 USERS' MANUAL FOR CSTES  
  FINITE ELEMENT PROGRAM  

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1. INTRODUCTION

CSTES program is for the stress and deformation analysis of two dimensional elasticity problems by Finite Element Techniques. This approach discretizes the planar continuum by using macro-components, "Finite Elements". In the formulation "Constant Stress Triangle" (CST) type finite elements are used (Ref. 1). Program is based on E. L. Wilson's study on the finite element analysis of planar continuum by using CSTs (Ref. 2).

CSTES program can be used for the linear elastic analysis of two dimensional continuum subjected to gravity, body, thermal, distributed or concentrated forces and their combinations. Planar problems can be treated as plane stress or plane strain cases depending upon the input information provided by the user.
2. ANALYTIC FORMULATION

2.1 Displacement Field

Basic assumptions of finite element formulation requires the definition of displacement field. In CST type formulation displacement field is defined by

\[ u(x,y) = C_1 + C_2 x + C_3 y \]
\[ v(x,y) = C_4 + C_5 x + C_6 y \]

in \( x \) and \( y \) directions respectively. This assumption also presupposes the stress and strain variation within the element:

\[ \epsilon_x = \frac{\partial u}{\partial x} = C_2 \]
\[ \epsilon_y = \frac{\partial v}{\partial y} = C_6 \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_3 + C_5 \]

Thus the strains are constant throughout the element. Consequently the stresses within an element are also constant.

Furthermore, the assumed displacement field results in the continuity between the elements after the deformation; that is initially straight lines remain straight in their displaced position.
Stepwise change of stresses and strains in the continuum from element to element imposes a restriction for the types of problems that can be analyzed with given method. Utilization of CST approach for problems with high stress or strain gradients yields poor results. This is in part due to the approximation in the use of stepwise functions, discrete representation, for non-linear curves, stress gradients. Application of CST type formulation, such as CSTES, to high stress gradient problems should not be used if accurate assessment of stress and deformation variations are required.

2.2 Basic Formulation

Linkage between the finite elements is made at the vertices of the triangles which are the nodal points. If the reactions developed between the elements are shown by $F$, with components in $x$ and $y$ directions, and the corresponding nodal point displacements are shown by $\delta$ with similar components, then the following canonical equilibrium equation can be written:

$$ F = K \delta $$

Here $K$ is the global stiffness matrix of the problem under study.
2.3 Fundamental Matrices

In CST formulation there are three fundamental matrices.

(1) $B$, relates elemental strains to nodal point displacements

(2) $D$, relates elemental stresses to elemental strains, known as the Elasticity Matrix

(3) $k^e$, relates nodal point reactions to nodal point displacements, known as the Element Stiffness Matrix

Derivation of these fundamental matrices can be found in many sources (Ref. 1 and 3). However, for user's convenience, the general form of these matrices are listed.

(1) $B$:

$$
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = B \delta^e
$$

$$
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix}
b_i & 0 & b_j & 0 & b_k & 0 \\
0 & c_i & 0 & c_j & 0 & c_k \\
c_i & b_i & c_j & b_j & c_k & b_k
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i \\
u_j \\
v_j \\
u_k \\
v_k
\end{bmatrix}
$$
where
\[ b_i = y_j - y_k \quad \text{and} \quad c_i = x_k - x_j \]
\[ b_j = y_k - y_i \quad \text{and} \quad c_j = x_i - x_k \]
\[ b_k = y_i - y_j \quad \text{and} \quad c_k = x_j - x_i \]
\[ 2\Delta = c_k b_j - c_j b_k \]

(2) Elasticity Matrix \( \mathbf{D} \):
\[
\sigma = \mathbf{D} \varepsilon
\]
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

This relation holds for the plane stress formulation of isotropic materials.

(3) Element Stiffness Matrix, \( \mathbf{k}^e \):
\[
\mathbf{k}^e = \int \int \mathbf{B}^T \mathbf{D} \mathbf{B} \, dx \, dy = \mathbf{B}^T \mathbf{D} \mathbf{B} \, \Delta
\]
\[
\mathbf{F}^e = \mathbf{k}^e \mathbf{\delta}^e
\]
\[
\begin{bmatrix}
F_{i1}^e \\
F_{i2}^e \\
F_{i3}^e
\end{bmatrix} = \begin{bmatrix}
k_{ii}^e & k_{ij}^e & k_{ik}^e \\
k_{ji}^e & k_{jj}^e & k_{jk}^e \\
k_{ki}^e & k_{kj}^e & k_{kk}^e
\end{bmatrix} \begin{bmatrix}
\delta_{i1}^e \\
\delta_{i2}^e \\
\delta_{i3}^e
\end{bmatrix}
\]
where \( k_{rs}^e \) are 2 x 2 submatrices of the following form.

\[
\begin{bmatrix}
F_{xr}^e \\
F_{yr}^e
\end{bmatrix} =
\begin{bmatrix}
k_{xrxs}^e & k_{xyrs}^e \\
k_{yrxs}^e & k_{yyrs}^e
\end{bmatrix}
\begin{bmatrix}
\delta_{xs}^e \\
\delta_{ys}^e
\end{bmatrix}
\]

Because of their simplicity the \( B \) and \( D \) matrices are directly generated in the computer program while the element stiffness matrix \( K^e \) is calculated numerically.

2.4 Global Stiffness Matrix

One of the major problems in a Finite Element program is the assemblage of the global stiffness matrix \( K \) of the structure.

\[
F = K \delta
\]

\[
\begin{bmatrix}
F_1 \\
F_r \\
F_s \\
F_n
\end{bmatrix} =
\begin{bmatrix}
k_{11} & \cdots & k_{1r} & \cdots & k_{1s} & \cdots & k_{1n} \\
k_{r1} & \cdots & k_{rr} & \cdots & k_{rs} & \cdots & k_{rn} \\
k_{s1} & \cdots & k_{sr} & \cdots & k_{ss} & \cdots & k_{sn} \\
k_{n1} & \cdots & k_{nr} & \cdots & k_{ns} & \cdots & k_{nn}
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_r \\
\delta_s \\
\delta_n
\end{bmatrix}
\]

\( k_{rs} \) are 2 x 2 submatrices which can be obtained by the following summation:
\[ k_{rs} = \sum_{e=1}^{m} k_{rs}^e \]

\( n \) = total number of nodal points
\( m \) = total number of elements

Obviously \( k_{rs} \) is only then different from zero, if \( r = s \) or \( r \) and \( s \) are adjacent nodal points. The restriction of the CSTES program that a nodal point may not have more than 8 adjacent nodal points thus is equivalent to the statement that every row of the global stiffness matrix does not have more than 8 non-zero off-diagonal terms.

The following arrays are used for the assemblage and storage of the \( K \) matrix:

- NST (350,9) "Bookkeeping" matrix, containing Col. 1: the nodal point number
  Col. 2-9: The number of adjacent nodal points
- FXX (350,10) arrays containing the elements of the \((2 \times 2)\) submatrices
- FXY (350,10) Col. 1: Elements of the main diagonal submatrices \( k_{rr} \)
  Col. 2-9: Elements of the non-zero off-diagonal submatrices \( k_{rs} \)
Column 10 contains the elements of the inverted main diagonal submatrices \( f_{rr} \) = nodal point flexibility matrix) which are required for the iteration procedure.

In the assemblage procedure the proper storage location in the "bookkeeping" matrix for every nodal point of every element is determined by means of indirect addressing. Subsequently the elements of the element stiffness matrix are added to the global stiffness matrix.

It should be noted that an over-relaxation approach (for solving the system of simultaneous equations) is used in this program. The storage of the rather large global stiffness matrix can therefore be done in a very efficient way and problems concerning the "band width" of this matrix will not come into the picture.

2.5 Solution Procedure

Solution of the canonical equilibrium equation system can be solved by using Gauss-Seidel type iterative procedure. Equilibrium equations can be re-written for the ease of this iterative approach:

\[
\delta_n^{(j+1)} = k_{nn}^{-1} \left[ F_n - \sum_{i=1}^{n-1} k_{ni} \delta_i^{(j+1)} - \sum_{i=n+1}^{N} k_{ni} \delta_i^{(j)} \right]
\]
Superscript \(-j\) denotes the iteration counter. Since the stiffness matrix \(K\) is positive definite, the iterative approach will always converge. By introducing the over-relaxation factor \(-\beta\), the expression can be modified,

\[
\delta_n^{(j+1)} = \delta_n^{(j)} + \beta k^{-1} n \left[ F_n - \sum_{i=1}^{n-1} \delta_i^{(j+1)} \right] - \sum_{i=n}^{N} K_{ni} \delta_i^{(j)}
\]

Iteration can be terminated

(1) If specified maximum number of iterations are reached, or

(2) if the iteration scheme converges within specified tolerance limit.

Convergence of the problem is checked by summing the absolute values of unbalanced forces at each iteration cycle and comparing the sums with the sum from the first cycle. The formula used for each cycle is:

\[
c(j) = \sum_{i=1}^{N} \left[ \left| f_{xi} \Delta \delta_{xi}^{(j)} \right| + \left| f_{yi} \Delta \delta_{yi}^{(j)} \right| \right]
\]

Expressions in absolute value operators corresponds to the unbalanced forces in \(x\) and \(y\) directions due to the displacement
increments $\Delta \delta_x^i$ and $\Delta \delta_y^i$. Here superscript - (j) is the iteration counter, (N) number of nodal points and subscript -(i) denotes the nodal points.

2.6 Extension of CSTES

The CSTES program is flexible enough to be modified for different applications, such as non-linear materials. Possible non-standard methodologies are listed in References 3, 4, and 5.
3. **PROGRAM DESCRIPTION**

3.1 **Major Program Blocks**

To provide a better understanding of the program and also to explain the functions of program blocks for future modifications, if required, the breakdown of the program is included.

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<thead>
<tr>
<th>Card</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>9-15</td>
<td>dimension statements</td>
</tr>
<tr>
<td>18</td>
<td>over-relaxation factor (DATA statement)</td>
</tr>
<tr>
<td>23</td>
<td>input of control variables</td>
</tr>
<tr>
<td>24-31</td>
<td>print-out of control variables</td>
</tr>
<tr>
<td>32-33</td>
<td>input of element data</td>
</tr>
<tr>
<td>34-35</td>
<td>input of nodal point data</td>
</tr>
<tr>
<td>36-38</td>
<td>print-out of element data</td>
</tr>
<tr>
<td>39-41</td>
<td>print-out of nodal point data</td>
</tr>
<tr>
<td>42-48</td>
<td>initializing arrays for global stiffness and &quot;bookkeeping&quot; matrix</td>
</tr>
<tr>
<td>53-133</td>
<td>Do-loop (stiffness matrix)</td>
</tr>
<tr>
<td>54-56</td>
<td>indirect addressing</td>
</tr>
<tr>
<td>57-62</td>
<td>generating elements of B-matrix</td>
</tr>
<tr>
<td>63-65</td>
<td>calculating triangle area</td>
</tr>
<tr>
<td>66-68,75</td>
<td>calculating common factors</td>
</tr>
<tr>
<td>69-74</td>
<td>modification of nodal point loads</td>
</tr>
<tr>
<td>Card</td>
<td>Function</td>
</tr>
<tr>
<td>------</td>
<td>----------</td>
</tr>
<tr>
<td>76-81</td>
<td>initializing $B$, $D$ and $K^e$ matrices</td>
</tr>
<tr>
<td>85-90</td>
<td>generating $B$-matrix</td>
</tr>
<tr>
<td>91-93</td>
<td>generating $D$-matrix</td>
</tr>
<tr>
<td>94-109</td>
<td>calculating element stiffness matrix (matrix multiplication)</td>
</tr>
<tr>
<td>113-132</td>
<td>assembling element stiffness matrix in global stiffness matrix</td>
</tr>
<tr>
<td>137-142</td>
<td>calculating total number of adjacent nodal points for every point</td>
</tr>
<tr>
<td>143-149</td>
<td>calculating nodal point flexibility matrix</td>
</tr>
<tr>
<td>156-174</td>
<td>Do-loop (boundary conditions)</td>
</tr>
<tr>
<td>157</td>
<td>input of boundary conditions data</td>
</tr>
<tr>
<td>158</td>
<td>print-out of boundary conditions data</td>
</tr>
<tr>
<td>159-173</td>
<td>modification of nodal point flexibility</td>
</tr>
<tr>
<td>175-177</td>
<td>initializing iteration counters</td>
</tr>
<tr>
<td>181-198</td>
<td>iteration procedure</td>
</tr>
<tr>
<td>199-205</td>
<td>checking iteration counters</td>
</tr>
<tr>
<td>210</td>
<td>print-out of unbalanced force</td>
</tr>
<tr>
<td>211-212</td>
<td>print-out of nodal point displacements</td>
</tr>
<tr>
<td>214-246</td>
<td>Do-loop (element stresses)</td>
</tr>
<tr>
<td>215-217</td>
<td>indirect addressing</td>
</tr>
<tr>
<td>218-223</td>
<td>regenerating elements of $B$-matrix</td>
</tr>
<tr>
<td>224-233</td>
<td>calculating element stresses</td>
</tr>
<tr>
<td>234-244</td>
<td>calculating principal stresses and direction</td>
</tr>
</tbody>
</table>
Card | Function
---|---
245 | print-out of element stresses
251-293 | Do-loop (nodal point stresses)
252-253 | initializing variables
254-277 | averaging procedure
255-265 | determining the element numbers adjacent to every nodal point
266-276 | calculating weighting coefficients
278-280 | calculating nodal point stresses
281-291 | calculating principal stresses and direction
292 | print-out nodal point stresses
294-295 | checking iteration counters
296-298 | error messages
302-332 | output Format statements
333-336 | input Format statements

3.2 Input Information

For each problem the following data deck must be generated.

A. Control Card (614, 6X, E10.3)

<table>
<thead>
<tr>
<th>Cols.</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-4</td>
<td>Total number of elements</td>
</tr>
<tr>
<td>5-8</td>
<td>Total number of nodal points</td>
</tr>
<tr>
<td>9-12</td>
<td>Total number of restrained boundary points</td>
</tr>
<tr>
<td>Cols.</td>
<td>Input</td>
</tr>
<tr>
<td>------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>13-16</td>
<td>Cycle interval for the print of the force unbalance</td>
</tr>
<tr>
<td>17-20</td>
<td>Cycle interval for the print of the displacements and stresses</td>
</tr>
<tr>
<td>21-24</td>
<td>Maximum number of cycles</td>
</tr>
<tr>
<td>31-40</td>
<td>Convergence limit for unbalanced forces</td>
</tr>
</tbody>
</table>

B. Element Data - 1 Card per element (3I4, 8X, 5F10.0)

<table>
<thead>
<tr>
<th>Cols.</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-4</td>
<td>Nodal point number-i</td>
</tr>
<tr>
<td>5-8</td>
<td>Nodal point number-j</td>
</tr>
<tr>
<td>9-12</td>
<td>Nodal point number-k</td>
</tr>
<tr>
<td>21-30</td>
<td>Modulus of elasticity</td>
</tr>
<tr>
<td>31-40</td>
<td>Density of element</td>
</tr>
<tr>
<td>41-50</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>51-60</td>
<td>Coefficient of thermal expansion</td>
</tr>
<tr>
<td>61-70</td>
<td>Temperature change within the element</td>
</tr>
</tbody>
</table>

C. Nodal Point Data - 1 Card per nodal point (6F10.0)

<table>
<thead>
<tr>
<th>Cols.</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-10</td>
<td>X - Ordinate</td>
</tr>
<tr>
<td>11-20</td>
<td>Y - Ordinate</td>
</tr>
<tr>
<td>21-30</td>
<td>X - Load</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Cols.</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>31-40</td>
<td>Y - Load</td>
</tr>
<tr>
<td>41-50</td>
<td>X - Displacement (*)</td>
</tr>
<tr>
<td>51-60</td>
<td>Y - Displacement (*)</td>
</tr>
</tbody>
</table>

Note: (*) On free nodal points these are initial guesses, on restrained nodal points these are specified displacements.

D. Boundary Point Data - 1 Card per restrained boundary point (I4, 3X, Il, 2X, Fl5.0)

<table>
<thead>
<tr>
<th>Cols.</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-4</td>
<td>Number of restrained boundary points</td>
</tr>
<tr>
<td>8</td>
<td>0 if nodal point is fixed in both directions</td>
</tr>
<tr>
<td></td>
<td>1 if nodal point is fixed in x - direction</td>
</tr>
<tr>
<td></td>
<td>2 if nodal point is free to move along a line of slope S</td>
</tr>
<tr>
<td>11-25</td>
<td>Slope S (type 2 boundary point only)</td>
</tr>
</tbody>
</table>

Note: Cards of type B and C have to be placed in their natural sequence. To avoid errors the element or nodal point number should be punched in columns 71-80.
3.3 **Output Information**

The following information is printed by the program.

**A. Input Data**
1. Control variables
2. Element data
3. Nodal point data
4. Boundary point data

**B. Force unbalance**
(cycle interval INUNB)

**C. Displacements and Stresses**
(cycle interval INSTR)
1. Cycle number, force unbalance
2. Nodal point displacements
3. Element stresses
   (element numbers, $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $\sigma_1$, $\sigma_2$, direction of $\sigma_1$)
4. Nodal point stresses
   (nodal point numbers, $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $\sigma_1$, $\sigma_2$, direction of $\sigma_1$)

3.4 **Limitations and Assumptions**

The listed assumptions and remarks should always be considered in the preparation of input and in the interpretation of output.
1. Limitations
   (a) Maximum number of elements: 600
   (b) Maximum number of nodal points: 350
   (c) Maximum number of nodal points adjacent to a certain point: 8
   (d) Elements and nodal points have to be numbered in the natural sequence (Data cards have to be placed in the same order).

2. Assumptions
   (a) Over-relaxation factor: 1.84
   (b) Body forces: gravity is assumed to act in the negative direction of Y.

3. Remarks
   (a) Initial guesses for displacements
       For boundary condition type 2 (nodal point is forced to move along a line of slope S):
       Initial guesses of displacements perpendicular to the prescribed slope will be treated as imposed displacements.

   (b) Nodal point stresses
       The nodal point stresses are calculated as weighted average of stresses in all elements adjacent to a particular nodal point. The method used for this average procedure is described in Ref. 1. For nodal points at the interface of different types of materials
the nodal point stresses lose their meaning, since
the computations are based on stress rather than
strain values.

(c) Direction of principal stresses
The calculated value is the angle between the positive
x-axis and the direction of the maximum positive (or
minimum negative) principal stress. In case that both
values \((\sigma_x - \sigma_y)\) and \(\tau_{xy}\) are very small, this angle
becomes meaningless.

(d) Required field length
CM - 100 000
By using CM - 140 000 the program can handle up to 560
nodal points and 1120 elements.

(e) Plane strain problems
With the following modified values for \(E\), \(v\), and \(\alpha\) plane
strain problems can be transformed into plane stress
problems:

\[
E^* = E \frac{(1 - v)^2}{(1 - 2v)} = \frac{1}{(1 - v^*)^2}
\]

\[
v^* = \frac{v}{1 - v}
\]

\[
\alpha^* = (1 + v)\alpha
\]
3.5 Nomenclature

Nomenclature should be referred to in program modifications:

A. Arrays Related to Elements

- **NPI (600)** Nodal point numbers i, j, k
- **NPJ (600)** (1st level indirect addressing)
- **NPK (600)**
- **E (600)** modulus of elasticity
- **DE (600)**
  - (a) density
  - (b) common factor
- **P (600)** Poisson's ratio
- **AL (600)**
  - (a) coefficient of thermal expansion
  - (b) element stress in x direction
- **DT (600)**
  - (a) temperature change
  - (b) element stress in y direction
- **TXY (600)** Element shear stress

B. Arrays Related to Nodal Points

- **XORD (350)** X - coordinate
- **YORD (350)** Y - coordinate
- **XLOAD (350)** Nodal point load in x direction
- **YLOAD (350)** Nodal point load in y direction
- **DX (350)** Nodal point displacement in x direction
- **DY (350)** Nodal point displacement in y direction
- **FXX (350,10)** Elements of global stiffness matrix
- **FXY (350,10)** Col. 1: elements of main diagonal submatrices
FYX (350,10)  
Col. 2-9: elements of off-diagonal submatrices

FYY (350,10)  
Col. 10: elements of nodal point flexibility matrices

NST (350,9)  
"Bookkeeping" matrix

Col. 1: (a) number of nodal points  
(b) total number of adjacent nodal points

Col. 2-9: numbers of adjacent nodal points

C. Other Arrays

<table>
<thead>
<tr>
<th>Array</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B (6,6)</td>
<td>B - matrix</td>
</tr>
<tr>
<td>D (6,6)</td>
<td>D - matrix</td>
</tr>
<tr>
<td>S (6,6)</td>
<td>k_e - matrix</td>
</tr>
<tr>
<td>LM (3)</td>
<td>Scratch vector (2nd level indirect addressing)</td>
</tr>
</tbody>
</table>

D. Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>REL</td>
<td>Over-relaxation factor</td>
</tr>
<tr>
<td>NELEM</td>
<td>Total number of elements</td>
</tr>
<tr>
<td>NNOPO</td>
<td>Total number of nodal points</td>
</tr>
<tr>
<td>NREPO</td>
<td>Total number of restrained boundary points</td>
</tr>
<tr>
<td>NUNB</td>
<td>Cycle interval for the print-out of unbalanced force</td>
</tr>
<tr>
<td>INSTR</td>
<td>Cycle interval for the print-out of displacements and stresses</td>
</tr>
</tbody>
</table>
MAXCY  Maximum number of cycles
TOLER  Convergence limit for unbalanced force
BI, BJ, BK  Elements of B - matrix
CI, CJ, CK
AREA  Area (signed value)
ARE  Area (absolute value)
BODY, THE, FAC  Common factors
NBP  Number of restrained boundary points
NTYPE  Type of boundary condition
SLOPE  Slope of line along a restrained boundary point is allowed to move
NCYCLE  Iteration counter
IUB  Cycle number for the print-out of unbalanced force
IST  Cycle number for the print-out of displacements and stresses
DDX, DDY  Displacement (in x and y direction) during iteration cycle
SUM  Unbalanced force
EPX
EPY  Element strains
GAM
X, Y, XY  Stresses
XMAX, XMIN  Principal stresses
FI  Direction of maximum principal stress
SRX, SRY, R  Weighting coefficients for nodal point stress averaging procedure
4. REFERENCES

1. Zienkiewicz, O. C.

2. Wilson, E. L.
   "Finite Element Analysis of Two-Dimensional Structures", Ph.D. Dissertation, University of California, Berkeley, 1963


4. Felippa, C. A.
   "Refined Finite Element Analysis of Linear and Non-Linear Two-Dimensional Structures", SESM 66-22, University of California, Berkley, 1966

5. Kostem, C. N.
5. PROGRAM LISTING
CSTES 1

PROGRAM CST(INPUT,TAPE1=INPUT,OUTPUT,TAPE2=OUTPUT)

CSTES 2

*FINITE ELEMENT ANALYSIS OF PLANAR CONTINUUM*

CSTES 3

*THIS IS THE MODIFIED VERSION OF THE PROGRAM LISTED*

CSTES 4

*IN E.L. WILSON'S DISSERTATION, BERKELEY-1963*

CSTES 5

CSTES 6

CSTES 7

CSTES 8

DIMENSION NPI(600),NPJ(600),NPK(600),E(600),DE(600),P(600)

CSTES 9

DIMENSION AL(600),DT(600),TXY(600)

CSTES 10

DIMENSION XORD(350),YORD(350),XLOAD(350),YLOAD(350)

CSTES 11

DIMENSION DX(350),DY(350)

CSTES 12

DIMENSION FXX(350,10),FXY(350,10),FYX(350,10),FYY(350,10)

CSTES 13

DIMENSION NST(350,9)

CSTES 14

DIMENSION B(6,6),D(6,6),S(6,6),LM(3)

CSTES 15

IN=1

CSTES 16

I0=2

CSTES 17

DATA REL/1,84/

CSTES 18

INPUT DATA AND LISTING

CSTES 19

WRITE(IO,300)

CSTES 20

READ(IN,400)NELEM,NNPO,NREPO,INUNB,INSTR,MAXCY,TOLER

CSTES 21

WRITE(IO,301)NELEM

CSTES 22

WRITE(IO,302)NNPO

CSTES 23

WRITE(IO,303)NREPO

CSTES 24

WRITE(IO,304)INUNB

CSTES 25

WRITE(IO,305)INSTR

CSTES 26

WRITE(IO,306)MAXCY

CSTES 27

WRITE(IO,307)TOLER

CSTES 28

WRITE(IO,308)RE

CSTES 29

READ(IN,401)M,NPI(M),NPJ(M),NPK(M),E(M),DE(M),P(M),AL(M),DT(M),M=1CSTES 30

1,NELEM)

READ(IN,402)XORD(N),YORD(N),XLOAD(N),YLOAD(N),DX(N),DY(N),N=1;NCSTES 31

10PO)

WRITE(IO,369)

CSTES 32

WRITE(IO,310)M,NPI(M),NPJ(M),NPK(M),E(M),DE(M),P(M),AL(M),DT(M),M=1CSTES 33

1,NELEM)

WRITE(IO,311)

CSTES 34

WRITE(IO,312)N,XORD(N),YORD(N),XLOAD(N),YLOAD(N),DX(N),DY(N),N=1CSTES 35

1,NNPO)

DO 101 N=1,NNPO

CSTES 36

DO 100 MM=1,9

CSTES 37

FXX(N,MM)=FXY(N,MM)=FYX(N,MM)=FYY(N,MM)=0.0

CSTES 38

NST(N,MM)=0

CSTES 39

100 CONTINUE

CSTES 40

NST(N,1)=N

CSTES 41

101 CONTINUE

CSTES 42

CSTES 43

CSTES 44

CSTES 45

100 CONTINUE

CSTES 46

CSTES 47

CSTES 48

CSTES 49

CSTES 50

CSTES 51

CSTES 52

CSTES 53

CSTES 54

CSTES 55

CSTES 56

CSTES 57
BODY=AREA*DE(M)/3.0
DE(M)=E(M) * AL(V) * DT(M)/(1.0 - P(M))
THE=AREA*DE(M)/(2.0 * AREA)
XLOAD(I)=XLOAD(I) + THE * B(I)
XLOAD(J)=XLOAD(J) + THE * B(J)
XLOAD(K)=XLOAD(K) + THE * B(K)

DO 102 MA=1,6
DO 102 MB=1,6
B(MA,MB)=0.0
D(MA,MR)=0.0
S(MA,MB)=0.0

102 CONTINUE

* FORMATION OF ELEMENT STIFFNESS MATRICES *

B(1,1)=R(3,2)=EI
B(1,3)=R(3,4)=RJ
B(1,5)=R(3,6)=RK
B(2,2)=R(3,1)=CI
B(2,4)=R(3,3)=CJ
B(2,6)=R(3,5)=CK
D(1,1)=O(2,2)=FAC
D(2,1)=O(1,2)=FAC * P(M)
D(3,1)=FAC * (1.0 - P(M)) / 2.0

DO 103 J=1,6
DO 103 I=1,3
S(I,J)=0.0
DO 103 K=1,3
S(I,J)=S(I,J) + D(I,K) * B(K,J)

103 CONTINUE

DO 104 I=1,3
DO 104 J=1,6
D(I,J)=S(I,J)

104 CONTINUE

DO 105 J=1,6
DO 105 I=1,6
S(I,J)=0.0
DO 105 K=1,3
S(I,J)=S(I,J) + D(I,K) * B(K,J)

105 CONTINUE

* FORMATION OF MCDAL POINT STIFFNESS MATRICES *

LM(1)=NPJ(M)
LM(2)=NPJ(M)
LM(3)=NPK(M)
DO 108 L=1,3
DO 108 K=1,3
LX=LM(L)
MX=0

106 MX=MX+1
IF(MX.GE.10) GO TO 136
IF((NST(LX+MX)-LM(K)).EQ.0) GO TO 107
IF(NST(LX,MX).NE.0) GO TO 106

107 NST(LX,MX)=LM(K)
LB=2#K
LA=LB-1
KB=2#K
KA=KB-1
FXX(LX,MX)=FXX(LX,MX)+S(LA,KA)
FYX(LX,MX)=FYX(LX,MX)+S(LA,KB)
FYY(LX,MX)=FYY(LX,MX)+S(LB,KA)
FYX(LX,MX)=FYX(LX,MX)+S(LB,KB)

108 CONTINUE

INVERSION OF NODAL POINT STIFFNESS MATRICES, (FLEXIBILITY MATRIX)

DO 110 N=1,NNPO
NX=10
109 MX=NX-1
IF(NST(N,NX).EQ.0) GO TO 109
NST(N,1)=NX
110 CONTINUE

DO 111 N=1,NNPO
DET=FXX(N,1)#FYX(N,1)-FYX(N,1)#FXY(N,1)
FXX(N,10)=FYX(N,1)/DET
FYX(N,10)=FXY(N,1)/DET
FYY(N,10)=FXX(N,1)/DET
111 CONTINUE

MODIFICATION OF BOUNDARY POINT FLEXIBILITY MATRICES

ACCORDING TO THE PRESCRIBED BOUNDARY CONDITIONS

WRITE(10,313)
WRITE(10,314)
DO 116 L=1,NREPO
READ(IN*403) NBP,NTYPE,SLOPE
WRITE(10,325) NBP,NTYPE,SLOPE
IF(NTYPE.EQ.1) 114,113,112
112 DET=(FXX(NBP,10)#SLOPE-FXY(NBP,10))/(FXY(NBP,10)#SLOPE-FYY(NBP,10)
COF=1./DET#SLOPE
FXX(NBP,10)=(FXX(NBP,10)-DET#FYX(NBP,10))/COF
FYX(NBP,10)=(FYX(NBP,10)-DET#FXY(NBP,10))/COF
FYY(NBP,10)=FXX(NBP,10)#SLOPE
113 FYY(NBP,10)=FYX(NBP,10)#SLOPE-FXY(NBP,10)/FXX(NBP,10)
GO TO 116
114 FYY(NBP,10)=0.0
GO TO 115
115 FXX(NBP,10)=0.0
FXY(NBP,10)=0.0
FYY(NBP,10)=0.0
116 CONTINUE
  NCYCLE=NCYCLE+1
  IF(NCYCLE .GE. IUB) 121,122
  IUB=IUB+INUMB
121 WRITE(10,315) NCYCLE,SUM
122 IF(SUM .LE. TOLER) GO TO 123
  IF(NCYCLE .GE. MAXCY) GO TO 123
  IF(NCYCLE .LT. IST) GO TO 118

123 WRITE(10,316) NCYCLE,SUM
WRITE(10,317)
WRITE(10,319)
DO 127 M=1,NELEM
  J=NPI(M)
  K=NPJ(J)
  BT=YORD(J)-YORD(K)
  RJ=YORD(K)-YORD(J)
  HK=YORD(J)-YORD(K)
  CI=XORD(K)-XORD(J)
  CJ=XORD(J)-XORD(K)
  CK=XORD(K)-XORD(J)
  EP=BT*DX(J)+RJ*DX(K)
  EPY=CT*DY(J)+CJ*DY(K)
  GAMAG=CJ*DX(J)+CK*DX(K)+AI*DY(J)+BJ*DY(K)
  FAC=GT/(1.0-P(M)*P(Y))*(CK*BJ-CJ*SK))
  X=FAC*(EP*P(M)+EPY)-DE(W)
  Y=FAC*(EPY*P(M)+EPX)-DE(W)
  XY=FAC*GAMAG(1.0-P(M))/2.0
  AL(M)=X

CONTINUE
DT(M)=Y
TXY(M)=XY
SIG=(X+Y)/2.0
RIG=X-Y
R=SQRT(RIG*RIG/4.0+XY*XY)
XMAX=SIG+R
XMIN=SIG-R
T=SIGN(1.0,XY)
124 F1=T*45.0
GO TO 126
125 F1=2*R,.47890*ATAN(2.0*XY/RIG)
126 WRITE(10,320) X,Y,XY,XMAX,XMIN,F1
127 CONTINUE

NONAL POINT STRESSES
WRITE(10,321)
DO 134 N=1,NNOPO
X=Y=XY=0.0
SRX=SRY=R=0.0
DO 136 M=1,NELEM
I=NPI(M)
J=NPI(M)
K=NPK(M)
IF(N.EQ.1) GO TO 129
IF(N.EQ.J) GO TO 128
IF(N.EQ.K) GO TO 130
I=NPK(M)
GO TO 129
128 J=NPI(M)
129 AC=ABS(XORD(J)-XORD(I))*ABS(XORD(K)-XORD(I))
BC=ABS(YORD(I)-YORD(J))*ABS(YORD(K)-YORD(I))
AB=AC+BC
RX=AC/AB
SRX=SRX+RX
X=X+AL(M)*RX
RY=AC/AB
SRY=SRY+RY
Y=Y+DT(M)*RY
R=R+1.0
XY=XY+TXY(M)
130 CONTINUE
X=X/SRX
Y=Y/SRY
XY=XY/R
SIG=(X+Y)/2.0
RIG=X-Y
TR=SQRT(RIG*RIG/4.0+XY*XY)
XMAX=SIG+TR
XMIN=SIG-TR
T=SIGN(1.0,XY)
131 F1=T*45.0
GO TO 133
132 F1=28.647890*ATAN(2.0*XY/RIG)
   IF(RIG.LT.0.0) F1=1.090.0*F1
   CONTINUE

133 WRITE(10,322) N,X,Y,XY,XMAX,XMIN,F1
   CONTINUE

135 WRITE(10,323) M
   GO TO 500

136 WRITE(10,324) LX
   CONTINUE

* FORMAT STATEMENTS

300 FORMAT(1H1,/** INPUT DATA*,///)
301 FORMAT(* NUMBER OF ELEMENTS =*,I5,/)  CSTES302
302 FORMAT(* NUMBER OF NODAL POINTS =*,I5,/)  CSTES303
303 FORMAT(* NUMBER OF BOUNDARY POINTS =*,I5,/)  CSTES304
304 FORMAT(* CYCLE PRINT INTERVAL =*,I5,/)  CSTES305
305 FORMAT(* OUTPUT INTERVAL OF RESULTS =*,I5,/)  CSTES306
306 FORMAT(* CYCLE LIMIT =*,I5,/)  CSTES307
307 FORMAT(* TOLERANCE LIMIT =*,F13.3,/)  CSTES308
308 FORMAT(* OVERRELAXATION FACTOR =*,F13.3,/)  CSTES309
309 FORMAT(///,*/ ELEMENT#,I4,*,I*,7X,*,J,*,7X,*,K,*,6X,*,F-MODULUS*,I*,5X,*,DENSITIES*)
   I5ITY*,5X,*,POISSON*,8X,*,ALPHAS*,8X,*,DELTA, T,*,/)  CSTES310
   15ITY*,5X,*,POISSON*,8X,*,ALPHAS*,8X,*,DELTA, T,*,/)  CSTES311
310 FORMAT(1X,4(I4,4X),E11.3,2F12,4*F16.8,F12.3)  CSTES312
311 FORMAT(///,*/ POINT#,B8*,X-ORD#,10X,Y-ORD#,9X,X-LOAD#,9X,Y-LOAD#)  CSTES313
   19,10X,*,X-DISP#,11X,Y-DISP#/)  CSTES314
312 FORMAT(I4,1X,4F15.5,2F17.8)  CSTES315
313 FORMAT(1H1,/** BOUNDARY CONDITIONS*,///)  CSTES316
314 FORMAT(* POINT#,5X,*,TYPE#,6X,*,SLOPE#/)  CSTES317
315 FORMAT(3X,*,5X,*,F15.8)  CSTES318
316 FORMAT(1H1,/** CYCLE :*,I6,5X,*,FORCE UNBALANCE =*,3X,*,E15.8,////)  CSTES319
317 FORMAT(///,*/ POINT#,9X,*,X-DISPLACEMEN'T#,6X,*,Y-DISPLACEMENT#/)  CSTES320
318 FORMAT(I4,11X,*,E14.7,6X,E14.7)  CSTES321
319 FORMAT(///,*/ ELEMENT#,9X,*,X-STRESS#,12X,*,Y-STRESS#,11X,*,XY-STRESS#/)  CSTES322
   15*,10X,*,MAX-STRESS#,5X,*,MIN-STRESS#,6X,*,DIRECTION#/)  CSTES323
320 FORMAT(I12,3F20.4,5X,3F15.2)  CSTES324
321 FORMAT(///,*/ POINT#,9X,*,X-STRESS#,12X,*,Y-STRESS#,11X,*,XY-STRESS#/)  CSTES325
   15*,10X,*,MAX-STRESS#,5X,*,MIN-STRESS#,6X,*,DIRECTION#/)  CSTES326
322 FORMAT(I12,3F20.4,5X,3F15.2)  CSTES327
323 FORMAT(1H1,10X,*ZERO AREA ELEMENT NUMBER =*,I5)  CSTES328
324 FORMAT(1H1,*,10X,*MORE THAN 8 NODAL POINTS ADJACENT TO POINT =*,I5)  CSTES329
325 FORMAT(I4,9X,I1,F13.5)  CSTES330
326 FORMAT(1H1,10X,*CYCLE#,6X,*,FORCE UNBALANCE#,/)  CSTES331
332 FORMAT(1H1,10X,*,MAX-STRESS#,5X,*,MIN-STRESS#,6X,*,DIRECTION#,/)  CSTES332
327 FORMAT(1H1,10X,*,FORCE UNBALANCE#,/)  CSTES333
400 FORMAT(6I4,6X,E10.3)  CSTES334
401 FORMAT(3I4,*,5F10.0)  CSTES335
402 FORMAT(6F10.0)  CSTES336
403 FORMAT(I4,3X,I1,2X,F15.0)
500 CALL EXIT

END
6. ACKNOWLEDGMENTS

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