PRESTRESSED CONCRETE BRIDGE MEMBERS

PROBABLE FATIGUE LIFE

OF

PLAIN CONCRETE WITH STRESS GRADIENT

by

F. S. Ople, Jr.
Research Associate in Civil Engineering

C. L. Hulsbos
Research Professor of Civil Engineering

Fritz Engineering Laboratory

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SYNOPSIS

The work described in this paper is part of a research investigation into the fatigue life of prestressed concrete flexural members where crushing of the concrete compression block precedes the fracture of the tension steel reinforcement. The results of constant load cycle tests conducted on plain concrete specimens to study the effect of compressive stress gradient on fatigue life are presented and discussed.

Application of the results of the study for estimating beam fatigue life as limited by fatigue failure of the concrete in compression is briefly discussed. An approximate design check against the possibility of concrete failure in beams subjected to repeated flexural loads is formulated for a specified fatigue life \( N = 2,000,000 \) cycles and a probability "design limit" \( P \leq 0.00001 \).
INTRODUCTION

Previous attempts at developing methods for estimating beam fatigue life as limited by concrete failure have neglected the effect of compressive stress gradient on the fatigue life of plain concrete. An approximate method has been proposed\(^{(1,2,3,4)}\) for predicting whether or not a failure will occur in the concrete for a given range of loading when repeated for one million cycles. That procedure is based on the fatigue failure envelope of axially loaded plain concrete specimens together with the stress-moment relation for the top fiber at the critical section of the beam. However, in using the failure envelope to describe the fatigue properties of the concrete in the beam, several factors are not considered, which are: the effect of stress gradient, the scatter of fatigue test results, and the effect of "size". A means for obtaining a lower bound estimate of beam fatigue life has been described\(^{(5)}\) which accounts for the effects of scatter and size but neglects the presence of the compressive stress gradient in the beam. In this solution, the problem was reduced to a case of determining the fatigue life of a concrete element subjected to a uniform compressive stress equal to the extreme fiber stress in the beam. The necessary concrete fatigue data can be represented by an S-N-P (stress-fatigue life-probability of failure) relationship which can be obtained from tests of plain concrete specimens with cross sectional area equal to the area of the concrete stress block in the beam.

In the research\(^{(6)}\) described in this paper, the effect of compressive stress gradient on the fatigue life of plain concrete was investigated. Static and fatigue tests were conducted on 157 small plain concrete prismatic specimens. The main test variables considered were stress gradient and maximum stress level. The following variables were held constant: concrete composition, specimen size, frequency of cyclic loading, and minimum stress level. All fatigue tests were conducted with constant load cycles.
In the description of the concrete specimen tests and analysis of results which follow maximum and minimum stress levels are stated for convenience as percentages of the static ultimate stress.

**TEST SPECIMENS**

The test specimens were 4- by 6- by 12-in. concrete prisms which were manufactured in batches as listed in Table 1. At least twelve 6- by 12-in. cylinders were prepared with each batch of specimens.

The concrete used in the manufacture of specimens was made from 3/4-in. maximum size crushed limestone, fine Lehigh river sand, and Type I ordinary Portland cement. Grading curves of the aggregates are shown in Fig. 1. The fineness modulus of sand was 3.03. The specific gravities of the fine and coarse aggregates were 2.57 and 2.77, respectively.

The batch mix which yielded 6.5 cu. ft. of concrete was held constant with the following proportions: cement - 120 lb., fine aggregate - 402 lb., and coarse aggregate - 402 lb. The water added varied from 70 to 80 lb. per batch in order to maintain a slump of approximately 2-in.

The concrete prisms were vibrated; the cylinders were rodded. The specimens were stripped of the forms after 3 days and cured in a moist room until 28 days. After curing, the prisms and cylinders were capped with carbo-vitrobond and stored at room temperature until tested.

**TEST PROCEDURE**

The experimental program was divided into static and fatigue tests. Static tests were conducted on prisms and cylinders to determine the stress-strain and ultimate strength properties of the concrete. Constant load cycle tests were conducted on prisms to obtain stress-fatigue life data for three types of strain distribution.
The following nomenclature for the test groups was adopted:
Numerals 1 and 2 referred to the static and fatigue tests, respectively. Lower case letters a, b, and c referred to the type of strain distribution to which the prisms were subjected, thus:

Groups 1a, 2a - Tests with uniform strain distribution  
(e = 0)

Groups 1b, 2b - Tests with zero to maximum strain distribution (e = 1")

Groups 1c, 2c - Tests with one-half to maximum strain distribution (e = 1/3")

The specimens from each batch were randomly assigned to the different test groups shown in Table 1. One batch of specimens was tested within approximately 15 days.

**Static Tests**

Static ultimate tests on cylinders and prisms were conducted in a 300-kip Baldwin Universal Testing Machine. Deformation measurements on cylinders were made using a mechanical compressometer with dial readings of 0.0001-in. Six-inch SR-4 gages were used to measure the strains on the prism. Deformation measurements were recorded at equal increments of load without stopping the loading process. Total time of testing a cylinder or a prism was approximately five minutes.

In general the static tests were carried to failure. However, because of the limited number of specimens per batch, static tests of Group 1c were conducted to approximately 90 percent of the static ultimate stress, and immediately after, the same specimens were subjected to repeated loads (Group 2c).

**Fatigue Tests**

Repeated load tests were conducted in the fatigue test setup shown in Fig. 2. Cyclic loads at 500 cpm were applied by an Amsler
pulsator connected to a 110-kip hydraulic jack. End fixtures consisting of thick plates pivoting on cylindrical pins located at each end of the concrete prism allowed jack loads to be applied at different eccentricities along the 6-in. dimension. Fig. 2(b) shows a close-up of a specimen under load applied at an eccentricity of one inch. The maximum and minimum loads were applied and maintained without interruption until failure or 2,000,000 cycles, whichever occurred first.

Tests were replicated at discrete maximum stress levels which varied for each group as follows: Group 2a - 65 to 80, Group 2b - 85 to 95, and Group 2c - 77.5 to 87.5. The minimum stress level for all tests was 10 percent. The specimens were assigned and tested at the different stress levels in a random manner.

The maximum and minimum load levels used in the fatigue tests of prisms were determined from the load-stress-strain results of the static tests. The stresses corresponding to the loads were therefore referred to the initial stress condition of the concrete specimens. The method of establishing the loads is illustrated on Fig. 3 which shows the load-stress-strain curves of specimens from Batch E. The loads that will induce a stress 80 percent of the ultimate stress are given by $P_a = 124.2$ kips, $P_b = 65$ kips, and $P_c = 95.5$ kips, for Groups 1a, 1b, and 1c, respectively. Thus, for a stress level $S_{\text{max}} = 80$, the maximum loads required for fatigue tests of Batch E specimens corresponding to Groups 2a, 2b, and 2c, respectively, are given by the preceding values. The loading portion of the stress-strain relation shown on Fig. 3 is an average curve of Groups 1a and 1b test results. The stress-strain curve for Group 1a ($e = 0$) is obtained by direct conversion of load to stress ($f_c = P/A_c$). The stress-strain curve for Group 1b ($e = 1''$) is calculated using a numerical differentiation procedure.

### STATIC TEST RESULTS

**Concrete Stress-Strain Properties**

Typical load-strain curves obtained from the static tests
are shown in Fig. 3. The load-strain curves for Group lb exhibited zero strains at the neutral face of the prism up to approximately 0.90 $f'_c$, after which tensile strains were observed. The magnitude of the tensile strain increased rapidly as the load approached the failure value. The observed maximum tensile and compressive strains at failure load are contained in Table 2.

The mechanical properties of the concrete are listed in Table 3. The static ultimate stress of prisms was found to be consistently higher than the cylinder ultimate stress, the average prism stress being 6 percent larger. The range varied from a minimum of 1 percent to a maximum of 11 percent.

The loading portion of the stress-strain curve was approximated by a cubic parabola of the form

$$F = \alpha E + (3 - 2\alpha)E^2 + (\alpha - 2)E^3$$

(1)

where the non-dimensional terms $F$, $E$, and $\alpha$ are defined as follows

$$F = \frac{f'_c}{f'_c}, \quad E = \frac{\varepsilon'_c}{\varepsilon'_c}, \quad \text{and} \quad \alpha = E = \frac{\varepsilon'_c}{f'_c}$$

(2)

where $f'_c$ is the ultimate compressive stress, $\varepsilon'_c$ is the strain at ultimate stress, and $E_c$ is the modulus of elasticity. Equation 1 fitted the test results reasonably well with values of $\alpha$ of 2.20 and 1.85 for cylinders and prisms, respectively. The curves corresponding to these values of $\alpha$ are compared with average test results in Fig. 4.

The complete stress-strain curve can be obtained from the load-strain data of Group lb ($e = 1''$) by applying a numerical differentiation method of calculating the stress-strain relationship from flexure tests similar to that used by Hognestad, et al. (7) The stress-strain curve from tests of eccentrically loaded specimens compared very well with that of axially loaded specimens in the loading portion of the curve. Because of the flatness of the load-strain curve and the rapid increase of tensile strains, the unloading portion of the curve as
calculated by numerical differentiation was not too reliable. However, the computed results gave an approximate picture of the shape of the unloading portion of the stress-strain curve as indicated in Fig. 3.

**Analysis of Variance**

An analysis of variance \(^{(8)}\) was performed on the cylinder test results of batches A to H with 9 cylinder tests per batch to investigate the sources of variation of concrete strength. An estimated overall variance of 0.1368 was obtained. It was found that 71 percent of the overall variance was contributed by the batch-to-batch differences and only 6 percent from specimen-to-specimen (within batch) differences. The remainder was attributed to the residual. The large batch-to-batch variation was not critical since the applied fatigue loads were referred to the mean prism stress of each batch which was not affected by the variation between batches. The overall mean cylinder ultimate stress was 5.59 ksi.

**FATIGUE TEST RESULTS**

Fatigue test results are contained in Tables 4, 5, and 6, and shown graphically in Fig. 5. The results are summarized in Table 7.

The results obtained from the constant load cycle tests on small concrete specimens show quite vividly the non-reproducible aspect of fatigue testing. The scatter of test results tend to increase in magnitude with decreasing maximum stress level for the same type of stress distribution. In addition, the scatter of test points falling within the same range of N are approximately of the same order of magnitude in different stress distributions as indicated by the computed standard deviations contained in Table 7.

The effect of stress gradient on the fatigue strength of plain concrete is exhibited in Fig. 5. A significant difference in fatigue strength exists between uniformly stressed specimens and non-uniformly stressed specimens, the fatigue strength of the latter being higher. The fatigue strength of Groups 2a and 2b test results differ
by approximately 15 to 18 percent over a range of fatigue life of 40,000 to 1,000,000 cycles. The lower fatigue strength of uniformly stressed specimens forms the basis for using such data as a lower bound estimate of the fatigue life of flexural members as limited by concrete fatigue in compression.\(^{(1,5)}\) Thus, substantial improvement of the lower bound method can be accomplished by taking the effect of stress gradient into account.

Two important observations are easily noted with respect to the trend of the mean S-N curves, which have been approximated by straight lines, in Fig. 5:

(1) The mean S-N curves are approximately parallel to one another.

(2) The slopes of the curves are quite "flat".

The first observation suggests a possible existence of an empirical relationship between stress \(S\), fatigue life \(N\), and stress gradient \(\theta\). The second observation points out the importance of determining the stress level as accurately as possible because a small change in the value of the stress reflects a large change in fatigue life. A change in stress of only 7.5 and 5 percent for Groups 2a and 2b, respectively, is required to change the fatigue life from approximately 40,000 to 1,000,000 cycles.

Typical specimen failures are shown in Fig. 6. Prior to complete crushing of the specimen, cracks were observed on the surface of the prism. This was particularly evident for tests with low
maximum stress levels where the time interval between initiation of cracking and final failure was of long duration. In axially loaded specimens, cracking initiated at any one of the four vertical faces of the prism. A typical final failure mode is shown in Fig. 6(a). In eccentrically loaded specimens, cracks initiated at the highest strained surface and progressed toward the center of the specimen. At failure, a tensile crack appeared at the neutral surface, followed by spalling of a wedge-shaped section of the prism as illustrated in Fig. 6(b).

**S-N-P Relationship**

The statistical nature of fatigue is now generally recognized so much so that the prevalent practice of presenting fatigue data as a simple S-N (stress level-number of cycles) relationship is being supplanted by a more adequate representation in three dimensional form, S-N-P, where P is the probability of failure at a number of cycles equal to or less than N. The frequency of test results can be established by ranking the specimens in the order of cycles to failure and calculating the probability of failure P of each specimen by

\[
P_r = \frac{r}{n + 1}
\]

where \( r \) is the rank of the specimen and \( n \) is the total number of specimens tested at a particular stress level. The range of frequencies of test data listed in Tables 4, 5, and 6 is 0 \( \leq P \leq 1.0 \).

McCall used a mathematical model to describe the S-N-P relationship of fatigue data of plain concrete tested in reversed bending. The following equation was proposed:

\[
L = 10 - aR^b (\log N)^c
\]

with properties

\[ N = 1 \text{ for } L = 1 \]

\[ N \to \infty \text{ for } L \to 0 \]
and

\[ R = 0 \text{ for } L = 1 \]

\[ R \approx 1 \text{ for } L > 0 \]

where \( a, b, \) and \( c \) are experimental constants, \( R \) is the stress level expressed as a ratio of static ultimate stress, \( N \) is the fatigue life, and \( L \) is the probability of survival at or before \( N \) cycles, \( L = 1 - P \). The use of \( L \) instead of \( P \) simplifies the form of Eq. 4.

Equation 4 can be linearized by taking the logarithms of the logarithms of both sides of the equation, rearranging, and reducing it to the form

\[ Z = A + BX + CY \]  

(5)

where \( X = \log S, Y = \log(-\log L), Z = \log(\log N), \) and constants \( A, B, \) and \( C \). Instead of the stress ratio \( R \), the maximum stress level \( S \) in percent of the static ultimate stress was used in Eq. 5.

The experimental constants in Eq. 5 were evaluated by a regression analysis of the data for each test group. The following relationships were obtained:

Group 2a: \((1.8293 \equiv \log S \equiv 1.8751)\)

\[ \log(\log N) = 4.9092 - 2.2470(\log S) + 0.0538 \log(-\log L) \]  

(6a)

Group 2b: \((1.9294 \equiv \log S \equiv 1.9542)\)

\[ \log(\log N) = 9.3083 - 4.4076(\log S) + 0.0435 \log(-\log L) \]  

(6b)

Group 2c: \((1.9031 \equiv \log S \equiv 1.9294)\)

\[ \log(\log N) = 6.5413 - 3.0365(\log S) + 0.0467 \log(-\log L) \]  

(6c)

The corresponding equations expressed in the form of Eq. 4 are:

Group 2a: \((67.5 \equiv S \equiv 75)\)

\[ L = 10^{-4.97 \times 10^{-92} S^{41.80} (\log N)^{18.60}} \]  

(7a)
Group 2b: \(85 \leq S \leq 90\)

\[ L = 10^{-1.11 \times 10^{-214} S^{101.31} (\log N)^{22.98}} \]  
(7b)

Group 2c: \(80 \leq S \leq 85\)

\[ L = 10^{-8.93 \times 10^{-141} S^{65.01} (\log N)^{21.41}} \]  
(7c)

Test results and equations are compared in Fig. 7. A measure of the degree of association among the variables \(S\), \(N\), and \(L\) was obtained by calculating the multiple correlation coefficient of each test group. The correlation coefficients are 95.5, 99.1, and 97.3 for Groups 2a, 2b, and 2c, respectively. Thus, the S-N-P relationships of plain concrete subjected to different types of compressive stress distribution can be adequately represented by a mathematical equation of the form given in Eq. 4.

Further investigation of the S-N-P relationships of the test data was made assuming the frequency distribution of fatigue life as represented by the log-normal and extreme value probability functions. The results of the analysis are included in the Appendix.

**SIZE AND STRESS GRADIENT EFFECTS**

**Size Effect**

A statistical explanation of size effect was proposed by Weibull\(^{(10)}\) who verified his theory with the results of rotating-beam endurance tests of specimens with two different effective lengths. The distribution functions of two sizes of specimens which are geometrically similar and tested under the same levels and distribution of stress, are related by means of the following equation

\[ P_1 = 1 - (1 - P_0)^{v_1/v_0} \]  
(8)
where $P_o$ is the probability of failure of a specimen with volume $v_o$ and $P_1$ is the probability of failure of a specimen with volume $v_1$. From Eq. 8 it is seen that the probability of failure at or before $N$ cycles increases with size and that, for equal $P$, a lower fatigue life is associated with increased specimen size. Thus, it is only necessary to know the distribution function, say $P_o$, of a specimen with volume $v_o$ in order to obtain the distribution function of a different size specimen.

### Stress Gradient Effect

Fowler\(^{(14)}\) proposed a statistical approach to the stress gradient problem which Stulen\(^{(15)}\) described in his study of the endurance limit of steel specimens with three different effective volumes and subjected to non-uniform alternating stresses. In treating the problem of stress gradient, the specimen is thought of as consisting of small elementary volumes, for which the stresses may be determined. From Eq. 8, once the cumulative frequency distribution of uniformly stressed specimens of a given volume is known, the frequency distribution of the elementary volumes may be calculated. The probability of failure of the whole specimen subjected to non-uniform stress distribution is found by taking the products of the corresponding probabilities of the elementary volumes.

The effect of stress gradient observed in the test results follows on the basis of the statistical theory since a non-uniformly stressed specimen has different elementary volumes stressed at levels less than the maximum stress, thus sustaining longer fatigue life compared to a uniformly stressed specimen. However, it was found\(^{(6)}\) that Fowler's statistical theory did not apply to the concrete fatigue data because the tests were conducted at constant load instead of constant stress cycles. The loads were maintained throughout the fatigue test of a specimen while allowing the stresses to change with load repetition. The effect of the difference in rate of change of stress between uniformly stressed specimens is not included in the statistical theory of stress gradient.
S-N-P-Θ Relationship

The effect of compressive stress gradient on the fatigue life of plain concrete prismatic specimens can be accounted for by means of an empirical relationship between the variables S, N, P, and Θ. Since the stress varies monotonically in one direction only, an expression for the stress gradient Θ, defined as the slope of the stress function at the point of maximum stress, can be derived knowing the stress-strain equation

\[ F_x = \alpha E_x^2 + (3-2\alpha) E_x^3 + (\alpha-2) E_x^3 \]  

(1)

and the linear strain relationship

\[ E_x = \frac{E_{\text{max}}}{t} \]  

(9)

where \( F_x \) and \( E_x \) are the non-dimensional stress and strain at \( x \), respectively, and \( E_{\text{max}} \) is the non-dimensional maximum strain corresponding to the maximum stress at \( t \). The distance \( x \) and \( t \) are measured from the point of zero strain or stress. (Reference can be made to Fig. 9(b).)

The above equations can be combined to give

\[ F_x = \alpha \left( \frac{E_{\text{max}}}{t} \right)^2 + (3-2\alpha) \left( \frac{E_{\text{max}}}{t} \right)^3 + (\alpha-2) \left( \frac{E_{\text{max}}}{t} \right)^3 \]  

(10)

Differentiating \( F_x \) with respect to \( x \) and putting \( x = t \)

\[ \Theta = \left( \frac{dF_x}{dx} \right)_{x=t} = \frac{E_{\text{max}}}{t} \left[ \alpha + 2(3-2\alpha)E_{\text{max}}^2 + 3(\alpha-2)E_{\text{max}}^2 \right] \]  

(11)

Note that the expression inside the brackets can be obtained from Eq. 1 by taking \( \frac{dF_x}{dE_x} \) and letting \( E_x = E_{\text{max}} \). Thus, the stress gradient \( \Theta \) can be expressed as

\[ \Theta = \frac{E_{\text{max}}}{t} \left( \frac{dF}{dE}_{\text{max}} \right) \]  

(12)
The value of $Q$ vanishes for each of the following conditions:

1. $E_{\text{max}} = 0$
2. $\left(\frac{dP}{dE}\right)_{E_{\text{max}}} = 0$
3. $t = \infty$

The first condition is a trivial case when no strain is applied. The second condition occurs when $E_{\text{max}}$ is equal to the strain at ultimate stress. A specimen subjected to a stress gradient and tested at a maximum stress level equal to the ultimate stress will fail at a very low number of cycles which is below the range of fatigue life considered in this investigation. The third condition corresponds to a case of uniform stress distribution where $t$ may be taken as infinite in magnitude.

The stress gradient $Q$ was calculated for specified values of $F$ and $t$ using the stress-strain relation in Eq. 1 with $\alpha = 1.85$ ($\alpha$-value for prisms). The results are plotted as $Q$ versus $S_{\text{max}}$ in Fig. 8 corresponding to the three types of stress distribution used in this investigation. Knowing the $S$-$N$-$P$ relationships for the different stress distributions, as given by Eqs. 6 (a, b, c), it is possible to superimpose on Fig. 8 curves of equal fatigue life $N$ for a specified probability level $P$. In effect, Fig. 8 is a three-dimensional representation of $S$-$N$-$Q$ for a specified value of $P = 0.50$. Since other similar families of $N$-curves can be drawn for probability values other than $P = 0.50$, an $S$-$N$-$P$-$Q$ relationship is obtained. As a first approximation therefore, the $S$-$N$-$P$-$Q$ relationship of concrete prismatic specimens subjected to stress varying monotonically in one direction only may be determined by means of this graphical approach.

The empirical approach to the stress gradient effect may be generalized to apply to plain concrete prismatic specimens by using the following approximation to account for size effect. A change in size
(depth) in the direction of the stress variation is already accounted for in the stress gradient expression. For instance, an increase in $t$ decreases the fatigue life $N$. A change in size in the other two directions (width and length) may be corrected for by using the statistical theory of size effect. Equation 8 is reduced to the following form

$$P_1 = 1 - (1 - P_0)^{u \cdot w}$$

where $u$ and $w$ are the width and length ratios, respectively, of the two different sizes of specimens.

**APPLICATIONS OF FATIGUE DATA**

**Estimation of Beam Fatigue Life**

A method for estimating the probable life of beams as limited by concrete fatigue in compression has been proposed\(^6\) based on the concrete fatigue data presented in this paper. The concrete top fiber stresses induced by repeated loads on the beam can be determined from stress-moment curves which can be calculated by a theoretical analysis.\(^5\) The stress gradient of the compression block can be evaluated by using Eq. 12 where $t$ may be approximated by the depth $k_d$ of the compression block.

For prestressed concrete beams, however, the use of $k_d$ in Eq. 12 gives conservative estimates of fatigue life. A modified depth less than $k_d$ should be used to account for the difference in the minimum stress distribution in the concrete test specimen and in the compression block due to prestressing in the beam. This is illustrated in Fig. 9 where the solid and dash lines indicate the stress distributions under maximum and minimum load levels, respectively; hence, the cross-hatched regions show the range of stress variation under repeated loading. For the same maximum stress, $f_{c', \text{max}}^t = (f_{c', \text{max}}^c)$, and $t = k_d$, the region subjected to stress variation in the beam is less than that of the concrete specimen.
The difference in size between the concrete beam section and the test specimen can be accounted for by using the following equation

\[ Q = 1 - (1 - P)^u \]  

(14)

where \( Q \) is the probability of beam failure at or before \( N \) cycles, \( P \) is the corresponding probability of specimen failure, and \( u \) is the ratio of the width of the beam to the width of the test specimen.

An estimate of the beam fatigue life can then be made having obtained values for the stress level \( (S) \), stress gradient \( (\theta) \), and probability level \( (P) \). The beam fatigue life \( (N) \) is obtained graphically by using an \( S-N-P-\theta \) relationship such as shown in Fig. 8. A detailed numerical example for estimating beam fatigue life of an actual test beam is included in Ref. 6.

**Approximate Design Check**

An approximate design check against the possibility of concrete fatigue failure of beams subjected to repeated flexural loading can be formulated from the fatigue data. A relationship between the maximum concrete top fiber stress and the depth \( k_d \) of the compression block in the beam can be established for specified values of fatigue life \( N = 2,000,000 \) cycles and probability levels \( P = 0.00001 \) and \( P = 0.01 \), such as shown in Fig. 10. The curves were obtained by extrapolation of the \( S-N-P-\theta \) relation and by expressing \( \theta \) in terms of \( k_d \) using Eq. 12.

In Fig. 10, it is seen that the fatigue strength of concrete in compression varies inversely with the depth \( k_d \) and becomes a minimum for \( k_d = \infty \). The minimum values are 51 and 59 percent corresponding to the curves with \( P = 0.00001 \) and \( P = 0.01 \), respectively, and in effect, represent the fatigue strength of uniformly stressed specimens \((k_d = \infty)\) for the \( N \) and \( P \) values.

For design purposes, a probability level as close to zero as possible must be specified to insure against fatigue failure. Furthermore, for probability levels approaching zero, size effect becomes
Freudenthal has quoted a "design limit" $P = 0.01$. In order to account for other uncertainties such as extrapolation of fatigue data, a probability level $P \leq 0.00001$ is assumed. Hence, for a specified "design limit" $P \leq 0.00001$, the "safe" region in Fig. 10 is the area below the solid curve. Thus, the probability of concrete fatigue failure occurring at or before $N = 2,000,000$ cycles associated with a point plotting within this region is equal to or less than 1 in 100,000.

The fatigue strength versus compressive depth $kd$ relationship presented on Fig. 10 was established for a minimum extreme fiber stress of 10 percent of the ultimate compressive stress. It is known that the fatigue strength of concrete increases with increasing minimum stress level, thus the design check will give conservative results when applied to cases where the minimum top fiber stress is greater than 10 percent. For prestressed concrete beams, the minimum stress condition at the top fibers would be taken as the stress caused by the combination of dead load and prestressing which is usually not less than 10 percent.

The calculation of the concrete top fiber stress and depth $kd$ can be accomplished by using any conventional stress analysis procedure. Thus the information presented in Fig. 10 may be incorporated into current design practice quite readily.

In Fig. 10 the maximum top fiber stress is expressed as percent of the ultimate compressive stress of the concrete in the beam, $k_3 f'_c$. In order to account for any difference in concrete strength between the beam and the test specimen, $k_3 = 0.85$ is assumed. The current AASHO allowable concrete stress in compression of $0.40 f'_c$ corresponds to $0.47 k_3 f'_c$ which is less than the minimum fatigue strength ($kd = \infty$) of $0.51 k_3 f'_c$. Indeed, according to Fig. 10, for realistic values of the compressive depth $kd$, say $kd \leq 15$ in., the fatigue strength is greater than $0.60 k_3 f'_c$ which would permit a stress at the top fibers of $0.51 f'_c$. 

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CONCLUSIONS

On the basis of the work described in this paper, the following conclusions may be drawn:

(1) Stress gradient has a significant effect on the fatigue strength of plain concrete in compression. The fatigue strength of non-uniformly stressed specimens of Group 2b is higher than that of uniformly stressed specimens of Group 2a by approximately 17 percent of the static ultimate stress.

(2) Concrete fatigue life is highly sensitive to small changes in maximum stress levels. A change in stress of only 7.5 and 5 percent for Groups 2a and 2b, respectively, causes the fatigue life to change from approximately 40,000 to 1,000,000 cycles.

(3) Statistical treatment of test data is necessary due to the large degree of variability associated with plain concrete fatigue life.

(4) An approximate empirical relationship between the variables — stress level, fatigue life, probability of failure, and stress gradient — can be established from the fatigue data and may be generalized to apply to specimens of different sizes by using a statistical explanation of size effect.

(5) The results of the plain concrete specimen tests can be used to develop a method for estimating beam fatigue life as limited by concrete fatigue in compression. The solution takes into particular consideration the effect of compressive stress gradient on fatigue life of concrete.

(6) An approximate design check against the possibility of concrete fatigue failure in beams subjected to repeated flexural loading can be formulated for specified values of fatigue life and probability "design limit".
(7) The current AASHO allowable concrete compressive stress of $0.40f'_c$ is a conservative estimate of the fatigue strength of concrete in compression. The results of this investigation indicate that for a fatigue life $N = 2,000,000$ cycles, a probability of failure $P = 0.00001$, and a minimum compressive top fiber stress of $0.10f'_c$, a maximum compressive stress of $0.50f'_c$ may be permitted at the top fibers of pre-stressed flexural members subjected to repeated loading.

ACKNOWLEDGEMENTS

The work described in this paper was carried out in the Department of Civil Engineering at the Fritz Engineering Laboratory, under the auspices of the Institute of Research of Lehigh University, as part of an investigation currently sponsored by: Pennsylvania Department of Highways; U. S. Department of Commerce, Bureau of Public Roads; and Reinforced Concrete Research Council.

The authors wish to express their thanks to Mr. Wilfred F. Chen for his assistance during the experimental phase of the project.

APPENDIX

Further Analysis of S-N-P Relationships

Recent studies have been made to investigate the frequency distributions that may be associated with the phenomenon of fatigue failure. The logarithmic-normal distribution has been found by several investigators (5, 9) to approximate the distribution of fatigue test results satisfactorily. Another type of distribution proposed (10, 11, 12) is based on the statistical theory of extreme values. The logarithmic-normal and extreme value distributions both fit fatigue test data satisfactorily in the vicinity of the central value, $P = 0.50$; however, in the limiting values of $P$, such as $P \to 1.0$ and $P \to 0$, the extreme value distribution may represent the fatigue data better.
Logarithmic-Normal Distribution

The S-N-P relationship of fatigue data based on an assumed log-normal distribution is expressed in the form of the cumulative distribution function

\[ P(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]  

where \( X = \log N \). The parameters \( \mu \) and \( \sigma \) are estimated by the mean \( \log N \) and standard deviation \( D_{\log N} \), respectively, of the \( \log N \) values for a specified stress level \( S \).

The "goodness-of-fit" of the log-normal distribution is graphically demonstrated in Fig. 11, where test results plotted in logarithmic-normal probability paper, are randomly distributed about a straight line for each stress level. In addition, the lines are approximately parallel within the range of stress levels shown in Fig. 11.

Equations giving the relationship between \( S \) and \( \log N \), and likewise, between \( S \) and \( D_{\log N} \), were obtained by the least squares method:

Group 2a: \( (67.5 \leq S \leq 75) \)

\[ \log N = 18.2001 - 0.1814 S \]  

\[ D_{\log N} = 1.0770 - 0.0115 S \]  

Group 2b: \( (85 \leq S \leq 90) \)

\[ \log N = 28.4768 - 0.2647 S \]  

\[ D_{\log N} = 1.6841 - 0.0165 S \]
Group 2c: \(77.5 \leq S \leq 85\)

\[
\log N = 20.1920 - 0.1844 S
\]

No equation for \(D_{\log N}\) was obtained for Group 2c because only two values of standard deviation could be calculated from the data. The equations for mean fatigue life are plotted on Fig. 5. The S-N-P relationships of the fatigue data obtained in this investigation can therefore be approximated by Eqs. 1, 2, and 3.

**Extreme Value Distribution**

Studies by Freudenthal and Gumbel\(^{(11,12)}\) and Weibull\(^{(10)}\) have shown that the S-N-P relationship of metals can be represented by the extreme value distribution. An asymptotic probability function of the type

\[
L(N) = e^{-\left(\frac{N-N_0}{\beta} \right) \left(\frac{V}{S-N_0} \right)}
\]

with properties

\[
L(V_s) = 1/e \quad \text{and} \quad L(N_0) = 1
\]

where \(L(N) = 1 - P(N)\), has been proposed. If \(N_0\) is assumed equal to zero, Eq. 4 reduces to

\[
L(N) = e^{-\left(\frac{N}{V_s} \right)^{\beta}}
\]

with boundary conditions \(L(0) = 1\) and \(L(\infty) = 0\). The parameters \(V_s\), \(\beta\), and \(N_0\) are defined as follows: \(V_s\) or "characteristic number", is the mode of log \(N\) values, \(1/\beta\) is the "geometric standard deviation", and \(N_0\), or "minimum life", is the lower limit of \(N\) below which
failure will not occur at a given stress amplitude.

Equation 5 is the probability function used in the so-called "linear theory" \(^{(11)}\) because it plots as a straight line on extremal probability paper. Equation 4, used in the "general theory" \(^{(12)}\), plots as a curve and becomes asymptotic to the limiting value \(N_0\). Freudenthal and Gumbel \(^{(12)}\) showed that fatigue data of nickel tested in reversed bending agreed well with the linear theory at high stress levels and with the general theory at low stress levels.

In the analysis of the plain concrete fatigue data only the linear theory was used. Equation 5 was expressed in linear form

\[
Y = a + bX
\]  

where \(Y = \log(-\log L)\), \(X = \log N\) and the parameters \(a\) and \(b\) were determined from the experimental data by the least squares method. The following equations were obtained:

**Group 2a:**

\[
\begin{align*}
S &= 75.0: \quad \log(-\log L) = -10.8079 + 2.2153 (\log N) \\
S &= 72.5: \quad \log(-\log L) = -8.0592 + 1.4822 (\log N) \quad (7a) \\
S &= 70.0: \quad \log(-\log L) = -8.6731 + 1.5096 (\log N) \\
S &= 67.5: \quad \log(-\log L) = -8.4486 + 1.3180 (\log N)
\end{align*}
\]

**Group 2b:**

\[
\begin{align*}
S &= 90.0: \quad \log(-\log L) = -9.0160 + 1.8033 (\log N) \\
S &= 87.5: \quad \log(-\log L) = -10.6945 + 1.9116 (\log N) \quad (7b) \\
S &= 85.0: \quad \log(-\log L) = -8.1422 + 1.2536 (\log N)
\end{align*}
\]

No equations were obtained for Group 2c because of the limited test replications available.
Equations and test results are plotted on extremal probability paper in Fig. 12. For each stress level, the test values are scattered randomly about the straight line. Except for $S = 75$ of Group 2a and $S = 85$ of Group 2b, the requirement that the functions should be parallel is fairly well satisfied. In addition, the linear theory is acceptable if the observed and estimated values of the geometric standard deviations are the same, at least within the errors of random sampling. A comparison between estimated and observed values is made in Table 8. Except for $S = 67.5$ of Group 2a and $S = 87.5$ and 90 of Group 2b, agreement is reasonably good.
Table 1. Distribution of Prisms into the Different Test Groups

<table>
<thead>
<tr>
<th>Batch</th>
<th>Static Tests</th>
<th>Fatigue Tests</th>
<th>Total Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1a</td>
<td>1b</td>
<td>1c</td>
</tr>
<tr>
<td>A</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>2</td>
<td>(2)</td>
</tr>
<tr>
<td>F</td>
<td>3</td>
<td>2</td>
<td>(2)</td>
</tr>
<tr>
<td>G</td>
<td>3</td>
<td>(2)</td>
<td>(2)</td>
</tr>
<tr>
<td>H</td>
<td>4</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>13</td>
<td>-</td>
</tr>
</tbody>
</table>

Notes: ( ) - Specimens not loaded to failure and were later tested in fatigue.

Groups 1a, 2a: e = 0
Groups 1b, 2b: e = 1"
Groups 1c, 2c: e = 1/3"
Table 2. Compressive and Tensile Strains at Failure - Group 1b (e = 1"")

<table>
<thead>
<tr>
<th>Batch</th>
<th>Load (kips)</th>
<th>Strains x10^{-6} in/in</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Maximum $\varepsilon_c$</td>
<td>Maximum $\varepsilon_t$</td>
</tr>
<tr>
<td>A</td>
<td>87.8</td>
<td>2850</td>
<td>380</td>
</tr>
<tr>
<td></td>
<td>89.5</td>
<td>3330</td>
<td>380*</td>
</tr>
<tr>
<td></td>
<td>92.5</td>
<td>2690</td>
<td>280</td>
</tr>
<tr>
<td>B</td>
<td>86.2</td>
<td>2350</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>88.3</td>
<td>2850</td>
<td>460</td>
</tr>
<tr>
<td>C</td>
<td>91.3</td>
<td>2840</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>92.0</td>
<td>3090</td>
<td>420</td>
</tr>
<tr>
<td>D</td>
<td>95.8</td>
<td>3110</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>98.8</td>
<td>2940</td>
<td>--</td>
</tr>
<tr>
<td>E</td>
<td>100.5</td>
<td>2500</td>
<td>200*</td>
</tr>
<tr>
<td></td>
<td>109.2</td>
<td>3200</td>
<td>300*</td>
</tr>
<tr>
<td>F</td>
<td>(104.0)+</td>
<td>(2590)</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>(105.0)+</td>
<td>(2800)</td>
<td>--</td>
</tr>
<tr>
<td>G</td>
<td>(90.0)+</td>
<td>(2240)</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>(93.5)+</td>
<td>(2200)</td>
<td>--</td>
</tr>
<tr>
<td>H</td>
<td>(93.0)+</td>
<td>(2260)</td>
<td>--</td>
</tr>
</tbody>
</table>

* Extrapolated  ( )+ - Less than failure load
Table 3. Mechanical Properties of Concrete

<table>
<thead>
<tr>
<th>Batch</th>
<th>Cylinders</th>
<th>Prisms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f'_c$ (ksi)</td>
<td>$\varepsilon'_c$ (in/in)</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>A</td>
<td>4.84</td>
<td>5.49</td>
</tr>
<tr>
<td>B</td>
<td>4.62</td>
<td>5.26</td>
</tr>
<tr>
<td>C</td>
<td>4.55</td>
<td>5.44</td>
</tr>
<tr>
<td>D</td>
<td>4.84</td>
<td>5.74</td>
</tr>
<tr>
<td>E</td>
<td>4.91</td>
<td>5.89</td>
</tr>
<tr>
<td>F</td>
<td>4.78</td>
<td>6.14</td>
</tr>
<tr>
<td>G</td>
<td>4.52</td>
<td>5.41</td>
</tr>
<tr>
<td>H</td>
<td>4.50</td>
<td>5.47</td>
</tr>
<tr>
<td>Ave.</td>
<td>4.70</td>
<td>5.59</td>
</tr>
</tbody>
</table>

1. Average of 3 cylinder tests
2. Average of 9 cylinder tests
3. Range of age of prisms from commencement to completion of fatigue tests per batch
4. Average of Group 1a and 1b tests
Table 4. Results of Group 2a (e = 0)

<table>
<thead>
<tr>
<th>r</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 75</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 72.5</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 70</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 67.5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Pr</td>
<td>N</td>
<td>Pr</td>
</tr>
<tr>
<td>1</td>
<td>17</td>
<td>0.091</td>
<td>39</td>
<td>0.111</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.182</td>
<td>60</td>
<td>0.222</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>0.273</td>
<td>107</td>
<td>0.333</td>
</tr>
<tr>
<td>4</td>
<td>39</td>
<td>0.364</td>
<td>110</td>
<td>0.445</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>0.455</td>
<td>130</td>
<td>0.556</td>
</tr>
<tr>
<td>6</td>
<td>47</td>
<td>0.545</td>
<td>136</td>
<td>0.667</td>
</tr>
<tr>
<td>7</td>
<td>53</td>
<td>0.637</td>
<td>192</td>
<td>0.778</td>
</tr>
<tr>
<td>8</td>
<td>59</td>
<td>0.728</td>
<td>275</td>
<td>0.889</td>
</tr>
<tr>
<td>9</td>
<td>65</td>
<td>0.819</td>
<td></td>
<td></td>
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<tr>
<td>10</td>
<td>70</td>
<td>0.910</td>
<td></td>
<td></td>
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<tr>
<td>11</td>
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<td>0.786</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td>492</td>
<td>0.858</td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td>2000+</td>
<td>0.929</td>
</tr>
</tbody>
</table>

n = 10 8 13 12

Table 5. Results of Group 2b (e = 1")

<table>
<thead>
<tr>
<th>r</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 90</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 87.5</th>
<th>S&lt;sub&gt;max&lt;/sub&gt; = 85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Pr</td>
<td>N</td>
</tr>
<tr>
<td>1</td>
<td>28</td>
<td>0.111</td>
<td>81</td>
</tr>
<tr>
<td>2</td>
<td>31</td>
<td>0.222</td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td>35</td>
<td>0.333</td>
<td>131</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>0.445</td>
<td>141</td>
</tr>
<tr>
<td>5</td>
<td>46</td>
<td>0.556</td>
<td>156</td>
</tr>
<tr>
<td>6</td>
<td>58</td>
<td>0.667</td>
<td>180</td>
</tr>
<tr>
<td>7</td>
<td>61</td>
<td>0.778</td>
<td>190</td>
</tr>
<tr>
<td>8</td>
<td>129</td>
<td>0.889</td>
<td>226</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td>242</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td>317</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td>351</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td>527</td>
</tr>
</tbody>
</table>

n = 8 12 9

Notes: S<sub>min</sub> = 10  P<sub>r</sub> = \frac{r}{n+1}  N - Number of cycles in thousands  + - No failure.  * - Statically loaded prior to fatigue loading.
Table 6. Results of Group 2c (e = 1/3")

<table>
<thead>
<tr>
<th>r</th>
<th>N</th>
<th>P_r</th>
<th>N</th>
<th>P_r</th>
<th>N</th>
<th>P_r</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>0.143</td>
<td>108</td>
<td>0.111</td>
<td>*464</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>*26</td>
<td>0.286</td>
<td>206</td>
<td>0.222</td>
<td>888</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>35</td>
<td>0.429</td>
<td>*224</td>
<td>0.333</td>
<td>*941</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>37</td>
<td>0.571</td>
<td>249</td>
<td>0.445</td>
<td>1198</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>*46</td>
<td>0.712</td>
<td>270</td>
<td>0.556</td>
<td>2000+</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>*65</td>
<td>0.859</td>
<td>364</td>
<td>0.667</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>*542</td>
<td>0.778</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>*2000+</td>
<td>0.889</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes:  
S_{min} = 10  
P_r = \frac{r}{n + 1}  
N - Number of cycles in thousands  
+ - No failure  
* - Statically loaded prior to fatigue loading
Table 7. Summary of Test Results

<table>
<thead>
<tr>
<th>Test Group</th>
<th>$S_{\text{max}}$ (% $f_C$)</th>
<th>No. of Spec.</th>
<th>Fatigue Life</th>
<th>Log Fatigue Life</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{N}$</td>
<td>$D_N$</td>
</tr>
<tr>
<td>2a</td>
<td>80.0</td>
<td>3</td>
<td>16</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>10</td>
<td>45</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>72.5</td>
<td>8</td>
<td>131</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>70.0</td>
<td>12*</td>
<td>249</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td>67.5</td>
<td>9*</td>
<td>890</td>
<td>460</td>
</tr>
<tr>
<td></td>
<td>65.0</td>
<td>5</td>
<td>2000+</td>
<td>--</td>
</tr>
<tr>
<td>2b</td>
<td>95.0</td>
<td>3</td>
<td>16</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>92.5</td>
<td>3</td>
<td>35</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>90.0</td>
<td>8</td>
<td>54</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>87.5</td>
<td>12</td>
<td>222</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>85.0</td>
<td>8*</td>
<td>1167</td>
<td>825</td>
</tr>
<tr>
<td>2c</td>
<td>85.0</td>
<td>6</td>
<td>38</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>80.0</td>
<td>7*</td>
<td>280</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>77.5</td>
<td>4*</td>
<td>873</td>
<td>--</td>
</tr>
</tbody>
</table>

Notes: $S_{\text{min}} = 10$

- $\bar{N}$ - Mean fatigue life x $10^3$ cycles
- $\log N$ - Mean of log $N$
- $D_N$ - Standard deviation of $N$ x $10^3$ cycles
- $D_{\log N}$ - Standard deviation of log $N$
- * - Run-outs discarded
- + - No failure
Table 8. Comparison of Geometric Standard Deviation
(Linear Theory)

<table>
<thead>
<tr>
<th>Test Group</th>
<th>$S_{\text{max}}$ (% $f'_{C}$)</th>
<th>$V_s$</th>
<th>$\frac{D_N}{V_S}$</th>
<th>$D_{\log N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Estimated</td>
</tr>
<tr>
<td>2a</td>
<td>75.0</td>
<td>52</td>
<td>0.3017</td>
<td>0.1704</td>
</tr>
<tr>
<td></td>
<td>72.5</td>
<td>156</td>
<td>0.4782</td>
<td>0.2882</td>
</tr>
<tr>
<td></td>
<td>70.0</td>
<td>320</td>
<td>0.4250</td>
<td>0.2531</td>
</tr>
<tr>
<td></td>
<td>67.5</td>
<td>1366</td>
<td>0.3367</td>
<td>0.1939</td>
</tr>
<tr>
<td>2b</td>
<td>90.0</td>
<td>63</td>
<td>0.5159</td>
<td>0.3123</td>
</tr>
<tr>
<td></td>
<td>87.5</td>
<td>254</td>
<td>0.4921</td>
<td>0.2972</td>
</tr>
<tr>
<td></td>
<td>85.0</td>
<td>1608</td>
<td>0.5130</td>
<td>0.3105</td>
</tr>
</tbody>
</table>

Note: Estimated $D_{\log N}$ from Refs. 11 and 12.
Fig. 1  Grading curves for aggregates

(a) Overall view

(b) Close-up view

Fig. 2  Fatigue test setup
Fig. 3  Load-stress-strain curves (Batch E)
Fig. 4  Concrete stress-strain curves

Fig. 5  Maximum stress vs. fatigue life
Fig. 6 Typical fatigue failures

(a) Uniformly stressed specimen

(b) Non-uniformly stressed specimen
Fig. 7a  S-N-P diagram (Group 2a)

Fig. 7b  S-N-P diagram (Group 2b)
Z = 6.5413 - 3.0365X + 0.0467Y

where X = \log S

L = 1 - P

Fig. 7c S-N-P diagram (Group 2c)

Fig. 8 S-N-\theta diagram for P = 0.50
(a) Prestressed Concrete Beam

(b) Concrete Test Specimen

Fig. 9 Stress and strain distributions
N = 2.0 x 10⁶ Cycles

Minimum Top Fiber Stress = 10%
P - Probability of Failure

Fig. 10 Fatigue strength vs. compressive depth kd
Fig. 11  S-N-P diagrams - Logarithmic-normal
Fig. 12  S-N-P diagrams - Extreme value (linear theory)
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