I. INTRODUCTION

1. The Problem - to determine the strength of metal columns as a function of lengths normally encountered in engineering structures.

2. Limitations

The column is considered to be pin-ended, centrally loaded, of uniform cross-section along the length, and of homogenous material. (steel and aluminum columns will be considered).

3. Scope

The following aspects of the problem will be considered:

1. Slender columns (elastic solution)
2. "Short" columns (inelastic solution)
   a. Basmst solution - Reduced modulus - Tangent modulus (Slamley)
3. Determining the "column curve"
4. Residual Stress
5. Working formulas and the factor of safety

4. Criterion of Strength: The condition of stability of the equilibrium position (indifferent or stable in the deflected configuration).

Although column strength is not a function of stress as such, it is recognized, of course that it is a function of the stress-strain diagram for the material. On the one hand, in elastic stability (seminar #2), column strength is a function of $E$. On the other hand, a short block loaded in compression will yield prior to reaching an unstable condition. During this seminar series with the exception of the "transition range" for the lateral buckling of beams (Seminar 5), Fig. 1 has provided adequate definition of the stress-strain relationship.

\[
\sigma = \frac{P}{A}
\]

\[
\frac{d\sigma}{d\epsilon} = E
\]

Average Stress

\[\epsilon = \text{strain} \rightarrow\]
To obtain an answer to the present problem, it will be necessary to consider the complete elastic and inelastic stress-strain behavior (Fig. 2).

5. Terminology: To avoid misunderstanding, the terminology shown in Appendix A will be adhered to. In particular it is suggested that the term, "buckling load" only be used in the classical sense, if at all.

II. SLENDER COLLUMS (Review of Elastic Solution)

1. Euler Load Using the statical criterion of stability, the differential equation of the bent column at the Euler load is, approximately, from \[ \frac{k}{EI} = \frac{1}{R} = \frac{\partial^2 y}{\partial x^2} \]

\[ \frac{\partial^2 y}{\partial x^2} + \frac{P}{EI} y = 0 \]  

from which

\[ P_e = \frac{\pi^2 EI}{L^2} \]  

More exactly

\[ \frac{d\theta}{ds} + \frac{P}{EI} y = 0 \]  

Using this expression, it is found that the load must increase above \( P_e \). The centerline deflection is given approximately by

\[ \frac{\gamma_m}{L} = \frac{4\sqrt{\gamma P/P_e - 1}}{\pi} \]  

(4)
and the load is plotted against the deflection in Fig. 3. (Timoshenko suggests on Page 74 of Reference No. 3 a different approximate expression; it gives similar results).

2. **Limitation** Deflection of column increases stresses on the concave side. Thus, solution suggested by Eq. 4 is not valid after yield point is reached. The behavior is sketched in portion B-C, Fig. 3.

3. **Column Curve** By dividing both sides of Eq. 2 by \( A \) an expression is obtained in terms of average critical stress,

\[
\frac{P_e}{\varepsilon} = \frac{\pi^2E}{A} = \frac{\pi^2E}{(L/r)^2}
\]

The column curve is the plot of this equation, shown in Fig. 5, valid until the slenderness ratio is such that the average stress exceeds the proportional limit (Point B in Fig. 4).

4. **Historical Notes**

1756 - Van Musschenbroek formulated the statement that the strength of a column varies inversely as the square of the length and directly to width and depth square.

1759 - Euler presented \( P = \frac{\pi^2E \times \text{kk}}{L^2} \). As an incident to this development, Euler presented the equation, \( d^2y/dx^2 = li/Ekk \).

1830 - The factor kk was defined as 1 (appr.)

III. "SHORT" COLUMNS (Inelastic Solution) Since columns used in structures commonly have slenderness ratios less than 100, the region of inelastic buckling is a practical one.
1. Ehrasier Solution. In 1889, while Diller was being discredited by those who incorrectly attempted to apply his formula to "short columns", Ehrasier presented the Tangent Modulus Theory. Since the critical load of an elastic column is a direct function of \( E \), he reasoned that in shorter columns the critical load would be a function of \( E_t \), the tangent modulus. Fig. 5

In his theory it is assumed that the deflection of the column may be infinitely small. Thus \( E_t \) would be uniform along the length of the member and the column would bend in a sine curve.

The differential equation is

\[
\frac{d^2 y}{dx^2} + \frac{P}{E_t I} y = 0
\]

and thus

\[
P = \frac{\pi^2 E_t I}{L^2}
\]

The fallacy: The differential equation preceding Eq. 6 is based on an assumed stress and strain distribution shown in Fig. 6 that is incorrect. If there is constant axial load and strain reversal, Fig. 5 (Point B) shows that unloading strains are a function of \( E \) not \( E_t \) as assumed in the theory.

2. Reduced Modulus. In 1910 Von Kármán apparently substantiated by tests the Reduced or Double Modulus Theory (two different moduli govern buckling).
Assumptions: 1. Displacements are very small
2. Strain is proportional to distance from neutral axis
3. Independent action of fibres as governed by $\sigma - \varepsilon$ diagram
4. Bending plane is plane of symmetry
5. Static criterion for stability is used; assumes $P$ is constant during bending

When the average stress is greater than $\sigma_p$ (Fig. 5) and the column is straight, the strain distribution is shown in Fig. 6 by line A-B. Upon bending, (concave to left) strain distribution $E$-F is obtained.

Rate of increase of strain to left is proportional to $E_1$; rate of decrease on right is proportional to $E$. Thus the stress distribution will appear as in Fig. 6.

Since it is assumed that $P$ is constant, equilibrium of normal force and bending moment requires that

\[ \int_0^{h_1} S_1 dA = \int_0^{h_2} S_2 dA \quad (7) \]

\[ \int_0^{h_1} S_1 z_1 dA + \int_0^{h_2} S_2 z_2 dA = P y \quad (8) \]

Making use of the assumptions above, the relationships of Figs. 7 and 8, and

\[ \phi = \frac{1}{\beta} = -\frac{d^2 y}{dx^2}, \]

there is obtained

(7a)

where $Q$ is the statical moment, and

\[ -\frac{d^2 y}{dx^2} \left[ \frac{E}{S} \int_0^{h_1} z_1^2 dA + \frac{E_1}{S_1} \int_0^{h_1} z_2^2 dA \right] = P y \quad (9a) \]
Introducing the reduced modulus \( E_I = E_1 + E_1 I^2 \),

\[
\frac{\partial^2 y}{\partial x^2} + \frac{P}{E_I} y = 0 \quad \text{(9)}
\]

Since the deflection \( y \) is assumed as infinitely small, \( E \) may be assumed as constant along the length. Then as before

\[
y = C \sin \left( \sqrt{\frac{P_1}{E_1 I}} \right) x ,
\]

valid only when \( \sqrt{\frac{P_1}{E_1 I}} = \frac{D}{L} \).

Thus

\[
P_r = \frac{\pi^2 E_I}{L^2} \quad \text{(10)}
\]

The difficulty: The results of carefully conducted column tests invariably showed strengths less than \( P_r \). (Investigators attributed this to errors in test technique, unavoidable eccentricities).

C. Tangent modulus (Shanley)

It was not until 1946 that Shanley pointed out the fallacy of reasoning leading to the Reduced Modulus: The column cannot remain straight up to load \( P_r \) and then bend at that load. There exist stable deflected configurations at loads lower than \( P_r \), and the lowest point at which the equilibrium position "bifurcates" is \( P_{1,2} \). Shanley's principal conclusions:

1. Bending curvatures at the tangent modulus load with an increase in load,
2. The maximum load lies between \( P_t \) and \( P_r \), the latter as the upper limit.

Von Kármán's acknowledgement of Shanley's contribution is of value here (see Ref. No. 2, page 19).

Possible stable positions are shown diagramatically in Fig. 9. The case is somewhat analogous to the "exact" elastic solution which resulted in Fig. C.

Solution: Shanley assumed the idealized bent column of Fig. 10 with plastic action only at the mid-height.

\[ \text{Fig. 9} \]
He showed that transition to bent position was accompanied by an increase in load (necessary for stability and for the validity of Eq. 6 for $P_t$) by the following formula for a load $P$ greater than $P_t$,

$$P = P_t \left[ 1 + \frac{1}{\frac{b}{2d} + \frac{1 + \tau}{1 - \tau}} \right]$$  \hspace{1cm} (11)

where $\tau = E_t/E_r$. If deflection $d = 0$, $P = P_t$; if a deflection exists, $P > P_t$. Thus diagramatic Fig. 9 is verified.

Knowing that $P > P_t$ when bending starts, strain reversal does not immediately occur on the convex side (right). Instead, strain increases in each fibre according to the tangent modulus; said in a different way, $E_t$ "controls" over the entire cross-section at the instant bending commences. Fig. 11 shows to expanded scale the strain distribution under increasing load. Since $E_t$ is the same for each fibre and is constant for the length of the member, the solution is similar to the elastic solution and

$$P_t = \frac{\pi^2 E_t I}{L^2}$$  \hspace{1cm} (6)

As shown in Fig. 11 by $A_2 - B_2$, eventually strain reversal occurs and thus $P_t$ is approached.

The increase of load from $P_t$ to $P_r$ is slight (Fig. 9 exaggerates it). Thus it is reasonable to use $P_t$ as the critical load end to base column design upon it.

4. Comparison with test results: \hspace{1cm} (Figs. 12 - 13*)

a. Excellent agreement in the elastic range.
b. The reduced modulus predicts higher loads than Tangent Modulus.
c. Test results usually are closer to tangent modulus value, although slightly above it.

* Von Karman tests
FIG. 12 - COLUMN CURVES FOR ALUMINUM ALLOY

FIG. 13 - COLUMN CURVES FOR STEEL BAR.
IV. DETERMINATION OF COLLU H CURVE FROM STRESS-STRAIN DIAGRAM

a. Plot stress-strain diagram to suitable scale. (Fig. 14 as an example)
b. Obtain Euler curve directly from Eq. 5
\[ \sigma = \frac{\pi^2 E}{(L/r)^2} \] (Fig. 16, B-C)
c. Obtain \( E_t \) at various stresses above \( \sigma_p \) (Fig. 14, D)
d. Plot Fig. 15 (not essential but desirable)
e. With a given \( \sigma \), obtain \( E_t \) from Fig. 15 and solve Eq. 13 which is obtained from Eq. 6.
\[ \sigma_t = \frac{\pi^2 E_t}{(L/r)^2} \] (12)
\[ L/r = \pi \frac{E_t}{\sigma_t} \] (13)
f. Plot \( \sigma_t \) vs. \( L/r \) in Fig. 16 (A-B)

Generalization: For a boundary condition other than pin-ends replace \( L \) (column length), by \( KL \) (equivalent length) in Fig. 13. Thus, for fixed or restrained ends:
\[ \sigma_t = \frac{\pi^2 E_t}{(KL/r)^2} \] (12a)
The strain-hardening range of Structural Steel

Stress-strain relationship for structural steel is shown in Fig. 17. Strain hardening commences at $\epsilon_s \geq 15 \epsilon_y$, the slope $E_{ts}$ being about $E/60$. Now, in ideally plastic steel ($E_{ts} = 0$), the column curve would be as shown by A-B-C-D, Fig. 18. The line A-C represents in this case the upper limit of column strength, since the slightest eccentricity of loading causes instability.

Theories are available and are confirmed by experiments to show that for sufficiently short members column strength is represented by line B0, Fig. 18. This is a plot of Eq. 12, where $E_t$ is the tangent modulus in the strain-hardening range. $L/r$ at point B may be determined from Eq. 13

$$L/r = \pi \left( \frac{E_{ts}}{\sigma_y} \right)$$

On a theoretical basis, the complete column curve for structural steel is O-B-C-D, Fig. 18.

V. INFLUENCE OF RESIDUAL STRESS

Except for the strain hardening effect (which covers an impractical range of slenderness ratio), it would first appear that Tangent Modulus Theory is of little value in the case of structural steels.

Since residual stresses due to cooling after rolling are present in all structural steel shapes, it develops that Tangent Modulus Theory is of direct application.

1. Form of Residual Stress: For WF shape: Fig. 19. In particular the flange edges are in compression. An average value for flange-tip stress is 12,000 psi.

2. The Solution

Yielding will start when

$$\sigma_p = \sigma_y - \sigma_{rc}$$
When the load is increased above this pseudo-proportional limit, \( \sigma_p \), portions of the cross-section yield as shown in Fig. 20. If the yielded parts are perfectly plastic, their bending stiffness is reduced to zero. The buckling strength will then be equivalent to that of a new column, whose moment of inertia is \( I_e \), the moment of inertia of the elastic part,

\[
\sigma_{cr} = \frac{\pi^2 E I_e}{(I_e/r)^2} \quad (14)
\]

Dividing both sides by \( A \),

\[
\sigma_{cr} = \frac{\pi^2 E I_e}{(I_e/r)^2} \quad (14)
\]

Now \( \frac{E_t}{E} = \frac{d\sigma}{d\varepsilon} = E \frac{A_e}{A} \). But

in a rectangular element bent about its weak axis (2-2 in Fig. 21) \( I_{e}/I = A_e/A \). For flexure about the 1-1 axis,

\[
I_{e}/I = \left( \frac{A_e}{A} \right)^3 \quad (15)
\]

Then, since \( \frac{A_e}{A} = \frac{E_t}{E} = \tau \)

\[
\sigma_{cr} = \frac{\pi^2 E \tau}{(I_e/r)^2} \quad (axis \ x-x) \quad (15)
\]

\[
\sigma_{cr} = \frac{\pi^2 E \tau^3}{(I_e/r)^2} \quad (axis \ y-y) \quad (16)
\]
AVERAGE COLUMN CURVES AND COLUMN TEST RESULTS

Fig. 22
These equations are approximately true for WF shapes (reflecting the web) and show that \( E_t \) is of direct application. Column curves are shown in Fig. 22 (attached), test results also being shown. \( E_t \) may also be obtained by compressing a full cross-section containing residual stresses. The resulting \( \sigma-\epsilon \) diagram for structural steel WF shapes is thus similar to Fig. 23.

VI. WORKING FORMULAS AND FACTOR OF SAFETY

If the effect of residual stress is included, a fair representation of the column curve in the inelastic range is given by a straight line joining \( \sigma_y \) and \( \sigma_r \) (Fig. 24). If \( \sigma_y \) is assured at 12 ksi, \( \sigma_r = 23 \) ksi for steel with a yield point of 33 ksi. Then \( L/r \) at \( \sigma_y \geq 12 \).

For WF shapes, then (assuming further research substantiates present results)

\[
L/r > 120 : \quad \sigma_{cr} = \frac{E}{(L/r)^2}
\]

\[
L/r < 120 : \quad \sigma_{cr} = 33,000 - 100 L/r
\]

The factor of safety is arrived at by analysis of the factors that influence column strength (i.e., material properties, end restraint, crookedness, eccentric loading). Bleich suggests a uniform factor of safety over the entire range of slenderness ratio but does not suggest a value, other than that a more cautious viewpoint is needed than in tension members.

The factor of safety implied in the AISC column formula,

\[
\sigma_w = 17,000 - 0.15 \times (L/r)^2
\]

when compared to the column curve is a minimum at about \( L/r = 80 \) (1.8) and a maximum at \( L/r = 120 \) (2.1).
**APPENDIX A**

**Terminology -- Columns (Tentative)**

**Classical Definitions**

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td><strong>Bifurcation of the equilibrium position</strong></td>
<td>The phenomenon by which there are neighboring positions of equilibrium to the straight configuration.</td>
</tr>
<tr>
<td><strong>D buckling Load</strong></td>
<td>The point at which bifurcation of the equilibrium position becomes possible.</td>
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</table>

**For Engineering Applications**

<table>
<thead>
<tr>
<th>Term</th>
<th>Formula/Description</th>
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<tbody>
<tr>
<td>Euler Load</td>
<td>$P_e = \frac{n^2 \pi^2 E I}{L^2}$</td>
</tr>
<tr>
<td>Tangent Modulus Load</td>
<td>$P_t = \frac{n^2 \pi^2 E_t I}{L^2}$</td>
</tr>
<tr>
<td>Reduced Modulus Load</td>
<td>$P_r = \frac{n^2 \pi^2 E_r I}{L^2}$</td>
</tr>
<tr>
<td>Critical Load</td>
<td>$P_{cr}$ -- A limit of structural usefulness. It must be defined in each application.</td>
</tr>
<tr>
<td>Recommended: The load which a column will carry without too much deflection.</td>
<td></td>
</tr>
<tr>
<td>Maximum Load</td>
<td>$P_{max}$ -- Maximum load a column will carry.</td>
</tr>
<tr>
<td>Statistical Criterion of Stability</td>
<td>$P_y = \sigma_y \cdot A$ -- A column has neutral (indifferent) stability if, infinitely near the straight equilibrium position, there exist other equilibrium configurations under the same load. From this, the Euler load is the load required to keep an originally straight member in the bent configuration.</td>
</tr>
</tbody>
</table>

Note: This term should not be confused with $P_{max}$. 

$P = \frac{V}{A}$
REFERENCES


5. Thurliein, "The Problem of Structural Stability" (first seminar) 1953

6. Shuck, "The Mathematics of Elastic Buckling" (second seminar) 1953