INFLUENCE SURFACES OF ORTHOTROPIC PLATES

by

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SYNOPSIS

Modern developments of reinforced concrete structures have presented many problems in the field of theory of elasticity. Especially in the case of plate and shell structures, theoretical investigations based on the theory of elasticity have become indispensable for a safe and economical design. The application of plate theory, that is, influence surfaces of plates has been taking more and more important roles in the design of bridge floor slabs.

In this dissertation, the extension of the theory of influence surfaces to orthotropic plates are made, the approach being based on the mathematical concept of "Green's Function" for the deflection of a plate.

Solutions for the moments of semi-infinite strips as well as infinite strips with various boundary conditions are derived mostly in closed form.

Such a solution in closed form will render numerical computations much easier than series solutions as presented by Pucher and other investigators. A general discussion of the singularities of the surfaces are presented with several numerical examples.
CHAPTER I

Introduction

1.1 The Importance of Influence Surfaces in the Design of Bridge Floors

The use of influence lines for the design of bridges subjected to live loads has become a standard practice, even to the extent that no further method is accepted. The influence lines allow to determine the maximum moment, shearing force, axial load, etc. for a given section in a bridge member under live loads.

A logical extension of this method to the design of bridge slabs is the development of influence surfaces (two-dimensional influence lines). They allow the determination of the maximum moment (and shearing force, twisting moment, etc. if desired) at a given point of the slab subjected to concentrated wheel loads. The proper detailing of the slab can readily be handled, once the extreme moment values are known.

In this chapter, the fundamental equation of an orthotropic plate will be introduced first. Then the engineering concept of influence surfaces will be described. Finally, some important theorems as well as properties of influence surfaces will be listed without proof.

1.2 Bending of Orthotropic Plates (for example, (1) p.188)

It is assumed that the material of the plate has three planes of symmetry with respect to its elastic properties.

Taking these planes as the coordinate planes, the relations between the stress and strain components for a case of plane stress in the xy-plane can be represented by the following equations: (Fig.(1-1))
It is seen that in the case of plane stress, four constants $E_x'$, $E_y'$, $E''$ and $G$ are needed to characterize the elastic properties of a material.

Considering the bending of a plate made of such a material, it is assumed that linear elements perpendicular to the middle plane ($x-y$-plane) of the plate before bending remain straight and normal to the deflection surface of the plate after bending. Hence, the usual expressions for the components of strain can be used (1.2).\[
\varepsilon_x = -2 \frac{\partial^2 W}{\partial x^2}, \quad \varepsilon_y = -2 \frac{\partial^2 W}{\partial y^2}, \quad \gamma_{xy} = -2\frac{\partial^2 W}{\partial x \partial y}
\] (1.2)

The corresponding stress components, are

\[
\sigma_x = -2 \left( E_x' \frac{\partial^2 W}{\partial x^2} + E'' \frac{\partial^2 W}{\partial y^2} \right)
\]

\[
\sigma_y = -2 \left( E_y' \frac{\partial^2 W}{\partial y^2} + E'' \frac{\partial^2 W}{\partial x^2} \right)
\]

\[
\tau_{xy} = -2GZ \frac{\partial^2 W}{\partial x \partial y}
\]

With these expressions for the stress components the bending and twisting moments are

\[
M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x Z \, dZ = - \left( D_x \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} \right)
\]

(1.4)
\( M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z \, dz = -\left( D_y \frac{\partial^2 W}{\partial y^4} + D_z \frac{\partial^2 W}{\partial x^2} \right) \) \\
\( M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z \, dz = 2 D_{xy} \frac{\partial^2 W}{\partial x \partial y} = -M_{yx} \) 

in which

\[
D_x = \frac{E_x h^3}{12}, \quad D_y = \frac{E_y h^3}{12}, \quad D_z = \frac{E_z h^3}{12}, \quad D_{xy} = \frac{G h^3}{12} \tag{1.5}
\]

Substituting expressions (1.4) into the equations of equilibrium for a differential element in \(x, y\) and \(z\) directions. (Fig. 1-2)

\[
\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_y}{\partial x} - Q_x = 0 \\
\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_x}{\partial y} + Q_y = 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + g = 0 \tag{1.6}
\]

the equation for an orthotropic plate is obtained

\[
D_x \frac{\partial^4 W}{\partial x^4} + 2 \frac{h}{2} \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = g \tag{1.7}
\]

where

\[
H = D_1 + 2 D_{xy}
\]

In the particular case of isotropy,

\[
E' = E'' = \frac{E}{1-\nu^2}, \quad G = \frac{E}{2(1+\nu)}
\]

Hence

\[
D_x = D_y = \frac{E h^3}{12(1-\nu^2)} = D \\
H = D_1 + 2 D_{xy} = \frac{h^3}{12} \left( \frac{\nu E}{1-\nu^2} + \frac{E}{1+\nu} \right) = \frac{E h^3}{12(1-\nu^2)} = D
\]

Therefore equation (1.7) reduces to the ordinary plate equation:

\[
D \Delta \Delta W = g \tag{1.8}
\]

where

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]
In addition to equation (1.5) and equation (1.7), the expressions for the shearing force $Q_x$, $Q_y$ and the boundary shear $V_x$, $V_y$ are collected here:

$$
\begin{align*}
Q_x &= \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} = -\frac{\partial}{\partial x} \left( D_x \frac{\partial^2 W}{\partial x^2} + H \frac{\partial^2 W}{\partial y^2} \right) \\
Q_y &= \frac{\partial M_{yx}}{\partial y} - \frac{\partial M_{yx}}{\partial x} = -\frac{\partial}{\partial y} \left( H \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} \right) \\
V_x &= Q_x - \frac{\partial M_{yx}}{\partial y} = -\frac{\partial}{\partial x} \left( D_x \frac{\partial^2 W}{\partial x^2} + (2H-D_y) \frac{\partial^2 W}{\partial y^2} \right) \\
V_y &= Q_y - \frac{\partial M_{yx}}{\partial x} = -\frac{\partial}{\partial y} \left( (2H-D_y) \frac{\partial^2 W}{\partial x^2 \partial y} + D_y \frac{\partial^2 W}{\partial y^2} \right)
\end{align*}
$$

1.3 **Engineering Concept of Influence Function for the Deflection of a Plate**

Consider a plate of any shape with prescribed boundary conditions subjected to a concentrated load $P=1$ acting at the point $(x,y)$. (Fig. 1-3) The deflection $W(u,v;x,y)$ of a point $(u,v)$ is called the Green's function (influence function) for the deflection of the given plate.

The influence function $W(u,v;x,y)$ depends on the four variables $u,v$ and $x,y$. For the graphical presentation of the function, a two-dimensional contour line system will be employed. For instance, if $(u,v)$ is fixed ($(u,v)$ being the influence point), the function depends upon $x$ and $y$, therefore $W(u,v;x,y)$ will form a surface. This surface, $W(x,y)$, will be called influence surface for the deflection of point $(u,v)$. On the other hand, if $x,y$ is fixed ($(x,y)$ being the loading point) the function, $W(u,v)$ represents another surface, which is the deflection surface of the plate under a concentrated load $P=1$ at $(x,y)$. The theory of influence surfaces is based on the ordinary theory of plate. Therefore, following assumptions made in section (1.2) apply:
1. The plate thickness \( h \) is assumed to be constant and small compared to other dimensions.

2. The material is orthotropic and follows Hooke's law.

3. The deflection of plates is small against the thickness \( h \).

1.4 Some Important Theorems and Properties of Influence Functions

It is not the purpose of this section to introduce the general theory of influence surfaces developed by A. Fucher. However, several fundamental theorems and properties of influence surfaces will be pointed out.

(a) The influence function for the deflection of a plate \( W(u,v;x,y) \) consists of two functions, that is,

\[
W(u,v;x,y) = W_0(u,v;x,y) + W_1(u,v;x,y)
\]

where \( W_0(u,v;x,y) \) is the particular solution of the differential equation:

\[
D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q(x,y)
\]

and \( W_1(u,v;x,y) \) is the homogeneous solution of the above equation whose constants are determined such that \( W(u,v;x,y) \) will fulfill the prescribed boundary conditions. \( W_0(w,v;x,y) \) contains the singular solution corresponding to \( r^2 \log \frac{r}{r_0} \) in case of isotropic plates.\(^{(2)}\) The corresponding solution for orthotropic plates has been derived by Mossakowski.\(^{(11)}\)

It is this part which plays the important role for the singular behavior of influence surfaces as will be shown later.

(b) The influence function \( F(u,v;x,y) \) for any effect in a plate (such as bending moment, twisting moment, shearing force, etc.) at a given point \( (u,v) \) is obtained by differentiating the influence function for the deflection, \( W(u,v;x,y) \), with respect to \( u \) and \( v \).
Following are the formulae for the derivation of such influence functions:

**Bending Moments**

\[
M_x(u, v; x, y) = -\left(D_x \frac{\partial^2 W}{\partial u^2} + D_l \frac{\partial^2 W}{\partial V^2}\right)
\]
\[
M_y(u, v; x, y) = -\left(D_l \frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial V^2}\right)
\]

**Twisting Moments**

\[
M_{xy}(u, v; x, y) = 2D_y \frac{\partial^2 W}{\partial u \partial V} = -M_{yx}(u, v; x, y)
\]

**Shearing Forces**

\[
Q_x(u, v; x, y) = -\frac{\partial}{\partial u} \left(D_x \frac{\partial^2 W}{\partial u^2} + H \frac{\partial^2 W}{\partial V^2}\right)
\]
\[
Q_y(u, v; x, y) = -\frac{\partial}{\partial V} \left(H \frac{\partial^2 W}{\partial u^2} + D_x \frac{\partial^2 W}{\partial V^2}\right)
\]

**Boundary Shear**

\[
\tau_x(u, v; x, y) = -\frac{\partial}{\partial u} \left[D_x \frac{\partial^2 W}{\partial u^2} + (2H-D_l) \frac{\partial^2 W}{\partial V^2}\right]
\]
\[
\tau_y(u, v; x, y) = -\frac{\partial}{\partial V} \left[(2H-D_l) \frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial V^2}\right]
\]

The function \(F(u, v; x, y)\), can be used in two different ways. If the point \((u, v)\) is fixed (this point \((u, v)\) will be called from now on influence point), the function will represent the influence surface for the particular effect (for example, bending moment, etc.) with respect to the influence point \((u, v)\) and will be written \(f(x, y)\).

On the other hand if the point \((x, y)\), the loading point, is fixed the function determines the distribution of the effect over the plate due to the load \(P\) acting at \((x, y)\). For example,
in case of $M_u(u,v;x,y)$ it represents the $M_x$-moment surface due to a concentrated load $P=1$. It will be written as $F(u,v)$.

(c) From section (b) it can be concluded that the influence function $F(u,v;x,y)$ for any effect in a plate is a solution of

$$D_x \frac{\partial^4 F}{\partial x^4} + 2H \frac{\partial^4 F}{\partial x^3 \partial y} + D_y \frac{\partial^4 F}{\partial y^4} = 0$$

with a singularity at the influence point $(u,v)$. The function $F(u,v;x,y)$ fulfills the same prescribed boundary condition as $W(u,v;x,y)$. In references (5) (6) some cases were solved directly for moments using this principle instead of deriving $W(u,v;x,y)$. However, in this dissertation, $W(u,v;x,y)$ is always thought first and $M_x, M_y$ are obtained through differentiations. This is done for the following two reasons.

(i) Once, $W(u,v;x,y)$ is determined, any other influence function is obtained quickly by simple differentiation.

(ii) $W(u,v;x,y)$ can be successfully applied to solve other important problems such as eigen value problems of plates (Vibration, buckling), dynamical behavior of plates due to impulsive loading, etc.

(d) Magnitude of particular effect in a plate under arbitrary loading:

The magnitude is given by the following expression

$$F = \sum_i P_i f(u,v;x_i,y_i) + \int \rho(s) f(u,v;x(s),y(s)) ds$$

$$+ \iint \nu(x,y) f(u,v;x,y) dx dy$$
where

\[ F_i : \text{concentrated loads acting at } (x,y) \]
\[ p(s) : \text{line load distributed along some line} \]
\[ p(x,y) : \text{distributed load over some area}. \]

With the use of influence surface diagrams, this computation can be done graphically and numerically.

(e) Influence surfaces are generally controlled by following four conditions:

(i) location of the influence point \((u,v)\)

(ii) shape of plate boundaries

(iii) boundary conditions

(iv) material properties of plates: that is, the two parameters: \[ \lambda = \frac{H}{D_y}, \quad \mu = \sqrt{\frac{D_x}{D_y}} \]

(f) All influence functions \(f(u,v;x,y)\) have singularities at the influence point \((u,v)\) with the exception of the one for deflection. Values of \(M_x,M_y\) for interior points of plates, edge moments along free edge become infinitely large at the influence point \((u,v)\). Though other influence functions show singular behaviors at the influence point \((u,v)\), the corresponding values stay finite. In the vicinity of the influence point \((u,v)\), the singular part of the solution \(F_0(u,v;x,y)\) becomes predominant.

(g) In order to clarify the adopted definitions and notation they are summarized in the following table:

(i) For the influence function \(W(u,v;x,y)\) of the deflection \((u,v)\) and \((x,y)\) are completely interchangeable (Maxwell's Law). However, for the influence function of any effect \(F(u,v;x,y)\)
obtained through differentiation from $W(u,v;x,y)$, such a reciprocity does not apply in general.

(ii)

<table>
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<tr>
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<th>Loading Point $(x,y)$</th>
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<td>$F(u,v;x,y)$</td>
<td>Influence function for any effect in a plate at a given point $(u,v)$ to a unit concentrated load $P=1$ at $(x,y)$</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>$F(u,v)$</td>
<td>The distribution of any effect over the plate due to the unit load $P=1$ acting at $(x,y)$ example: $M_x(u,v)$, Moment surface for Bending Moment $M_x$</td>
<td>Variable</td>
<td>Fixed</td>
</tr>
<tr>
<td>$f(x,y)$</td>
<td>Influence surface for any effect with respect to the influence point $Q(u,v)$ example: $m_x(x,y)$, influence surface for bending moment $M_x$ at point $u,v$.</td>
<td>Fixed</td>
<td>Variable</td>
</tr>
<tr>
<td>$F$</td>
<td>The magnitude of any effect at $(u,v)$ due to specific loads.</td>
<td>Fixed</td>
<td>Fixed</td>
</tr>
</tbody>
</table>

1.5 Application of the Theory of Orthotropic Plates to Actual Bridge Floor Systems

There are quite a few specific cases to which the theory of orthotropic plate is applicable: two-way reinforced concrete slabs, stiffened plates, corrugated plates, gridwork systems, plywood plates, etc. are typical examples of orthotropic plates. In order to study the behavior of such plates, applying the theory of
orthotropic plates, elastic constants, $D_x, D_y$, $H$ must be determined either by experiment or on the basis of theoretical consideration.

As stated in (1.4,e) the shape of an influence surface of an orthotropic plate is controlled by the two ratios of the elastic constants: $\lambda = \frac{H}{D_y}, \mu = \sqrt{\frac{H}{D_y}}$. It is very important to study the methods to determine these constants. Since Huber's work on reinforced concrete slabs a great number of investigation have been carried out on this particular problem. However it may be premature to say that accurate methods for the determination of $\lambda$ and $\mu$ have been established. It is a problem beyond the scope of this dissertation. However, in order to get a picture on the variation of $\lambda$ and $\mu$ as encountered in practice, numerical data on actual bridge floor systems have been collected and represented in Fig(1-4) (See also Chapter XI, References (19)-(26)).

These data were obtained either by theoretical analysis or by direct tests. The domain of $\lambda - \mu$ diagram is bisected by the $\lambda = \mu$ line, and most of the points ($\lambda, \mu$) are located in the domain $\lambda < \mu$, with several points ((2), (3), (4), (14)) are very close to $\mu$-axis.

Along the $\mu$-axis, $\lambda = 0$, or, in other words, $H = 0$. This is the case for gridwork system for which the torsional rigidity of the floor may be negligible. On the other hand, along $\lambda$-axis $\mu = 0$, that is, $D_x = 0$ this is the case of articulated plates (26). In general, for actual orthotropic plates, $\lambda$ and $\mu$ values can be limited.

$$0 \leq \lambda \leq \lambda_0$$
$$0 \leq \mu \leq \mu_0$$
where \( \lambda_0, \mu_0 \) present some maximum upper limits\(^*\). The other limit \( \lambda=\mu=0 \) is practically less important, because the structure is effectively reduced to a system of beams side by side without connection (\( D_x=H=0 \)).

It is also interesting that the case \( \lambda<\mu \) is quite common as far as bridge floor systems are concerned. However, it is the more complicated case for practical computation of influence surfaces as will be seen later.

1.6 Historical Review of Investigation on Influence Surfaces

Since theory of influence surfaces is essentially the theory of Green's functions associated with the linear fourth order partial differential plate equation the problem is closely related to the bending of plates in the theory of elasticity. The first solution of the problem of bending of a simply supported rectangular plate with the use of double trigonometric series is due to Navier in 1820. This famous solution in case of a single concentrated load \( P \) is actually the Green function for this particular plate in double series form of eigen functions.\((1) , p.117\)

In discussing problems of bending of rectangular plates with two opposite edges simply supported M. Levy suggested the single series solution in 1899. Thus, the Green's function of this problem has become possible to be expressed in a single series form (Levy's solution)\((1),p.125\)

Almost, at the same time, J.H. Michell has derived the Green's function for a circular plate whose boundary is clamped, using the method of inversion in 1901.\((7)\)

\(^*\)For the numerical discussion of the singularities of influence surfaces in Chapter \( \Pi \) \( \lambda_0=\mu_0=10 \) is assumed and twelve values of \( \lambda \) and \( \mu \) are considered.
However, the first attempt to compute influence surfaces for the stresses in slabs was probably made by Westergaard(8). Realizing the reciprocity between the bending moment at point 
$(u,v)$ due to a load at $(x,y)$ and vice-versa in the case of a simply supported plate strip he obtained a moment influence surface.

Subsequent investigators(9),(10) followed the same line of reasoning by basing the influence surfaces on Maxwell's reciprocity theorem. However, this theorem on the reciprocity of deflections, if applied to moments holds for a limited number of cases only (that is, simply supported plate strip, simply supported rectangular plates, etc.).

Pucher has developed the general theory of influence surfaces in 1935(5) and he furnished a great number of important results in form of contour line diagrams.(4) But his work and that of work made by other investigators is confined to the case of isotropic plates.

The extension of the theory of influence surfaces to the case of orthotropic plates is presented in this dissertation.

Incidentally, a recent literature review disclosed that such work has been started independently in Poland by Nowacki, Mossakowski and others since 1950(11),(12),(13). It should be pointed out that some minor results developed in this dissertation have been already derived by these investigators, employing methods similar to the ones in this dissertation.
CHAPTER II

Practical Application of Influence Surfaces

The practical application of influence surfaces will be discussed shortly in this chapter. Since the influence surfaces are generally presented in the form of contour line diagrams, it is important to know how to use these surfaces in order to get accurate results. Furthermore, the consistency between theory and experiments will be discussed.

2.1 Application of Influence Surfaces to Actual Problems

As pointed out in (1.4,d) already, the determination of any effect (bending moment, shearing force, etc) at a given point due to an arbitrary load, requires only the computation of simple area or volume integrals by making use of influence surfaces. (similar to influence lines).

(i) for a distributed load \( p(x,y) \)

\[
F = \iint p(x,y) f(u,v;x,y) \, dx \, dy
\]

(ii) for a line load \( p(s) \)

\[
F = \int p(s) f(u,v;x(s),y(s)) \, ds
\]

(iii) for several concentrated loads \( P_i(x,y) \)

\[
F = \sum_i P_i(x_i,y_i) f(u,v;x_i,y_i)
\]

In actual computation, (for case (i)) surfaces are sectioned by horizontal or vertical planes and for each section, the area is computed using a planimeter or applying Simpson's Rule. The volume can be computed by repeating Simpson's Rule on the areas.
At the influence point the value of the influence function very often grows to infinity. In numerical computations the volume in the immediate neighborhood of this singular point is usually neglected. In order to justify this practice the following example is given:

Consider the singular part of $m_x(u,v)$ in the vicinity of the influence point $(u,v)$. (Fig. 2-1) Since the singular part of $m_x$ is predominant around this point the volume of neglected portion of the surface $\Delta V$ is essentially governed by this singular part and can hence be computed as follows.

In the case of an isotropic plate the singular part is:

$$(m_x)_o = -\frac{1}{8\pi} \left( 2 \log \frac{r}{r_0} + 2 \cos^2 \varphi + 1 \right)$$

assuming $(m_x)_o = X$

$$\log \frac{r}{r_0} = -\frac{1}{2} \left( 8 \pi X + 2 \cos^2 \varphi + 1 \right)$$

$$\therefore \quad r(X, \varphi) = r_0 e^{-\frac{8\pi X + 1}{2}} \cos^2 \varphi$$

This is the equation of a section $(m_x)_o = X$ of the surface. The area of the section follows to:

$$A(X) = \frac{1}{2} \int_0^{2\pi} r^2 \, d\varphi = \frac{1}{2} r_0^2 e^{-\left(8\pi X + 1\right)} \int_0^{2\pi} e^{-2\cos^2 \varphi} \, d\varphi$$

$$= \frac{1}{2} \times 2.926 \ r_0^2 \ e^{-\left(8\pi X + 1\right)}$$

Therefore the volume $V(X)$ of the surface above plane $X$ is obtained:

$$V(X) = \int_X^{\infty} F(x) \, dx = \frac{r_0^2}{2e} \times 2.926 \int_x^{\infty} e^{-\frac{8\pi X}{e}} \, dx = \frac{r_0^2}{2e} \times 2.926 e^{-8\pi X}$$

$$\therefore \quad V(X) = 0.02146 \ r_0^2 e^{-8\pi X}$$

*by numerical integration

$$\int_0^{2\pi} e^{-2\cos^2 \varphi} \, d\varphi = 2.926$$
Using \( V(X) \), \( \Delta V \) is easily estimated

\[
\Delta V = \mathcal{V} \left( \frac{7}{8\pi} \right) = 0.02146 \gamma_0 \epsilon^{-7} = 1.957 \times 10^{-5}
\]

such that it can be usually neglected in the computation of \( M_x \).

In case of orthotropic plates, magnitude of \( \Delta V \) will change depending upon \( \lambda \) and \( \mu \), however it is still of order \( 10^{-5} \).

Since influence surfaces have singularities at the influence point, careful consideration must be paid to the computation in the vicinity of that point.

Further details concerning practical computation will be found in Pucher's book.\(^4\) Careful computation yields always very accurate results (max. error = 5%).

2.2 Consistency Between Theory and Experiments

Since the theory of influence surfaces is based on the ordinary theory of plates, results obtained are certainly correct within the limitation of the theory of elasticity. Therefore it can be expected that corresponding results are much superior than present semi-empirical formulae given in specifications such as AASHO. Theory of plates subjected to concentrated loads and hence the theory of influence surfaces has been checked experimentally. Especially Dutch investigators have recently carried out a very successful experimental study of slabs subjected to concentrated loads.\(^{14}\)

The experiments were conducted on a steel model to obtain information about the stress-strain distribution in slabs, subjected to concentrated loads.
(i) Investigation of influence of the size of the loading surface (the concentration of the load) on the bending moments in the slab.

The load was in succession transmitted by a ball (which gave a contact area with a diameter of about 0.45 cm) and by circular distribution pads with diameters D of 1.6 cm, 3.6 cm, 5.4 cm and 7.6 cm. The ratios e/a (radius of distributor pad/span) were respectively 0.0024, 0.0087, 0.0195, 0.0293 and 0.0411. For these measurements investigations on the influence of various intermediate layers such as, 3 mm cardboard and rubber with various thicknesses were also made.

(ii) Investigation of the stress-distribution in the slab as a function of the boundary conditions and the locations of the load. (Fig. 2-3)

Summarizing the test results, the following conclusions were drawn:

(a) Outside the immediate neighborhood of the load there was a good agreement between the experiments and the elementary theory of plates.

(for concentration e/a=0.0024 to e/a=0.0411 no noticeable influence was found outside an area with a radius of 5 cm (about \( \frac{1}{18} \) of the span) around the center of gravity of the load)

(b) For the bending moments under the load, the correction presented by Westergaard\(^{(8)}\) was in good agreement with the experiments. (Fig. 2-4 and 2-5).

As will be seen later, influence functions for any effect except the deflection exhibit singular behavior in the neighborhood
of the influence point. This is due to the assumption of an idealized concentrated load. Actually, this ideal concentration of load cannot be realized.

Instead, a small portion of the plate just under the load must be subjected to rather high compressive pressure because of highly localized loads.

Therefore it is impossible to apply the ordinary plate theory in the vicinity of the applied loads. Nadai, Woinowsky-Krieger, Westergaard, and other, have investigated the stress distribution directly under the loads (theory of thick plates). Nevertheless, such a disturbance has such localized effects that the accuracy of the theory is practically not affected (by St. Venant's Principle), because, the volume of influence surfaces above the certain limiting values is usually negligible as stated before.
CHAPTER III

Deflections, Moments And Influence Functions For
The Infinite Plate Strip With Simply Supported

Parallel Edges

3.1 Method of Solution

In order to obtain the solution, the usual approach solving directly differential equation will be employed. Although the deflection surface is obtained in a series form, bending moments twisting moment, shearing forces can be expressed in closed form as will be seen later. The expressions consist of a singular part due to the particular solution of the generalized Biharmonic equation and a regular part due to homogeneous solution of

\[ D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = 0 \]

3.2 Formation of the Problem and Derivation of the Solution

Consider an infinite plate strip with simply supported parallel edges (Fig. 3-1).

The problem consists of deriving the deflection surface and hence the influence function for deflections (Green's function) of this infinite plate strip. The deflection surface must satisfy the following differential equation

\[ D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = 0 \]  

(3.1)

except at the point where the concentrated load P=1 is applied.
The corresponding boundary conditions are as follows:

\[ \begin{align*}
    x = 0 & : \quad W = 0 \quad M_x = -(D_x \frac{\partial^2 W}{\partial x^2} + D_l \frac{\partial W}{\partial y}) = 0 \quad (1) \\
    x = a & : \quad W = 0 \quad M_x = 0 \quad \text{or} \quad \frac{\partial^2 W}{\partial x^2} = 0 \quad (11) \\
    y \to \pm \infty & : \quad W \to 0 \\
\end{align*} \]

And \[ \frac{\partial W}{\partial y} = 0 \quad Q_y(d, \theta) = -\frac{P}{2}, \quad P = 1 \quad (iii) \quad \theta > 0 \]

Condition (iii) assures that the deflection surface is symmetrical with respect to the x-axis and the shearing force \( Q_y \) disappears except at the loading point \((d, 0)\).

Assuming the deflection surface

\[ W(x, y) = \sum_{n=1}^{\infty} \gamma_n(y) \sin \frac{n \pi x}{a} \quad (3.3) \]

and substituting equation \((3.3)\) into equation \((3.1)\), gives the following expression for the \( n \) th-term

\[ D_y \gamma_n'''' - 2H \left( \frac{n \pi}{a} \right)^2 \gamma_n'' + D_x \left( \frac{n \pi}{a} \right)^4 \gamma_n = 0 \quad (n = 1, 2, 3, ...) \quad (3.4) \]

Taking \[ \gamma_n(y) = e^{\lambda_n y} \] and substituting it into equation \((3.4)\):

\[ D_y \lambda_n^4 - 2H \left( \frac{n \pi}{a} \right)^2 \lambda_n^2 + D_x \left( \frac{n \pi}{a} \right)^4 = 0 \]

The roots of the corresponding characteristic equation are:

\[ \lambda_n = \pm \left( \frac{n \pi}{a} \right) \sqrt{\frac{H}{D_y} \pm \sqrt{\left( \frac{H}{D_y} \right)^2 - \frac{D_x}{D_y}}} \quad (3.5) \]

The following three cases must be considered:

\[ \begin{align*}
    (1) \quad H^2 - D_x D_y & > 0 \\
    (2) \quad H^2 - D_x D_y &= 0 \\
    (3) \quad H^2 - D_x D_y & < 0 \quad (3.6)
\end{align*} \]
In the first case all the roots of equation (3.5) are real. However, in the second case, the characteristic equation has two double roots, and the function $Y_n$ has the same form as in the case of an isotropic plate. In the third case, the roots of the characteristic equation are imaginary, and $Y_n$ are expressed by trigonometric functions.

For the time being, the first case is considered. All the roots of the characteristic equation (3.6) are real. Considering the part of the plate with positive $y$ and observing that the deflection $w$ and its derivatives must vanish at large distances from the load (Boundary condition (3.2,ii)), only the negative roots can be retained.

Using the notation

\[
K_1 = \sqrt{\frac{H}{D_y}} + \sqrt{\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}} = \sqrt{\lambda + \lambda^2 - \mu^2}
\]

\[
K_2 = \sqrt{\frac{H}{D_y} - \left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}} = \sqrt{\lambda - \lambda^2 - \mu^2}
\]

(3.7)

where

\[\lambda = \frac{H}{D_y}, \quad \mu^2 = \frac{D_x}{D_y} \quad \text{and} \quad \lambda^2 - \mu^2 > 0\]

The integral of equation (3.4) becomes

\[Y_n(y) = A_n \, e^{-\frac{\pi y}{a}} + B_n \, e^{-\frac{\pi y}{a}}\]

and expression (3.3) can be represented as follows:

\[W(x, y) = \sum_{n=1}^{\infty} \left( A_n \, e^{-\frac{\pi y}{a}} + B_n \, e^{-\frac{\pi y}{a}} \right) \sin \frac{n \pi x}{a}\]  

(3.8)

Since it is easily seen that the boundary condition (i),(ii) of (3.2) are satisfied already, the coefficients $A_n$ and $B_n$ must be determined by (3.2,iii).
From \[ (-\frac{\partial W}{\partial y})_{y=0} = 0 \]

\[ A_n K_1 + B_n K_2 = 0 \]

The other condition \( (Q_y)_{y=0} = -\frac{1}{2} \) can be written as follows

\[-\frac{\partial}{\partial y} (D_y \frac{\partial^2 W}{\partial y^2} + \pi \frac{\partial^2 W}{\partial x^2}) = -\frac{1}{2} \]

Expanding the term of external load \( P = 1 \) into a Fourier Sine series, that is,

\[ P = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \]

where \[ b_n = \frac{2P}{2\pi} \tan \frac{n\pi x}{a} \]

and substituting for \( W \) its expression (3.8) and using (3.7)

\[ A_n K_2 + B_n K_1 = \frac{b_n}{2(\frac{\pi}{a})^2 D_x D_y} \]

or \[ A_n K_2 + B_n K_1 = \frac{2\pi^2 \pi \sin \frac{n\pi d}{a}}{2(\frac{\pi}{a})^2 D_x D_y} \sin \frac{n\pi x}{a} \]

Thus \( A_n, B_n \) are determined,

\[
\begin{align*}
A_n &= \frac{\sum_{n=1}^{\infty} \frac{K_1 a^2}{2 \pi^2 D_y \sqrt{\lambda^2 - \mu^2}}}{\sum_{n=1}^{\infty} \frac{K_1 a^2}{2 \pi^2 D_y \sqrt{\lambda^2 - \mu^2}}} \\
B_n &= \frac{\sum_{n=1}^{\infty} \frac{2\pi^2 \pi \sin \frac{n\pi d}{a}}{2(\frac{\pi}{a})^2 D_x D_y}}{\sum_{n=1}^{\infty} \frac{2\pi^2 \pi \sin \frac{n\pi d}{a}}{2(\frac{\pi}{a})^2 D_x D_y}}
\end{align*}
\]

Equation (3.8) and equation (3.9) the solution \( W(x,y) \)

becomes finally

\[ W(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left( K_1 e^{\frac{-n\pi x}{a}} - K_2 e^{\frac{-n\pi x}{a}} \sin \frac{n\pi d}{a} \right) \]

(\( \lambda > \mu \))

Differentiating the solution for \( W(x,y) \) in equation (3.10) the bending moments \( M_x(x,y), M_y(x,y) \), the twisting moment \( M_{xy}(x,y) \) and shearing forces \( Q_x(x,y), Q_y(x,y) \) are easily derived.
This series solution can be expressed in closed form by making use of the summation formulae listed in the Appendix.

\( \mathbf{M}_x = - (D_x \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2}) \)

\[
= \frac{1}{2 \pi \mu \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n} \left[ D_x \left( K_1 e^{\frac{-n \pi k y}{a}} - K_2 e^{\frac{-n \pi k y}{a}} \right) - 
D_y K_1 K_2 \left( K_1 e^{\frac{-n \pi k y}{a}} - K_2 e^{\frac{-n \pi k y}{a}} \right) \right] \sin \frac{n \pi y}{a} \sin \frac{n \pi x}{a}
\]

Similarly

\( \mathbf{M}_y = - (D_x \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2}) \)

\[
= \frac{1}{2 \pi \mu \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n} \left[ D_y \left( K_1 e^{\frac{-n \pi k y}{a}} - K_2 e^{\frac{-n \pi k y}{a}} \right) -
D_x K_1 K_2 \left( K_2 e^{\frac{-n \pi k y}{a}} - K_1 e^{\frac{-n \pi k y}{a}} \right) \right] \sin \frac{n \pi x}{a} \sin \frac{n \pi y}{a}
\]

\( \mathbf{M}_{xy} = 2 D_{xy} \left( \frac{\partial^2 W}{\partial x \partial y} \right) \)

\[
= \frac{D_{xy}}{\pi D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n} \left( e^{\frac{-n \pi k y}{a}} - e^{\frac{-n \pi k y}{a}} \right) \sin \frac{n \pi y}{a} \sin \frac{n \pi x}{a}
\]
\[
Q_x = -\frac{\partial}{\partial x} \left( D_x \frac{\partial^2 W}{\partial x^2} + H \frac{\partial^2 W}{\partial y^2} \right)
\]

\[
= \frac{1}{2a/\lambda^2 - \mu^2} \sum_{n=1}^{\infty} \left[ (\mu_k - \lambda_k) e^{\frac{\pi n y}{a}} - (\mu_k - \lambda_k) e^{-\frac{\pi n y}{a}} \right] \sin \frac{\pi n d}{a} \sin \frac{\pi n y}{a}
\]

\[
= \frac{1}{8a/\lambda^2 - \mu^2} \left[ (\mu_k - \lambda_k) \left\{ \frac{\sin \frac{\pi n y}{a}(x+d)}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x+d)} - \frac{\sin \frac{\pi n y}{a}(x-d)}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x-d)} \right\} \right.
\]

\[
- (\mu_k - \lambda_k) \left\{ \frac{\sin \frac{\pi n y}{a}(x+d)}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x+d)} - \frac{\sin \frac{\pi n y}{a}(x-d)}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x-d)} \right\} \right]
\]

\[
Q_y = -\frac{\partial}{\partial y} \left( H \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} \right)
\]

\[
= \frac{1}{2a/\lambda^2 - \mu^2} \sum_{n=1}^{\infty} \left[ \chi (e^{-\frac{\pi n y}{a}} - e^{-\frac{\pi n y}{a}}) - (\mu_k e^{-\frac{\pi n y}{a}} - \lambda_k e^{-\frac{\pi n y}{a}}) \right] \sin \frac{\pi n d}{a} \sin \frac{\pi n x}{a}
\]

\[
= \frac{1}{8a} \left[ \left\{ \frac{\sinh \frac{\pi n y}{a}}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x+d)} - \frac{\sinh \frac{\pi n y}{a}}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x-d)} \right\} \right]
\]

\[
- \left\{ \frac{\sinh \frac{\pi n y}{a}}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x+d)} - \frac{\sinh \frac{\pi n y}{a}}{\cosh \frac{\pi n x}{a} - \cos \frac{\pi n x}{a}(x-d)} \right\} \right]
\]
Turning to case (3) \( H^2 - D_x D_y < 0 \), or \( \lambda < \mu \) the following abbreviations are introduced:

\[
K_3 = \sqrt{\frac{D_x D_y + H}{2 D_y}} = \sqrt{\frac{1}{2} (\mu + \lambda)}
\]
\[
K_4 = \sqrt{\frac{D_x D_y - H}{2 D_y}} = \sqrt{\frac{1}{2} (\mu - \lambda)}
\]  

Observing the following relations.

\[
K_1 = K_3 + i K_4, \quad K_2 = K_3 - i K_4
\]

the solution \( W(x, y) \) can be easily derived.

\[
W(x, y) = \frac{a^2}{\pi^2 \mu D_y \sqrt{\mu^2 - \lambda}} \sum_{n=1}^{\infty} \frac{e^{-\frac{n \pi y}{a}}}{n^2} \left( K_4 \cos \frac{n \pi x}{a} + K_3 \sin \frac{n \pi x}{a} \right) x
\]  

For case (2) \( H^2 - D_x D_y = 0 \), or \( \lambda = \mu \), \( \lambda \) approaches the \( \mu \) in (3.12). Taking the limit, the solution \( W(x, y) \) becomes:

\[
W(x, y) = \frac{a^2}{2 \pi^2 D_y \sqrt{\chi^2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( 1 + \frac{n \pi x y}{a} \right) e^{-\frac{n \pi x y}{a}} x
\]  

Likewise, closed form expression for \( M_x, M_y, M_{xy}, Q_x \) and \( Q_y \) can be derived for both cases \( \lambda < \mu \) and \( \lambda = \mu \).

So far the point where the load \( P=1 \) is applied has been located on the \( x \)-axis. However, it is quite simple to derive the expressions for the general case. Assuming that the load \( P=1 \) is applied at \( (x, y) \) and the influence point is \( (u, v) \) then \( y \) is replaced by \( \pm (v - y) \) (upper sign for \( V \geq y \), lower sign for \( V < y \)) and \( x \) is replaced by \( u \pm x \). (Fig. 3-2)

*Hereafter this rule should be applied to any double sign, unless otherwise noted.*
Furthermore, for simplicity, non-dimensional coordinates are introduced:

\[
\frac{\pi x}{a} = \xi, \quad \frac{\pi y}{a} = \eta; \quad \frac{\pi u}{a} = \alpha, \quad \frac{\pi v}{a} = \beta
\]

Using the above notation, several important functions are defined in Table I.

Referring to these functions general expressions for the influence functions of an infinite strip are obtained.

(I) **Deflection** \(W(\alpha, \beta, \xi, \eta)\)

(i) \(\lambda > \mu\)

\[
\frac{a^2}{2 \pi^2 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( K_1 e^{i n K_1 (\beta - \eta)} - K_2 e^{i n K_1 (\beta - \eta)} \right) \sin \alpha \sin \eta \sin \xi
\]

(ii) \(\lambda < \mu\)

\[
\frac{a^2}{\pi^2 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \frac{\epsilon^{i n K_1 (\beta - \eta)}}{n^2} \left( K_1 \cos(n K_1 (\beta - \eta)) + K_2 \sin(n K_1 (\beta - \eta)) \right) \sin \alpha \sin \eta \sin \xi
\]

(iii) \(\lambda = \mu\)

\[
\frac{a^2}{2 \pi^2 D_y \sqrt{\lambda^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 + n \sqrt{\lambda(\beta - \eta)} \right) \epsilon^{i n K_1 (\beta - \eta)} \sin \alpha \sin \eta \sin \xi
\]

(II) **Bending Moments** \(M_x(\alpha, \beta; \xi, \eta)\), \(M_y(\alpha, \beta; \xi, \eta)\)

(i) \(\lambda > \mu\)

\[
\begin{cases}
M_x = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ (K_1 \mu - K_1 (\frac{D_1}{D_y})) R_2 - (K_2 \mu - K_1 (\frac{D_1}{D_y})) R_1 \right] \\
M_y = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ (\frac{K_1 D_1}{\mu D_y} - K_1) R_2 - (\frac{K_1 D_1}{\mu D_y} - K_1) R_1 \right]
\end{cases}
\]

(ii) \(\lambda < \mu\)

\[
\begin{cases}
M_x = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ K_4 (\mu + \frac{D_1}{D_y}) R_3 + 2 K_3 (\mu - \frac{D_1}{D_y}) R_4 \right] \\
M_y = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ K_4 (\frac{D_1}{\mu D_y} + 1) R_3 + 2 K_3 (\frac{D_1}{\mu D_y} - 1) R_4 \right]
\end{cases}
\]

(3.15)
(iii) \( \lambda = \mu \)

\[
M_x = \frac{1}{8\pi} \left[ (\sqrt{x} + \frac{1}{\sqrt{x}} (\frac{D_x}{D_y})) R_s \mp (\lambda - \frac{D_x}{D_y}) (\beta - \eta) S_1 \right] \\
M_y = \frac{1}{8\pi} \left[ (\frac{1}{\sqrt{x}} + \sqrt{x} (\frac{D_y}{D_x})) R_s \mp (\frac{1}{\lambda} (\frac{D_y}{D_x}) - 1) (\beta - \eta) S_1 \right]
\]

(III) **Twisting Moment** \( M_{xy}(\alpha, \beta, \xi, \gamma) \)

(i) \( \lambda > \mu \)

\[
= \frac{\pm D_{xy}}{2\pi D_y \sqrt{\lambda^3 - \mu^3}} (R_7 - R_6)
\]

(ii) \( \lambda < \mu \)

\[
= \frac{\pm D_{xy}}{4\pi D_y \sqrt{\mu^2 - \lambda^2}} R_8
\]

(iii) \( \lambda = \mu \)

\[
= \frac{\pm (\beta - \eta) D_{xy}}{4\pi D_y \lambda} S_2
\]

(IV) **Shearing Forces** \( Q_x(\alpha, \beta; \xi, \gamma), Q_y(\alpha, \beta; \xi, \gamma) \)

(i) \( \lambda > \mu \)

\[
Q_x = \frac{1}{8a \sqrt{\lambda^3 - \mu^3}} \left[ (K_1 \mu - K_2 \lambda) S_4 - (K_2 \mu - K_3 \lambda) S_5 \right] \\
Q_y = \frac{1}{8a} (S_5 + S_6)
\]

(ii) \( \lambda < \mu \)

\[
Q_x = \frac{1}{8a} (K_3 S_7 + K_4 S_9) \\
Q_y = \frac{1}{8a} S_9
\]

(iii) \( \lambda = \mu \)

\[
Q_x = \frac{\sqrt{x}}{4a} S_2 \\
Q_y = \frac{1}{4a} S_1
\]
In the case of isotropic plate \((D_x = D_y = H = D)\)

\[
\lambda = \mu = 1 \quad \frac{D_x}{D_y} = \nu \quad \frac{D_{xy}}{D_y} = \frac{1-\nu}{2}
\]

and the above expressions reduce as follows:

\[
W = \frac{a^2}{2\pi^2D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(1 \mp n(\beta-\eta)\right) e^{\pm n(\beta-\eta)} \sin n\alpha \sin n\beta
\]

\[
M_x = \frac{1}{8\pi} \left[ (1+\nu) \log \frac{\cosh(\beta-\eta) - \cosh(\alpha+\xi)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \pm (1-\nu)(\beta-\eta) \times \right.
\]

\[
\left. \left( \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta) - \cosh(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \right) \right]
\]

upper sign for \(M_x\)

lower sign for \(M_y\)

\[
M_{xy} = \frac{\pm(1-\nu)(\beta-\eta)}{8\pi} \left[ \frac{\sin(\alpha+\xi)}{\cosh(\beta-\eta) - \cosh(\alpha+\xi)} - \frac{\sin(\alpha-\xi)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \right]
\]

\[
Q_x = \frac{1}{4a} \left[ \frac{\sin(\alpha+\xi)}{\cosh(\beta-\eta) - \cosh(\alpha+\xi)} - \frac{\sin(\alpha-\xi)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \right]
\]

\[
Q_y = \frac{\pm 1}{4a} \left[ \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta) - \cosh(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \right]
\]
CHAPTER IV

Influence Surfaces For The Semi-Infinite Plate Strips With Simply-Supported Parallel Edges

4.1 General Method to Obtain the Solutions

In Chapter III, the solution for the infinite plate strip was obtained. It will constitute the particular solution \( W_0(\alpha, \beta; \xi, \eta) \) for solutions of semi-infinite plate strips or rectangular plates.

Taking the solution \( W(\alpha, \beta; \xi, \eta) = W_0(\alpha, \beta; \xi, \eta) + W_1(\alpha, \beta; \xi, \eta) \) with \( W_0 \) as the particular solution and \( W_1 \) as a general integral of the homogeneous plate equation, the sum must satisfy all the boundary conditions. The homogeneous solution for a plate strip is generally expressed as follows:

\[
W_1 = \begin{cases} 
\frac{a^2}{2\pi^2\mu x^4\lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n^3} (A_n e^{-n^2\beta} + B_n e^{-n^2\eta}) \sin n\alpha & (\lambda > \mu) \\
\frac{a^2}{\pi^2\mu x^4\mu^2 - \lambda^2} \sum_{n=1}^{\infty} \frac{1}{n^3} (A_n \cos n\lambda\beta + B_n \sin n\lambda\beta) e^{-n^2\beta} \sin n\alpha & (\lambda < \mu) \\
\frac{a^2}{2\pi^2\mu x^4\lambda^2} \sum_{n=1}^{\infty} \frac{1}{n^3} (A_n + B_n (n^2\lambda^2\beta)) e^{-n^2\beta} \sin n\alpha & (\lambda = \mu) 
\end{cases}
\]

Since the particular solution and homogeneous solutions satisfy the boundary conditions imposed on the parallel edges:

\[
\alpha = 0 \quad W = 0 , \quad \frac{\partial^2 W}{\partial \alpha^2} = 0
\]

\[
\alpha = \pi \quad W = 0 , \quad \frac{\partial^2 W}{\partial \alpha^2} = 0
\]

the boundary condition of the third edge, that is, \( \beta = 0 \) will determine the unknown constants \( A_n, B_n \) of the homogeneous solution (4.1)

In this dissertation 3 different cases are considered, that is, (a) simply supported (b) clamped (c) free edge. (Fig. 4-1)
4.2 Influence Functions for the Simply-Supported Strip

(i) The particular solution \( W_0(\alpha, \beta; \xi, \eta) \) is rewritten here

\[
W_0 = \begin{cases} \\
\frac{a^2}{2 \pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ K_1 e^{\pm \eta K_1(\beta - \eta)} - K_2 e^{\pm \eta K_1(\beta + \eta)} + K_1 e^{- \eta K_1(\beta - \eta)} \right] \sin \eta \sin \xi \\
\end{cases} 
\]

\( (\lambda > \mu) \)

\[
\begin{align*}
\frac{a^2}{2 \pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ K_4 \cos \eta K_4(\beta - \eta) + K_3 \sin \eta K_3(\beta - \eta) \right] \sin \eta \sin \xi \\
(\lambda < \mu) \\
\frac{a^2}{2 \pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ (1 + \eta K_2) e^{\pm \eta K_2(\beta - \eta)} \sin \eta \sin \xi \\
(\lambda = \mu) \\
\end{align*}
\]

Assuming the solution \( W(\alpha, \beta; \xi, \eta) = W_0(\alpha, \beta; \xi, \eta) + W_1(\alpha, \beta; \xi, \eta) \) and applying the boundary condition along the \( \alpha \) axis:

\[
\beta = 0 : W = 0, \quad \frac{\partial^2 W}{\partial \beta^2} = 0 
\]

(Fig. 4-1)

Thus \( A_n, B_n \) are determined.

\[
A_n = \begin{cases} \\
- K_1 e^{- \eta K_1(\beta - \eta)} \sin \eta \sin \xi \\
(\lambda > \mu) \\
- e^{- \eta K_1(\beta - \eta)} (K_3 \sin \eta K_3(\beta - \eta) + K_4 \cos \eta K_4(\beta - \eta)) \sin \eta \sin \xi \\
(\lambda < \mu) \\
- (1 + \eta K_2) e^{- \eta K_2(\beta - \eta)} \sin \eta \sin \xi \\
(\lambda = \mu) \\
\end{cases} 
\]

\( (4.2) \)

\[
B_n = \begin{cases} \\
+ K_2 e^{- \eta K_2(\beta + \eta)} \sin \eta \sin \xi \\
(\lambda > \mu) \\
- e^{- \eta K_2(\beta + \eta)} (K_3 \cos \eta K_3(\beta + \eta) + K_4 \sin \eta K_4(\beta + \eta)) \sin \eta \sin \xi \\
(\lambda < \mu) \\
- e^{- \eta K_2(\beta + \eta)} \sin \eta \sin \xi \\
(\lambda = \mu) \\
\end{cases} 
\]

Substituting equation (4.2) into equation (4.1), the general solution for the deflection are derived.

(I) Influence Functions for the deflection \( W(\alpha, \beta; \xi, \eta) \)

(i) \( \lambda > \mu \)

\[
= \frac{a^2}{2 \pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ e^{\pm \eta K_1(\beta - \eta)} - K_1 e^{\pm \eta K_1(\beta + \eta)} + K_1 e^{- \eta K_1(\beta - \eta)} - K_2 e^{- \eta K_2(\beta + \eta)} \right] \sin \eta \sin \xi 
\]

(ii) \( \lambda < \mu \)

\[
= \frac{a^2}{\pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ e^{\pm \eta K_2(\beta - \eta)} (K_4 \cos \eta K_4(\beta - \eta) + K_3 \sin \eta K_3(\beta - \eta)) \right] \sin \eta \sin \xi 
\]

(4.3)
(iii) \( \lambda = \mu \)

\[
\frac{a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{1}{1+n^2} \left[ 1 + n \sqrt{\lambda} (\beta - \eta) \right] e^{n \sqrt{\lambda} (\beta - \eta)} - \frac{1}{1+n^2} \left[ 1 + n \sqrt{\lambda} (\beta + \eta) \right] e^{-n \sqrt{\lambda} (\beta + \eta)} \right] e^{-n \sqrt{\lambda} (\beta + \eta)} \sin \alpha \sin \beta x
\]

(II) Influence Functions for the Bending Moments \( M_x(\alpha, \beta; \xi, \eta) \), \( M_y(\alpha, \beta; \xi, \eta) \)

Bending moments, twisting moments, etc. can be derived by differentiating equation (4.3) and summing up the series solution into closed form expression as explained in Chapter III.

Here only the final results are summarized without showing the intermediate mathematical operations.

(i) \( \lambda > \mu \)

\[
M_x = \frac{1}{8 \pi \sqrt{\lambda - \mu^2}} \left[ (K_1 \mu - K_2 (\frac{D_1}{D_y})) (R_2 - \bar{R}_2) \right.
\]
\[
- (K_1 \mu - K_2 (\frac{D_1}{D_y})) (R_1 - \bar{R}_1) \left. \right] 
\]
\[
M_y = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ (\frac{K_1 D_1}{\mu D_y}) - K_2 \right] (R_2 - \bar{R}_2)
\]
\[
- (\frac{K_2 D_1}{\mu D_y}) - K_1 (R_1 - \bar{R}_1) \right] 
\]

(ii) \( \lambda < \mu \)

\[
M_x = \frac{1}{8 \pi \sqrt{\mu^2 - \lambda^2}} \left[ K_4 (\mu + \frac{D_1}{D_y}) (R_3 - \bar{R}_3) \right.
\]
\[
+ 2K_3 (\mu - \frac{D_1}{D_y}) (R_4 - \bar{R}_4) \left. \right] 
\]
\[
M_y = \frac{1}{8 \pi \sqrt{\mu^2 - \lambda^2}} \left[ K_4 (\frac{D_1}{\mu D_y} + 1) (R_3 - \bar{R}_3) \right.
\]
\[
+ 2K_3 (\frac{D_1}{\mu D_y} - 1) (R_4 - \bar{R}_4) \left. \right] 
\]
(111) $\lambda = \mu$

\[
M_x = \frac{1}{8 \pi x^3} \left\{ (\lambda + \frac{D_1}{D_2})(R_s - \bar{R}_s) - (\lambda - \frac{D_1}{D_2}) \right\} 
\pm \sqrt{\alpha (\beta - \eta)} S, + \sqrt{\alpha (\beta + \eta)} \bar{S}, \right\} \]

\[
M_y = \frac{1}{8 \pi x^3} \left\{ (\frac{D_1}{D_2} + \lambda)(R_s - \bar{R}_s) + (\lambda - \frac{D_1}{D_2}) \right\} 
\pm \sqrt{\alpha (\beta - \eta)} S, + \sqrt{\alpha (\beta + \eta)} \bar{S}, \right\} \]

(III) Influence Function for the Twisting Moments $M_{xy}(\alpha, \beta; \xi, \eta)$

(i) $\lambda > \mu$

\[
= \frac{D_{xy}}{2 \pi D_2 \sqrt{\lambda^2 - \mu^2}} \left[ \pm (R_7 - R_6) + (\bar{R}_7 - \bar{R}_6) \right]
\]

(ii) $\lambda < \mu$

\[
= \frac{D_{xy}}{2 \pi D_2 \sqrt{\mu^2 - \lambda^2}} \left( \pm R_2 + \bar{R}_2 \right)
\]

(iii) $\lambda = \mu$

\[
= \frac{D_{xy}}{4 \pi \lambda^2 D_2} \left[ \mp (\beta - \eta) S_2 + (\beta + \eta) \bar{S}_2 \right]
\]

IV. Influence Surface for Corner Reaction $r(\xi, \eta)$

In order to prevent the uplifting of the plate at the corners (for example, origin $\alpha = \beta = 0$) concentrated corner reaction must exist acting downward. According to geometrical consideration and observing that the angle of the corner is equal to $\frac{\pi}{2}$ so that $M_{xy} = -M_{yx}$,
it is concluded that

\[ Y(\xi, \eta) = M_{xy}(0, 0; \xi, \eta) - M_{yx}(0, 0; \xi, \eta) = 2M_{xy}(0, 0; \xi, \eta) \]

Therefore the corresponding influence surfaces are easily derived.

(i) \( \lambda > \mu \)

\[ = \frac{4D_{xy}}{\pi D y \lambda^2 - \mu^2} \left[ \tan^{-1} \left( \frac{\sin \xi}{e^{k_1 \eta - \cos \xi}} \right) - \tan^{-1} \left( \frac{\sin \xi}{e^{k_1 \eta - \cos \xi}} \right) \right] \]

(ii) \( \lambda < \mu \)

\[ = \frac{2D_{xy}}{\pi D y \mu^2 - \lambda^2} \log \frac{\cosh k_1 \eta - \cos (\xi - k_1 \eta)}{\cosh k_1 \eta - \cos (\xi + k_1 \eta)} \]

(iii) \( \lambda = \mu \)

\[ = \frac{2D_{xy}}{\pi \lambda D y} \left( \frac{\sin \xi}{\cosh k_1 \eta - \cos \xi} \right) \]

For the case of an isotropic plate, \( \lambda = \mu = 1 \), the expressions simplify considerably:

\[ W = \frac{a^2}{2 \pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \left\{ 1 + n(\beta - \eta) \right\} e^{\pm(\beta - \eta)} - \left\{ 1 + n(\beta + \eta) \right\} e^{-n(\beta + \eta)} \right] \sin \alpha \sin n \xi \]

\[ \begin{bmatrix} M_x \\ M_y \end{bmatrix} = \frac{1}{8 \pi} \left[ (1+\nu) \log \frac{\cosh(\beta-\eta) - \cosh(\alpha+\xi)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \left\{ \cosh(\beta+\eta) - \cosh(\alpha+\xi) \right\} \right] \\
 \quad \pm (1-\nu)(\beta-\eta) \left\{ \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cosh(\alpha+\xi)} \right\} \]

\[ - (1-\nu)(\beta+\eta) \left\{ \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cosh(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cosh(\alpha+\xi)} \right\} \]

upper sign for \( M_x \)
lower sign for \( M_y \)
\[
M_{xy} = \frac{1 - \nu}{8\pi} \left[ \pm (\beta - \eta) \left\{ \frac{\sin(\omega - \xi)}{\cosh(\beta - \eta) - \cosh(\omega - \xi)} - \frac{\sin(\omega + \xi)}{\cosh(\beta - \eta) - \cosh(\omega + \xi)} \right\} \\
- (\beta + \eta) \left\{ \frac{\sin(\omega - \xi)}{\cosh(\beta + \eta) - \cosh(\omega - \xi)} - \frac{\sin(\omega + \xi)}{\cosh(\beta + \eta) - \cosh(\omega + \xi)} \right\} \right]
\]

\[
\gamma = \frac{1 - \nu}{2\pi} \cdot \frac{\eta \sin \xi}{\cosh \eta - \cosh \xi}
\]

4.3 Influence Functions for the Clamped Edge

The corresponding boundary conditions are (Fig. 4-1)

\[
\beta = 0 : \quad W = 0, \quad \frac{\partial W}{\partial \beta} = 0
\]

The general solutions can be derived by determining the two constants \( A_n, B_n \).

(i) Influence Function for the Deflection \( W(\alpha, \beta; \xi, \eta) \)

\[ (i) \quad \lambda > \mu \]

\[
= \frac{a^2}{2\pi^2D_y\sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ K_1 e^{\pm nK_1(\beta - \eta)} - K_3 e^{\pm nK_3(\beta - \eta)} + \frac{K_1(K_1 + K_3)}{K_5 - K_1} e^{-nK_1(\beta + \eta)} - \frac{2K_1K_3}{K_5 - K_1} e^{-n(\xi + K_2 + K_3)} + \frac{K_1(K_1 + K_3)}{K_5 - K_1} e^{-nK_3(\beta + \eta)} \right] \sin n\alpha \sin n\xi
\]

\[ (ii) \quad \lambda < \mu \]

\[
= \frac{a^2}{\pi^2\mu D_y \sqrt{\mu^2 - \lambda^2}} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ e^{\pm nK_1(\beta - \eta)} (K_4 \cos nK_4(\beta - \eta) + K_1 \sin nK_1(\beta - \eta)) + \frac{2K_1}{K_4} e^{-nK_1(\beta + \eta)} (K_2 \cos nK_2(\beta + \eta) - K_4 \sin nK_4(\beta + \eta)) + \frac{2(K_1^2 + K_3)}{K_4} e^{-nK_1(\beta + \eta)} \cos nK_4(\beta - \eta) \right] \sin n\alpha \sin n\xi
\]

\[ (iii) \quad \lambda = \mu \]

\[
= \frac{a^2}{2\pi^2D_y \sqrt{\alpha^2}} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ (1 + \eta \sqrt{\alpha} (\beta - \eta)) e^{\pm \eta \sqrt{\alpha}(\beta - \eta)} - (1 + \eta \sqrt{\alpha}(\beta + \eta)) e^{-\eta \sqrt{\alpha}(\beta + \eta)} \right] \sin n\alpha \sin n\xi
\]
Only the influence surfaces for bending moments $M_x(\alpha, \beta; \xi, \eta)$ and $M_y(\alpha, \beta; \xi, \eta)$ will be derived in this case. The corner reactions disappear as one of the edges is clamped.

(II) Influence Function for the Bending Moments $M_x(\alpha, \beta; \xi, \eta)$ and $M_y(\alpha, \beta; \xi, \eta)$

\(\lambda > \mu\)

\[
M_x = \frac{1}{8\pi \sqrt{\lambda^2 - \mu^2}} \left[ \left( K_1 \mu - \frac{K_3}{D_y} \right) R_3 - \left( K_3 \mu - \frac{K_1}{D_y} \right) R_1 + \frac{K_3}{K_2} \left( \frac{K_1 - K_3}{K_2 - K_1} \right) K_1 R_1 \right] \nonumber
\]

\[
M_y = \frac{1}{8\pi \sqrt{\lambda^2 - \mu^2}} \left[ \left( K_2 \mu - \frac{K_1}{D_y} \right) R_3 - \left( K_3 \mu - \frac{K_1}{D_y} \right) R_1 + \frac{K_1}{K_2} \left( \frac{K_2 - K_3}{K_2 - K_1} \right) K_1 R_1 \right] \nonumber
\]

\(\lambda < \mu\)

\[
M_x = \frac{1}{8\pi \sqrt{\mu^2 - \lambda^2}} \left[ K_4 (\mu + \frac{D_1}{D_y}) R_3 - 2 K_3 (\mu - \frac{D_1}{D_y}) R_4 + \frac{K_3^2}{K_4} (\mu - \frac{D_1}{D_y}) R_3 \right] + 2 K_3 (\mu + \frac{D_1}{D_y}) R_4 + \frac{1}{K_4} (\mu - \frac{D_1}{D_y}) R_1 - 4 K_4 (\frac{D_1}{D_y}) \nonumber
\]

\[
M_y = \frac{1}{8\pi \sqrt{\mu^2 - \lambda^2}} \left[ K_4 (-\frac{D_1}{D_y} - 1) R_3 - 2 K_3 (\frac{D_1}{D_y} - 1) R_4 + \frac{K_3^2}{K_4} (\frac{D_1}{D_y} - 1) R_3 \right] + 2 K_3 (-\frac{D_1}{D_y} - 1) R_4 + \frac{1}{K_4} (\frac{D_1}{D_y} - \lambda) R_1 - 4 K_4 (\frac{D_1}{D_y}) \nonumber
\]

\(\lambda = \mu\)

\[
M_x = \frac{1}{8\pi \lambda} \left[ (\lambda + \frac{D_1}{D_y}) R_3 + (\lambda - \frac{D_1}{D_y}) (\lambda \beta - \eta) S_i + (\lambda + \frac{D_1}{D_y}) S_i \right] + \left\{ (\lambda + 3(\frac{D_1}{D_y})) \eta + (\lambda - \frac{D_1}{D_y}) \beta \right\} \beta \lambda - 2 \lambda \beta \eta \left( \frac{D_1}{D_y} - \lambda \right) T_i \nonumber
\]

\[
M_y = \frac{1}{8\pi \lambda^2} \left[ (\frac{D_1}{D_y} + \lambda) (R_3 - \lambda S_i) + (\frac{D_1}{D_y} - \lambda) (\lambda \beta - \eta) S_i \right] + \left\{ (\frac{D_1}{D_y} - \lambda) \beta + (\frac{D_1}{D_y} + 3 \lambda) \eta \right\} \beta \lambda - 2 \lambda \beta \eta \left( \frac{D_1}{D_y} - \lambda \right) T_i \nonumber
\]
(iv) $\lambda = \mu = | \text{(isotropic)}$

$$W = \frac{a^2}{2\pi^2 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ (1 \mp n(\beta-\gamma)) e^{\pm n(\beta-\gamma)} - (1+n\eta) e^{-n(\beta+\eta)} - \eta \beta (1+2n\eta) e^{-n(\beta+\eta)} \right] \sin \alpha \sin n\xi$$

$$M_x = \frac{1}{8\pi} \left[ (1+\nu) \log \frac{\cosh(\beta-\eta) - \cos(\alpha+\xi)}{\cosh(\beta-\eta) - \cos(\alpha-\xi)} - (1-\nu)(\beta-\gamma) \right]$$

$$- \left\{ \frac{\sinh(\beta-\gamma)}{\cosh(\beta-\eta) - \cos(\alpha+\xi)} - \frac{\sinh(\beta-\gamma)}{\cosh(\beta-\eta) - \cos(\alpha-\xi)} \right\} - (1+\nu) \log \frac{\cosh(\beta+\eta) - \cos(\alpha+\xi)}{\cosh(\beta+\eta) - \cos(\alpha-\xi)}$$

$$- (1+3\nu)(\beta+\gamma) \left\{ \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cos(\alpha+\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cos(\alpha-\xi)} \right\}$$

$$- 2(1-\nu)(\beta-\eta) \left\{ \frac{\cosh(\beta+\eta) \cos(\alpha+\xi) - 1}{(\cosh(\beta+\eta) - \cos(\alpha+\xi))^2} - \frac{\cosh(\beta+\eta) \cos(\alpha+\xi) - 1}{(\cosh(\beta+\eta) - \cos(\alpha-\xi))^2} \right\} \right]$$

$$M_y = \frac{1}{8\pi} \left[ (1+\nu) \log \frac{\cosh(\beta-\eta) - \cos(\alpha+\xi)}{\cosh(\beta-\eta) - \cos(\alpha-\xi)} + (1-\nu)(\beta-\gamma) \right]$$

$$- \left\{ \frac{\sinh(\beta-\gamma)}{\cosh(\beta-\eta) - \cos(\alpha+\xi)} - \frac{\sinh(\beta-\gamma)}{\cosh(\beta-\eta) - \cos(\alpha-\xi)} \right\} - (1+\nu) \log \frac{\cosh(\beta+\eta) - \cos(\alpha+\xi)}{\cosh(\beta+\eta) - \cos(\alpha-\xi)}$$

$$- (1+3\nu)(\beta+\gamma) \left\{ \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cos(\alpha+\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta) - \cos(\alpha-\xi)} \right\}$$

$$+ 2(1-\nu)(\beta-\eta) \left\{ \frac{\cosh(\beta+\eta) \cos(\alpha+\xi) - 1}{(\cosh(\beta+\eta) - \cos(\alpha+\xi))^2} - \frac{\cosh(\beta+\eta) \cos(\alpha+\xi) - 1}{(\cosh(\beta+\eta) - \cos(\alpha-\xi))^2} \right\} \right]$$

(4.10)
4.4 Influence Functions for the Free Edge

The boundary conditions for this case are (Fig. 4-1)

\[ \beta = 0 : \quad M_y = - \left( D_1 \frac{\partial^2 W}{\partial x^2} + D_2 \frac{\partial^2 W}{\partial y^2} \right) = 0 \]
\[ \nabla_y = - \left( (H + 2D_x) \frac{\partial^2 W}{\partial x \partial \beta} + D_y \frac{\partial^2 W}{\partial \beta^2} \right) = 0 \]

The general solution for \( W(\alpha, \beta; \xi, \eta) \) is obtained as follows.

(I) Influence Functions for the deflection \( W(\alpha, \beta; \xi, \eta) \)

(i) \( \lambda > \mu \)

\[
\frac{a^2}{2 \pi^3 \mu D_y} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ K_1 e^{\pm m_k(\beta-\eta)} - K_2 e^{\pm m_k(\beta+\eta)} + \frac{K_1 (K_1 + K_2) M}{(K_1 - K_2) L} e^{-m_k(\beta+\eta)} \right] \\
- \frac{2K_1K_2N}{(K_1 - K_2) L} e^{-m_k(\beta+\eta)} + \frac{K_2 (K_1 + K_2) M}{(K_2 - K_1) L} e^{-m_k(\beta+\eta)}
\]

(ii) \( \lambda < \mu \)

\[
\frac{a^2}{\pi^3 \mu D_y} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ e^{m_k(\beta-\eta)} (K_2 \cos n_k(\beta-\eta) = K_2 \sin n_k(\beta-\eta)) \right. \\
+ \left. \frac{K_3 M}{K_4 L} e^{-m_k(\beta+\eta)} (K_2 \cos n_k(\beta+\eta) - K_4 \sin n_k(\beta+\eta)) + \frac{N(K_2^2 + K_4^2)}{K_4 L} e^{-m_k(\beta+\eta)} \xi \cos n_k(\beta+\eta) \right]
\]

(iii) \( \lambda = \mu \)

\[
\frac{a^2}{2 \pi^3 D_y} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ (1 + \eta \sqrt{5}(\beta-\eta)) e^{\pm m_k(\beta-\eta)} + \frac{(2H^2 - 2D_x D_y H + D_y^2) + D_x^2 (\eta \sqrt{5} \eta)}{D_y (2H - D_x)} e^{-m_k(\beta+\eta)} \right] \\
+ \frac{D_{xy}(1 + 2\eta \sqrt{5} \eta)(\eta \sqrt{5} \eta)}{2H - D_{xy}} e^{-m_k(\beta+\eta)} \right] \sin \mu \sin \mu \sin \xi
\]

where

\[ L = 4D_{xy}\sqrt{D_x D_y} - D_1^3 + D_x D_y \]
\[ M = -4D_{xy}\sqrt{D_x D_y} - D_1^3 + D_x D_y \]
\[ N = 4D_1 D_{xy} + D_1^2 - D_x D_y \]
(II) Influence Functions for Bending Moment $M_x(\alpha, \beta; \xi, \eta)$, $M_y(\alpha, \beta; \xi, \eta)$

(i) $\lambda > \mu$

\[ M_x = \frac{1}{8\pi\sqrt{\lambda^2 - \mu^2}} \left[ \left( K_1 \mu - \frac{K_3 D_y}{D_y} \right) R_2 + \frac{M(K_1 + K_2)}{L(K_2 - K_1)} R_2 - \frac{2K_1 N}{L(K_2 - K_1)} R_{10} \right] \]
\[ - \left( K_2 \mu - \frac{K_3 D_y}{D_y} \right) \left[ R_1 - \frac{M(K_1 + K_2)}{L(K_2 - K_1)} R_1 + \frac{2K_1 N}{L(K_2 - K_1)} R_{11} \right] \]

\[ M_y = \frac{1}{8\pi\sqrt{\lambda^2 - \mu^2}} \left[ \left( \frac{D_1}{K_1 D_y} - K_3 \right) \left[ R_2 + \frac{M(K_1 + K_2)}{L(K_2 - K_1)} R_2 - \frac{2K_1 N}{L(K_2 - K_1)} R_{10} \right] \right] 
\[ - \left( \frac{D_1}{K_1 D_y} - K_3 \right) \left[ R_1 - \frac{M(K_1 + K_2)}{L(K_2 - K_1)} R_1 + \frac{2K_1 N}{L(K_2 - K_1)} R_{11} \right] \]

(ii) $\lambda < \mu$

\[ M_x = \frac{1}{8\pi\sqrt{\mu^2 - \lambda^2}} \left[ K_4 \left( \frac{D_1}{D_y} + 1 \right) R_3 - 2K_3 \left( \frac{D_1}{D_y} - \frac{D_3}{D_3} \right) R_4 + \frac{K_3 M}{K_4 L} \left( \frac{D_1}{D_y} - \frac{D_3}{D_3} \right) R_3 \right] 
\[ + \frac{4K_3 D_y N}{2H - D_{xy}} R_{12} - \frac{4K_3 D_y N}{L} R_{13} \]

\[ M_y = \frac{1}{8\pi\sqrt{\mu^2 - \lambda^2}} \left[ K_4 \left( \frac{D_1}{D_y} + 1 \right) R_3 - 2K_3 \left( \frac{D_1}{D_y} - \frac{D_3}{D_3} \right) R_4 + \frac{K_3 M}{K_4 L} \left( \frac{D_1}{D_y} - \frac{D_3}{D_3} \right) R_3 \right] 
\[ + \frac{4K_3 D_y N}{2H - D_{xy}} R_{12} - \frac{4K_3 D_y N}{L} R_{13} \]

(iii) $\lambda = \mu$

\[ M_x = \frac{1}{8\pi\sqrt{\lambda^2}} \left[ \left( \lambda - \frac{D_1}{D_y} \right) R_5 \pm \left( \lambda - \frac{D_1}{D_y} \right) \sqrt{\lambda} (\beta - \gamma) S_1 \right] \]
\[ + \frac{\left( 2H - 2D_{xy} \right) \lambda - 2 \left( \frac{D_1}{D_y} \right) D_{xy}^2}{2H - D_{xy}} \frac{\sqrt{\lambda} D_{xy} \left( \lambda - \frac{D_1}{D_y} \right)}{2H - D_{xy}} A_5 \]
\[ + \frac{2D_{xy} \left( \lambda - \frac{D_1}{D_y} \right)}{2H - D_{xy}} \lambda \beta \eta T_1 \]

\[ M_y = \frac{1}{8\pi\sqrt{\lambda^2}} \left[ \left( \frac{D_1}{D_y} + \lambda \right) R_5 \pm \left( \frac{D_1}{D_y} - \lambda \right) \sqrt{\lambda} (\beta - \gamma) S_1 \right] 
\[ + \frac{\left( 2H - 2D_{xy} \right) \lambda - 2 \left( \frac{D_1}{D_y} \right) D_{xy}^2}{2H - D_{xy}} \frac{\sqrt{\lambda} D_{xy} \left( \lambda - \frac{D_1}{D_y} \right)}{2H - D_{xy}} A_5 \]
\[ + \frac{2D_{xy} \left( \lambda - \frac{D_1}{D_y} \right)}{2H - D_{xy}} \lambda \beta \eta T_1 \]

(4.12)
(iv) $\lambda = \mu = 1$ (isotropic)

\[ W = \frac{a^2}{2 \pi^2 D} \sum_{n=1}^{\infty} \frac{1}{\eta^3} \left[ (1 + n(\beta - \eta)) e^{\pm \pi \eta (\beta - \eta)} + \left\{ \frac{5 + 2 \nu + \nu^2}{(3 + \nu)(1 - \nu)} + \frac{1 - \nu}{3 + \nu} \right\} (n(\beta + \eta) + 2 n^2 \beta \eta) \right] \sin \alpha \sin n \xi \]

\[ M_x = \frac{1}{8 \pi} \left[ (1 + \nu) \log \frac{\sinh(\beta - \eta) - \cosh(\alpha + \xi)}{\cosh(\beta - \eta) - \cosh(\alpha - \xi)} + (1 - \nu)(\beta - \eta) \right] \]

\[ \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cosh(\alpha - \xi)} - \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cosh(\alpha + \xi)} \right\} + \left( \frac{5 + \nu(1 - \nu)}{3 + \nu} \right) x \]

\[ \log \frac{\cosh(\beta + \eta) - \cosh(\alpha + \xi)}{\cosh(\beta + \eta) - \cosh(\alpha - \xi)} - \frac{1 - \nu}{3 + \nu} ((1 + 3 \nu) \eta + (1 - \nu) \beta) \]

\[ \left\{ \frac{\cosh(\beta + \eta) \cosh(\alpha - \xi) - 1}{(\cosh(\beta + \eta) - \cosh(\alpha - \xi))^2} - \frac{\cosh(\beta + \eta) \cosh(\alpha + \xi) - 1}{(\cosh(\beta + \eta) - \cosh(\alpha + \xi))^2} \right\} \]

\[ M_y = \frac{1}{8 \pi} \left[ (1 + \nu) \log \frac{\cosh(\beta - \eta) - \cosh(\alpha + \xi)}{\cosh(\beta - \eta) - \cosh(\alpha - \xi)} - (1 - \nu)(\beta - \eta) \right] \]

\[ \frac{-\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cosh(\alpha - \xi)} - \frac{-\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cosh(\alpha + \xi)} \right\} + \left( \frac{5 + \nu(1 - \nu)}{3 + \nu} \right) x \]

\[ \log \frac{\cosh(\beta + \eta) - \cosh(\alpha + \xi)}{\cosh(\beta + \eta) - \cosh(\alpha - \xi)} - \frac{1 - \nu}{3 + \nu} ((1 + 3 \nu) \eta + (1 - \nu) \beta) \]

\[ \left\{ \frac{\cosh(\beta + \eta) \cosh(\alpha - \xi) - 1}{(\cosh(\beta + \eta) - \cosh(\alpha - \xi))^2} - \frac{\cosh(\beta + \eta) \cosh(\alpha + \xi) - 1}{(\cosh(\beta + \eta) - \cosh(\alpha + \xi))^2} \right\} \]

(4.13)
From the practical point of view, the most general case is the case where the third edge is elastically supported. The corresponding boundary conditions of the third edge are:

\[ u = 0, \quad M_y = EI \omega \frac{\partial^5 W}{\partial u^4 \partial V} - GK_t \frac{\partial^3 W}{\partial u^3 \partial V} \]

or

\[ D_t \frac{\partial^3 W}{\partial u^3} + D_t \frac{\partial^3 W}{\partial V^3} = GK_t \frac{\partial^3 W}{\partial u^3 \partial V} - EI \omega \frac{\partial^5 W}{\partial u^4 \partial V} \]

where

- \( EI \): Bending stiffness of the edge beam.
- \( GK_t \): St. Venant's torsional rigidity of the edge beam.
- \( EI_w \): Warping rigidity of the edge beam.

The solution can be obtained in the same way as illustrated before, though it may be very complicated.

Three cases treated in this chapter are actually the special cases of this particular problem.
CHAPTER V

Influence Function for a Rectangular Plate
With Simply Supported Edges

5.1 Method of Solution

The influence surface for the deflection of a rectangular plate with simply supported edges will be derived in double Fourier series form (Navier's Solution) and thereafter it will be converted into a single series form (Levy's solution).

It turns out to be a simpler way to find the solution than the ordinary method illustrated in Chapter IV.

As far as the influence surfaces for bending moments $M_x$ and $M_y$ are concerned, influence functions can be expressed in terms of Jacobi's elliptic functions in this particular case.

Making the length of one side, say $b$, infinitely large, the solutions for semi-infinite as well as infinite plate strip will be derived again with the aid of Fourier's integrals.

5.2 Navier's Solution for a Rectangular Plate with Simply Supported Edges

Consider a rectangular plate whose sides are $a$ and $b$ respectively (Fig. 5.1). The concentrated load $P=1$ acting at $(x,y)$ can be expressed in the following double Fourier Series:

$$ P(u,v) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi u}{a} \sin \frac{n\pi v}{b} $$

where

$$ a_{mn} = \frac{4}{ab} \int_0^a \int_0^b P(u,v) \sin \frac{m\pi u}{a} \sin \frac{n\pi v}{b} \, du \, dv $$

$$ = \frac{4}{ab} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} $$

Assuming the solution $W(u,v;x,y)$

$$ W(u,v;x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin \frac{m\pi u}{a} \sin \frac{n\pi v}{b} $$

(5.2)
It is easily seen that all boundary conditions are satisfied by equation (5.2). Substituting equation (5.2) into the original partial differential equation, \( b_{mn} \) can be determined.

\[
\begin{align*}
b_{mn} &= \frac{4}{ab} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \left( \frac{m \pi}{a} \right)^4 D_x + 2 \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 H + \left( \frac{n \pi}{b} \right)^4 D_y
\end{align*}
\]

Therefore the solution can be written as follows:

\[
W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)}{\lambda_{mn}^2} \varphi_{mn}(u,v)
\]

where

\[
\varphi_{mn}(u,v) = \frac{2}{\sqrt{ab}} \sin \frac{m \pi u}{a} \sin \frac{n \pi v}{b}
\]

\[
\lambda_{mn}^2 = D_x \left( \frac{m \pi}{a} \right)^4 + 2 H \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + D_y \left( \frac{n \pi}{b} \right)^4
\]

This is the solution for rectangular or orthotropic plate corresponding to Navier's solution for an isotropic plate.

5.3 Transformation of Navier's Solution into Levy's Solution

Navier's solution can be transformed into Levy's solution with the aid of the following summation formulae (See (2) p.198 Appendix)

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{(n^2 + k^2)^2} = -\frac{1}{2k^2} + \frac{\pi^2}{4k^2} \frac{\cosh kx}{\sinh k\pi} + \frac{\pi}{4k^2} \frac{\cosh k(\pi - x)}{\sinh k\pi}
\]

\[
+ \frac{\pi x}{4k^2} \frac{\sinh k(\pi - x)}{\sinh k\pi} \quad (0 \leq x \leq \pi)
\]

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{(n^2 + k^2)(n^2 + k^2')} = \frac{1}{k^2 - k'^2} \sum_{n=1}^{\infty} \left( \frac{\cos nx}{n^2 + k^2} - \frac{\cos nx}{n^2 + k^2'} \right)
\]

\[
= \frac{1}{k^2 - k'^2} \left[ \frac{\pi}{2k} \frac{\cosh k(x - \pi)}{\sinh k\pi} - \frac{1}{2k} - \frac{\pi}{2k'} \frac{\cosh k(x - \pi)}{\sinh k\pi} + \frac{1}{2k^2} \right] \quad (0 \leq x \leq \pi)
\]
Taking the case of \( \lambda > \mu \), the transformation will be illustrated briefly. From equation (5.4):

\[
W = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{a} \sin \frac{n\pi y}{b}}{(m^2 \pi^2 + 2H \frac{2\lambda}{a} \pi^2 \frac{b^4}{a^2} + n^2 \pi^2) D_x + 2H (\frac{2\lambda}{a} \pi^2 \frac{b^4}{a^2}) + n^2 \pi^2 D_y}
\]

where \( K_1, K_2 \) are the constants defined in Chapter III.

Applying the second formula of equation (5.5) to the series of \( y \) in equation (5.6):

\[
W = \frac{2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{a} \sin \frac{n\pi y}{b}}{(m^2 \pi^2 + (\frac{mbk_1}{a})^2 + (\frac{mbk_2}{a})^2)} \left\{ \cos \frac{n\pi}{b} (y - \nu) - \cos \frac{n\pi}{b} (y + \nu) \right\}
\]

This solution for an orthotropic rectangular plate corresponds to Levy's solution for an isotropic plate. Without repeating the mathematical operation, the results obtained are summarized as follows. Again, non-dimensional coordinates as defined in Chapter IV are employed with another new parameter \( \gamma = \frac{\pi b}{a} \).
(I) Influence Functions for the Deflection $W(\alpha, \beta; \xi, \eta)$

(i) \( \lambda > \mu \)

\[
\frac{a^2}{2\pi^2 \mu D_y \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[ \frac{\cosh k_3 (\beta - \eta \tau)}{k_3 \sinh k_3 \tau} - \frac{\cosh k_3 (\beta - \eta \tau)}{k_3 \sinh k_3 \tau} \right] \sin \eta \sin \xi
\]

(ii) \( \lambda < \mu \)

\[
\frac{a^2}{\pi^2 \mu D_y \mu^2 - \lambda^2} \sum_{n=1}^{\infty} \frac{\sin \eta \sin \xi}{n^3 (\sinh k_3 \tau + \sinh k_3 \eta)} \left[ \frac{\cosh k_3 (\beta - \eta \tau) \cosh k_3 (\beta - \eta \tau)}{k_3 \sinh k_3 \eta} \right] \left[ \sinh k_3 (\beta + \eta - \xi) - \sinh k_3 (\beta + \eta - \xi) \right] (k_3 \sinh k_3 \eta \cosh k_3 \xi - k_3 \cosh k_3 \xi \sinh k_3 \eta)
\]

(iii) \( \lambda = \mu \)

\[
\frac{a^2}{\pi^2 D_y \sqrt{\lambda^2 - \lambda^2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{\sinh n\pi (\xi - \beta)}{\sinh n\pi \xi} \left( \cosh n\pi \eta - \cosh n\pi \eta \cosh n\pi \eta \right) + \right. \\
\left. \frac{\sinh n\pi \eta}{\sinh n\pi \eta} \left( \cosh n\pi \beta \cosh n\pi (\xi - \beta) - \frac{\sinh n\pi \beta}{\sinh n\pi \xi} \right) \sin \eta \sin \xi \right]
\]

\[
\left. \sin \eta \sin \xi \right] (\beta \leq \eta)
\]

\[
\frac{a^2}{\pi^2 D_y \sqrt{\lambda^2 - \lambda^2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{\sinh n\pi \beta}{\sinh n\pi \xi} \left( \cosh n\pi (\xi - \eta) - \cosh n\pi (\xi - \eta) \cosh n\pi (\xi - \eta) \right) + \right. \\
\left. \frac{\sinh n\pi (\xi - \eta)}{\sinh n\pi \xi} \left( \cosh n\pi (\xi - \beta) \cosh n\pi \beta - \frac{\sinh n\pi \beta}{\sinh n\pi \xi} \right) \sin \eta \sin \xi \right]
\]

\[
\left. \sin \eta \sin \xi \right] (\beta \leq \eta)
\]
5.4 Representation of $M_x, M_y$ -Influence Functions by Jacobi's θ-Functions

Differentiating the solution equation (5.7) with respect to $\alpha, \beta$ twice, the influence function for the bending moments can be obtained in single series form.

However, the theory of elliptic function shows that such series can be expressed in terms of Jacobi's θ-functions. (18)

The illustration will be made here only in case of $\lambda > \mu$.

Assuming $\beta > \eta$ and carrying out the differentiation of $W(\alpha, \beta; \xi, \eta)$ with respect to $\alpha, \beta$ and forming $M_x(\alpha, \beta; \xi, \eta)$

$$M_x = -\left( D_1 \frac{\partial^2 W}{\partial \alpha^2} + D_1 \frac{\partial^2 W}{\partial \beta^2} \right)$$

$$= \frac{1}{4\pi^2 \lambda^2 - \mu^2} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left( \frac{\mu^2}{k^2} - \frac{K_0}{\rho^2} \right) \left( \frac{\cosh k_1 (\beta-\eta-\xi) - \cosh k_1 (\beta+\eta+\xi)}{\sinh n k_1 \delta} \right) \right\} (5.8)$$

Using the relation:

$$\cosh x = \cos i x$$

new complex variables are introduced:

$$u + i k_1 v = a \xi_0$$

$$\alpha + i k_1 \beta = \pi \xi_0$$

$$\xi + i k_1 \eta = \pi \xi_0$$

so that

$$\alpha + i k_1 \beta = \pi \xi_0$$

$$\xi + i k_1 \eta = \pi \xi_0$$

$$\xi + i k_1 \eta = \pi \xi_0$$
Using the new notation equation (5.9), equation (5.8) is rewritten:

\[ M_x = \frac{1}{8\pi^2 \lambda^2} \sum_{n=1}^{\infty} \frac{1}{n} \left[ \frac{k_0^2 - k_0^2}{\sinh nk_0^2} \right] \cos n\pi \left( \frac{z - z_o + iKz}{\pi} \right) + \cos n\pi \left( z - z_o - \frac{iKz}{\pi} \right) 
- \cos n\pi \left( z' - z_o + iKz \right) - \cos n\pi \left( z' - z_o - \frac{iKz}{\pi} \right) 
- \cos n\pi \left( z' + z_o + \frac{iKz}{\pi} \right) + \cos n\pi \left( z' + z_o + \frac{iKz}{\pi} \right)

The theory of elliptic function furnishes the following mathematical relations:

\[ \log \nu_0(z) = \sum_{n=1}^{\infty} \log(1 - 8^{2n}) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{8^{2n} \cos 2n\pi z}{n(1 - 8^{2n})} \quad (5.10) \]

and

\[ \nu_0(z \pm \frac{1}{2}\tau) = \pm i 8^{-\frac{1}{2}} e^{\frac{i\pi\tau}{8}} \nu_0(z) \]

where

\[ \theta = e^{i\pi\tau} \quad \tau = \text{the period of } \nu_0(z) \]

Using equation (5.10)

\[ \sum_{n=1}^{\infty} \cos n\pi \left( z - z_o + iKz \right) \right] \frac{2 \theta^{2n}}{n(1 - \theta^{2n})} = \sum_{n=1}^{\infty} \frac{2 \theta^{2n}}{n(1 - \theta^{2n})} \cos n\pi \left( z - z_o + iKz \right) 
= \frac{1}{2} \sum_{n=1}^{\infty} \log \left(1 - 8^{2n}\right) - \frac{1}{2} \log \nu_0 \left( \frac{z - z_o + \tau'}{2} \right) \quad (5.11) \]

where

\[ \tau' = \frac{iKz}{\pi} \quad \theta = e^{2\pi} = e^{-\frac{Kz}{\pi}} \]

Performing the mathematical operation indicated in (5.11)
$$M_x = \frac{1}{8\pi\sqrt{\lambda^2 - \mu^2}} \text{Re} \left[ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \right.$$ 

$$+ \log \vartheta_0\left( \frac{\bar{r} - \bar{s}' + t}{2} \right) + \log \vartheta_0\left( \frac{\bar{r} - \bar{s}' - t'}{2} \right) + \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' + t'}{2} \right)$$ 

$$+ \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' - t}{2} \right) - \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' + t}{2} \right) - \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' - t}{2} \right) \right]$$ 

$$+ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{\bar{r} - \bar{s}' - t}{2} \right)}{\vartheta_1\left( \frac{\bar{r} + \bar{s}' + t}{2} \right)} + \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' + t}{2} \right) + \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' - t}{2} \right)$$ 

$$+ \log \vartheta_0\left( \frac{\bar{r} - \bar{s}' - t}{2} \right) + \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' + t}{2} \right) + \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' - t}{2} \right)$$ 

$$- \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' + t}{2} \right) - \log \vartheta_0\left( \frac{\bar{r} + \bar{s}' - t}{2} \right) \right]$$ 

$$\text{where} \quad \bar{z} = \frac{\imath \kappa}{\lambda}$$ 

$$= \frac{1}{4\pi\sqrt{\lambda^2 - \mu^2}} \text{Re} \left[ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \right.$$ 

$$\begin{bmatrix} \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \\ \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \end{bmatrix}$$ 

\text{(5.12)}

$$M_y = \frac{1}{4\pi\sqrt{\lambda^2 - \mu^2}} \text{Re} \left[ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \right.$$

$$\begin{bmatrix} \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \\ \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \end{bmatrix}$$

\text{(5.12)}$$

In the same way, influence functions for the other cases of bending moments can be expressed in terms of Jacobi's elliptic function. The results obtained are summarized as follows:

(i) $\lambda > \mu$

$$M_x = \frac{1}{4\pi\sqrt{\lambda^2 - \mu^2}} \text{Re} \left[ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \right.$$ 

$$\begin{bmatrix} \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \\ \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \end{bmatrix}$$ 

$$\text{(5.12)}$$

$$M_y = \frac{1}{4\pi\sqrt{\lambda^2 - \mu^2}} \text{Re} \left[ \left( \frac{\mu^2}{K_i} - \frac{K_D}{D_y} \right) \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \right.$$ 

$$\begin{bmatrix} \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \\ \log \frac{\vartheta_1\left( \frac{r - \bar{r} + t}{2} \right)}{\vartheta_1\left( \frac{r + \bar{r} - t}{2} \right)} \end{bmatrix}$$ 

\text{(5.12)}$$
(ii) \( \lambda < \mu \)

Remembering the relations:

\[
K_1 = K_3 + iK_4, \quad K_2 = K_3 - iK_4
\]

Expressions for \( M_x \) and \( M_y \) for the case \( \lambda > \mu \) can be used.

(iii) \( \lambda = \mu \)

\[
M_x = \frac{1}{4\pi} \text{Re} \left[ (\sqrt{\lambda} + \frac{D_1}{\lambda \partial_y}) \log \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} \right] - i(\lambda - \frac{D_1}{\lambda \partial_y}) \left\{ (\beta + \eta) \left( \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} - \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} \right) \right\}
\]

\[
M_y = \frac{1}{4\pi} \text{Re} \left[ \left( \frac{D_1}{\lambda \partial_y} + \frac{1}{\sqrt{\lambda}} \right) \log \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} \right] - i(\frac{D_1}{\lambda \partial_y} - 1) \left\{ (\beta + \eta) \left( \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} - \frac{\mathcal{N}^1(\frac{w-w_y}{\lambda})}{\mathcal{N}^1(\frac{w-w_y}{\lambda})} \right) \right\}
\]

where

\[
a \omega_0 = u + i \sqrt{\lambda} \quad \omega = x + i \sqrt{\lambda} \quad \omega_0 = u - i \sqrt{\lambda} \quad \omega = x - i \sqrt{\lambda}
\]

\[
\varphi = e^{-\sqrt{\lambda} x}
\]

5.5 Some Remarks on the Computation of \( M_x(\alpha, \beta; \xi, \eta) \) and \( M_y(\alpha, \beta; \xi, \eta) \)

According to the theory of elliptic functions an expansion formulae for \( \mathcal{N}^1(z) \) exists:

\[
\mathcal{N}^1(z) = 2 \sum_{n=1}^{\infty} (-1)^n q^{(2n+1)^2} \sin(2n+1) \pi z = 2 \frac{1}{z} (\sin z - \sin 3 \pi z + 3 \sin 5 \pi z - \cdots )
\]

where \( \pi z = \alpha + i \beta \).
In order to investigate the convergence of this series, a value for $\gamma / \pi$ is assumed, with $\gamma / \pi = b/\alpha = 1.5$:

$$\begin{align*}
\gamma &= e^{-1.5\pi} = 0.00898 \quad \text{(isotropic)}
\end{align*}$$

Therefore series (5.13) converges so rapidly that the sum of first two terms will give a very accurate result. So

$$V_1(\varepsilon) \sim 2 \frac{1}{\gamma} \left[ (\sin \alpha \cosh \beta + i \cos \alpha \sinh \beta) - 8^2 (\sin \alpha \cosh \beta + i \cos \alpha \sinh \beta) \right]$$

Putting

$$\begin{align*}
V_1 &= \varphi_1 + i \psi_1 \\
\varphi_1 &\sim 2 \frac{1}{\gamma} (\sin \alpha \cosh \beta - 8 \sin \alpha \cosh \beta) \\
\psi_1 &\sim 2 \frac{1}{\gamma} (\cos \alpha \sinh \beta - 8 \cos \alpha \sinh \beta)
\end{align*}$$

Using this expansion formulae, very accurate results of $M_x$ and $M_y$ can be obtained. The influence functions for $M_x$ or $M_y$ of semi-infinite plate strip can be deduced from (5.12) making $b \to \infty$

Consider $M_x$, in case of $\lambda > \mu$.

With the aid of expansion formula (5.15)

$$\text{Re} \left[ \log \frac{\mathcal{V}_1(\xi - \eta)}{\mathcal{V}_1(\xi + \eta)} \right] = \frac{1}{2} \left[ \log \left| \mathcal{V}_1(\xi - \eta) \right|^2 \left| \mathcal{V}_1(\xi + \eta) \right|^2 \right]$$

Since $b \gg 0$, $\gamma \approx 0$

$$\begin{align*}
|\mathcal{V}_1(\xi - \eta)|^2 &\sim 4 \frac{1}{\gamma} (\cosh K_2 (\beta + \gamma) - \cosh (\alpha - \xi)) \\
|\mathcal{V}_1(\xi + \eta)|^2 &\sim 4 \frac{1}{\gamma} (\cosh K_2 (\beta - \eta) - \cosh (\alpha + \xi)) \\
|\mathcal{V}_1(\xi - \eta)|^2 &\sim 4 \frac{1}{\gamma} (\cosh K_2 (\beta + \gamma) - \cosh (\alpha - \xi)) \\
|\mathcal{V}_1(\xi + \eta)|^2 &\sim 4 \frac{1}{\gamma} (\cosh K_2 (\beta - \eta) - \cosh (\alpha + \xi))
\end{align*}$$
Similarly
\[ \text{Re} \left[ \log \frac{J_1 \left( \frac{\xi + \xi'}{2} \right) J_1 \left( \frac{\xi - \xi'}{2} \right)}{J_1 \left( \frac{\xi' - \xi}{2} \right) J_1 \left( \frac{\xi + \xi}{2} \right)} \right] \sim \frac{1}{2} \left( R_1 - \bar{R}_1 \right) \]

Making \( b \to \infty \) so that \( q \to 0 \).

\[ M_x = \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ (\kappa_1 \mu - \frac{k_0}{B}) (R_2 - \bar{R}_2) - (\kappa_2 \mu - \frac{k_0}{B}) (R_1 - \bar{R}_1) \right] \]

This corresponds to the result obtained previously for a semi-infinite strip.

### 5.6 Application of Fourier Integrals for the Solution of Semi-Infinite Plate Strips.

Another simple way to find the solution for the semi-infinite plate strip is to start from the Navier's solution for a rectangular plate. As an example an isotropic plate is considered whose influence function for the deflection is

\[ W(u,v;x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Phi_m(u,v) \Phi_n(x,y)}{\lambda n} \]

\[ = \frac{2}{abD} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{m\pi y}{a} \sum_{n=1}^{\infty} \frac{\cos \frac{2\pi}{b}(y-n) - \cos \frac{2\pi}{b}(y+n)}{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{a} \right)^2} \]

Making \( b \to \infty \) the summation of series with respect to \( y \) is replaced by an integral

\[ W(u,v;x,y) = \frac{2}{a\pi D} \sum_{m=1}^{\infty} \sin \frac{m\pi u}{a} \sin \frac{m\pi x}{a} \int_0^{\infty} \frac{\cos \frac{2\pi}{b}(y-n) - \cos \frac{2\pi}{b}(y+n)}{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \, dp \quad (5.16) \]

Using the relation:

\[ \int_0^{\infty} \frac{\cos mx}{(x^2 + a^2)^2} \, dx = \frac{\pi e^{-ma}}{4a^3} \left( 1 + ma \right) \quad (m \geq 0) \quad (5.17) \]
(5.16) can be transformed to:

\[
W(\alpha, \beta; x, y) = \frac{a^2}{2\pi^3D} \sum_{m=1}^{\infty} \left\{ \left( 1 + \frac{m\pi}{a}(y - \nu) \right) e^{\pm \frac{m\pi y}{a}} \right. \\
\left. - (1 + \frac{m\pi}{a}(y + \nu)) e^{-\frac{m\pi y}{a}} \right\} \sin \frac{m\pi x}{a} \sin \frac{m\pi y}{a}
\]

This checks the results obtained in Chapter IV.

It is apparent that the first series represents the influence function of an infinite plate strip. The second series is due to an anti-symmetric load \( P \) with respect to the \( x \)-axis (Mirror Method).

Further applications of the Fourier integral will be discussed in Chapter VII.

5. Other Boundary Value Problems of Rectangular Plates

If a rectangular plate has two parallel edges simply supported solutions in product form as illustrated in Chapter IV are applicable (5-1). However, for other conditions solutions can be obtained by superposition, taking equation (5.4) as the particular solution of the problem. Unfortunately the solution leads to an infinite number of simultaneous equations for which only approximate solutions are possible. (Fig. 5-4)

By making the length of one edge infinitely long in those solutions obtained so that changing the summation to an integral, solutions for semi-infinite plate strip can be derived in Fourier integral form. (Fig. 5-5).
An Infinite Plate Strip With Simply Supported Parallel Edge (Fig. 6.1)

At \( y = 0 \) the plate is continuous over an elastic cross beam with a constant bending stiffness \( EI \). The coordinates of a point on the cross beam are taken as \((z,0)\) -- \( z \) being the \( x \)-coordinate -- in order to distinguish this point from a general point \((u,v)\), referred to as the influence point. The deflection of a general point \((u,v)\) due to a concentrated load \( P \) at point \( P(x,y) \) can be expressed by the following integral equation:

\[
W(x,f,\eta) = P G(\alpha,\beta;\xi,\eta) - \int_0^\pi EI(\pi) \frac{\partial^2 W(\xi,0;\xi,\eta)}{\partial \xi^2} G(\alpha,\beta;\xi,0) d\xi \quad (6.1)
\]

Here again non-dimensional coordinates defined in Chapter III are introduced with a new parameter

\[
\zeta = \frac{\pi z}{a}
\]

The function \( G(\alpha,\beta;\xi,\eta) \) is Green's function for the deflection of point \((\alpha,\beta)\) of an infinite plate strip with simply supported edges. (It is given in Chapter III, p.25).

The first term under the integral sign in equation (6.1)

\[
EI(\pi) \frac{\partial^2 W(\xi,0;\xi,\eta)}{\partial \xi^2}
\]

expresses the distributed reaction of the cross beam acting on the plate.

When multiplied by Green's function \( G(\alpha,\beta;\xi,0) \) and integrated over the length of the cross beam the integral constitutes the influence of this beam on the deflection at point \((\alpha,\beta)\).

Assuming the deflection surface \( W \) in the form

\[
W(\alpha,\beta;\xi,\eta) = \phi(\alpha,\beta;\xi,\eta) + P G(\alpha,\beta;\xi,\eta) \quad (6.2)
\]
The function \( \phi \) is determined by substituting (6.2) into (6.1)

\[
\phi(\alpha, \beta; \xi, \eta) = -\frac{\pi^2 EI}{a^3} \int_0^\pi \frac{\partial^4 W(\xi, 0; \xi, \eta)}{\partial \xi^4} G(\alpha, \beta; \xi, 0) \, d\xi
\]  

(6.3)

Since \( \phi \) is a continuous function with respect to \( \alpha \) and \( \beta \), it can be developed into eigen-functions associated with Green's function \( G \) as follows:

\[
\phi(\alpha, \beta; \xi, \eta) = \sum_{n=1}^\infty a_n(\xi, \eta) \varphi_n(\alpha, \beta)
\]  

(6.4)

Confining the discussion to the case \( \lambda > \mu \):

\[
\varphi_n(\alpha, \beta) = (K_1 e^{\pi n k_1 \beta} - K_2 e^{\pi n k_1 \beta}) \sin n \alpha
\]

Substituting into (6.3) and replacing \( G \) by equation (3.14) gives:

\[
\sum_{n=1}^\infty a_n(\xi, \eta) \varphi_n(\alpha, \beta) = -\frac{\pi^2 EI}{a^3} \int_0^\pi \frac{\partial^4 W(\xi, 0)}{\partial \xi^4} \left\{ \frac{a^2}{2\pi^2 \mu \beta^{1/2}} \sum_{m=1}^\infty \frac{1}{m^3} \left[ (K_1 e^{\pi m k_1 \beta} - K_2 e^{\pi m k_1 \beta}) \sin m \alpha \sin m \eta \right] \right\} d\xi
\]

(6.5)

Multiplying both sides by \( \sin n \alpha \) and integrating with respect to \( \alpha \) from 0 to \( \pi \), the orthogonality relations simplify equation (6.5) considerably.

\[
\frac{\pi}{2} a_n(\xi, \eta) (K_1 e^{\pi n k_1 \beta} - K_2 e^{\pi n k_1 \beta}) = -\frac{\pi}{2} \frac{EI}{2\pi^2 \mu \beta^{1/2}} \sum_{m=1}^\infty \frac{1}{m^3} \int_0^\pi \frac{\partial^4 W(\xi, 0)}{\partial \xi^4} \sin m \xi \sin m \eta \, d\xi
\]

(6.6)

with the substitution

\[
\frac{\partial^4 W(\xi, 0)}{\partial \xi^4} = \frac{\partial^4}{\partial \xi^4} \left[ \phi(\xi, 0; \xi, \eta) + PG(\xi, 0; \xi, \eta) \right]
\]

\[
= \sum_{m=1}^\infty \left[ m^4 a_m(\xi, \eta) \sin m \xi + \frac{m a^2}{2\pi^2 \mu \beta^{1/2}} \left( K_1 e^{\pi m k_1 \eta} - K_2 e^{\pi m k_1 \eta} \right) \sin m \xi \sin m \eta \right]
\]
the function $a_n$ can be determined. Again use of the orthogonality relations is made. Introducing the parameter

$$\frac{1}{\rho} = \frac{(K_1 - K_2)\pi EI}{4a\mu D_y \lambda^3 - \mu^3}$$

(6.7)

$$a_n = -\frac{a^2}{2n^3\mu^3(K_1 - K_2)D_y \lambda^3 - \mu^3}(n + \rho)(k_1 e^{\pm n\kappa\eta} - k_1 e^{\mp n\kappa\eta}) \sin\eta \sin n\xi$$

(6.8)

The non-dimensional parameter $\rho$ depends on the ratio of the bending stiffness of the plate in $y$ direction $D_y$ and the bending stiffness of the cross beam $EI$ as well as $\lambda$ and $\mu$. Substituting the pertinent values into (6.2) with $\rho = 1$ yields the influence function for the deflection:

$$W(\alpha, \beta; \xi, \eta) = \frac{a^2}{2\pi^3\mu D_y \lambda^3 - \mu^3} \sum_{n=1}^{\infty} \frac{1}{n^2} (k_1 e^{\pm n\kappa\eta} - k_1 e^{\mp n\kappa\eta})$$

$$- \frac{1}{n^2(m + \rho)(k_1 - k_2)}(k_1 e^{-n\kappa\eta} - k_1 e^{n\kappa\eta})(k_1 e^{-n\kappa\eta} - k_1 e^{-n\kappa\eta}) \sin\omega \sin n\xi$$

(6.9)

The first term within the parenthesis represents the influence surface for the deflection of point $(\alpha, \beta)$, of a simply supported plate strip without cross beam. The second term expresses the influence of this beam. If the cross beam is infinitely rigid, that is, $EI \to \infty$ and $\rho \to 0$, the coefficient of the second term reduces to:

$$\lim_{\rho \to 0} \frac{1}{n^2(m + \rho)(k_1 - k_2)} = \frac{1}{m^2(k_1 - k_2)}$$
On the other hand, in the absence of a cross beam, $EI \to 0$, and 
$p \to \infty$ such that

$$
l \lim_{p \to \infty} \frac{1}{n^2(n+p)(k_1-k_2)} = 0
$$

and the second term will disappear entirely (reduced to the case
of infinite plate strip). The following results were obtained
in a similar manner:

(i) Influence Surfaces for the deflection $W(\alpha, \rho; \xi, \eta)$

\[(i) \quad \lambda > \mu\]

\[
\lambda \frac{a^2}{2 \pi^2 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \left[ \frac{1}{n^2} \left( K_1 e^{\pm nk_1(\beta-\eta)} - K_2 e^{\pm nk_1(\beta+\eta)} \right) \right] \sin \alpha \sin \xi
\]

where

\[
\frac{1}{\rho} = \frac{\pi EI (k_1-k_2)}{4a \mu D_y \sqrt{\lambda^2 - \mu^2}}
\]

(ii) $\lambda < \mu$

\[
\lambda \frac{a^2}{\pi^2 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \left[ \frac{e^{\pm nk_1(\beta-\eta)}}{n^2} \left( K_4 \cos nk_4(\beta-\eta) + K_2 \sin nk_4(\beta-\eta) \right) \right] (6,10)
\]

\[
- \frac{e^{-nk_1(\beta+\eta)}}{n^2(n+p)k_4} (K_4 \cos nk_4^2 + K_2 \sin nk_4)(K_4 \cos nk_4 + K_2 \sin nk_4) \sin \alpha \sin \xi
\]

where

\[
\frac{1}{\rho} = \frac{K_4 EI}{2a \mu D_y \sqrt{\lambda^2 - \mu^2}}
\]

(iii) $\lambda = \mu$

\[
\lambda \frac{a^2}{2 \pi^2 D_y \sqrt{\lambda^2}} \sum_{n=1}^{\infty} \left[ \frac{1}{n^3} \left( \frac{1 + \sqrt{n} \xi (\beta-\eta)}{n^{1/n} (1 + \sqrt{n} \xi) (\beta-\eta)} \right) \right] \sin \alpha \sin \xi
\]

where

\[
\frac{1}{\rho} = \frac{\pi EI}{4a D_y \sqrt{\lambda^2}}
\]
(II) Influence Surfaces for Moments $M_x(\alpha, \beta; \xi, \eta)$ and $M_y(\alpha, \beta; \xi, \eta)$

(i) $\lambda > \mu$

\[
M_x = \frac{1}{2 \pi \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \left[ (K_i \mu - \frac{K_i D_i}{D_y}) \left\{ \frac{e^{\pm nk_1(x - \mu)}}{n} + \frac{K_i e^{-nk_1(x - \mu)}}{(n+\rho)(K_2 - K_1)} \right\} \right. \\
- \left( K_i \mu - \frac{K_i D_i}{D_y} \right) \left\{ \frac{e^{\pm nk_1(x - \mu)}}{n} + \frac{K_i e^{-nk_1(x - \mu)}}{(n+\rho)(K_2 - K_1)} \right\} \right] \sin(n \xi) \sin(n \eta)
\]

\[
M_y = \frac{1}{2 \pi \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \left[ (K_i \mu - \frac{K_i D_i}{D_y}) \left\{ \frac{e^{\pm nk_1(y - \mu)}}{n} + \frac{K_i e^{-nk_1(y - \mu)}}{(n+\rho)(K_2 - K_1)} \right\} \right. \\
- \left( K_i \mu - \frac{K_i D_i}{D_y} \right) \left\{ \frac{e^{\pm nk_1(y - \mu)}}{n} + \frac{K_i e^{-nk_1(y - \mu)}}{(n+\rho)(K_2 - K_1)} \right\} \right] \sin(n \xi) \sin(n \eta)
\]

(ii) $\lambda < \mu$

\[
M_x = \frac{1}{\pi \sqrt{\mu^2 - \lambda^2}} \sum_{n=1}^{\infty} \left[ \frac{e^{\pm nk_1(x - \mu)}}{n} \right] K_4(\mu + \frac{D_i}{D_y}) \cos(nK_4(x - \mu)) + K_3(\mu - \frac{D_i}{D_y}) x \sin(nK_4(x - \mu)) \right\} \right. \\
- \left\{ \frac{e^{-nk_1(x - \mu)}}{n} \right\} (K_4 \cos(nK_4 + K_3 \sin(nK_4)) \right\} \right]
\]

\[
M_y = \frac{1}{\pi \sqrt{\mu^2 - \lambda^2}} \sum_{n=1}^{\infty} \left[ \frac{e^{\pm nk_1(y - \mu)}}{n} \right] K_4(\mu + \frac{D_i}{D_y} + 1) \cos(nK_4(y - \mu)) + K_3(\mu - \frac{D_i}{D_y} - 1) x \sin(nK_4(y - \mu)) \right\} \right. \\
- \left\{ \frac{e^{-nk_1(y - \mu)}}{n} \right\} (K_4 \cos(nK_4 + K_3 \sin(nK_4)) \right\} \right]
\]

(iii) $\lambda = \mu$

\[
M_x = \frac{1}{2 \pi \sqrt{\lambda}} \sum_{n=1}^{\infty} \left[ \frac{1}{n} \right] \left( \lambda + \frac{D_i}{D_y} \right) \pm (\lambda - \frac{D_i}{D_y}) n \sqrt{x} (\eta - \mu) \left\{ \frac{e^{\pm n \sqrt{x}(\eta - \mu)}}{n} \right\} \right. \\
- \left\{ \frac{1}{n+\rho} (1 + n \sqrt{x}) \right\} (\lambda + \frac{D_i}{D_y} \pm (\lambda - \frac{D_i}{D_y}) n \sqrt{x} \eta) \left\{ \frac{e^{\pm n \sqrt{x} \eta}}{n} \right\} \right] \sin(n \xi) \sin(n \eta)
\]

\[
M_y = \frac{1}{2 \pi \sqrt{\lambda}} \sum_{n=1}^{\infty} \left[ \frac{1}{n} \right] \left( \lambda + \frac{D_i}{D_y} \right) \pm (\lambda - \frac{D_i}{D_y}) n \sqrt{y} (\eta - \mu) \left\{ \frac{e^{\pm n \sqrt{y}(\eta - \mu)}}{n} \right\} \right. \\
- \left\{ \frac{1}{n+\rho} (1 + n \sqrt{y}) \right\} (\lambda + \frac{D_i}{D_y} \pm (\lambda - \frac{D_i}{D_y}) n \sqrt{y} \eta) \left\{ \frac{e^{\pm n \sqrt{y} \eta}}{n} \right\} \right] \sin(n \xi) \sin(n \eta)
\]
upper sign for $\beta \leq \eta$
lower sign for $\beta \geq \eta$

and if $\eta < 0$; the signs preceding $\eta$ must be changed in the second series of above equations.

In general, it appears to be impossible to sum the series of the equation (6.11). However, for several specific values of $\beta$, such a summation can be made. (15)(16)(17) The results obtained for the case of a rigid cross beam, that is, $\rho = 0$ are tabulated as follows.

(III) $M_x, M_y$ - Influence Surfaces for the Case of the Rigid Cross Beam ($\rho = 0$)

(i) $\lambda > \mu$

$$M_x = \frac{1}{8\pi \sqrt{\lambda^3 - \mu^3}} \left[ \left( \frac{\lambda}{\mu} \right)^2 \gamma \left( \begin{array}{c} \gamma R_1 - 1 \\ - \lambda \end{array} \right) \right]$$

$$M_y = \frac{1}{8\pi \sqrt{\lambda^3 - \mu^3}} \left[ \left( \frac{\lambda}{\mu} \right)^2 \gamma \left( \begin{array}{c} \gamma R_1 - 1 \\ - \lambda \end{array} \right) \right]$$

(ii) $\lambda < \mu$

$$M_x = \frac{1}{4\pi \sqrt{\mu^3 - \lambda^3}} \left[ \left( \frac{\lambda}{\mu} \right)^2 \gamma \left( \begin{array}{c} \gamma R_1 - 1 \\ - \lambda \end{array} \right) \right]$$

$$M_y = \frac{1}{4\pi \sqrt{\mu^3 - \lambda^3}} \left[ \left( \frac{\lambda}{\mu} \right)^2 \gamma \left( \begin{array}{c} \gamma R_1 - 1 \\ - \lambda \end{array} \right) \right]$$
\[ M_x = \frac{1}{8\pi V\lambda} \left[ (\lambda + \frac{D_t}{D_y}) R_5 \mp (\lambda - \frac{D_t}{D_y}) \sqrt{\lambda(\beta - \eta)} S_1 \right. \\
- (\lambda + \frac{D_t}{D_y}) R_5 - \sqrt{\lambda} \eta (\lambda + \frac{D_t}{D_y}) S_1 - \lambda \beta \eta (\lambda - \frac{D_t}{D_y}) T_1 \\
- \sqrt{\lambda} \beta (\lambda - \frac{D_t}{D_y}) S_1, \right] \\
M_y = \frac{1}{8\pi V\lambda^2} \left[ (\lambda + \frac{D_t}{D_y}) R_5 \mp (\lambda - \frac{D_t}{D_y}) \sqrt{\lambda(\beta - \eta)} S_1 \right. \\
- (\lambda + \frac{D_t}{D_y}) R_5 - \sqrt{\lambda} \eta (\lambda + \frac{D_t}{D_y}) S_1 + \lambda \beta \eta (\lambda - \frac{D_t}{D_y}) T_1 \\
+ \sqrt{\lambda} \beta (\lambda - \frac{D_t}{D_y}) S_1, \right] \\
IV. \lambda = \mu = 1 \quad \text{(isotropic)} \\
W = \frac{a^2}{2\pi^4 D} \sum_{n=1}^{\infty} \left[ \frac{1}{\eta^2} (1 + n(\beta - \eta)) e^{\pm n(\beta - \eta)} - \frac{1}{\eta^2(n+\rho)} (1 + n(\beta - \eta)) e^{\pm n(\beta - \eta)} \right] \sin\alpha \sin\beta \\
where \quad \frac{1}{\rho} = \frac{\pi EI}{4AD} \\
M_x = \frac{1}{8\pi} \left[ (1 + \nu) \log \frac{\cosh(\beta - \eta) - \cos(\alpha + \xi)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} \\
\left( \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cos(\alpha + \xi)} \right) - (1 + \nu) \times \\
\log \frac{\cosh(\beta - \eta) - \cos(\alpha + \xi)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - \{ (1 + \nu) \eta + (1 - \nu) \beta \} \left( \frac{\sinh(\beta + \eta)}{\cosh(\beta + \eta) - \cos(\alpha + \xi)} \right) \\
- \frac{\sinh(\beta + \eta)}{\cosh(\beta + \eta) - \cos(\alpha - \xi)} \right] - (1 - \nu) \beta \eta \left( \frac{\cosh(\beta + \eta) \cosh(\alpha - \xi) - 1}{(\cosh(\beta + \eta) - \cos(\alpha - \xi))^2 \left( \cosh(\beta + \eta) - \cos(\alpha + \xi) \right)^2} \right) \\
M_y = \frac{1}{8\pi} \left[ (1 + \nu) \log \frac{\cosh(\beta - \eta) - \cos(\alpha + \xi)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - (1 - \nu) \beta \eta \left( \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} \right) \\
- \frac{\sinh(\beta - \eta)}{\cosh(\beta - \eta) - \cos(\alpha + \xi)} \right] - (1 + \nu) \log \frac{\cosh(\beta + \eta) - \cos(\alpha - \xi)}{\cosh(\beta + \eta) - \cos(\alpha + \xi)} \\
\{ (1 + \nu) \eta - (1 - \nu) \beta \} \left( \frac{\sinh(\beta + \eta)}{\cosh(\beta + \eta) - \cos(\alpha + \xi)} - \frac{\sinh(\beta + \eta)}{\cosh(\beta + \eta) - \cos(\alpha - \xi)} \right) \\
+ (1 - \nu) \beta \eta \left( \frac{\cosh(\beta + \eta) \cosh(\alpha - \xi) - 1}{(\cosh(\beta + \eta) - \cos(\alpha - \xi))^2 \left( \cosh(\beta + \eta) - \cos(\alpha + \xi) \right)^2} \right) \]
CHAPTER VII
Application of Fourier Integrals and Complex Variables

7.1 Alternative Methods of Solution

In this chapter, methods other than the ordinary methods employed so far in Chapters III and IV will be discussed briefly, particularly the application of Fourier integrals to the boundary value problem of a semi-infinite as well as an infinite plate strip, the application of conformal mapping to isotropic plates whose boundaries are simply supported.

Rather than solving any particular problem, brief discussion on the general approach of problems involved will be given.

7.2 Application of Fourier Integrals to Problems of Plate Strips

For simplicity, only isotropic plates will be considered.

(i) Influence function of plate strip with simply supported edges in form of Fourier Integral: Levy's solution obtained in equation (5.7) can readily be rewritten in the form of a y-sine series (Fig. 5-1):

\[ W = \sum_{n=1}^{\infty} \frac{1}{b D (\frac{n \pi}{b})^3} \sin \frac{n \pi y}{b} \sin \frac{n \pi v}{a} \left[ \frac{\sinh \frac{n \pi (a-u)}{b}}{\sinh \frac{n \pi a}{b}} \right] \left( \frac{\sinh \frac{n \pi x}{b}}{\sinh \frac{n \pi a}{b}} \right) - \frac{n \pi x}{b} \frac{\sinh \frac{n \pi x}{b}}{\sinh \frac{n \pi a}{b}} \left( \frac{n \pi u}{a} \frac{\cosh \frac{n \pi u}{b}}{\cosh \frac{n \pi a}{b}} (a-u) - \frac{\frac{n \pi a}{b} \sinh \frac{n \pi u}{b}}{\sinh \frac{n \pi a}{b}} \right) \]

Making \( b \to \infty \), the summation will turn into an integral. Introducing

\[ \frac{n \pi}{b} = \xi, \quad \frac{n \pi}{b} = \rho \]
Equation (7.1) represents the influence function of deflection for a semi-infinite isotropic plate strip with simply supported edges in integral form. For an infinite strip, the corresponding solution is obtained by simply replacing \( \sin y \sin v \) by \( 2 \cos (v-y)p \), because \( \sin y \sin v = \frac{1}{2} (\cos (v-y)p - \cos (v+y)p) \), and the latter is the image of the former with respect to the x-axis (Fig. 7-1).

Next the homogeneous solution of \( \Delta \Delta W = 0 \) will be obtained in Fourier integral form.

(ii) Homogeneous solution of \( \Delta \Delta W = 0 \) in Fourier integral form.

It is easy to see that

\[
(A \cosh p x + B \sinh p x + C x \cosh p x + D x \sinh p x) \cos (y-v) p
\]

satisfies the equation \( \Delta \Delta W = 0 \). Therefore, the general expression of the homogeneous solution can be written as

\[
W_0(u,v;x,y) = \int_0^\infty A(p) \cosh x p + B(p) \sinh y p + C(p) x p \cosh x p + D(p) x p \sinh y p \cos (y-v) p \, dp
\]  

(7.2)

where \( A(p), B(p), C(p), D(p) \), are arbitrary functions of \( p \).
(iii) The solution for the infinite plate strip with clamped parallel edges.

Combining equation (7.1) and equation (7.2)

\[ W(u,v;x,y) = W_0(u,v;z,y) + W_1(u,v;x,y) \]

with the boundary conditions

\[ \begin{align*}
  x = 0 & \quad W = 0 & \frac{\partial W}{\partial x} &= 0 \\
  x = a & \quad W = 0 & \frac{\partial W}{\partial x} &= 0 
\end{align*} \]

These four boundary conditions determine the functions \( A(p), B(p), C(p), \) and \( D(p) \) in equation (7.1).

For actual computation of influence functions, the theory of residues or methods of numerical integration must be employed.

(iv) Infinite Plate Resting on an Elastic Foundation:

The differential equation corresponding to this case is:

\[ D \Delta \Delta W + kW = q(x,y) \quad (7.3) \]

\( kW \) is the reaction of the foundation. The coefficient \( k \) is usually expressed in pounds per square inches per inch of deflection. This quantity is generally referred to as modulus of the foundation.

The influence function for the deflection of a simply supported rectangular plate on an elastic foundation is given in reference (1), p. 252 in double Fourier series form.

\[ W(u,v;x,y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi mx}{a} \cdot \sin \frac{\pi ny}{b}}{\pi D \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + k^2} \sin \frac{\pi mx}{a} \sin \frac{\pi ny}{b} \]

\[ = \frac{1}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\cos \frac{\pi mx}{a} \cdot \cos \frac{\pi nx}{b} - \cos \frac{\pi mx}{a} \cdot \cos \frac{\pi nx}{b} \left( \cos \frac{\pi my}{b} \cdot \cos \frac{\pi ny}{b} \right) }{\pi D \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + k^2} \right) \]
Making $a, b$ infinitely large, writing $\frac{\pi}{a} = d\rho$, $\frac{n\pi}{b} = \rho$, $\frac{\pi}{b} = d\theta$, the double Fourier integral form can be derived

$$W(u, v; x, y) = \frac{1}{\pi^2 D} \int_0^\infty \int_0^\infty \frac{1}{(\rho^2 + \gamma^2)^2 + \chi^2} \left( \cos \rho (u-x) \cos \gamma (v-y) - \cos \rho (u+x) \cos \gamma (v+y) + \cos \rho (u+x) \cos \gamma (v+y) \right) d\rho d\gamma$$

Equation (7.5) represents the influence function of an infinite wedge plate whose opening angle is $\frac{\pi}{2}$ (Fig. 7-2). Observing that edges of the wedge are simply supported, it can be concluded that the first integral represent the influence function for this particular problem. The other three terms are nothing but the image of the first term with respect to either $x$-axis or $y$-axis.

$$W(u, v; x, y) = \frac{1}{\pi^2 D} \int_0^\infty \int_0^\infty \frac{\cos (x-u) \cos (v-y) \rho}{(\rho^2 + \gamma^2)^2 + \chi^2} d\rho d\gamma$$

This is the solution for this particular case.

The deflection under the load can be easily computed

$$\left( \frac{W}{x=x} \right)_{y=y} = \frac{1}{\pi^2 D} \int_0^\infty \int_0^\infty \frac{d\rho d\gamma}{(\rho^2 + \gamma^2)^2 + \chi^2} = \frac{1}{\pi^2 D} \frac{\pi}{2} \int_0^\infty \frac{d\rho}{\sqrt{\rho^2 + \chi^2}} \frac{\chi^2}{2} = \frac{1}{8\sqrt{\kappa D}}$$

Equation (7.6) is the fundamental solution for the influence functions of the infinite plate on the elastic foundation.

The method illustrated so far in this chapter can be easily extended to the case of orthotropic plates. However, general solutions of such problems will not be treated here.
7.3 Application of Conformal Mapping

As mentioned in 7.1, if the shape of an isotropic plate is bounded by straight lines and the edges are simply supported, conformal mapping can be successfully applied to find the influence functions for $M_x$ and $M_y$.

Consider the moment sum $M_x + M_y = M$ in Cartesian coordinate.

$$M = M_x + M_y = -D(1+\nu)\Delta W$$

so that

$$D\Delta W = -\frac{1}{1+\nu} \Delta M = g(x,y)$$

Therefore, the fourth order plate equation reduces to a second order equation in $M$. The influence functions of the bending moment $M_x, M_y$ can be easily obtained as shown subsequently, once $M$ is derived. Since $M_x, M_y, M_x y$ are integrals of linearly varying stresses, $\sigma_x, \sigma_y, \tau_{xy}$ over the thickness of the plate they have the same tensor character as a two dimensional stress field. $M$ is an invariant of the system.

Assuming the edges of the plate to be straight segments and simply supported, $M$ will disappear along the boundary:

$$M = 0$$

Therefore (7.7) and (7.8) constitute the boundary value problem in a two dimensional moment field. Actually the influence functions for $M$ is directly proportional to Green's function for the deflection of membranes of equal shape.

Since $M$ satisfies Laplace's equation except at the loading point, it is possible to apply conformal mapping to find $M$ in a given domain from Green's function for $M$ of the unit circle. The theory of harmonic functions furnishes the Green's function $g(r, \theta; \rho, \varphi)$ for the unit circle:

$$g(r, \theta; \rho, \varphi) = \frac{1}{2} \log \frac{1-2\rho r \cos(\theta-\varphi)+\rho^2 r^2}{r^2-2\rho r \cos(\theta+\varphi)+\rho^2}$$

(7.9)
Observing the similarity between \( g(r, \theta; \rho, \varphi) \) and \( M(r, \theta; \rho, \varphi) \) the parameter relating the two effects is determined such that

\[
M(r, \theta; \rho, \varphi) = \frac{1+\nu}{4\pi} \log \frac{1-2\rho \cos(\theta-\varphi) + \rho^2}{\gamma^2 - 2\rho \cos(\theta-\varphi) + \rho^2} \tag{7.10}
\]

\( M(r, \theta; \rho, \varphi) \) for the semi-circular domain with unit radius can be derived, taking the image of (7.10) with respect to the line of \( \theta = 0 \) (Fig. 7-3).

\[
M(r, \theta; \rho, \varphi) = \frac{1+\nu}{4\pi} \log \frac{(1-2\rho \cos(\theta-\varphi) + \rho^2)(\gamma^2 - 2\rho \cos(\theta+\varphi) + \rho^2)}{(\gamma^2 - 2\rho \cos(\theta-\varphi) + \rho^2)(1-2\rho \cos(\theta+\varphi) + \rho^2)} \tag{7.11}
\]

Applying the conformal mapping \( Z = e^{i\omega} \) to equation (7.11)

\[
Z = r e^{i\theta}, \quad \omega = \alpha + i\beta
\]

\[
\gamma = e^{-\beta} \quad \theta = \alpha
\]

\[
\rho = e^{-\gamma} \quad \varphi = \xi \tag{7.12}
\]

Substituting (7.12) into (7.11):

\[
M(\alpha, \beta; \xi, \eta) = \frac{1+\nu}{4\pi} \log \frac{\cosh(\beta+\eta) - \cosh(\alpha-\xi)}{\cosh(\beta-\eta) - \cosh(\alpha+\xi)} \tag{7.13}
\]

This is the expression of \( M \) for a semi-infinite plate strip. \( M_x, \) \( M_y \) can be easily obtained using following relations:

\[
M_x = \frac{1}{2} \left[ M - \frac{1-\nu}{1+\nu} \gamma \frac{\partial M}{\partial \gamma} \right] \tag{7.14}
\]

\[
M_y = \frac{1}{2} \left[ M + \frac{1-\nu}{1+\nu} \gamma \frac{\partial M}{\partial \gamma} \right]
\]

where \( \gamma = (\beta \pm \eta) \)

Making \( \beta \to \infty, \eta \to \infty \), \( M \) for an infinite plate strip can also be derived

\[
M = \frac{1+\nu}{4\pi} \log \frac{\cosh(\beta-\eta) - \cosh(\alpha+\xi)}{\cosh(\beta-\eta) - \cosh(\alpha-\xi)} \tag{7.15}
\]
This checks the result obtained by Nadai((3), p. 89) using also conformal mapping but in a different way.

Further solutions of M for a rectangular plate or a wedge-shaped plate could be obtained with the aid of Schwarz-Christoffel's transformation.
CHAPTER VIII

Discussion of Singularities of Influence Surfaces

8.1 Singular Behavior of Influence Surfaces at the Influence Point

In general, influence surfaces exhibit singular behavior at the influence point; singularities are due to the singular (particular) part of the solutions. Since regular part of the solution does not show any singularity, they can be discarded as far as the discussion of the singularities are concerned.

In this chapter, a general discussion of the singularities of influence surfaces will be given, that is, singularities of the influence surfaces $m_x, m_y, m_{xy}, q_x, q_y$ at an interior point of the plate, of the corner reaction $r$ of a simply supported rectangular plate, of the boundary moment $m_y$ of a clamped edge, of the boundary moment $m_x$ of a free edge and of support moment $m_x, m_y$ of slabs continuous over a flexible cross beam. Numerical values will be presented so that the general appearance of surfaces can be easily visualized.

8.2 Derivation of Singularities of Influence Surfaces

In the case of an isotropic plate the singular solution of plate equation $D\Delta W = q(x,y)$ is $r^3 \log r$ where $r$ is the distance between the influence point and the loading point.

These singularities can be obtained considering the neighborhood of the influence point $(\alpha, \beta)$ only. Taking the coordinates in this neighborhood as

\[
\xi = \alpha + \varepsilon
\]
\[
\eta = \beta + \delta
\]
with $\varepsilon \neq 0$ and $\delta \neq 0$ the terms of the influence functions are expanded into series. Neglecting higher order terms in the singular part of the solutions and discarding the regular part entirely expressions for the singularity are obtained.

(I) Singularities of Influence Surfaces $m_x, m_{xy}, q_y$ at the Interior Point of the Plate

Since $m_y$ and $q_x$ show the same singular behavior as $m_x$ and $q_y$ respectively, $m_x$ and $q_y$ will be discussed only. In order to distinguish the singular part of the influence function, suffix 0 will be used in every case.

Taking the solution given in (3.15) the vicinity of the influence point $(\alpha, \beta, \xi, \eta)$ can be expressed as follows:

$$\begin{align*}
\xi &= \alpha + \varepsilon, \quad \eta = \beta + \delta \quad (\varepsilon \neq 0, \delta \neq 0)
\end{align*}$$

Consider the case of $\lambda > \mu$. Since $\varepsilon \neq 0, \delta \neq 0$

$$\begin{align*}
\cosh k_1(\beta - \eta) - \cosh(\alpha + \xi) &\approx 1 - \cosh \alpha \\
\cosh k_2(\beta - \eta) - \cosh(\alpha - \xi) &\approx \frac{1}{2} \left( (\alpha - \xi)^2 + k_2^2(\beta - \eta)^2 \right) = \frac{1}{2} \left( \varepsilon^2 + k_2^2 \delta^2 \right)
\end{align*}$$

\[ R_1 \sim \log \frac{1 - \cosh \alpha}{\frac{1}{2} \left( \varepsilon^2 + k_2^2 \delta^2 \right)} \quad R_2 \sim \log \frac{1 - \cosh \alpha}{\frac{1}{2} \left( \varepsilon^2 + k_2^2 \delta^2 \right)} \]

Introducing polar coordinate

$$\begin{align*}
\xi &= r \cos \theta, \quad \delta = r \sin \theta, \quad r \neq 0
\end{align*}$$

and discarding the regular part of the influence function:

$$\log 2(1 - \cos 2\alpha),$$

following final result is obtained

$$\begin{align*}
(M_x)_0 &\sim \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left[ (k_1 \mu - k_i(\frac{\partial}{\partial \xi})) \log r^2 (\cos \theta + k_2 \sin \theta) \right. \\
&\left. \quad - (k_1 \mu - k_i(\frac{\partial}{\partial \eta})) \log r^2 (\cos \theta + k_2 \sin \theta) \right]
\end{align*}$$
Similarly, \((m_{xy})_0, (q_y)_0\) can be obtained.

The results obtained are summarized as follows.

(i) \(\lambda > \mu\)

\[
(M_x)_0 \sim \frac{1}{8\pi\sqrt{\lambda^2 - \mu^2}} \left[ (K_2\mu - K_1(\frac{D_x}{D_y})) \log r^2 (\cos^2 \theta + K_2 \sin^2 \theta) 
- (K_1\mu - K_2(\frac{D_x}{D_y})) \log r^2 (\cos^2 \theta + K_1 \sin^2 \theta) \right]
\]

\[
(m_{xy})_0 \sim \frac{D_{xy}}{2\pi D_y \sqrt{\lambda^2 - \mu^2}} \left[ \tan^{-1} \left( \frac{\cot \theta}{K_2} \right) - \tan^{-1} \left( \frac{\cot \theta}{K_1} \right) \right]
\]

\[
(q_y)_0 \sim -\frac{1}{4ar} \left[ \frac{K_2 \sin \theta}{\cos^2 \theta + K_2 \sin^2 \theta} \right]
\]

(ii) \(\lambda < \mu\)

\[
(M_x)_0 \sim \frac{1}{8\pi\sqrt{\mu^2 - \lambda^2}} \left[ 2K_2(\mu - \frac{D_x}{D_y}) \tan^{-1} \left( \frac{2K_2 \lambda \sin \theta}{\cos^2 \theta + 2K_2 \lambda \sin \theta} \right) 
- K_1(\mu + \frac{D_x}{D_y}) \log r^4 (\cos^2 \theta + 2\lambda \sin \theta \cos \theta + \mu \sin \theta) \right]
\]

\[
(m_{xy})_0 \sim \frac{D_{xy}}{4\pi D_y \sqrt{\mu^2 - \lambda^2}} \log \left| \frac{\sin \theta}{\cos^2 \theta - 2K_2 \lambda \sin \theta \cos \theta + \mu \sin \theta} \right|
\]

\[
(q_y)_0 \sim -\frac{1}{4ar} \left( \frac{K_2 \sin \theta}{\cos^2 \theta - 2K_2 \lambda \sin \theta \cos \theta + \mu \sin \theta} \right)
\]

(iii) \(\lambda = \mu\)

\[
(M_x)_0 \sim \frac{1}{8\pi \left( \sqrt{\lambda} + \frac{D_x}{\lambda D_y} \right)} \log r^2 (\cos^2 \theta + \lambda \sin^2 \theta)
+ \frac{2\sqrt{\lambda}(\lambda - \frac{D_x}{D_y}) \sin \theta}{\cos^2 \theta + \lambda \sin^2 \theta}
\]

\[
(m_{xy})_0 \sim \frac{D_{xy}}{2\pi \lambda D_y} \frac{\sin \theta \cos \theta}{\cos^2 \theta + \lambda \sin^2 \theta}
\]

\[
(q_y)_0 \sim -\frac{1}{4ar} \left( \frac{\sqrt{\lambda} \sin \theta}{\cos^2 \theta + \lambda \sin^2 \theta} \right)
\]
(II) Corner Reaction of Simply Supported Edge \((Y)\),

(i) \(\lambda > \mu\)

\[
\frac{4D_{xy}}{\pi D_y \sqrt{\lambda^2 - \mu^2}} \left[ \tan\left(\frac{\cot \theta}{K_1}\right) - \tan\left(\frac{\cot \theta}{K_2}\right) \right]
\]

(ii) \(\lambda < \mu\)

\[
\frac{2D_{xy}}{\pi D_y \sqrt{\mu^2 - \lambda^2}} \log \frac{\cot \theta - 2K_3 \sin \theta \cos \theta + \mu \sin^2 \theta}{\cot \theta + 2K_3 \sin \theta \cos \theta + \mu \sin^2 \theta}
\]

(iii) \(\lambda = \mu\)

\[
\frac{4D_{xy}}{\pi D_y} - \frac{\sin \theta \cos \theta}{\cos^2 \theta + \lambda \sin^2 \theta}
\]

(III) Boundary Moment of Clamped Edge \((m_x)_0\), \((m_y)_0\)

(i) \(\lambda > \mu\)

\[
(M_x)_0 \sim \frac{(K_1 + K_2) D_I}{4\pi D_y \sqrt{\lambda^2 - \mu^2}} \log \frac{\cot \theta + K_2 \sin^2 \theta}{\cot \theta + K_1 \sin^2 \theta}
\]

\[
(M_y)_0 \sim \frac{(K_1 + K_2)}{4\pi \sqrt{\lambda^2 - \mu^2}} \log \frac{\cot \theta + K_2 \sin^2 \theta}{\cot \theta + K_1 \sin^2 \theta}
\]

(ii) \(\lambda < \mu\)

\[
(M_x)_0 \sim -\frac{K_2 D_I}{\pi D_y \sqrt{\mu^2 - \lambda^2}} \tan\left(\frac{2K_3 K_2 \sin^2 \theta}{\cot \theta + \lambda \sin^2 \theta}\right)
\]

\[
(M_y)_0 \sim -\frac{K_3}{\pi \sqrt{\mu^2 - \lambda^2}} \tan\left(\frac{2K_3 K_2 \sin^2 \theta}{\cot \theta + \lambda \sin^2 \theta}\right)
\]

(iii) \(\lambda = \mu\)

\[
(M_x)_0 \sim -\frac{\sqrt{\lambda} D_I}{\pi D_y} \frac{\sin \theta}{\cos^2 \theta + \lambda \sin^2 \theta}
\]

\[
(M_y)_0 \sim -\frac{\sqrt{\lambda}}{\pi} \frac{\sin \theta}{\cos^2 \theta + \lambda \sin^2 \theta}
\]
(IV) Boundary Moment of Free Edge \((m_x)_o\)

(i) \( \lambda > \mu \)

\[
(m_x)_o \sim \frac{1}{8 \pi \sqrt{\lambda^2 - \mu^2}} \left\{ \frac{(K_1 \mu - KD_y)}{\mu} \right\} \left\{ \left[ (K_1 \mu - KD_y) \left( 1 + \frac{M(K_1 + K_2)}{L(K_1 - K_2)} \right) \right] + \right.
\]

\[
x \log r^2 \left( \cos^2 \theta + K_1^2 \sin^2 \theta \right) - \left\{ (K_1 \mu - \frac{KD_y}{L(K_1 - K_2)}) \left( 1 - \frac{M(K_1 + K_2)}{L(K_1 - K_2)} \right) \right.
\]

\[
+ \left( K_2 \mu - \frac{KD_y}{L(K_1 - K_2)} \right) \left( \frac{2KN}{L(K_1 - K_2)} \right) \right\} \log r^2 \left( \cos^2 \theta + K_2^2 \sin^2 \theta \right)
\]

(ii) \( \lambda < \mu \)

\[
(m_x)_o \sim \frac{1}{8 \pi \sqrt{\mu^2 - \lambda^2}} \left\{ \frac{N}{K_2L} \left( \mu^2 - \frac{KD_y}{D_y} - K_2 (\mu + \frac{D_1}{D_y}) - \frac{K_1^2 M}{K_2L} (\mu - \frac{D_3}{D_y}) \right) \right\} \times
\]

\[
x \log r^2 \left( \cos^2 \theta + 2 \cos \theta \sin^2 \theta + \mu^2 \sin^2 \theta \right) + \left\{ 2K_1 (\mu - \frac{D_2}{D_y}) + \frac{2KM}{L} (\mu + \frac{D_2}{D_y}) \right.
\]

\[
+ \frac{4KD_yN}{D_yL} \right\} \tan^{-1} \left( \frac{2K_2 \sin \theta}{\cos \theta + \lambda \sin \theta} \right)
\]

(iii) \( \lambda = \mu \)

\[
(m_x)_o \sim \frac{1}{8 \pi \sqrt{\lambda^2}} \left\{ \left( \frac{2H^2 - 2D_2y + D_3y}{D_1y} \right) (\lambda - \frac{D_1}{D_y}) + \left( \lambda + \frac{D_3}{D_y} \right) \right\} \times
\]

\[
x \log r^2 \left( \cos^2 \theta + \lambda \sin^2 \theta \right) + \left\{ (\lambda - \frac{D_1}{D_y}) + \frac{D_3y (1 + \frac{D_3}{D_y})}{2H - D_2y} \right\} \times \frac{2\lambda \sin^2 \theta}{\cos \theta + \lambda \sin \theta}
\]

(V) Support Moments \((m_x)_o, (m_y)_o\) of Slabs Continuous Over a Flexible Cross Beam

(i) \( \lambda > \mu \)

\[
(m_x)_o \sim \frac{1}{2\pi \sqrt{\lambda^2 - \mu^2}} \left[ (K_1 + K_2) \frac{D_y}{4D_y} \log \frac{\cos^2 \theta + K_1^2 \sin^2 \theta}{\cos^2 \theta + K_1^2 \sin^2 \theta} + (K_1 - K_2)(\mu + \frac{D_2}{D_y}) J(r) \right]
\]

\[
(m_y)_o \sim \frac{1}{2\pi \sqrt{\lambda^2 - \mu^2}} \left[ (K_1 + K_2) \frac{D_y}{4D_y} \log \frac{\cos^2 \theta + K_1^2 \sin^2 \theta}{\cos^2 \theta + K_1^2 \sin^2 \theta} + (K_1 - K_2)(\mu \frac{D_2}{D_y} + 1) J(r) \right]
\]

where \( r = \frac{2\mu \sqrt{\lambda^2 - \mu^2}}{K_1 - K_2} \frac{4\alpha D_y}{\pi EI} \)
(ii) $\lambda < \mu$

$$(M_x)_0 \sim \frac{1}{\pi \sqrt{\mu^2 - \lambda^2}} \left[ - \frac{k_1 D_1}{2 \rho_1} \tan^{-1} \left( \frac{2 k_1}{k_1} \tan \theta \right) + k_1 \left( \frac{D_1}{D_1} + 1 \right) J(\rho) \right]$$

$$(M_y)_0 \sim \frac{1}{\pi \sqrt{\mu^2 - \lambda^2}} \left[ - \frac{k_2}{2} \tan^{-1} \left( \frac{2 k_2}{k_2} \tan \theta \right) + k_2 \left( \frac{D_2}{\mu D_2} + 1 \right) J(\rho) \right]$$

where

$$\rho = \frac{\mu}{k_1} \sqrt{\mu^2 - \lambda^2} \cdot \frac{4aD_y}{\pi EI}$$

(iii) $\lambda = \mu$

$$(M_x)_0 \sim \frac{1}{2 \pi \sqrt{\lambda}} \left[ - \frac{\lambda (\frac{D_2}{D_2}) \tan \theta}{\cos \theta + \lambda \sin \theta} + (\lambda + \frac{D_1}{D_1}) J(\rho) \right]$$

$$(M_y)_0 \sim \frac{1}{2 \pi \sqrt{\lambda}} \left[ - \frac{\lambda \tan \theta}{\cos \theta + \lambda \sin \theta} + (\frac{D_2}{\lambda \mu D_2} + 1) J(\rho) \right]$$

where

$$\rho = 2 \sqrt{\lambda} \cdot \frac{2aD_y}{\pi EI}$$

if $\rho=0$, $(M_x)_0, (M_y)_0$ are the support moments in case of a rigid cross beam. Furthermore, it is easily seen that $2(M_x)_0, 2(M_y)_0$ are exactly identical with the boundary moments for a clamped edge.

The function $J(\rho)$ introduced here is defined as follows:

$$J_\rho(\rho) = \sum_{n=1,2,5}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right)$$

$$= \frac{1}{2} \left( \Psi(\rho) + \gamma + \log 2 + \frac{1}{2} \left( \Psi(\frac{\rho+1}{2}) - \Psi(\frac{\rho}{2}) \right) \right)$$

where $\Psi(\rho)$ is the Psi-function introduced by Gauss(28).

$$\Psi(\rho) = \frac{\Gamma(\rho)}{\Gamma(\rho)} = \int_0^{\infty} \left( e^{\alpha} - \frac{1}{(1+\alpha)^\rho} \right) \frac{d\alpha}{\alpha}$$

* Derivation is given in Appendix.
and \( \gamma = 0, 5772156649 \) --- (Euler's constant)

For practical computation of \( J(\rho) \), the following two mathematical formulas are used.

\[
\bar{V}(\rho) + \varphi' = -\frac{\pi}{2} \cot \frac{\pi}{n} + 2 \sum_{n=1}^{\infty} \left\{ \frac{\cos(\frac{2\pi n}{n}) \log \sin \left( \frac{\pi}{n} \right)}{n} \right\} - \log(2n) \\
\bar{V}(\rho) + \varphi' = \sum_{n=1}^{\infty} \frac{1}{n} \quad \left( \rho = 1, 2, 3, \ldots \right)
\]

8.3 General Appearance of Singularities

In order to visualize the general appearance of singularities, the isotropic case \( \lambda = \mu = 1 \) is considered here.

(a) \( (m_x)_0, (m_y)_0, (q_y)_0 \) at the interior point of a slab.

\[
(m_x)_0 \sim \frac{1}{8\pi} \left[ -(1+\nu) \log \gamma + 2(1-\nu) \sin^2 \theta \right]
\]

\[
(m_{xy})_0 \sim \frac{(1-\nu)}{8\pi} \sin 2\theta
\]

\[
(q_y)_0 \sim -\frac{1}{4\alpha_r} \sin \theta
\]

(b) Corner reaction \( (\gamma)_0 \) of simply supported edge

\[
(\gamma)_0 \sim \frac{(1-\nu)}{\pi} \sin 2\theta
\]

(c) Boundary moments \( (m_x)_0, (m_y)_0 \) of clamped edge

\[
(m_x)_0 \sim -\frac{\nu}{\pi} \sin^2 \theta
\]

\[
(m_y)_0 \sim -\frac{1}{\pi} \sin^2 \theta
\]

(d) Boundary moment \( (m_x)_0 \) of free edge

\[
(m_x)_0 \sim \frac{1}{\pi(1+\nu)} \left[ - \log \gamma^2 + (1-\nu^2) \sin^2 \theta \right]
\]
(e) Support moments \((m_x)_0\), \((m_y)_0\) of a continuous slab

\[
(m_x)_0 \sim -\frac{1}{2\pi}[-\nu \sin^2 \theta + (1+\nu) J(p)] \\
(m_y)_0 \sim -\frac{1}{2\pi}[-\nu \sin^2 \theta + (1+\nu) J(p)]
\]

The above equations, except (e), were already obtained by Pucher(4). Fig. 8-1 gives a graphical representation of these singularities.*

Knowing the singular behavior of the influence functions, their general appearance in specific cases can be easily drawn as shown in Figs. 8-2 and 8-3.

A three dimensional view of the \((m_x)_0\) surface at the interior point of a slab is also given in Fig. 2-2.

8.4 Discussion on the Singularities of Orthotropic Plates

As pointed out in Chapter II (4), the domain \(0 \leq \lambda \leq 10\) \(0 \leq \mu \leq 10\) is of practical importance. Therefore, numerical computations were made for several cases listed in Fig. 8-4.

Generally, the influence functions take completely different mathematical expressions depending on the relation: \(\lambda \leq \mu\). However, results of numerical computation show that the influence surfaces will change their shape as well as their numerical values continuously according to the value of \(\lambda\) and \(\mu\).

The domain \(\lambda < \mu\) is the case where the mathematical expressions take their most complicated form. However, it is exactly this domain where most of the data of actual bridge slabs.

* \((m_x)_0\), for the interior point, \((m_y)_0\) for the free edge become infinitely large at the influence point. In computing the contours shown in Fig. 8-1, the assumption \(\nu = 0\) was made. Furthermore, for the cases where the singularity tends to infinity a value of the influence function equal to zero was assumed. As all contours are similar this assumption does not influence their general shape.
fall, (Fig. 1-4), (especially for bridge slabs, cases \( \lambda = 0, \lambda < \mu \) (points on the \( \mu \)-axis) are of importance).

Results of numerical computation are collected in Figs. 8-5, 8-6, 8-7, 8-8, 8-9, 8-10, 8-11.

It is easy to understand how mountains (positive zone) and valleys (negative zone) will change their shapes, contracting or expanding depending on the value of \( \lambda \) and \( \mu \).

Some of the mathematical aspects of the singularities of orthotropic plates have recently been discussed by Mossakowski (11) using a Fourier Integral transform.
CHAPTER IX

Summary

In this dissertation mathematical expressions for the influence surfaces of orthotropic rectangular plates are derived. The principal results of the investigation can be divided into four parts:

(1) Cases Solved

The Green's function for the deflection of an infinite orthotropic plate strip with simply supported parallel edges is solved as a fundamental case (Chapter III). Combining this solution with the homogeneous solution for orthotropic rectangular plate and determining its coefficients such that the combination fulfills the boundary conditions at the third edge, the influence functions for the semi-infinite plate strip with simply supported parallel edges are derived in Chapter IV.

Using a solution in double Fourier series form (corresponding to Navier's solution for isotropic plate) rectangular plate with simply supported edges is treated (Chapter V). Through summation a solution in simple series form is developed. Finally, in Chapter VI, the plate strip continuous over a flexible cross beam is studied.

(2) Closed Form Solutions

In this dissertation, most solutions are carried through to a closed form by making use of several mathematical summation formulae. Thus, the discussion of the singularities of the influence functions become possible and the general appearance of influence surfaces around the singularities is made clear. Many previous solutions for isotropic plates are in series form which
converge very slowly in the vicinity of the influence point and are divergent for the point itself. They do not allow a discussion of singular points.

(3) Discussion of the Singularities

Discarding the regular part as well as higher order terms of the singular part in the vicinity of the influence point, the solutions are obtained for various cases. Assuming various values for the orthotropic parameters $\lambda$ and $\mu$ a general investigation of the singular behavior of the influence surfaces is made.

(4) Practical Application

In practical application the orthotropic parameters $\lambda$ and $\mu$ seem to be limited as follows:

$$0 \leq \lambda \leq 10$$
$$0 \leq \mu \leq 10$$

This square domain covers such cases as two-way reinforced concrete slabs, grid work systems, corrugated sheets, plywood plates, stiffened plates, etc. Orthotropic bridge slabs fall generally in the domain $\lambda < \mu$ and even $\lambda = 0$ as shown in (Fig. 1-4).

Assuming twelve values of $\lambda$ and $\mu$, numerical computation of the singularities was carried out, and the results were represented in contour line diagrams.

The change of the shapes as well as numerical values of influence surfaces due to changes of $\lambda$ and $\mu$ are easily visualized. Since the change of influence surfaces in shape and numerical value is continuous depending upon the change of $\lambda$ and $\mu$, an interpolation between the computed surfaces is admissible.
1. Mathematical Formulae for the Summation of the Series of the Type
\[ \sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos nx \]

If \( Z \) is a complex variable and \( |Z| < 1 \), the following expansion holds.

\[ \frac{1}{1-Z} = 1 + Z + Z^2 + \ldots = \sum_{n=0}^{\infty} Z^n \quad \text{(A)} \]

Expressing \( Z \) in polar coordinates

\[ Z = r e^{i\theta} = r (\cos \theta + i \sin \theta) \]

and its conjugate

\[ \overline{Z} = r e^{-i\theta} = r (\cos \theta - i \sin \theta) \]

yields

\[ \frac{1}{1-Z} = \frac{1-\overline{Z}}{(1-Z)(1-\overline{Z})} = \frac{1-\gamma \cos \theta + i \gamma \sin \theta}{1-2 \gamma \cos \theta + \gamma^2} \quad \text{(B)} \]

\[ \sum_{n=1}^{\infty} Z^n = \sum_{n=1}^{\infty} r^n \cos n \theta + i \sum_{n=1}^{\infty} r^n \sin n \theta \quad \text{(C)} \]

Comparing equations (A), (B) and (C) the following expressions can be derived, provided \( 0 \leq r < 1 \)

\[ \sum_{n=0}^{\infty} \gamma^n \cos n \theta = \frac{1-\gamma \cos \theta}{1-2 \gamma \cos \theta + \gamma^2} \]

and

\[ \sum_{n=1}^{\infty} \gamma^n \cos n \theta = \frac{1-\gamma \cos \theta}{1-2 \gamma \cos \theta + \gamma^2} - 1 = \frac{1}{2} \left( \frac{1-\gamma^2}{1-2 \gamma \cos \theta + \gamma^2} - 1 \right) \quad \text{(D)} \]

\[ \sum_{n=1}^{\infty} \gamma^n \sin n \theta = \frac{\gamma \sin \theta}{1-2 \gamma \cos \theta + \gamma^2} \quad \text{(E)} \]
Integrating equation (A)

\[- \log(1 - \varepsilon) = \varepsilon + \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n} \]  

(F)

\[1 - \varepsilon = (1 - r \cos \theta) + i r \sin \theta = \sqrt{1 - 2r \cos \theta + r^2} \, e^{i \tan^{-1}\left(\frac{r \sin \theta}{1 - r \cos \theta}\right)} \]  

(G)

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos n \theta = \sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos n \theta + i \sum_{n=1}^{\infty} \frac{\gamma^n}{n} \sin n \theta \]  

(H)

From (F), (G) and (H)

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos n \theta = -\frac{1}{2} \log(1 - 2r \cos \theta + r^2) \]  

(I)

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \sin n \theta = \tan^{-1}\left(\frac{r \sin \theta}{1 - r \cos \theta}\right) \]  

(J)

In the same way several other formulae can be derived. Only the final expressions are given:

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos n \theta = -\frac{1}{2} \log(1 - 2r \cos \theta + r^2) \]

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \sin n \theta = \tan^{-1}\left(\frac{r \sin \theta}{1 - r \cos \theta}\right) \]

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \cos n \theta = \frac{1}{4} \log \frac{1 + 2r \cos \theta + r^2}{1 - 2r \cos \theta + r^2} \]

\[\sum_{n=1}^{\infty} \frac{\gamma^n}{n} \sin n \theta = \tan^{-1}\left(\frac{2r \sin \theta}{1 - r^2}\right) \]

\[\sum_{n=1}^{\infty} \gamma^n \cos n \theta = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2} - 1 = \frac{1}{2} \left(\frac{1 - r^2}{1 - 2r \cos \theta + r^2} - 1\right) \]

\[\sum_{n=1}^{\infty} \gamma^n \sin n \theta = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2} \]

\[\sum_{n=1}^{\infty} \gamma^n \cos n \theta = \frac{\gamma \left(1 + r^2 \cos \theta - 2r\right)}{(1 - 2r \cos \theta + r^2)^2} \]
(2) Mathematical Formulae for the Summation of the Series of the Type

\[ \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + k^2} \]

Expanding \(e^{kx}\) into Fourier series in the range \([0, 2\pi]\)

\[
\frac{\pi e^{kx}}{e^{2k\pi} - 1} = \frac{1}{2k} + \sum_{n=1}^{\infty} \frac{k \cos nx}{n^2 + k^2} - \sum_{n=1}^{\infty} \frac{n \sin nx}{n^2 + k^2} \quad (A)
\]

changing \(k\) to \(-k\)

\[
\frac{\pi e^{-kx}}{e^{-2k\pi} - 1} = -\frac{1}{2k} - \sum_{n=1}^{\infty} \frac{k \cos nx}{n^2 + k^2} - \sum_{n=1}^{\infty} \frac{n \sin nx}{n^2 + k^2} \quad (B)
\]

Combining equation (A) and (B)

\[
\frac{\pi e^{k(x-\pi)}}{e^{k\pi} - e^{-k\pi}} = \frac{1}{2k} + \sum_{n=1}^{\infty} \frac{k \cos nx}{n^2 + k^2} - \sum_{n=1}^{\infty} \frac{n \sin nx}{n^2 + k^2} \quad (C)
\]

\[
\frac{\pi e^{-k(x-\pi)}}{e^{k\pi} - e^{-k\pi}} = \frac{1}{2k} + \sum_{n=1}^{\infty} \frac{k \cos nx}{n^2 + k^2} + \sum_{n=1}^{\infty} \frac{n \sin nx}{n^2 + k^2} \quad (D)
\]

Adding equations (C) and (D)

\[
\frac{\pi \cosh k(x-\pi)}{\sinh k\pi} = \frac{1}{k} + 2k \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + k^2}
\]

or

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + k^2} = \frac{\pi \cosh k(x-\pi)}{\sinh k\pi} - \frac{1}{2k^2} \quad (E)
\]
Differentiating equation (E) with respect to $k$

\[-2K \sum_{n=1}^{\infty} \frac{\cos nx}{(n^2 + K^2)^{1/2}} = \frac{\pi}{2K^2} \cdot \coth K(x - \pi) + \frac{1}{4K^3} \]

\[+ \frac{\pi (x - \pi) \sinh K(x - \pi) \sinh K\pi - \pi \cosh K\pi \cosh K(2\pi)}{\sinh^2 K\pi} \]

\[- \sum_{n=1}^{\infty} \frac{\cos nx}{(n^2 + K^2)^{1/2}} = \frac{1}{2K^4} + \frac{\pi}{4K^3} \cdot \coth K(x - \pi) \]

\[+ \frac{\pi^2}{4K^2} \cdot \frac{\cosh K(x - \pi) \cosh K\pi}{\sinh^2 K\pi} - \frac{\pi (x - \pi)}{4K^2} \cdot \frac{\sinh K(x - \pi)}{\sinh K\pi} \]

\[0 \leq x \leq 2\pi \]

In the same way many useful summation formulae of series can be obtained.

(3) Derivation of $J(\rho)$

\[J(\rho) = \sum_{n=1,3,5,\ldots}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right)(1 - (-1)^n) \]

\[= \frac{1}{2} \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+\rho} \right] \]

However the theory of Gamma functions furnishes the following relationships ((28)p, 458)).

\[\frac{d}{d\rho} \log \Gamma(\rho) + \rho = \sum_{k=0}^{\infty} \left( \frac{1}{1+k} - \frac{1}{\rho+k} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) - \frac{1}{\rho} \]
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+p} \right) = \frac{\Gamma(p)}{\Gamma(p)} + \gamma + \frac{1}{p} \]

Making use of the relations\(^{28}\)

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2 \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n+p} = \int_0^1 \frac{x^{p-1}}{1+x} \, dx - \frac{1}{p} = \frac{1}{2} \left[ \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} - \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \right] - \frac{1}{p} \]

\( J(p) \) becomes

\[ J(p) = \frac{1}{2} \left[ \frac{\Gamma'(p)}{\Gamma(p)} + \gamma + \log 2 + \frac{1}{2} \left( \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} - \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \right) \right] \]

\[ = \frac{1}{2} \left[ \Psi'(p) + \gamma + \log 2 + \frac{1}{2} \left( \Psi\left(\frac{p+1}{2}\right) - \Psi\left(\frac{p}{2}\right) \right) \right] \]

where

\[ \Psi'(p) = \frac{\Gamma'(p)}{\Gamma(p)} = \int_0^\infty \left( e^\alpha - \frac{1}{(1+\alpha)^p} \right) \frac{d\alpha}{\alpha} \]

\[ \gamma = 0.5772156649 \ldots \] (Euler's Constant)

\( J(p) \) is represented graphically in the following figure.
\[ J(\rho) = \frac{1}{2} \left[ \psi'(\rho) + \gamma + \log \rho + \frac{1}{2} \left( \psi\left(\frac{\rho+1}{2}\right) - \psi\left(\frac{\rho}{2}\right) \right) \right] \]

\[ \psi'(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_{0}^{\infty} \left( e^{-\alpha} - \frac{1}{\Gamma(\rho)} \right) \frac{d\alpha}{\alpha} \]

\[ \gamma = 0.577216 \quad (\text{Euler's Constant}) \]
CHAPTER XI

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CHAPTER XII

Nomenclature

a  Width of strips
b  Length of a rectangular plate
D  Flexural rigidity of an isotropic plate
$D_x, D_y$  Flexural rigidity of an orthotropic plate in the $x$- and $y$-axes respectively
$D_{xy}$  Torsional rigidity of an orthotropic plate
$D_1$  Some elastic constant of an orthotropic plate
E  Modulus of elasticity in tension and compression
$E_x, E_y, E''$  Elastic constants to characterize the properties of an orthotropic material
G  Modulus of elasticity in shear
H  Torsional rigidity of an orthotropic plate, $H = D_1 + 2D_{xy}$
h  Thickness of a plate
I  Bending rigidity of a beam
$I_w$  Warping rigidity of the beam
$K_t$  Torsional rigidity of the beam
$k_1, k_2, k_3, k_4$  Some constant controlling elastic properties of an orthotropic plate (Section (3.2))
$L, M, N$  Some elastic constants associated with free edge boundary (Section (4.4))
$M_x, M_y$  Bending moment per unit length of sections of a plate perpendicular to $x$- and $y$-axes, respectively
$M_{xy}$  Twisting moment per unit length of section of a plate perpendicular to $x$-axis
$m_x, m_y, m_{xy}$  Influence surfaces for $M_x, M_y, M_{xy}$, respectively
$Q_x, Q_y$  Shearing forces parallel to $z$-axis per unit length of sections of a plate perpendicular to $x$- and $y$-axes respectively
$q_x, q_y$  Influence surfaces for $Q_x$ and $Q_y$ respectively
q  Intensity of a distributed load
Some transcendental functions defined in Table (I)

Influence surface for reaction of a simply supported rectangular corner (Section (4.2; IV))

Polar coordinates

Rectangular coordinates of influence point

Boundary shears corresponding to $Q_x$ and $Q_y$, respectively.

Deflection of a plate in z-axis

Rectangular coordinates

Non-dimensional coordinates of the influence point (p. 25)

Aspect ratio of a rectangular plate (p. 42)

Shearing strain component in rectangular coordinates

Unit elongation in x- and y-directions

Normal components of stress parallel to x- and y-axes

Shearing stress component in rectangular coordinates

Half periods of $\nu_1$-functions (p. 46)

Ratio of bending rigidity of a cross beam and bending rigidity of a plate in y-direction (eq. (6.7))

Parameters controlling anisotropy of a plate (eq. (3.7))

Poisson's ratio

Non-dimensional coordinates in complex variable (eq. (5.9))
Displacements of the point Z

\[ W_z = W \]
\[ u_z = -z \frac{\partial W}{\partial x} \]
\[ v_z = -z \frac{\partial W}{\partial y} \]

Strains

\[ \varepsilon_x = \frac{\partial u_z}{\partial x} = -z \frac{\partial^2 W}{\partial x^2} \]
\[ \varepsilon_y = \frac{\partial v_z}{\partial y} = -z \frac{\partial^2 W}{\partial y^2} \]
\[ \sigma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial v_z}{\partial x} = -z \frac{\partial^2 W}{\partial x \partial y} \]

Figure 1-1 Transversely Loaded Plate
Figure 1-2 EQUILIBRIUM OF THE PLATE ELEMENT

\[ Q_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} T_{xz} \, dz \]

\[ Q_y = \int_{-\frac{1}{2}}^{\frac{1}{2}} T_{yz} \, dz \]
Basic Differential Equation

\[ D_x \frac{\partial^4 W}{\partial x^4} + 2 H \frac{\partial^4 W}{\partial x^3 \partial y} + D_y \frac{\partial^4 W}{\partial y^4} = \delta(x,y) \]

\[ q(x,y) \text{: external load acting on the plate and in this case} \]

\[ q(x,y) = \begin{cases} 
0 & \text{(for any point other than (x,y))} \\
P=1 & \text{(at (x,y))} 
\end{cases} \]

with prescribed boundary conditions. (either statical or geometrical conditions)

---

Figure 1-3  GREEN'S FUNCTION W(u,v;x,y) FOR THE DEFLECTION OF AN ORTHOTROPIC PLATE
Figure 1-4  EXAMPLES OF ORTHOTROPIC SLABS
Figure 2-1 \((m_x)_0\) INFLUENCE SURFACE IN THE VICINITY OF THE INFLUENCE POINT

Figure 2-2 THREE-DIMENSIONAL APPEARANCE OF \(m_x(u,v)\) INFLUENCE SURFACES
SLAB SIMPLY SUPPORTED AT THE LONG EDGE   SLAB CLAMPED AT THE LONG EDGES

(a) \[ \text{Load in Center of Slab} \]

(b) \[ \text{Load Near Clamped Edge} \]

(c) \[ \text{Load Near Simply Supported Edge} \]

(d) \[ \text{Load Near the Free Edge} \]

(e) \[ \text{Load in Center of Slab} \]

(f) \[ \text{Load Near Clamped Edge} \]

(g) \[ \text{Load Near Simply Supported Edge} \]

(h) \[ \text{Load Near the Free Edge} \]

--- simply supported

--------- free

--- clamped

Figure 2-3  THE DIFFERENT CASES OF BOUNDARY CONDITIONS
According to the elementary theory.

According to test results.

Figure 2-5

According to elementary theory of plates ($\nu=0.3$)

According to test results.

According to Westergaard ($\nu=0.3$)

Figures 2-4 and 2-5 CONSISTENCY BETWEEN THEORY AND EXPERIMENT for isotropic plates. (Dutch investigator)
Figure 3-1 INFINITELY LONG STRIP WITH A CONCENTRATED LOAD

Figure 3-2 SIMPLY SUPPORTED INFINITE STRIP
Figure 4-1  SEMI-INFINITE STRIPS
Boundary Conditions

\begin{align*}
\alpha = 0 & \quad W = 0, \quad \frac{\partial^2 W}{\partial x^2} = 0 \\
\alpha = a & \quad W = 0, \quad \frac{\partial^2 W}{\partial x^2} = 0 \\
y = 0 & \quad W = 0, \quad \frac{\partial^2 W}{\partial y^2} = 0 \\
y = b & \quad W = 0, \quad \frac{\partial^2 W}{\partial y^2} = 0
\end{align*}

Figure 5-1  RECTANGULAR PLATE WITH SIMPLY SUPPORTED EDGES

Figure 5-2  NON-DIMENSIONAL COORDINATE SYSTEM

Figure 5-3  POSSIBLE BOUNDARY VALUE PROBLEMS OF A RECTANGULAR PLATE WITH TWO PARALLEL EDGES SIMPLY SUPPORTED
Figure 5-4 POSSIBLE BOUNDARY VALUE PROBLEMS OF A RECTANGULAR PLATE WITH ALL EDGES EITHER SIMPLY SUPPORTED OR CLAMPED

Figure 5-5 BOUNDARY VALUE PROBLEMS OF SEMI-INFINITE PLATE STRIPS WITH THE THIRD EDGE SIMPLY SUPPORTED
Non-Dimensional Coordinates

\[
\frac{\pi u}{a} = \alpha, \quad \frac{\pi v}{a} = \beta, \quad \frac{\pi x}{a} = \xi, \quad \frac{\pi y}{a} = \gamma, \quad \frac{\pi z}{a} = \zeta
\]

Figure 6-1 PLATE STRIP CONTINUOUS OVER FLEXIBLE CROSS BEAM

Figure 7-1 SIMPLY SUPPORTED SEMI-INFINITE STRIP
simply supported

Figure 7-2 INFINITE WEDGE-SHAPED PLATE
(opening angle = $\frac{\pi}{2}$)

Figure 7-3 CORRESPONDENCY BETWEEN THE UNIT SEMI-CIRCLE AND SEMI-INFINITE PLATE STRIP
Bending Moment \( (M_x)_0 \) (Interior Point)

\[ Y = e^{\alpha \sin \theta} \]

Twisting Moment \( (M_y)_0 \) (Interior Point)

\[ 8 \pi (M_y)_0 = \sin 2\theta \]

Shearing Force \( (q_y)_0 \) (Interior Point)

\[ 8 q(y)_0 = -\frac{2 \sin \theta}{r} = 1 \]

Corner Reaction \( (r)_0 \) (Simply Supported Edge)

\[ 8 \pi (r)_0 = 8 \sin 2\theta \]

Boundary Moment \( (M_y)_0 \) (Built-on Edge)

\[ 8 \pi (M_y)_0 = -8 \sin \theta \]

Boundary Moment \( (M_z)_0 \) (Free Edge)

\[ Y = e^{\frac{1}{2} \sin \theta} \]

Support Moment \( (M_y)_0 \) (Continuous Plate)

\[ 8 \pi (M_y)_0 = 4 \left( 0.693 - \sin \theta \right) \]

Fig 8-1
Various Singularities of Influence Surfaces
\( (\lambda = \mu = 1: \text{isotropic}) \)
Figure 8-2 GENERAL APPEARANCE OF INFLUENCE SURFACES
Figure 8-3  INFLUENCE SURFACE FOR SUPPORT MOMENT m_y OF INFINITE PLATE STRIP CONTINUOUS OVER ONE CROSS BEAM
Figure 8-4  SEVERAL COMPUTED CASES
Fig 8-5 \((m_s)\) as a Function of \(\lambda\) and \(\mu\) (interior point)
Fig 8-7 \( g_z \) as a Function of \( \lambda \) and \( \mu \). (Interior point)
Fig 8-8 $8\pi(r)$ as a function of $\lambda$ and $\mu$. 
Fig 8-9  \( 8\pi\mu y_0 \) as a Function of \( \lambda \) and \( \mu \) (clamped edge)
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<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig 8-10 ($m_x$) as a Function of $\lambda$ and $\mu$ (free edge)
<table>
<thead>
<tr>
<th>( \frac{H}{L} )</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig 8-11 \( 8\pi(m_{\nu}) \) as a Function of \( \lambda \) and \( \nu \) (Continuous Plate)
TABLE I

DEFINITION OF SOME IMPORTANT TRANSCENDENTAL FUNCTIONS

Several functions which constitute the influence functions of orthotropic plates are defined in the following table.

Following remarks should be observed for the application of this table.

(i) In order to avoid complexity, every function is written without showing four independent variables \( \alpha, \beta, \xi, \eta \).

For example,

\[
M_x = M_x(\alpha, \beta, \xi, \eta)
\]

\[
R_1 = R_1(\alpha, \beta, \xi, \eta), \text{ etc.}
\]

This rule should be applied to any influence functions unless otherwise noted.

(ii) The functions \( \tilde{R}_1 \) is defined as follows:

\[
\tilde{R}_1 = R_1(\alpha, \beta, \xi, \eta)
\]

(iii) If a function has \( \pm \) sign, the following sign convention should be observed:

- upper sign (+) for \( \beta \leq \eta \)
- lower sign (-) for \( \beta > \eta \)

For example: if \( \beta > 0, \eta > 0 \).

\[
\bar{R}_\varepsilon = R_\varepsilon(\alpha, \beta, \xi, -\eta)
\]

\[
= \tan^{-1}\left(\frac{\sin(\alpha+\varepsilon)}{e^{K_1(\beta+\eta) - \cos(\alpha+\varepsilon)}}\right) - \tan^{-1}\left(\frac{\sin(\alpha-\varepsilon)}{e^{K_1(\beta+\eta) - \cos(\alpha-\varepsilon)}}\right)
\]

(iv)

\[
K_1 = \sqrt{\lambda + \sqrt{\lambda^2 - \mu^2}} \quad , \quad K_2 = \sqrt{\lambda - \sqrt{\lambda^2 - \mu^2}} \quad (\lambda > \mu)
\]

\[
K_3 = \sqrt{\frac{\mu + \lambda}{2}} \quad , \quad K_4 = \sqrt{\frac{\mu - \lambda}{2}} \quad (\lambda < \mu)
\]
<table>
<thead>
<tr>
<th>Functions</th>
<th>Series and Closed Form Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$4 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \sin n \xi \sin \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \log \frac{\cosh K_1(\beta-\eta) - \cos(\alpha + \xi)}{\cosh K_1(\beta-\eta) - \cos(\alpha - \xi)}$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$4 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \sin n \xi \sin \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \log \frac{\cosh K_1(\beta-\eta) - \cos(\alpha + \xi)}{\cosh K_1(\beta-\eta) - \cos(\alpha - \xi)}$</td>
</tr>
<tr>
<td>$R_3$</td>
<td>$8 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \cosh K_2(\beta-\eta) \sin n \xi \sin \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \log \frac{\cosh K_1(\beta-\eta) - \cos(\alpha + \xi)}{\cosh K_1(\beta-\eta) - \cos(\alpha - \xi)}$</td>
</tr>
<tr>
<td>$R_4$</td>
<td>$4 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \sin n \xi \sin \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \tan^{-1}\left(\frac{\sin(\alpha + \xi \pm K_1(\beta-\eta))}{e^{\frac{\eta K_1(\beta-\eta)}{2}} - \cos(\alpha \pm K_1(\beta-\eta))}\right) + \tan^{-1}\left(\frac{\sin(\alpha - \xi \pm K_1(\beta-\eta))}{e^{\frac{\eta K_1(\beta-\eta)}{2}} - \cos(\alpha \pm K_1(\beta-\eta))}\right)$</td>
</tr>
<tr>
<td>$R_5$</td>
<td>$4 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \sin n \xi \sin \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \log \frac{\cosh K_2(\beta-\eta) - \cos(\alpha + \xi)}{\cosh K_2(\beta-\eta) - \cos(\alpha - \xi)}$</td>
</tr>
<tr>
<td>$R_6$</td>
<td>$2 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm \eta K_1(\beta-\eta)} \sin n \xi \cos \alpha$</td>
</tr>
<tr>
<td></td>
<td>$= \tan^{-1}\left(\frac{\sin(\alpha + \xi)}{e^{\frac{\eta K_1(\beta-\eta)}{2}} - \cos(\alpha + \xi)}\right) - \tan^{-1}\left(\frac{\sin(\alpha - \xi)}{e^{\frac{\eta K_1(\beta-\eta)}{2}} - \cos(\alpha - \xi)}\right)$</td>
</tr>
<tr>
<td>Functions</td>
<td>Series and Closed Form Expressions</td>
</tr>
<tr>
<td>-----------</td>
<td>-----------------------------------</td>
</tr>
</tbody>
</table>
| $R_7$     | \[ 2 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm nk_\eta (\beta - \eta)} \sin n \xi \cos n \alpha \]  
|           | \[ = \tan^{-1} \left( \frac{\sin (\omega + \xi)}{e^{\pm k_\eta (\beta - \eta)} \cos (\omega + \xi)} \right) - \tan^{-1} \left( \frac{\sin (\omega - \xi)}{e^{\pm k_\eta (\beta - \eta)} \cos (\omega - \xi)} \right) \] |
| $R_8$     | \[ 8 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm nk_\eta (\beta - \eta)} \sin n \xi \cos n \alpha \]  
|           | \[ = \log \left| \frac{\cosh k_\eta (\beta - \eta) - \cosh (\omega + \xi + k_\eta (\beta - \eta))}{\cosh k_\eta (\beta - \eta) - \cosh (\omega - \xi + k_\eta (\beta - \eta))} \right| \left| \frac{\cosh k_\eta (\beta - \eta) - \cosh (\omega + \xi - k_\eta (\beta - \eta))}{\cosh k_\eta (\beta - \eta) - \cosh (\omega - \xi - k_\eta (\beta - \eta))} \right| \] |
| $R_9$     | \[ 2 \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm nk_\eta (\beta - \eta)} \sin n \xi \cos n \alpha \]  
|           | \[ = \tan^{-1} \left( \frac{\sin (\omega + \xi)}{e^{\pm k_\eta (\beta - \eta)} \cos (\omega + \xi)} \right) - \tan^{-1} \left( \frac{\sin (\omega - \xi)}{e^{\pm k_\eta (\beta - \eta)} \cos (\omega - \xi)} \right) \] |
| $R_{10}$  | \[ 4 \sum_{n=1}^{\infty} e^{-nk_\beta + k_\eta} \sin n \xi \sin n \alpha \]  
|           | \[ = \log \frac{\cosh (k_\beta + k_\eta) - \cosh (\omega + \xi)}{\cosh (k_\beta + k_\eta) - \cosh (\omega - \xi)} \] |
| $R_{11}$  | \[ 4 \sum_{n=1}^{\infty} e^{-nk_\beta + k_\eta} \sin n \xi \sin n \alpha \]  
|           | \[ = \log \frac{\cosh (k_\beta + k_\eta) - \cosh (\omega + \xi)}{\cosh (k_\beta + k_\eta) - \cosh (\omega - \xi)} \] |
| $R_{12}$  | \[ 8 \sum_{n=1}^{\infty} \frac{1}{n} e^{-nk_\eta (\beta - \eta)} \cosh n k_\eta (\beta - \eta) \sin n \xi \sin n \alpha \]  
<p>|           | [ = \log \left| \frac{\cosh k_\eta (\beta - \eta) - \cosh (\omega + \xi + k_\eta (\beta - \eta))}{\cosh k_\eta (\beta - \eta) - \cosh (\omega - \xi + k_\eta (\beta - \eta))} \right| \left| \frac{\cosh k_\eta (\beta - \eta) - \cosh (\omega + \xi - k_\eta (\beta - \eta))}{\cosh k_\eta (\beta - \eta) - \cosh (\omega - \xi - k_\eta (\beta - \eta))} \right| ] |</p>
<table>
<thead>
<tr>
<th>Functions</th>
<th>Series and Closed Form Expressions</th>
</tr>
</thead>
</table>
| $R_{13}$  | 4 \[ \sum_{n=1}^{\infty} \frac{\pm i e^{-n k_i(\beta-\eta)}}{n} \sin n k_i(\beta-\eta) \sin m e - \sin m \xi \]  

= \tan^{-1} \left( \frac{\sin \left( \alpha - \delta \pm k_i(\beta-\eta) \right)}{e^{k_i(\beta-\eta)} - \cos \left( \alpha - \delta \pm k_i(\beta-\eta) \right)} \right) + \tan^{-1} \left( \frac{\sin \left( \alpha + \delta \pm k_i(\beta-\eta) \right)}{e^{k_i(\beta-\eta)} - \cos \left( \alpha + \delta \pm k_i(\beta-\eta) \right)} \right)  

- \tan^{-1} \left( \frac{\sin \left( \alpha - \delta \mp k_i(\beta-\eta) \right)}{e^{k_i(\beta-\eta)} - \cos \left( \alpha - \delta \mp k_i(\beta-\eta) \right)} \right) - \tan^{-1} \left( \frac{\sin \left( \alpha + \delta \mp k_i(\beta-\eta) \right)}{e^{k_i(\beta-\eta)} - \cos \left( \alpha + \delta \mp k_i(\beta-\eta) \right)} \right)  

---

| $S_1$      | 4 \[ \sum_{n=1}^{\infty} e^{\pm i n k_i(\beta-\eta)} \sin n \xi \sin m \alpha \]  

= \tan \left[ \frac{\sinh \left( \alpha - \delta \pm k_i(\beta-\eta) \right)}{\cosh \left( \alpha - \delta - k_i(\beta-\eta) \right)} - \frac{\sinh \left( \alpha - \delta \mp k_i(\beta-\eta) \right)}{\cosh \left( \alpha - \delta \mp k_i(\beta-\eta) \right)} \right]  

---

| $S_2$      | 4 \[ \sum_{n=1}^{\infty} e^{\pm i n k_i(\beta-\eta)} \sin n \xi \cos m \alpha \]  

= \frac{\sin \left( \alpha + \delta \right)}{\cosh \left( \alpha - \delta + k_i(\beta-\eta) \right) - \cos \left( \alpha + \delta \right)} - \frac{\sin \left( \alpha - \delta \right)}{\cosh \left( \alpha - \delta - k_i(\beta-\eta) \right) - \cos \left( \alpha - \delta \right)}  

---

| $S_3$      | 4 \[ \sum_{n=1}^{\infty} e^{\pm i n k_i(\beta-\eta)} \sin n \xi \cos m \alpha \]  

= \frac{\sin \left( \alpha + \delta \right)}{\cosh \left( \alpha - \delta + k_i(\beta-\eta) \right) - \cos \left( \alpha + \delta \right)} - \frac{\sin \left( \alpha - \delta \right)}{\cosh \left( \alpha - \delta - k_i(\beta-\eta) \right) - \cos \left( \alpha - \delta \right)}  

---

| $S_4$      | 4 \[ \sum_{n=1}^{\infty} e^{\pm i n k_i(\beta-\eta)} \sin n \xi \cos m \alpha \]  

= \frac{\sin \left( \alpha + \delta \right)}{\cosh \left( \alpha - \delta + k_i(\beta-\eta) \right) - \cos \left( \alpha + \delta \right)} - \frac{\sin \left( \alpha - \delta \right)}{\cosh \left( \alpha - \delta - k_i(\beta-\eta) \right) - \cos \left( \alpha - \delta \right)}  

---

| $S_5$      | 4 \[ \sum_{n=1}^{\infty} e^{\pm i n k_i(\beta-\eta)} \sin m \alpha \sin n \xi \]  

= \tan \left[ \frac{\sinh k_i(\beta-\eta)}{\cosh k_i(\beta-\eta) - \cos \left( \alpha - \delta \right)} - \frac{\sinh k_i(\beta-\eta)}{\cosh k_i(\beta-\eta) - \cos \left( \alpha + \delta \right)} \right]  

---
<table>
<thead>
<tr>
<th>Functions</th>
<th>Series and Closed Form Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_6$</td>
<td>$4 \sum_{n=1}^{\infty} e^{\pm nk_1(\beta-\eta)} \sin n \xi \sin m \alpha$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$= \pm \left[ \frac{\sinh k_2(\beta-\eta)}{\cosh k_2(\beta-\eta) - \cos (\alpha-\xi)} - \frac{\sinh k_2(\beta-\eta)}{\cosh k_2(\beta-\eta) - \cos (\alpha+\xi)} \right]$</td>
</tr>
<tr>
<td>$S_7$</td>
<td>$8 \sum_{n=1}^{\infty} e^{\pm nk_3(\beta-\eta)} \cos n k_4(\beta-\eta) \sin n \xi \cos m \alpha$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$= \frac{\sin (\alpha+\xi + k_4(\beta-\eta))}{\cosh k_4(\beta-\eta) - \cos (\alpha+\xi + k_4(\beta-\eta))} - \frac{\sin (\alpha-\xi + k_4(\beta-\eta))}{\cosh k_4(\beta-\eta) - \cos (\alpha-\xi + k_4(\beta-\eta))}$</td>
</tr>
<tr>
<td>$S_8$</td>
<td>$\pm 8 \sum_{n=1}^{\infty} e^{\pm nk_5(\beta-\eta)} \sin n k_6(\beta-\eta) \sin n \xi \sin m \alpha$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$= \frac{\sinh k_5(\beta-\eta)}{\cosh k_5(\beta-\eta) - \cos (\alpha+\xi + k_6(\beta-\eta))} + \frac{\sinh k_5(\beta-\eta)}{\cosh k_5(\beta-\eta) - \cos (\alpha-\xi + k_6(\beta-\eta))} - \frac{\sinh k_5(\beta-\eta)}{\cosh k_5(\beta-\eta) - \cos (\alpha-\xi - k_6(\beta-\eta))}$</td>
</tr>
<tr>
<td>$S_9$</td>
<td>$8 \sum_{n=1}^{\infty} e^{\pm nk_7(\beta-\eta)} \cos n k_8(\beta-\eta) \sin n \xi \sin m \alpha$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$= \pm \left[ \frac{\sinh k_7(\beta-\eta)}{\cosh k_7(\beta-\eta) - \cos (\alpha+\xi + k_8(\beta-\eta))} + \frac{\sinh k_7(\beta-\eta)}{\cosh k_7(\beta-\eta) - \cos (\alpha-\xi + k_8(\beta-\eta))} - \frac{\sinh k_7(\beta-\eta)}{\cosh k_7(\beta-\eta) - \cos (\alpha-\xi - k_8(\beta-\eta))} \right]$</td>
</tr>
<tr>
<td>$T_1$</td>
<td>$4 \sum_{n=1}^{\infty} e^{\pm nk_9(\beta-\eta)} \sin n \xi \sin m \alpha$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$= \frac{\cosh k_9(\beta+\eta) \cosh (\alpha-\xi) - 1}{(\cosh k_9(\beta+\eta) - \cosh (\alpha-\xi))^2} - \frac{\cosh k_9(\beta+\eta) \cosh (\alpha+\xi) - 1}{(\cosh k_9(\beta+\eta) - \cosh (\alpha+\xi))^2}$</td>
</tr>
</tbody>
</table>
TABLE II

VARIOUS SINGULARITIES OF INFLUENCE SURFACES AS FUNCTIONS OF λ and μ

Figures 8-5 to 8-11 are graphical representation of the equations of stress singularities given in this Table (II).

In derivation of these equations following assumptions were made:

(i) \( D_1 = 0 \) \quad \( H = 2D_{xy} \) \quad \( \frac{D_{xy}}{D_y} = \frac{A}{2} \)

(ii) For the case (G), \( \frac{4\lambda D_y}{\pi E_1} = 1 \)

Except cases (A) and (F), limit values of the surfaces stay finite.

In cases (A) and (F), \( (m_x)_0 = 0 \) is assumed since every contour lines are similar to each other. In case (C) \( (q_y)_0 = 1 \) is also assumed.
(A) Bending Moment \((m_x)_0\) (interior point)

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation ((m_x)_0 = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(\log y - \sin^2\theta = 0)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(\log y + 0.683 \log(1+0.366 \sin^2\theta) - 0.183 \log(1-0.866 \sin^2\theta) = 0)</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(y \cos \theta = 1)</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(\log y + 0.5 \log(1-0.5 \sin^2\theta) - \frac{\sin^2\theta}{1 + \cos^2\theta} = 0)</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>(\log y + 0.25 \log(1-2 \sin^2\cos^2\theta) - 0.5 \tan^{-1}(\tan^2\theta) = 0)</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(\log y + 0.25 \log(1-\sin^2\cos^2\theta) - 0.866 \tan^{-1}\left(\frac{1.232 \sin^2\theta}{1 + \cos^2\theta}\right) = 0)</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(\log y + 0.25 \log(1+1.25 \sin^2\theta) - 1.118 \tan^{-1}(1.118 \sin^2\theta) = 0)</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(\log y + 0.5 \log(1+0.5 \sin^2\theta) - \frac{3 \sin^2\theta}{2 + \sin^2\theta} = 0)</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(\log y + 0.809 \log(1-0.618 \sin^2\theta) - 0.309 \log(1+0.618 \sin^2\theta) = 0)</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(y \cos \theta = 1)</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(\log y + 0.5 \log(1+9 \sin^2\theta) - \frac{10 \sin^2\theta}{1+9 \sin^2\theta} = 0)</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>(\log y + 0.25 \log(10 \sin^2\theta + 100 \sin^2\theta) - 0.5 \tan^{-1}(10 \tan^2\theta) = 0)</td>
</tr>
</tbody>
</table>
(B) Twisting Moment \((m_{xy})_o\) (interior point)

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation: (8\pi(m_{xy})_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(\sin 2\theta)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(2.309 \left[ \tan^{-1}(2.732 \cot \theta) - \tan^{-1}(0.732 \cot \theta) \right])</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(3.142 - 2 \tan^{-1}(0.707 \cot \theta))</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(\frac{\sin 2\theta}{1 - 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(0.577 \log \frac{1 + \sin \theta \cos \theta}{1 - \sin \theta \cos \theta})</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(0.895 \log \frac{\cot \theta + \sin \theta \cos \theta + 1.5 \sin^2 \theta}{\cot \theta - \sin \theta \cos \theta + 1.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(\frac{\sin 2\theta}{1 + 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(2.683 \left[ \tan^{-1}(1.618 \cot \theta) - \tan^{-1}(0.618 \cot \theta) \right])</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(3.142 - 2 \tan^{-1}(0.224 \cot \theta))</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(\frac{\sin 2\theta}{1 + 9 \sin^2 \theta})</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>
(C) Shearing Force \((q_y)_0\) (interior point)

<table>
<thead>
<tr>
<th>Case</th>
<th>(\chi)</th>
<th>(\mu)</th>
<th>Equation: (8a(q_y)_0 = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(\gamma = -2 \sin \theta)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(\gamma = -2 \left(\frac{0.366 \sin \theta}{1 - 0.866 \sin^3 \theta} + \frac{1.366 \sin \theta}{1 + 0.866 \sin^3 \theta}\right))</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(\gamma = -\frac{2.828 \sin \theta}{1 + \sin^2 \theta})</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(\gamma = -\frac{1.414 \sin \theta}{1 - 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>(\gamma = -\frac{2.828 \sin \theta}{1 - 2 \sin^2 \theta \cos \theta})</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(\gamma = -\frac{3.464 \sin \theta}{1 - \sin^2 \theta \cos \theta})</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(\gamma = -2.226 \left(\frac{\sin \theta}{\cos \theta - \sin \theta \cos \theta + 1.5 \sin \theta} + \frac{\sin \theta}{\cos \theta + \sin \theta \cos \theta + 1.5 \sin \theta}\right))</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(\gamma = -\frac{2.45 \sin \theta}{1 + 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(\gamma = -2 \left(\frac{0.618 \sin \theta}{1 - 0.618 \sin \theta} + \frac{1.618 \sin \theta}{1 + 1.618 \sin^2 \theta}\right))</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(\gamma = -\frac{8.944 \sin \theta}{1 + 19 \sin^2 \theta})</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(\gamma = -\frac{6.324 \sin \theta}{1 + 9 \sin^2 \theta})</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>(\gamma = -\frac{8.944 \sin \theta (1 + 9 \sin^2 \theta)}{\cos \theta + 100 \sin^2 \theta})</td>
</tr>
</tbody>
</table>
### Corner Reaction \((r)_o\) of Simply Supported Rectangular Edges

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation: (8\pi(r)_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(8\sin 2\theta)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(18.475 \left[ \tan^{-1}(2.732 \cot \theta) - \tan^{-1}(0.732 \cot \theta) \right])</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(25.133 - 16 \tan^{-1}(0.707 \cot \theta))</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(\frac{8\sin 2\theta}{1 - 0.5\sin^2 \theta})</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(4.619 \log \frac{1 + \sin \theta \cos \theta}{1 - \sin \theta \cos \theta})</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(7.156 \log \frac{\cos^2 \theta + \sin^2 \theta \cos \theta + 1.5 \sin \theta}{\cos^2 \theta - \sin^2 \theta \cos \theta + 1.5 \sin \theta})</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(\frac{8\sin 2\theta}{1 + 0.5\sin^2 \theta})</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(21.467 \left[ \tan^{-1}(1.618 \cot \theta) - \tan^{-1}(0.618 \cot \theta) \right])</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(25.133 - 16 \tan^{-1}(0.224 \cot \theta))</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(\frac{8\sin 2\theta}{1 + 9\sin^2 \theta})</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>
### (E) Boundary Moment \((m_y)_o\) of Clamped Edge

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation: (8\pi(m_y)_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1 1</td>
<td></td>
<td>(-8 \sin^2 \theta)</td>
</tr>
<tr>
<td>(2)</td>
<td>1 0.5</td>
<td></td>
<td>(4 \log \frac{1 - 0.866 \sin^2 \theta}{1 + 0.866 \sin^2 \theta})</td>
</tr>
<tr>
<td>(3)</td>
<td>1 0</td>
<td></td>
<td>(2.848 \log \frac{\cos^2 \theta}{1 + \sin^2 \theta})</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5 0.5</td>
<td></td>
<td>(-\frac{5.658 \sin^3 \theta}{1 - 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(5)</td>
<td>0 1</td>
<td></td>
<td>(-8 \tan^{-1}(\tan^2 \theta))</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5 1</td>
<td></td>
<td>(-8 \tan^{-1}\left(\frac{0.866 \sin \theta}{1 - 0.5 \sin^2 \theta}\right))</td>
</tr>
<tr>
<td>(7)</td>
<td>1 1.5</td>
<td></td>
<td>(-8 \tan^{-1}(1.118 \sin^2 \theta))</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5 1.5</td>
<td></td>
<td>(-\frac{9.796 \sin^2 \theta}{1 + 0.5 \sin^2 \theta})</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5 1</td>
<td></td>
<td>(4 \log \frac{1 - 0.618 \sin^2 \theta}{1 + 0.618 \sin^2 \theta})</td>
</tr>
<tr>
<td>(10)</td>
<td>10 0</td>
<td></td>
<td>(-0.894 \log \frac{\cos^2 \theta}{1 + 19 \sin^2 \theta})</td>
</tr>
<tr>
<td>(11)</td>
<td>10 10</td>
<td></td>
<td>(-\frac{25.3 \sin^2 \theta}{1 + 9 \sin^2 \theta})</td>
</tr>
<tr>
<td>(12)</td>
<td>0 10</td>
<td></td>
<td>(-1.789 \tan^{-1}(10 \tan^2 \theta))</td>
</tr>
</tbody>
</table>
### Boundary Moment \((m_x)\) of Free Edge

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation: ((m_x)_0 = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(\log r - 0.5 \sin^2 \theta = 0)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(\log r - 0.039 \log (1+0.866 \sin^2 \theta) - 0.539 \log (1-0.866 \sin^2 \theta) = 0)</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(r \cos \theta = 1)</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(\log r + 0.5 \log (1-0.5 \sin^2 \theta) - \frac{0.25 \sin^2 \theta}{1-0.5 \sin^2 \theta} = 0)</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>(\log r + 0.25 \log (\cos^2 \theta + 2 \sin^2 \theta) = 0)</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(\log r + 0.25 \log (1-\sin^2 \theta) - 0.289 \tan^{-1}(0.866 \sin \theta) = 0)</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(\log r + 0.25 \log (1+1.25 \sin^2 \theta) - 0.447 \tan^{-1}(1.118 \sin \theta) = 0)</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(\log r + 0.5 \log (1+0.5 \sin^2 \theta) - \frac{0.75 \sin \theta}{1+0.5 \sin^2 \theta} = 0)</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(\log r - 0.085 \log (1+1.618 \sin^2 \theta) + 0.585 \log (1-0.618 \sin^2 \theta) = 0)</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(r \cos \theta = 1)</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(\log r + 0.5 \log (1+9 \sin^2 \theta) - \frac{5 \sin^2 \theta}{1+9 \sin^2 \theta} = 0)</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>(\log r + 0.25 \log (\cos^2 \theta + 100 \sin^2 \theta) = 0)</td>
</tr>
</tbody>
</table>
(G) Support Moment \((m_y)_0\) of a Slab Continuous Over Flexible Cross Beam

<table>
<thead>
<tr>
<th>Case</th>
<th>(\lambda)</th>
<th>(\mu)</th>
<th>Equation: (8\pi (m_y)_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>1</td>
<td>(2.772 - 4\sin^2\theta)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>0.5</td>
<td>(1.811 + 2\log\frac{1-0.866\sin^2\theta}{1+0.866\sin^2\theta})</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
<td>(1.414 \log\frac{\cos^2\theta}{2-\cos^2\theta})</td>
</tr>
<tr>
<td>(4)</td>
<td>0.5</td>
<td>0.5</td>
<td>(1.935 - \frac{2.829\sin^2\theta}{1-0.5\sin^2\theta})</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1</td>
<td>(3.128 - 2.828\tan^{-1}(\tan^2\theta))</td>
</tr>
<tr>
<td>(6)</td>
<td>0.5</td>
<td>1</td>
<td>(2.938 - 4\tan^{-1}\left(\frac{0.866\sin^2\theta}{1-0.5\sin^2\theta}\right))</td>
</tr>
<tr>
<td>(7)</td>
<td>1</td>
<td>1.5</td>
<td>(3.256 - 4\tan^{-1}(1.118\sin^2\theta))</td>
</tr>
<tr>
<td>(8)</td>
<td>1.5</td>
<td>1.5</td>
<td>(3.122 - \frac{4.898\sin^2\theta}{1+0.5\sin^2\theta})</td>
</tr>
<tr>
<td>(9)</td>
<td>1.5</td>
<td>1</td>
<td>(2.637 + 2\log\frac{1-0.618\sin^2\theta}{1+1.618\sin^2\theta})</td>
</tr>
<tr>
<td>(10)</td>
<td>10</td>
<td>0</td>
<td>(0.447 \log\frac{\cos^2\theta}{1+19\sin^2\theta})</td>
</tr>
<tr>
<td>(11)</td>
<td>10</td>
<td>10</td>
<td>(2.988 - \frac{12.65\sin^2\theta}{1+9\sin^2\theta})</td>
</tr>
<tr>
<td>(12)</td>
<td>0</td>
<td>10</td>
<td>(3.917 - 0.894\tan^{-1}(10\tan^2\theta))</td>
</tr>
</tbody>
</table>
VITA

The author was born as the second child of Kanjiro and Shizue Kawai on February 20, 1926 in Tokyo, Japan.

In April, 1949 he entered the University of Tokyo and in March 1952 was awarded the degree of B.S. in Naval Architecture. Thereafter, he continued his study in the graduate school at the University of Tokyo until August 1954.

In September 1954 he accepted an appointment from Lehigh University as a Research Assistant at Fritz Engineering Laboratory.