the flange will be zero since it is simply supported at that point. Since ($M_2^F$) is a function of ($\psi'$), the latter can be set equal to zero at this point.
FIG. 5:17

NON-UNIFORM TORSION VARIABLE K
MODIFICATION NO. 2

(b) TORSION CONSTANT
ASSUMED VARIATION

(c) FLEXURAL CONSTANT
ASSUMED VARIATION

(d) LATERAL DEFLECTION
(TOP OF FLANGE
AFTER TWIST)

(e) ANGLE OF TWIST

(f) UNIT ANGLE OF TWIST

(g) DIVISION OF TORQUE

(h) FLANGE SHEAR

(i) LATERAL BENDING
. MOMENT IN FLANGE

(j) LATERAL BENDING
STRESS IN FLANGE
(4) Consideration of the physical deformation picture indicates that there should be no points of discontinuity or cusps in the angle of twist curve. The angle of twist in the range considered is directly proportional to the lateral deflection of the flange which must act as a cantilevered beam. Consequently, it is necessary to set the angle of twist for the segment AB at B equal to the angle for the segment BC at B, as shown in Fig. (e).

(5) The ordinate to the unit angle of twist curve at any point is equal to the slope of the angle of twist curve at the corresponding point. Since there is no cusp in the (ψ) curve at B, the tangents to the curve must coincide. It follows that the ordinates to the (θ) curve to the left and right of B must be equal.

The condition of continuity which applies is indicated in Fig. (f). Although no discontinuities can exist in this curve, a cusp indicating a change in the unit angle of twist at B will result. Since the torsional shear torque (T_θ) is equal to KG9, the slope of the (θ) curve at B will be inversely proportional to the effective (K). As a result, there will be a sharp break in the (T_S) curve, the (T_θ) curve, and the flange shear curve, as shown in Fig. (g) and (h).

(6) The flange bending moment is obtained by integrating the flange shear curve. Although the slope varies, it would be a physical impossibility to have a discontinuity in the (M^F) curve. Therefore, the last condition is that the moment to the left and right of point B must be equal, as in Fig. (i).
c. Evaluation of Arbitrary Constants

The system of coordinates and the notation used are shown in Fig. 5:19a. Subscript (1) is used in segment AB and (2) in segment BC. Segment AB extends from \( z = 0 \) to \( z = gL \), where \( g \) is some arbitrary fraction.

By using the six conditions established in part (b) of this section, the six arbitrary constants can be evaluated in terms of the section properties and the applied torque. The actual calculations are carried out in the following pages.

(1a) The three functions are expressed in general form for segment AB as equations (1)_3, (2)_3, and (3)_3. Substituting the proper boundary conditions at \( z = 0 \) leads to equations for constants \( A_1 \) and \( B_1 \). At the division point, the angle of twist can be expressed as equation (6)_3 by combining constant terms. Similarly, the unit angle of twist and the flange bending moment at the division point can be written as equations (7)_3 and (8)_3 respectively.

(1b) The three functions in general form for the segment BC are given by equations (9)_3, (10)_3, and (11)_3. At the free end, the condition of a simply supported flange leads to equation (12)_3 for \( D_2 \). The value of the functions at the division point are expressed in condensed form by equations (13)_3, (14)_3, and (15)_3.

(2) Substitutions of the appropriate expressions for the flange bending moments in the continuity condition at B yields equation (1) in terms of two arbitrary constants.
(3) The continuity of unit angles of twist at B results in the related equation (II).

(4) The equality of twist angles at B leads to an expression for $A_2$.

(5) Based on the solution of the simultaneous equations (I) and (II), constants $D_1$ and $B_2$ can be evaluated by equations (19) and (20).

(6) All six arbitrary constants can now be computed for any given section and torque. The constant terms are summarized as equations (21) through (34).

(7) The expressions in general form for each segment are given as equations (34) through (39) and are summarized on Fig. 5:18.

(8) The maximum values of the three functions are given by equations (40), (41), and (42).
Case III: Shaft Loading - Constant Torque $T$
Variable $K$ - Constant $EI_y$

Non-uniform Torsion - Shaft Loading - Variable $K$ - Derivation

(a) $T$
(b) $K_1$, $K_2$

General Solution:

Angle of Twist

$\psi = \frac{Tz}{GK} + A + Br \sinh nz + Dr \cosh nz$

(c) $\psi$

$[\psi_A = 0]$

Unit Angle of Twist

$\theta = \frac{d\psi}{dz} = \frac{T}{GK} + Br \cosh nz + Dr \sinh nz$

(d) $\theta$

$[\theta_A = 0]$

Moment in Flange

$M^F = \frac{EJ_y b}{4} \frac{d^2\psi}{dz^2} = \frac{EJ_y b}{4} \left[ Br^2 \sinh nz + Dr^2 \cosh nz \right]$

(e) $M^F$

$[M^F_c = 0]$

FIG. 5:19 VARIABLE $K$ DERIVATION
Case III - Variable K

1. Evaluate constants of integration

(a) Segment AB \(0 \leq z < g\):

\[
\psi_i = \frac{Te_i}{GK_i} + A_i + B_i \sinh n_i z + D_i \cosh n_i z \quad (1)
\]

\[
\theta_i = \psi_i' = \frac{T}{GK_i} + B_i n_i \cosh n_i z + D_i n_i \sinh n_i z \quad (2)
\]

\[
M_i^t = \frac{EI_i h}{4} \psi_i'' = \frac{EI_i h}{4} \left[ B_i n_i^2 \sinh n_i z + D_i n_i^2 \cosh n_i z \right] \quad (3)
\]

At \(z = 0\):

\[
\psi_A = 0 = 0 + A_i + B_i (0) + D_i (1) \quad \Rightarrow \quad A_i = -D_i \quad (4)
\]

At \(z = 0\):

\[
\psi_A' = 0 = \frac{T}{GK_i} + B_i n_i (1) + D_i (0) \quad B_i = -\frac{1}{GK_i n_i} \quad (5)
\]

At \(z = g\):

\[
(\psi_B)_l = \frac{T(g)}{GK_i} + A_i + B_i \sinh n_i (g) + D_i \cosh n_i (g)
\]

\[
= \frac{T(g)}{GK_i} - D_i - \frac{1}{GK_i n_i} \sinh n_i (g) + D_i \cosh n_i (g)
\]

\[
= D_i \left[ \cosh n_i (g) - 1 \right] + \frac{1}{GK_i} \left[ (g) - \frac{\sinh n_i (g)}{n_i} \right] T
\]

\[
\Rightarrow \psi_B' = D_i \{ e \} + \{ f \} T \quad (6)
\]

At \(z = g\):

\[
(\psi_B)_l = \frac{T}{GK_i} + B_i n_i \cosh n_i (g) + D_i n_i \sinh n_i (g)
\]

\[
= \frac{T}{GK_i} - \frac{1}{GK_i n_i} \cosh n_i (g) + D_i n_i \sinh n_i (g)
\]

\[
= D_i n_i \sinh n_i (g) + \frac{1}{GK_i} \left[ 1 - \cosh n_i (g) \right] T
\]

\[
\Rightarrow \psi_B' = D_i \{ e \} + \{ f \} T \quad (7)
\]

At \(z = g\):

\[
(M_B)_l = \frac{EI_i h}{4} \left[ B_i n_i^2 \sinh n_i (g) + D_i n_i^2 \cosh n_i (g) \right]
\]

\[
= \frac{EI_i h}{4} \left[ -\frac{T}{GK_i n_i} \sinh n_i (g) + D_i n_i^2 \cosh n_i (g) \right]
\]

\[
= \frac{EI_i h}{4} \left[ D_i \{ m \} - \{ e \} T \right] \quad (M_B)_l \quad (8)
\]
CASE III - Variable K

1. Evaluate constants of integration (cont'd)

(b) Segment $BC$ \( gL < z < L \)

\[ \psi_2 = \frac{Tz}{GK_2} + A_2 + B_2 \sinh n_2 z + D_2 \cosh n_2 z \]

\[ \theta_2 = \psi'_2 = \frac{T}{GK_2} + B_2 n_2 \cosh n_2 z + D_2 n_2 \sinh n_2 z \]

\[ M_2^F = \frac{EIy}{4} \psi_2'' = \frac{EIy}{4} \left[ B_2 n_2^2 \sinh n_2 z + D_2 n_2^2 \cosh n_2 z \right] \]

At $z = L$: \( M_C^F = 0 = \frac{EIy}{4} \left[ B_2 n_2^2 \sinh n_2 L + D_2 n_2^2 \cosh n_2 L \right] \)

\[ D_2 = -\left( \frac{n_2^2 \sinh n_2 L}{n_2^2 \cosh n_2 L} \right) B_2 = -B_2 \tanh n_2 L \]

At $z = \text{gl}$:

\[ (M_B^F)_L = \frac{EIy}{4} \left[ B_2 n_2^2 \sinh n_2 (\text{gl}) + D_2 n_2^2 \cosh n_2 (\text{gl}) \right] \]

\[ (M_B^F)_R = \frac{EIy}{4} \left[ B_2 n_2^2 \sinh n_2 (\text{gl}) - B_2 n_2^2 \tanh n_2 L \cosh n_2 (\text{gl}) \right] \]

\[ = \frac{EIy}{4} \left[ B_2 n_2 \left\{ \sinh n_2 (\text{gl}) - \tanh n_2 L \cosh n_2 (\text{gl}) \right\} \right] \]

At $z = \text{gl}$:

\[ (\psi_0)'_R = \frac{T}{GK_2} + B_2 n_2 \cosh n_2 (\text{gl}) + D_2 n_2 \sinh n_2 (\text{gl}) \]

\[ (\psi_0)'_R = \frac{T}{GK_2} + B_2 n_2 \cosh n_2 (\text{gl}) - B_2 n_2 \tanh n_2 L \cdot \sinh n_2 (\text{gl}) \]

\[ = \frac{T}{GK_2} + B_2 \left\{ \frac{1}{n_2} \cosh n_2 (\text{gl}) - n_2 \tanh n_2 L \cdot \sinh n_2 (\text{gl}) \right\} \]

\[ = \frac{T}{GK_2} + B_2 \left\{ \phi \right\} \]

(14)
Case III - Variable K

1. (b) Evaluate constants of integration - Segment BC - (Cont'd)

At z = gl : \( (\psi_B)_R = \frac{T(gl)}{GK_2} + A_2 + B_2 \sinh \nu_2(gl) + D_2 \cosh \nu_2(gl) \)

\[
(\psi_B)_R = \frac{T(gl)}{GK_2} + A_2 + B_2 \sinh \nu_2(gl) - B_2 \tanh \nu_2L \cosh \nu_2(gl)
\]

\[
= \frac{T(gl)}{GK_2} + A_2 + B_2 \left[ \sinh \nu_2(gl) - \tanh \nu_2L \cosh \nu_2(gl) \right]
\]

\[
= \frac{T}{t} B_2 \{ A \} + A_2 + T\{ t \} \quad \rightarrow \quad (\psi_B)_R \quad (15)_3
\]

2. Continuity condition at B : \( (M_B^f)_L = (M_B^f)_R \) : \( (8)_3 = (13)_3 \)

\[
\frac{EI_yh}{4} \left[ D_1 \{ m \} - T\{ k \} \right] = \frac{EI_yh}{4} \left[ B_2 \{ n \} \right]
\]

\[
D_1 \{ m \} - B_2 \{ n \} = T\{ k \} \quad \rightarrow \quad I \quad (16)_3
\]

3. Continuity condition at B : \( (\psi^f)_L = (\psi^f)_R \) : \( (7)_3 = (14)_3 \)

\[
D_1 \{ h \} + T\{ j \} = B_2 \{ q \} + T\{ p \}
\]

\[
D_1 \{ h \} - B_2 \{ q \} = T\{ p \} - T\{ j \} = T\{ u \} \quad \rightarrow \quad II \quad (17)_3
\]

4. Continuity condition at B : \( (\psi)_L = (\psi)_R \) : \( (6) = (15) \)

\[
D_1 \{ e \} + T\{ f \} = B_2 \{ a \} + A_2 + \{ t \} T
\]

\[
A_2 = D_1 \{ e \} - B_2 \{ a \} + T\{ f \} - T\{ t \} \quad \rightarrow \quad A_2 \quad (18)_3
\]
CASE III - Variable K

5. Solve Eqs. (I) & (II) simultaneously: by determinants

\[ D_1 \{ m \} - B_2 \{ n \} = T \{ k \} \quad (I) \]
\[ D_2 \{ h \} - B_2 \{ g \} = T \{ u \} \quad (II) \]

\[ D_1 = \begin{bmatrix} T_k & -n \\ T_u & -g \end{bmatrix} = \begin{bmatrix} -kq + nu \end{bmatrix} T = \begin{bmatrix} nu - kq \end{bmatrix} T \quad D_1 \]
\[ B_2 = \begin{bmatrix} m & T_k \\ h & T_u \end{bmatrix} = \begin{bmatrix} mu -hk \end{bmatrix} T = \begin{bmatrix} mu -hk \end{bmatrix} T \quad B_2 \]
Case III - Variable K

6. Summary - Constants

<table>
<thead>
<tr>
<th>Segment AB</th>
<th>Segment BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 = -D_1$</td>
<td>$A_2 = D_1 e - B_2 \Delta + (f - t) T$</td>
</tr>
<tr>
<td>$B_1 = -\frac{T}{GK_1 \lambda_1}$</td>
<td>$B_2 = \frac{[mu - h k] T}{nh - mg}$</td>
</tr>
<tr>
<td>$D_1 = \frac{[nu - kg] T}{nh - mg}$</td>
<td>$D_2 = -B_2 \tanh r_2 l$</td>
</tr>
</tbody>
</table>

Warping rigidity $W = \frac{EI_y h^2}{4}$ (WF beams) $W = \frac{GK_2}{W}$

\[ \lambda_1 = \frac{GK_1}{W} \]

\[ \{ e \} = \left[ \cosh r_1 (gl) - 1 \right] \]
\[ \{ f \} = \frac{1}{GK_1} \left[ (gl) - \sinh r_1 (gl) / \lambda_1 \right] \]
\[ \{ h \} = r_1 \sinh r_1 (gl) \]
\[ \{ j \} = \frac{1}{GK_1} \left[ 1 - \cosh r_1 (gl) \right] = \frac{1}{GK_1} \left\{ -e \right\} \]
\[ \{ k \} = \frac{\lambda_2}{GK_1} \sinh r_1 (gl) = \frac{1}{GK_1} \left\{ h \right\} \]
\[ \{ m \} = r_1^2 \cosh r_1 (gl) \]
\[ \{ n \} = r_2^2 \left[ \sinh r_2 (gl) - \tanh r_2 l \cdot \cosh r_2 (gl) \right] \]
\[ \{ p \} = \frac{1}{GK_2} \]
\[ \{ q \} = r_2 \left[ \cosh r_2 (gl) - \tanh r_2 l \cdot \sinh r_2 (gl) \right] \]
\[ \{ s \} = \left[ \sinh r_2 (gl) - \tanh r_2 l \cdot \cosh r_2 (gl) \right] = \frac{1}{r_2} \left\{ n \right\} \]
\[ \{ t \} = \frac{(gl)}{GK_2} = \left\{ p \right\} (gl) \]
\[ \{ u \} = \left\{ p \right\} - \left\{ j \right\} \]
CASE III - Variable \( K \)

7. Summary of Equations: Segment \( AB \) \( 0 < z < gL \) \( \rightarrow \) Segment \( BC \) \( gL < z < L \)

(a) Angle of Twist \( \psi \)

\[ \psi_1 = \frac{TE}{GK_1} + A_1 + B_1 \sinh \nu_1 z + D_1 \cosh \nu_1 z \] \hspace{1cm} (34)

\[ \psi_2 = \frac{TE}{GK_2} + A_2 + B_2 \sinh \nu_2 z + D_2 \cosh \nu_2 z \] \hspace{1cm} (35)

(b) Unit Angle of Twist \( \theta \)

\[ \theta_1 = \frac{T}{GK_1} + B_1 \nu_1 \cosh \nu_1 z + D_1 \nu_1 \sinh \nu_1 z \] \hspace{1cm} (36)

\[ \theta_2 = \frac{T}{GK_2} + B_2 \nu_2 \cosh \nu_2 z + D_2 \nu_2 \sinh \nu_2 z \] \hspace{1cm} (37)

(c) Moment in Flange \( M_F \)

\[ M_1 = \frac{EI_y h}{4} \left[ B_1 \nu_1^2 \sinh \nu_1 z + D_1 \nu_1^2 \cosh \nu_1 z \right] \] \hspace{1cm} (38)

\[ M_2 = \frac{EI_y h}{4} \left[ B_2 \nu_2^2 \sinh \nu_2 z + D_2 \nu_2^2 \cosh \nu_2 z \right] \] \hspace{1cm} (39)

8. Max. Values:

(a) \( \max (\psi)_{z=L} = \frac{TL}{GK_2} + A_2 + B_2 \sinh \nu_2 L + D_2 \cosh \nu_2 L \) \hspace{1cm} (40)

(b) \( \max (\theta)_{z=L} = \frac{T}{GK_2} + B_2 \nu_2 \cosh \nu_2 L + D_2 \nu_2 \sinh \nu_2 L \) \hspace{1cm} (41)

(c) \( \max (M_A^F)_{z=0} = \frac{EI_y h}{4} \left[ B_1 \nu_1^2 \right] \) \hspace{1cm} (42)

\[ = \frac{EI_y h}{4} \left[ \frac{(mn-kg)T}{nh-mq} \right] \nu_1^2 \]
Case III: Variable K Shaft Loading Modification No. 2

(a) Sketch

(b) $\psi$

(c) $\theta$

(d) $M^f$

(e) $\theta$

\[ \psi_1 = \frac{T}{GK_1} + A_1 + B_1 \sinh \gamma z + D_1 \cosh \gamma z \]

\[ \psi_2 = \frac{T}{GK_2} + A_2 + B \sinh \tau_2 z + D_2 \cosh \tau_2 z \]

\[ \theta_1 = \frac{T}{GK_1} + B_1 \cosh \gamma z + D_1 \sinh \gamma z \]

\[ \theta_2 = \frac{T}{GK_2} + B_2 \tau_2 \cosh \tau_2 z + D_2 \tau_2 \sinh \tau_2 z \]

\[ M_1^f = \frac{EI}{4} \left[ B_1 \gamma^2 \sinh \gamma z + D_1 \gamma^2 \cosh \gamma z \right] \]

\[ M_2^f = \frac{EI}{4} \left[ B_2 \tau_2^2 \sinh \tau_2 z + D_2 \tau_2^2 \cosh \tau_2 z \right] \]
d. Extension - Variable \( I_y \)

To take care of the case where the torsional and lateral bending constants change in value at the same point, it is only necessary to incorporate the \( I_y \) and \( h \) terms in the constants. The form of the expressions for the arbitrary constants will not change.

Where \( K \) and \( I_y \) are assumed to change at different points, more segments will be required with a corresponding increase in the number of constants and the computations involved. Otherwise, the procedure will be essentially the same.

Another case which can be handled in a similar manner is the one involving more than two values of \( K \), the number of arbitrary constants to be determined being equal to three times the number of segments of constant \( K \).
3. Numerical Example T3-8-II

a. General

To investigate the characteristics of this extended solution, it was applied to a typical built-up specimen T3-8. The range of \((k)\) was taken to be between \((K_1)\), the computed effective constant assuming integral action, and \((K_S)\), the computed separate action constant. The lateral bending constant \((I_y)\) was assumed to have the same value throughout.

b. Combination

It was assumed that there were two values of \((k)\) which changed abruptly at a certain point. Three combinations, designated Computations "E", "F", and "G", were investigated.

<table>
<thead>
<tr>
<th>Computation</th>
<th>Segment AB</th>
<th>Segment BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;E&quot;</td>
<td>(gL = 1/2) L</td>
<td>1/2 L</td>
</tr>
<tr>
<td>&quot;F&quot;</td>
<td>(gL = 7/8) L</td>
<td>1/8 L</td>
</tr>
<tr>
<td>&quot;G&quot;</td>
<td>(gL = 3/4) L</td>
<td>1/4 L</td>
</tr>
</tbody>
</table>

4. Discussion of Results

a. Computation "D"

For purposes of comparison, Computation "D" was made assuming that \((K_S)\) was effective over the whole span. These results would set the upper limit while Computation "G" based on computed \((K_1)\) would form the lower limit.

The effect of combining \((K_1)\) and \((K_S)\) in various proportions on the behavior of a built-up girder (T3-8) is shown graphically in Fig. 5:20, 5:21, and 5:22.
FIG. 5:20 VARIABLE K EXPERIMENTAL VS. COMPUTED $\sigma$ - T3-B-II
FIG. 5:21  VARIABLE K  EXPERIMENTAL VS. COMPUTED $\psi$  T3-B-II
FIG. 5:22 VARIABLE K EXPERIMENTAL VS. COMPUTED θ. T3-B-II
b. Normal Stress

As predicted, the lateral bending stress curves for cases of variable (K) lie above the lower envelope curve of constant (K₁) and progressively approach the (K₂) curve as the proportional part of (K₂) increases. Fig. 5:19 also shows a characteristic cusp in each curve at the transition point. In the case of specimen T3-B-II, a comparison of experimental with theoretical values shows that measured points come quite close to the Computation "F" curve.

c. Angle of Twist

The curves of (ψ) for the various combinations follow the expected pattern; a progressive increase from the case with constant (K₁) to that with constant (K₂) as shown in Fig. 5:20. No abrupt change in the slope of the curves is noticeable, confirming the anticipated behavior. Computation "F" gives results which approximate the experimentally measured values representing an average angle of twist for the web and flange.

d. Unit Angle of Twist

The computed unit angle of twist curves fall in line with the predicted behavior as shown in Fig. 5:21. The curves do not show a pronounced change in the slope of the tangents at the division point. This is probably due to the limited number of computed points in the vicinity of the transition point. Of the computed curves, combination "F" comes closest to the average measured value for web and flange.
e. Evaluation

At one time certain deviations in the experimental vs. computed curves were thought to be due to a possible variation of \((K)\) at the end. Although Computation "F" showed up extremely well when compared with measured values for one specimen, this modification of the Timoshenko solution is not offered as being applicable to the tests as conducted. Later a more reasonable and a less speculative explanation was found in Computation "H".

However, the possibility of a variation in \((K)\) must not be overlooked or discounted, since any reduction in effective \((K)\) is reflected in an increase in normal and shear stresses as well as in the angular distortions. Some further experimental work to evaluate this factor would be advisable.
CHAPTER VI. CONCLUSIONS

A. Evaluation - Test Program

In retrospect, the loading and support arrangements finally adopted worked satisfactorily. Once the inevitable "bugs" had been removed or rectified, the equipment gave good service.

The selection of specimens proved to be representative. The layout of the instrumentation gave adequate coverage, considering the available equipment. The use of a vertical bar against the sides of the flange to measure the angular deformation of the web neatly by-passed the possible complications due to web deformation.

The SR-4 and level-bar readings were reproducible with the exception of a limited number of low-level stress readings. In the case of the built-up sections, a plot of the observed data before conversion was found desirable.

The experimental results proved to be consistent. Apparent deviations in the measured normal stress near the free end could be explained by using the observed angular distortions.

B. Conclusions

On the basis of the observed behavior and the analytical study, the following conclusions were reached:

(1) The torsion constant \( K \), as obtained from the pure torsion test, when used in the available theories, led to a satisfactory check of the measured stresses and distortions due to non-uniform torsion.

(2) Since the test \( K \) can be approximated by the deVries assumption as established by Chang and Johnston, it follows that the computed value of \( K \) can be used in the available theories to predict
the behavior of shop-fabricated plate girders under non-uniform torsion.

(3) The lateral bending constant \( K_y \) of a plate girder, based on a modified rectangular section formed by the cover plates and the outstanding leg of the flange angles, can be used as a basis for computing the distance between flange centroids and the transverse shear stresses due to flange shear in the non-uniform torsion problem.

(4) Values suitable for design can be obtained by use of the Timoshenko solution provided that the constants are evaluated as recommended, the beam is of the usual proportions, and the end conditions approximate those that are assumed in the derivation.

(5) The use of flange connections in lieu of, or in conjunction with web framing angles will tend to increase the contribution of the flange shear which, in turn, will increase the normal bending stresses all along the span under these loading conditions.

(6) The angular distortions could not be predicted by the Timoshenko theory which assumes that the section twists as a unit. The actual angle of twist of the flanges lagged behind that of the web due to web distortion.

(7) The use of the measured unit angle of twist along the span generally improved the comparison of experimental and computed values of stresses, especially near the free end. This procedure, designated Computation "H", was devised as Modification No. 1.

(8) The two available cases based on the Goodier-Barton theory, while not directly applicable to the specimens as tested, provided an analytical means of improving the predicted values based on the corresponding Timoshenko solution. The former theory, broader in formulation, is, however, more complicated in application.
(9) The "long" beam solution gave an indication of the relative magnitude of the reduction of flange twist to be expected under shaft loading conditions.

(10) The "finite" beam solution, derived from the Goodier-Barton theory and based on slightly different end conditions, produced results which compared quite closely with the measured values.

(11) The improvement in the analytical check by use of the "finite" beam solution, in place of the Timoshenko theory, increased as the torsional behavior of the specimen digressed further from the ideal conditions.

C. Recommendations

Certain aspects of the non-uniform torsion tests uncovered in the course of this investigation are worthy of further study. An extension of this program could be laid out to investigate the following topics:

(1) The behavior of built-up plate girders of constant section can now be predicted by the available theories with reasonable accuracy in the elastic range. A possible extension of this investigation would be to consider the ultimate capacity of similar specimens both analytically and experimentally.

(2) The effect of variable $(k)$, as well as $(I_y)$, presented by a possible change in section, could be evaluated experimentally following the suggested procedure under uniform torsion conditions. This study could then be extended to determine the effect of the change in effective $(k)$ at the free end of built-up sections for the non-uniform torsion case. The experimental values could be compared with the analytical extension of the Timoshenko solution worked out as Modification No. 2.
(3) An analytical formulation could be derived to predict the stress condition under more general boundary conditions than possible with the available theories.
# APPENDIX A

## INDIVIDUAL CURVES - INDEX

Experimental vs. Computed Values

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Normal Stress</th>
<th>Shear Stress</th>
<th>Unit Angle of Twist</th>
<th>Angle of Twist</th>
<th>Flange Shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 1-R</td>
<td>A 1;1 (129)</td>
<td>A 1;2 (130)</td>
<td>A 1;3 (131)</td>
<td>A 1;4 (132)</td>
<td></td>
</tr>
<tr>
<td>T 2-R</td>
<td>A 2;1 (133)</td>
<td>A 2;2 (134)</td>
<td>A 2;3 (135)</td>
<td>A 2;4 (136)</td>
<td></td>
</tr>
<tr>
<td>T 3-B</td>
<td>A 3;1 (137)</td>
<td>A 3;2 (138)</td>
<td>A 3;3 (139)</td>
<td>A 3;4 (140)</td>
<td>A 3;6 (141)</td>
</tr>
<tr>
<td>T 5-R</td>
<td>A 5;1 (142)</td>
<td>A 5;2 (143)</td>
<td>A 5;3 (144)</td>
<td>A 5;4 (145)</td>
<td></td>
</tr>
<tr>
<td>T 7a-W</td>
<td>A 7;1 (146)</td>
<td>A 7;2 (147)</td>
<td>A 7;3 (148)</td>
<td>A 7;4 (149)</td>
<td></td>
</tr>
<tr>
<td>T 7b-W</td>
<td>A 8;1 (150)</td>
<td>A 8;2 (151)</td>
<td>A 8;3 (152)</td>
<td>A 8;4 (153)</td>
<td></td>
</tr>
<tr>
<td>T 9-WF</td>
<td>A 9;1 (154)</td>
<td>A 9;2 (155)</td>
<td>A 9;3 (156)</td>
<td>A 9;4 (157)</td>
<td>A 9;5 (158)</td>
</tr>
</tbody>
</table>

Note: Page number is given in parentheses.
Fig. A1:2  Experimental vs. Computed \( \tau \) (Web) - TI-R
Fig. A1:3 Experimental vs. Computed $\tau$ (Flange) - T1-R

T1-R
Riveted
T=40 in-kip
FRITZ 215A
Fig. A1: 4 EXPERIMENTAL VS. COMPUTED $\theta$ - TI-R

Unit Angle of Twist (Rad./In.) x 10^-5

$\theta$

L = 88"
Fig. A2:1  Experimental vs. Computed $\sigma$ : T2-R
**Fig. A2.2** Experimental vs. Computed $\tau$ (web) - T2-R
Fig. A2:3 Experimental vs. Computed $T$ (Flange) - T2-R
Fig. A2-4  Experimental vs. Computed $\theta$ - T2-R
Fig. A3:1  Experimental vs. Computed $\sigma$ - T3-B-II
Fig. A3:2  Experimental VS. Computed \( \gamma \) (Web) - T3-B
Fig. A3.3 Experimental vs. Computed γ (Flange) - T3-B
FIG. A3:4  EXPERIMENTAL vs. COMPUTED $\theta$ - T3-B
FIG. A3-6  DIVISION OF TORQUE T – COMPUTED  T3-B-II
Fig. A5.1  Experimental vs. Computed $\sigma$ : T5-R
Fig. A5: 2  Experimental vs. Computed $\tau$ (Web) - T5-R
Fig. A5:3  Experimental Vs. Computed $T$ (Flange) - T5-R
Fig. A5:4  Experimental vs. Computed $\theta$ - T5-R
Fig. A7:1  Experimental vs. Computed $\sigma$ : T7q-W
Fig. A7-2: Experimental vs. Computed \( \tau \) (Web) - T79-N

T79 W
WELDED
T=40IN-KIP
FRTZ215A

Shear Stress (Web) K.S.I.
Fig. A7:3  Experimental vs. Computed $\tau$ (Flange) - T72-W
Fig. A7:4  Experimental vs. Computed $\theta - T79-W$
Fig. A8: 2 Experimental vs. Computed $T$ (Web) - T76-W
Fig. A8-3  Experimental vs. Computed $T$ (Flange) - $T76-W$
Fig. A8:4  Experimental vs. Computed $\theta$-T76-W
FIG. A9:1  EXPERIMENTAL VS. COMPUTED $\sigma$  T9-WF
Fig. A9: 2  Experimental VS. Computed T (WEB) - T 9
Fig. A9:3  Experimental vs. Computed \( \gamma \) (Flange) - T.9
FIG. A9.4  EXPERIMENTAL VS. COMPUTED $\theta$  T9-WF

UNIT ANGLE OF TWIST

\( \text{RAD/IN.} \times 10^{-5} \)

T9 - WF
18 WF 77
T = 40 IN. KIP
FRITZ 215A

$\theta$ "A" - TEST K

EXP $\theta$ (AVG. W. & F)

STATION

7" AT 11" = 77"

4"
FIG. A9-5  EXPERIMENTAL VS. COMPUTED $\psi$  T9-WF
Case II: Goodier-Barton Theory

Part C: Finite Beam

(Flanges rigidly attached at both ends to plates which remain normal to axis)

SUMMARY OF DERIVATION:

1. Beam is twisted by torques applied to the plates to which the flange ends are fixed.

These plates rotate an equal amount \( \psi_{max} \) in opposite directions in their own plane.

2. With the origin at the middle of the span, \( \psi \) will be an odd function of \( z \) due to point symmetry.

3. Angular distortions:

   Angle of Twist: \( \psi = A_2 z + B_3 \sinh \lambda_z z + B_5 \sinh \lambda_z z \)

   Flange Reduction:

   \[ \alpha = (k_5 (\lambda_z)^{1/2} - k_6 (\lambda_z)^{3/2}) B_3 \sinh \lambda_z z + [k_5 (\lambda_z)^{1/2} - k_6 (\lambda_z)^{3/2}] B_5 \sinh \lambda_z z \]

   \[ = [1] B_3 \sinh \lambda_z z + [2] B_5 \sinh \lambda_z z \]

4. Boundary Conditions

   At \( z = L \) : \( \alpha = 0 \)
   At \( z = L \) : \( \psi' = 0 \)
   At \( z = L \) : \( \psi = \psi_{max} = \psi_m \)

5. Procedure:

   Express arbitrary constants \( A_2, B_3, B_5 \) in terms of \( \psi_{max} \) by use of boundary conditions.

   Obtain relation between \( \psi_{max} \) and the applied torque by means of equilibrium equation (15).

   Solve for arbitrary constants.

   Solve for angle of twist and flange bending moment.
CASE II  PART C  COMPUTATION \( J \)  FINITE BEAM
(BASED ON GOODIER-BARTON THEORY)

SPECIMEN TQ- WF  (Refer to Comp "I" for Constants)

1. Dimensionless Constants
\[ k_1 = 14.85 \quad k_2 = 0.149 \quad k_3 = 0.426 \]
\[ k_4 = 0.702 \quad k_5 = 0.106 \quad k_6 = 10.58 \]
\[ K_1 = 1.66 \quad K_2 = 0.16 \]

2. Roots of Quadratic: \( \lambda_1 = 0.0725 \quad \lambda_2 = 0.015 \)

3. \[
\begin{array}{l}
\left[ k_5 (\lambda_1 h)^2 - k_6 (\lambda_1 h)^4 \right] = [0.106 (1.59) - 10.58 (1.59)^2] =
\end{array}
\]
\[
\begin{array}{ll}
\text{[1]} & = [ 0.169 - 26.70 ] = -26.53
\end{array}
\]

4. \[
\begin{array}{l}
\left[ k_5 (\lambda_2 h)^2 - k_6 (\lambda_2 h)^4 \right] = [0.106 (0.066) - 10.58 (0.066)^2] =
\end{array}
\]
\[
\begin{array}{ll}
\text{[2]} & = [ 0.007 - 0.044 ] = -0.037
\end{array}
\]

5. \( L = 88" \quad \lambda_1 L = 0.0725 (88) = 6.45 \quad \lambda_2 L = 0.015 (88) = 1.34 \)

\( \sinh \lambda_1 L = 332.6 \quad \sinh \lambda_2 L = 1.778 \)

\( \cosh \lambda_1 L = 332.6 \quad \cosh \lambda_2 L = 2.040 \)

\( \tanh \lambda_1 L = 1.00 \)

6. ARBITRARY CONSTANTS

\[ B_3 = \frac{-B_5 \sinh \lambda_2 L \ [\text{[2]}]}{\sinh \lambda_1 L \ [\text{[1]}]} = \frac{-B_5 (1.778)(-0.037)}{(332.6)(-26.53)} \]
\[ = -7.48 \times 10^{-6} \ B_5 \]

\[ A_2 = B_5 \left[ \frac{\sinh \lambda_2 L \ [\text{[2]}]}{\tanh \lambda_1 L \ [\text{[1]}]} - \lambda_2 \cosh \lambda_2 L \right] \]
Comp. "J"  TF-WF  Arbitrary Constants (Cont'd)

\[
A_2 = B_5 \left[ \frac{(0.0725)(1.778)(-0.037)}{(1.00)(-26.53)} - 0.015(2.040) \right]
\]
\[
= B_5 \left[ +1.80 \times 10^{-4} - 3.06 \times 10^{-2} \right] = -0.0304 \ B_5
\]

\[
\psi_m = A_2 L + B_3 \sinh \lambda_1 L + B_5 \sinh \lambda_2 L
\]
\[
= (-0.0304 \ B_5)(88) - 7.48 \times 10^{-6} B_5 (332.57) + B_5 (1.778)
\]
\[
= -0.904 \ B_5
\]

\[
B_5 = -1.11 \ \psi_m
\]

\[
B_3 = (-7.48 \times 10^{-6})(-1.11 \ \psi_m) = +8.28 \times 10^{-6} \ \psi_m
\]

\[
A_2 = (-0.0304) (-1.11 \ \psi_m) = +0.0336 \ \psi_m
\]

Angular Distortions at \( z = 0 \)  \( \cosh \theta = 1.0 \)

\[
\psi' = A_2 + B_3 \lambda_1 \cosh \lambda_1(0) + B_5 \lambda_2 \cosh \lambda_2(0)
\]
\[
= +0.0336 \ \psi_m + 8.28 \times 10^{-6} \ \psi_m (0.0725)(1.0)
\]
\[
\quad + (-1.11 \ \psi_m)(0.015) (1.0)
\]
\[
= +0.0336 \ \psi_m + 6.0 \times 10^{-7} \ \psi_m - 1.67 \times 10^{-2} \ \psi_m
\]
\[
= +0.0169 \ \psi_m
\]

\[
\alpha' = B_3 \lambda_1 \left[ (1) \right] \cosh \lambda_1(0) + B_5 \lambda_2 \left[ (2) \right] \cosh \lambda_2(0)
\]
\[
= 8.28 \times 10^{-6} \ \psi_m (-26.53)(0.0725)(1.0)
\]
\[
\quad + (-1.11 \ \psi_m)(0.015)(-0.037)(1.0)
\]
\[
= -1.59 \times 10^{-5} \ \psi_m + 6.16 \times 10^{-4} \ \psi_m = +0.00060 \ \psi_m
\]

\[
\psi'' = B_3 \lambda_1^3 \cosh \lambda_1(0) + B_5 \lambda_2^3 \cosh \lambda_2(0)
\]
\[
= +8.28 \times 10^{-6} \ \psi_m (0.0725)^3(1.0) - 1.11 \ \psi_m (0.015)^3(1.0)
\]
\[
= +3.16 \times 10^{-9} \ \psi_m - 3.74 \times 10^{-6} \ \psi_m = -3.737 \times 10^{-6} \ \psi_m
\]
**COMP. "J" T9-WF (Cont'd.)**

8. Substitute in Equation of Torsion

\[ T = KG \psi' - 2K^2G\alpha' - \frac{EIyh^2}{4} \psi'' = 40 \text{ in.-kip} \]

\[ = (3.83)(1.15 \times 10^4) (+0.0169 \psi_m) \]
\[ - 2(1.63)(1.15 \times 10^9)(+0.00060 \psi_m) \]
\[ - \frac{(2.95 \times 10^4)(88.6)(17.38)^2}{4} (-3.73 \times 10^{-6} \psi_m) \]

\[ = (+744.0 - 22.5 + 735.0) \psi_m = 1456.5 \psi_m \]

\[ \psi_m = \frac{40.0}{1456.5} = 0.0275 \text{ rad.} \]

9. **ARBITRARY CONSTANTS**

\[ A_2 = +0.0336 \psi_m = +9.25 \times 10^{-4} \]
\[ B_3 = +8.28 \times 10^{-6} \psi_m = +2.28 \times 10^{-7} \]
\[ B_5 = -1.11 \psi_m = -3.06 \times 10^{-2} \]

10. **ANGLE OF TWIST**

\[ \psi = A_2 z + B_3 \sinh \lambda_1 z + B_5 \sinh \lambda_2 z \]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( z = 0 )</th>
<th>( z = \frac{L}{4} = 22^\circ )</th>
<th>( z = \frac{L}{2} = 44^\circ )</th>
<th>( z = \frac{3L}{4} = 66^\circ )</th>
<th>( z = L = 88^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_2 z )</td>
<td>0</td>
<td>+0.0204</td>
<td>+0.0407</td>
<td>+0.0610</td>
<td>+0.0814</td>
</tr>
<tr>
<td>( B_3 \sinh \lambda_1 z )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B_5 \sinh \lambda_2 z )</td>
<td>0</td>
<td>-0.0103</td>
<td>-0.0217</td>
<td>-0.0355</td>
<td>-0.0540</td>
</tr>
<tr>
<td>( \psi )</td>
<td>0</td>
<td>+0.0101</td>
<td>+0.0190</td>
<td>+0.0255</td>
<td>+0.0275</td>
</tr>
</tbody>
</table>

**Timoshenko Comp. "A" Test K Modified to conform to origin.**

| 0 | 0.0099 | +0.0186 | +0.0249 | +0.0274 |
### T9-WF Comp. "J" G-B. Theory Finite Beam

Flange Moment \( M^F = \frac{-EIh}{4} \left[ B_3 \lambda_1^2 \sinh \lambda_1 z + B_5 \lambda_2^2 \sinh \lambda_2 z \right] \)

Flange Bending Stress (Max.) \( \sigma = \frac{M^F c}{I_y} = \frac{M^F b}{I_y} \)

<table>
<thead>
<tr>
<th></th>
<th>( z = 0 )</th>
<th>( z = \frac{L}{4} = 22&quot; )</th>
<th>( z = \frac{L}{2} = 44&quot; )</th>
<th>( z = \frac{3L}{4} = 66&quot; )</th>
<th>( z = L = 88&quot; )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>0</td>
<td>1.60</td>
<td>3.18</td>
<td>4.78</td>
<td>6.4</td>
</tr>
<tr>
<td>( \sinh \lambda )</td>
<td>0</td>
<td>2.376</td>
<td>12.00</td>
<td>59.55</td>
<td>332.6</td>
</tr>
<tr>
<td>( \lambda^2 )</td>
<td>0</td>
<td>0.33</td>
<td>0.66</td>
<td>0.99</td>
<td>1.34</td>
</tr>
<tr>
<td>( \sinh \lambda^2 )</td>
<td>0</td>
<td>0.3360</td>
<td>0.7090</td>
<td>1.160</td>
<td>1.78</td>
</tr>
<tr>
<td>( B_3\lambda^2\sinh\lambda z )</td>
<td>0</td>
<td>(+2.84 \times 10^{-9})</td>
<td>(+1.44 \times 10^{-8})</td>
<td>(+7.14 \times 10^{-8})</td>
<td>(+3.98 \times 10^{-7})</td>
</tr>
<tr>
<td>( B_3\lambda^2\sinh\lambda z )</td>
<td>0</td>
<td>(-2.32 \times 10^{-6})</td>
<td>(-4.88 \times 10^{-6})</td>
<td>(-8.00 \times 10^{-6})</td>
<td>(-1.225 \times 10^{-5})</td>
</tr>
<tr>
<td>( (\ ) + (\ ) )</td>
<td>0</td>
<td>(-2.32 \times 10^{-6})</td>
<td>(-4.88 \times 10^{-6})</td>
<td>(-7.39 \times 10^{-6})</td>
<td>(-1.19 \times 10^{-5})</td>
</tr>
<tr>
<td>( -(EIh/4) )</td>
<td>(-1.135 \times 10^7)</td>
<td>(-1.135 \times 10^7)</td>
<td>(-1.135 \times 10^7)</td>
<td>(-1.135 \times 10^7)</td>
<td>(-1.135 \times 10^7)</td>
</tr>
<tr>
<td>( M^F )</td>
<td>0</td>
<td>2.54 \times 10 ^ 0</td>
<td>5.63 \times 10 ^ 0</td>
<td>0.90 \times 10 ^ 2</td>
<td>1.35 \times 10 ^ 2</td>
</tr>
<tr>
<td>( \sigma = \frac{b}{I_y} M^F ) (ksi)</td>
<td>0</td>
<td>2.49</td>
<td>5.42</td>
<td>8.83</td>
<td>13.3</td>
</tr>
</tbody>
</table>

**Timoshenko Comp. "A"**

- \( A_2 = +9.25 \times 10^{-4} \)
- \( B_3 = +2.28 \times 10^{-7} \)
- \( B_5 = -3.06 \times 10^{-2} \)
- \( \lambda_1 = 0.0725 \)
- \( \lambda_1^2 = 5.25 \times 10^{-3} \)
- \( \lambda_2 = 0.015 \)
- \( \lambda_2^2 = 2.25 \times 10^{-4} \)
- \( E = 29.5 \times 10^3 \) ksi
- \( I_y = 88.6 \) in^4
- \( h = 17.38" \)
- \( b = 8.70" \)
- \( \frac{b}{I_y} = 0.098 \)
1. Constants

Torsion Constant $K = C$

Web $C_w = 0.57$

Flange $C_f = 3.26$

Section (Test) $C = 3.83$ in.

Nominal $I_Y = 88.6$ in.$^4$

$I_Y = \frac{1}{2} I_Y$

Web $t_w = 0.48$ in.

Web Constant $D_w = \frac{E t_w^3}{12(1 - \nu^2)}$

$D_w = \frac{(29.5 \times 10^3)(0.48)^3}{12(1 - 0.29^2)} = 297.0$ (k-in)  \hfill (301)

2. Dimensionless Constants

$k_1 = \frac{E I_Y}{4 G C} = \frac{(29.5 \times 10^3)(88.6)}{4(115 \times 10^3)(3.83)} = 14.85$ \hfill (302)

$k_2 = \frac{C_w}{C} = \frac{0.57}{3.83} = 0.149$ \hfill (303)

$k_3 = \frac{C_f}{C} = \frac{1.63}{3.83} = 0.426$ \hfill (304)

$k_4 = \frac{6 D_w h}{G C} = \frac{6(297)(17.38)}{(115 \times 10^3)(3.83)} = 0.102$ \hfill (305)

$k_5 = \frac{k_3}{2k_4} = \frac{0.149}{2(0.102)} = 0.702$ \hfill (306)

$k_6 = \frac{k_1}{2k_4} = \frac{14.85}{2(0.102)} = 70.58$ \hfill (307)

$k_i = \frac{k_2 + k_4}{k_5} = \frac{0.149 + 0.702}{0.426}$

$= 0.010 + 1.65 = 1.66$ \hfill (308)
COMPUTATION 'I' (CONTINUED)

2 Dimensionless Constants (continued)

\[ K_2 = \frac{\frac{k_4}{k_3}}{1 + \frac{k_4}{k_3}} = \frac{4 \left( 2 + \frac{k_4}{k_3} \right)}{k_1} \]

\[ = 4 \frac{0.702}{14.86} (2 + \frac{0.426}{0.426}) = 0.089 (2 + 0.350) = 0.16 \text{ (309)} \]

(Note: Since \( K_2 \) is less than 1, roots are real.)

3 Roots of Quadratic

\[ (\lambda h)^2 = \frac{k_1}{k_2} \left[ 1 \pm \sqrt{1 - K_2} \right] = \frac{1.66}{2} \left[ 1 \pm \sqrt{1 - 0.16} \right] \]

\[ = 0.83 \left[ 1 \pm 0.92 \right] \]

\[ (\lambda_1 h)^2 = 0.83 \left[ 1.92 \right] = +1.59 \text{ (310)} \]

\[ (\lambda_2 h)^2 = 0.83 \left[ 0.08 \right] = +0.060 \text{ (311)} \]

\[ \lambda_1 h = \pm \sqrt{1.59} = \pm 1.26 \]

\[ \lambda_2 h = \pm \sqrt{0.060} = \pm 0.26 \]

\[ \lambda_1 = \pm \frac{1.26}{17.38} = \pm 0.0726 \text{ (312)} \]

\[ \lambda_2 = \pm \frac{0.26}{17.38} = \pm 0.015 \text{ (313)} \]

Discard positive exponentials in this "Long" Beam case.
**Computation I** (Continued)

4. Evaluate Arbitrary Constants @ T = 40 in-kips

\[ A_z = \frac{I}{GC} = \frac{40}{(11.5 \times 10^3)(3.83)} = +9.10 \times 10^{-4} \]  \hspace{1cm} (314)

\[ A_4 = \frac{A_z}{\lambda_z} \left[ \frac{\lambda_1 - k_5 (\lambda_1 h)^2 - k_6 (\lambda_1 h)^4}{\lambda_1} \right]^{-1} \]

\[ = \frac{A_z}{\lambda_2} \left[ \frac{\lambda_1 - k_5 (\lambda_1 h)^2 - k_6 (\lambda_1 h)^4}{\lambda_2} \right]^{-1} \]

\[ = \frac{9.10 \times 10^{-4}}{0.15 \left[ 0.0725 \left( \frac{106(1.59) - 10.58(1.59)^2}{0.105(106(0.066) - 10.58(0.06)^2) - 53.67} \right) \right]} \]

\[ = \frac{9.10 \times 10^{-4}}{0.15 \left[ 4.83 - \frac{126.53}{453} \right]} \]

\[ = \frac{9.10 \times 10^{-4}}{0.15 (-53.67)} = -11.3 \times 10^{-4} = -1.13 \times 10^{-3} \]  \hspace{1cm} (315)

\[ A_4 = \frac{A_2}{\lambda_1} \left[ \frac{\lambda_2 - k_5 (\lambda_2 h)^2 - k_6 (\lambda_2 h)^4}{\lambda_1} \right]^{-1} \]

\[ = \frac{9.10 \times 10^{-4}}{0.0725 \left( \frac{1207 + 453}{26.53} \right)} \]

\[ = \frac{9.10 \times 10^{-4}}{0.0725 (18.99)} \]

\[ = +6.61 \times 10^{-2} \]  \hspace{1cm} (316)

\[ A_1 = -(A_4 + A_6) \]

\[ = -(0.00113 + 0.0661) \]

\[ = -0.065 \approx 6.5 \times 10^{-2} \]  \hspace{1cm} (317)
**Computation 1** (continued)

Equations: Case 1 - Long Beam

\[ \beta = \psi = A_1 + A_2 z + A_4 e^{-\lambda_1 z} + A_6 e^{-\lambda_2 z} \]  
\[ \beta' = \psi' = A_2 - \lambda_1 A_4 e^{-\lambda_1 z} - \lambda_2 A_6 e^{-\lambda_2 z} \]  
\[ \alpha = [k_5 (\lambda_1 h)^2 - k_6 (\lambda_1 h)^3] A_4 e^{-\lambda_1 z} \]  
\[ + [k_5 (\lambda_2 h)^2 - k_6 (\lambda_2 h)^3] A_6 e^{-\lambda_2 z} \]

6. Angle of Twist of Web \( \beta (\psi) @ T = 40 \) in Kips

\[ \beta = A_1 + A_2 z + A_4 e^{-\lambda_1 z} + A_6 e^{-\lambda_2 z} \]
\[ = -6.5 \times 10^{-2} + 9.1 \times 10^{-4} z - (1.13 \times 10^{-3}) e^{-0.0725 z} + (6.61 \times 10^{-2}) e^{-0.015 z} \]
\[ \beta = -6.5 \times 10^{-2} + 0 - (1.13 \times 10^{-3}) 1 + (6.61 \times 10^{-2}) 1 = 0 \]  
CHECKS

\[ \beta = -6.5 \times 10^{-2} + 9.1 \times 10^{-4} \times 88 - 1.13 \times 10^{-3} \times 88 \times e^{-0.0725(88)} \]
\[ + 6.61 \times 10^{-2} \times 88 \times e^{-0.015(88)} \]
\[ = -0.065 + 0.0801 = 0.015 \]
\[ = 0.327 \times 10^{-2} \text{ radians} \]  

Angle of Twist - Uniform Torsion \( T = 40 \) in kips, \( l = 88'' \)

\[ \psi_{\text{MAX}} = \frac{TL}{KJ} \]
\[ = \frac{40,000 (88)}{3.83 (11.5 \times 10^6)} = 0.080 \text{ radians} \]  

Angle of Twist - Max. - Timoshenko

Test Constants "A" \( \kappa = 3.83 \), \( l = 88'' \)
\( a = 66.7 \), \( \tanh \frac{a}{\kappa} = 0.866 \)

\[ \psi_{\text{MAX}} = \frac{T}{KJ} \left[ l - a \tanh \frac{a}{\kappa} \right] = \frac{40,000}{3.83 (11.5 \times 10^6)} \left[ 88 - (66.7)(0.866) \right] \]
\[ \psi_{\text{MAX}} = 0.0275 \text{ radians} \]  

(320)
APPENDIX B

PART 2: COMPUTATION "H"

MODIFICATION NO. 1

BASED ON TEST θ

TYPICAL COMPUTATIONS

SPECIMEN T 9-WF

NORMAL STRESS

SHEAR STRESS
Case II - Variable K - Specimen T3-B - Constant T = 40 in.

### Computation of \( E \)
Assume \( g = \frac{1}{2} (88) = 44 \) in. units

\[
\begin{align*}
A & = gL = 0.5(88) = 44 \\
B & = (1-g) L = 0.5(88) = 44 \\
K_1 = K_2 &= 7.24 \text{ in}^4 \\
K_2 = K_5 &= 1.87 \text{ in}^4
\end{align*}
\]

\[
\begin{align*}
S_1 &= G K_1 = (11.5 \times 10^3)(7.24) = 8.32 \times 10^6 \\
S_2 &= G K_2 = (11.5 \times 10^3)(1.87) = 2.15 \times 10^6 + S_2 \\
W_1 &= \frac{E h^2}{4} = \frac{(29.5 \times 10^{-3})(99.7)(12.16)}{4} = 2.42 \times 10^8 = W_2 \\
\eta_1 &= \frac{S_1}{W_1} = 3.44 \times 10^{-4} \\
\eta_2 &= \frac{S_2}{W_2} = 0.89 \times 10^{-4} \\
\eta_1 &= 0.0185 = \frac{1}{a_1} \\
\eta_2 &= 0.0094 = \frac{1}{a_2}
\end{align*}
\]

\[
\begin{align*}
\text{Cone} \ n_1(gL) &= (0.0185)(44) = 0.815 \\
cosh n_1(gL) &= 1.350 \\
\text{Cone} \ n_2(gL) &= 0.0094(44) = 0.415 \\
cosh n_2(gL) &= 1.088 \\
\text{Cone} \ n_1(gL) &= 0.9083 \\
cosh n_1(gL) &= 0.427 \\
N_1 L &= (0.0185)(88) = 1.63 \\
N_2 L &= 0.825 \\
\tan \text{h} n_1 L &= 0.9361 \\
\tan \text{h} n_2 L &= 0.6176
\end{align*}
\]

\[
\begin{align*}
\{e\} &= \left[ \text{cosh} n_1(gL) - 1 \right] = [1.350 - 1.000] = +0.350 \\
\{f\} &= \frac{1}{G K_1} \left[ gL - \text{sinh} n_1(gL) \right] = \frac{-1}{8.32 \times 10^6} \left[ 44 - \frac{0.9083}{0.0185} \right] \\
&= (1.202 \times 10^{-5}) [44.0 - 49.0] = -6.03 \times 10^{-5} \\
\{h\} &= N_1 \text{sinh} n_1(gL) = 0.0185 (0.9083) = +1.68 \times 10^{-2} \\
\{j\} &= \frac{1}{G K_1} \{-e\} = (1.202 \times 10^{-5})(-0.350) = -4.2 \times 10^{-6} \\
\{k\} &= \frac{1}{G K_1} \{h\} = (1.202 \times 10^{-5})(1.68 \times 10^{-2}) = +2.03 \times 10^{-7} \\
\{m\} &= N_1^2 \text{cosh} n_1(gL) = (3.44 \times 10^{-4})(1.350) = +4.64 \times 10^{-4}
\end{align*}
\]
Case II - Variable K - T3: B - II - Comp. E

Constants (Cont'd)

\[
\{n\} = \frac{L^*}{N_2} \left[ \tanh N_2 (qL) - \tanh N_1 L \cdot \tanh N_2 (qL) \right]
\]
\[
= (0.89 \times 10^{-2}) \left[ 0.427 - (0.6776) (1.088) \right]
\]
\[
= (8.9 \times 10^{-5}) \left[ -0.310 \right] = -2.76 \times 10^{-5}
\]

\[
\{p\} = \frac{1}{GK_2} = \frac{i}{2.15 \times 10^2} = +4.65 \times 10^{-5}
\]

\[
\{q\} = \frac{N_2 \left[ \cos N_2 (qL) - \tanh N_1 L \cdot \tanh N_2 (qL) \right]}{N_2}
\]
\[
= (0.0997) \left[ 1.088 - 0.6776 (0.427) \right]
\]
\[
= (9.4 \times 10^{-3}) [0.799] = +7.5 \times 10^{-3}
\]

\[
\{a\} = \frac{1}{N_2} \{n\} = \frac{1}{8.9 \times 10^{-5}} (-2.76 \times 10^{-5}) = -0.310
\]

\[
\{t\} = \{p\} (qL) = (4.65 \times 10^{-5}) (0.44) = +2.05 \times 10^{-3}
\]

\[
\{u\} = \{p\} - \{j\} = 4.65 \times 10^{-5} - (-4.2 \times 10^{-5}) = +5.07 \times 10^{-5}
\]

Constants of Integration: Note: \( T = 40 \) in kip included in constants

\[
D_1 = \frac{[\mu \cdot u - k \cdot g]T}{nh^2 - m g} = \frac{-2.76 \times 10^{-5} (2.028 \times 10^{-3}) - (8.1 \times 10^{-6}) (7.5 \times 10^{-3})}{-2.76 \times 10^{-5} (1.68 \times 10^{-2}) - (4.64 \times 10^{-4}) (7.5 \times 10^{-3})}
\]
\[
= -5.60 \times 10^{-2} - 6.08 \times 10^{-8} = -11.68 \times 10^{-8} = 4.64 \times 10^{-7} - 3.48 \times 10^{-6} = -3.949 \times 10^{-6}
\]

\[
B_2 = \frac{[\mu \cdot u - k \cdot g]T}{nh^2 - m g} = \frac{+4.64 \times 10^{-4} (2.028 \times 10^{-3}) - (1.68 \times 10^{-4}) (8.1 \times 10^{-6})}{-2.76 \times 10^{-5} (1.68 \times 10^{-2}) - (4.64 \times 10^{-4}) (7.5 \times 10^{-3})}
\]
\[
= +9.4 \times 10^{-7} - 1.36 \times 10^{-7} = +8.04 \times 10^{-7} = -4.64 \times 10^{-7} - 3.48 \times 10^{-6} = -3.949 \times 10^{-6}
\]

\[
A_1 = -D_1 = -2.96 \times 10^{-2}
\]

\[
B_1 = -\frac{T}{GK_2 \cdot \frac{N_1}{n}} = \frac{-40.0}{8.32 \times 10^4 (0.0185)} = -2.6 \times 10^{-2}
\]
### Case II - Variable $K'$ - T3-B-II Comp "E" $T=40$ kip

#### Constants (Cont'd.)

Note: $T=40$ in-kip included in constants.

1. \[ A_2 = D_2 e - B_2 a + \left[ f - t \right] T \]  
   \[ = (2.96 \times 10^{-2})(0.350) - (-0.204)(-0.310) - 2.41 \times 10^{-3} - 8.20 \times 10^{-2} \]  
   \[ = + 1.035 \times 10^{-2} - 6.33 \times 10^{-2} - 0.241 \times 10^{-2} - 8.20 \times 10^{-2} \]  
   \[ = -13.736 \times 10^{-2} = \frac{-0.1374}{A_2} \]

2. \[ D_2 = - B_2 \tanh \ n \xi \xi = \left( -0.204 \right)(0.6776) = + 0.138 \]

3. Angle of Twist  
   \[ \psi = \frac{T}{GK} + A + B \sinh n \xi + D \cosh n \xi \]
   \[ AB: \quad \psi_1 = (4.81 \times 10^{-4}) \xi - 0.0296 - (0.026) \sinh n \xi + (0.0296) \cosh n \xi \]
   \[ BC: \quad \psi_2 = (1.86 \times 10^{-3}) \xi - 0.1374 - (0.204) \sinh n \xi + (0.138) \cosh n \xi \]

4. Unit Angle of Twist  
   \[ \theta = \frac{T}{GK} + Bn \tanh n \xi + Dn \sinh n \xi \]
   \[ AB: \quad \theta_1 = (4.81 \times 10^{-4}) - 0.026(0.0185) \cosh n \xi + 0.0296(0.0185) \sinh n \xi \]
   \[ \theta_1 = (4.81 \times 10^{-4}) - (4.81 \times 10^{-4}) \cosh n \xi + (5.48 \times 10^{-4}) \sinh n \xi \]
   \[ BC: \quad \theta_2 = (1.86 \times 10^{-3}) - 0.204(0.0094) \cosh n \xi + 0.138(0.0094) \sinh n \xi \]
   \[ \theta_2 = 1.86 \times 10^{-3} - (1.92 \times 10^{-3}) \cosh n \xi + (1.295 \times 10^{-3}) \sinh n \xi \]

5. Moment in Flange  
   \[ M_f = \frac{EI_{yy}}{2} \left[ B n^2 \sinh n \xi + D n^2 \cosh n \xi \right] \]
   \[ EI_{yy} = \frac{(29.5 \times 10^3)(99.7)(18.16)}{9} = 1.33 \times 10^7 \] (k-in³)

18: \[ M_1^f = (1.33 \times 10^7)[(-0.026)(0.0185)^2 \sinh n \xi + 0.0296(0.0185)^2 \cosh n \xi] \]
   \[ = (1.33 \times 10^7)[-8.9 \times 10^{-6}] \sinh n \xi + (1.01 \times 10^{-5}) \cosh n \xi \]

BC: \[ M_2^f = (1.33 \times 10^7)[(-0.204)(0.0094)^2 \sinh n \xi + (0.138)(0.0094)^2 \cosh n \xi] \]
   \[ = (1.33 \times 10^7)[(-1.805 \times 10^{-5}) \sinh n \xi + (1.22 \times 10^{-5}) \cosh n \xi] \]
Case II - Variable $K = T3-B-II$ Comp. "E" $T = 40$ in

6. Substitute particular values of $z$ to obtain the corresponding values of $\psi$, $\theta + M^2$ in each segment.

Summary of Results

<table>
<thead>
<tr>
<th>Assumed $g L = 44''$</th>
<th>$(1 - g) L = 44$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution</td>
<td></td>
</tr>
<tr>
<td>Restrainted End</td>
<td></td>
</tr>
<tr>
<td>$K_1 = K_5 = 7.24$ in$^4$</td>
<td>$K_2 = K_5 = 1.87$ in$^4$.</td>
</tr>
<tr>
<td>22''</td>
<td>22''</td>
</tr>
<tr>
<td>22''</td>
<td>22''</td>
</tr>
<tr>
<td>Free End</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Point Item</th>
<th>$A$</th>
<th>$A_1$</th>
<th>$B_L$</th>
<th>$B_R$</th>
<th>$Q_3$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>0</td>
<td>0.25</td>
<td>0.50</td>
<td>0.50</td>
<td>0.75</td>
<td>1.00</td>
</tr>
<tr>
<td>$z = g L$ (in)</td>
<td>0''</td>
<td>22''</td>
<td>44''</td>
<td>44''</td>
<td>66''</td>
<td>88''</td>
</tr>
<tr>
<td>$\psi$ (rad)</td>
<td>0</td>
<td>0.0023</td>
<td>0.0081</td>
<td>0.0081</td>
<td>0.0166</td>
<td>0.0266</td>
</tr>
<tr>
<td>$\theta$ (rad/ft)</td>
<td>0</td>
<td>1.91 x 10$^{-4}$</td>
<td>3.3 x 10$^{-4}$</td>
<td>3.3 x 10$^{-4}$</td>
<td>1.91 x 10$^{-4}$</td>
<td>1.91 x 10$^{-4}$</td>
</tr>
<tr>
<td>$M^F$ (in-kip)</td>
<td>136.0</td>
<td>99.0</td>
<td>75.1</td>
<td>75.1</td>
<td>37.1</td>
<td>0</td>
</tr>
<tr>
<td>$S$ (ksi)</td>
<td>10.05</td>
<td>7.32</td>
<td>5.56</td>
<td>5.56</td>
<td>2.54</td>
<td>0</td>
</tr>
<tr>
<td>$V^F$ (kip)</td>
<td>-2.2</td>
<td>-1.35</td>
<td>-0.68</td>
<td>-1.81</td>
<td>-1.71</td>
<td>-1.67</td>
</tr>
<tr>
<td>$T_w = V^F H$</td>
<td>40.0</td>
<td>24.5</td>
<td>12.2</td>
<td>32.8</td>
<td>31.0</td>
<td>30.3</td>
</tr>
<tr>
<td>$T_s = T - T_w$</td>
<td>0</td>
<td>15.5</td>
<td>27.8</td>
<td>7.2</td>
<td>9.0</td>
<td>9.7</td>
</tr>
</tbody>
</table>
APPENDIX B

PART 3: COMPUTATION "E"

MODIFICATION NO. 2

VARIABLE X

TYPICAL COMPUTATIONS

SPECIMEN T 3-B

NORMAL STRESS

FLANGE STRESS
<table>
<thead>
<tr>
<th>STATION</th>
<th>1-1</th>
<th>1-2</th>
<th>2-3</th>
<th>3-4</th>
<th>4-5</th>
<th>5-6</th>
<th>6-7</th>
<th>7-8</th>
<th>8-9</th>
<th>END</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.02</td>
<td>1.62</td>
<td>2.04</td>
<td>2.40</td>
<td>2.70</td>
<td>2.99</td>
<td>2.99</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.43</td>
<td>7.84</td>
<td>10.25</td>
<td>12.98</td>
<td>14.32</td>
<td>15.38</td>
<td>13.05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_F = \frac{T_W}{h}$</td>
<td>(2.20)</td>
<td>1.99</td>
<td>1.76</td>
<td>1.57</td>
<td>1.42</td>
<td>1.32</td>
<td>1.24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Torsional Shear Stress**

| WEB - MID-HEIGHT | $T_S^W = 0.84(T_S)^W$ | 0.856 | 1.360 | 1.713 | 2.015 | 2.270 | 2.510 | 2.510 |     |     |
| FLANGE - GAGE LINE | $T_S^F = 0.254(T_S)^F$ | 1.125 | 1.990 | 2.605 | 3.300 | 3.640 | 3.905 | 3.320 |     |     |

**Transverse Shear Stress**

| FLANGE - GAGE LINE | $T_S^F = 0.11(V)^F$ | 0.242 | 0.219 | 0.194 | 0.175 | 0.156 | 0.145 | 0.136 | 0.152 | 0.222 |

**Combined Shear Stress**

| FLANGE - TOP SIDE | $T_S^F = (T_S)^F + (T_S)^F$ | 1.344 | 2.184 | 2.780 | 3.456 | 3.785 | 4.041 | 3.472 |     |     |

\[
\text{Web: } (T_S)^W = \frac{(T_S)^W}{K_W} = 0.48(T_S)^W \\
\text{Flange: } (T_S)^F = \frac{(T_S)^F}{K_F} = 0.83(T_S)^F \\
\text{Flange: } (T_S)^F = \left[1 - \frac{x^2}{(h/2)^2}\right]\left[\frac{1.5V^F}{6.7F}\right] \\
\text{Flange: } (T_S)^F = \left[1 - \frac{3^2}{(8.79(0.83))^2}\right]\left[1.5\right]V^F \\
\text{Both Flanges Combined} = 0.11(V)^F
\]
V - "H" Test θ

σ - "H" Test θ

σ - Exp. θ

Flange Shear: Computed vs. Experimental θ
<table>
<thead>
<tr>
<th>STATION</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
<td>E-1</td>
</tr>
<tr>
<td>$\theta^w_{\times 10^{-4}} \text{ rad/in.}$</td>
<td>1.55</td>
<td>2.46</td>
<td>3.10</td>
<td>3.64</td>
<td>4.10</td>
<td>4.55</td>
<td>4.55</td>
<td>4.55</td>
<td>4.55</td>
</tr>
<tr>
<td>$\theta^f_{\times 10^{-4}} \text{ rad/in.}$</td>
<td>1.18</td>
<td>2.09</td>
<td>2.73</td>
<td>3.46</td>
<td>3.82</td>
<td>4.10</td>
<td>3.48</td>
<td>3.48</td>
<td>3.48</td>
</tr>
<tr>
<td>$(T_5)^w = (0.658 \times 10^4) (\theta)^w$</td>
<td>1.02</td>
<td>1.62</td>
<td>2.04</td>
<td>2.40</td>
<td>2.70</td>
<td>2.99</td>
<td>2.99</td>
<td>2.99</td>
<td>2.99</td>
</tr>
<tr>
<td>$(T_5)^f = (3.75 \times 10^4) (\theta)^f$</td>
<td>4.43</td>
<td>7.84</td>
<td>10.25</td>
<td>12.98</td>
<td>14.32</td>
<td>15.38</td>
<td>13.05</td>
<td>13.05</td>
<td>13.05</td>
</tr>
<tr>
<td>$T_5 = (T_5)^w + (T_5)^f$</td>
<td>1.80</td>
<td>5.45</td>
<td>9.46</td>
<td>12.29</td>
<td>15.38</td>
<td>17.02</td>
<td>18.37</td>
<td>16.04</td>
<td>16.04</td>
</tr>
<tr>
<td>$T_w = T - T_5 = 40 - T_5$</td>
<td>38.20</td>
<td>34.55</td>
<td>30.54</td>
<td>27.71</td>
<td>24.62</td>
<td>22.98</td>
<td>21.63</td>
<td>23.96</td>
<td>23.96</td>
</tr>
<tr>
<td>$V = \frac{T_w}{h} = \frac{T_w}{17.38}$</td>
<td>2.20</td>
<td>1.97</td>
<td>1.76</td>
<td>1.59</td>
<td>1.42</td>
<td>1.32</td>
<td>1.24</td>
<td>1.38</td>
<td>1.38</td>
</tr>
<tr>
<td>$\Delta Z$ (See V-Curve)</td>
<td>7&quot;</td>
<td>11&quot;</td>
<td>11&quot;</td>
<td>11&quot;</td>
<td>11&quot;</td>
<td>11&quot;</td>
<td>11&quot;</td>
<td>4&quot;</td>
<td>4&quot;</td>
</tr>
<tr>
<td>$V(\Delta Z)$</td>
<td>15.40</td>
<td>21.90</td>
<td>19.37</td>
<td>17.50</td>
<td>15.61</td>
<td>14.52</td>
<td>13.65</td>
<td>15.19</td>
<td>8.08</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STATION</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>End</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^w = \frac{V(\Delta Z)}{h}$</td>
<td>141.22</td>
<td>125.82</td>
<td>103.92</td>
<td>84.55</td>
<td>67.05</td>
<td>51.44</td>
<td>36.92</td>
<td>23.27</td>
<td>8.08</td>
</tr>
<tr>
<td>$\sigma = 0.098 M^f (ksi)$</td>
<td>13.81</td>
<td>12.32</td>
<td>10.19</td>
<td>8.30</td>
<td>6.57</td>
<td>5.04</td>
<td>3.62</td>
<td>2.28</td>
<td>0.79</td>
</tr>
</tbody>
</table>

**Notes:** Test $K = 3.83 \text{ in.}^4$  Nominal $I_y = 88.6$  Comp. $K = (K)^w + 2(K)^f = 0.59 + 3.32 = 3.91$

**Assume:** $(K)^w = \frac{0.59}{3.91} (3.83) = 0.572$  $(T_5)^w = K^w \theta^w = (0.572)(115 \times 10^3) \theta^w = 0.658 \times 10^4 (\theta)^w$

**Both Flanges Combined**  $2(K)^f = \frac{3.32}{3.91} (3.83) = 3.26$  $(T_5)^f = 2 K^f \theta^f = (3.26)(115 \times 10^3) \theta^f = 3.75 \times 10^4 (\theta)^f$
APPENDIX C

BIBLIOGRAPHY


(6) Timoshenko, S. "Theory of Bending, Torsion and Buckling of Thin-Walled Members of Open Cross Section." Journal of the Franklin Institute, April, 1945.


APPENDIX D

VITA

The author, Gerald G. Kubo, was born February 12, 1917, in Tacoma, Washington, the second son of Mr. and Mrs. Yukichi Kubo. After graduating from the public schools of Tacoma, he attended the College of Puget Sound for a year of pre-engineering. He then transferred to the University of Washington from which he graduated with the degree of Bachelor of Science in Civil Engineering in June, 1939.

Mr. Kubo continued his studies at the Massachusetts Institute of Technology on a graduate scholarship and received a Master of Science degree in Civil Engineering in 1939. After a year as a structural steel detailer with the American Bridge Company, he enrolled at Lehigh University as a graduate student on a research fellowship in September, 1940.

During the summer of 1941, Mr. Kubo worked for the firm of Howard, Needles, Tammen & Bergendoff, of New York City. He returned then to the American Bridge Company where he was employed as a structural designer-detailer until January, 1946. After working as a structural designer for George F. Hardy & Son, Consulting Engineers, of New York City, he joined the staff of New York University, as an Instructor in the Department of Engineering Mechanics in September, 1946.

Following summer employment with the firm of Parsons, Brinckerhoff, Hogan & MacDonald, of New York City, he became Assistant Professor of Civil Engineering at the University of Connecticut for the academic year 1947-48. Mr. Kubo spent the summer of 1948 as a Research Engineer with the Chicago Bridge & Iron Company. He returned to New York University as an Assistant Professor in the Department of Civil Engineering.
After being reinstated as a candidate for the Ph.D. degree, Mr. Kubo continued his graduate studies at Lehigh University and started his doctoral research project in the summer of 1949. Following the academic year teaching at New York University, he spent the summer at Lehigh University and Swarthmore College on an extension of the project. He returned to teach at New York University and then devoted the ensuing summer to a tied arch model study. He was promoted to the rank of Associate Professor of Civil Engineering at New York University in September, 1951.

Mr. Kubo is an Associate Member of the American Society of Civil Engineers and the American Concrete Institute. He is also a member of the American Society for Engineering Education, the Society for Experimental Stress Analysis, and the International Association for Bridge and Structural Engineering.

He is a member of Tau Beta Pi and an associate member of Sigma Xi. Mr. Kubo is a registered Professional Engineer in the State of New York.