Welded Continuous Frames and Their Components

ANALYSIS OF BEAM-AND-COLUMN SUBASSEMBLAGES IN PLANAR MULTI-STORY FRAMES

by
Victor Levi
George C. Driscoll, Jr.
Le-Wu Lu

Fritz Engineering Laboratory Report No. 273.11
Welded Continuous Frames and Their Components

ANALYSIS OF BEAM-AND-COLUMN SUBASSEMBLAGES IN PLANAR MULTI-STORY FRAMES

by

Victor Levi
George C. Driscoll, Jr.
Le-Wu Lu

This work has been carried out as part of an investigation sponsored jointly by the Welding Research Council and the Department of the Navy with funds furnished by the following:

American Institute of Steel Construction
American Iron and Steel Institute
Office of Naval Research (Contract Nonr. 610(03))
Bureau of Ships
Bureau of Yards and Docks

Reproduction of this report in whole or in part is permitted for any purpose of the United States Government.

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

July 1964

Fritz Engineering Laboratory Report No. 273.11
SYNOPSIS

Methods are developed for the analysis of beam-and-column subassemblages which constitute a part of a two-dimensional multi-story frame. A subassemblage consists essentially of a column which is framed at its two ends with beams of known properties. When such a structure is subjected to load, the beams will deform with the column and thus provide rotational restraint at the ends of the column. The strength of the subassemblage therefore depends on not only the strength of the column member but also that of the beams.

Two types of subassemblages are analyzed by the procedures presented herein; namely, the sway type and the non-sway type. A sway subassemblage is a subassemblage whose upper joint may translate with respect to the lower joint when subjected to load. Any horizontal force which is applied to the subassemblage is resisted by the column and the restraining beams. In a non-sway subassemblage, translation of the joints is assumed to be prevented and the lateral forces applied at the level of the joints are resisted directly by the supports. Thus, in general, the behavior of the two types of structures is quite different.

Another problem which is studied by the method developed for non-sway subassemblages is the analysis of beam-columns continuous over several supports. The problem has its practical importance in the analysis and design of columns in a multi-story frame.

The methods of analysis developed are graphical in nature and can be used to study both the elastic and inelastic behavior of the various
subassemblages. When the analysis is carried into the inelastic range, the ultimate strength of the subassemblages can also be determined. The methods are first presented for the general case of the two types of subassemblages and are subsequently simplified to provide solutions to some special cases. Numerical examples are included to illustrate the application of the methods.
# TABLE OF CONTENTS

SYNOPIS

1. INTRODUCTION
   1.1 Previous Research
   1.2 Scope of Development

2. ANALYSIS OF SUBASSEMBLAGES WITHOUT SIDESWAY
   2.1 General Definition and Sign Convention
   2.2 Method of Analysis
   2.3 Stability Criterion
   2.4 Criterion of Adequacy
   2.5 Analysis of Special Cases
   2.6 Columns Continuous Over Several Supports

3. ANALYSIS OF SUBASSEMBLAGES WITH SIDESWAY
   3.1 General Definition and Sign Convention
   3.2 Method of Analysis
   3.3 Analysis of Special Cases
   3.4 Buckling of Subassemblages

4. SUBASSEMBLAGES COMPRISED OF BEAMS AND COLUMNS

5. SUMMARY AND CONCLUSIONS

6. ACKNOWLEDGMENTS
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. APPENDICES</td>
<td>61</td>
</tr>
<tr>
<td>Appendix A</td>
<td>61</td>
</tr>
<tr>
<td>Appendix B</td>
<td>65</td>
</tr>
<tr>
<td>Appendix C</td>
<td>68</td>
</tr>
<tr>
<td>Appendix D</td>
<td>71</td>
</tr>
<tr>
<td>8. TABLES AND FIGURES</td>
<td>73</td>
</tr>
<tr>
<td>9. NOTATION</td>
<td>100</td>
</tr>
<tr>
<td>10. REFERENCES</td>
<td>102</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

In the past decade, the plastic method of designing steel structures has received wide acceptance by the structural engineering profession and a large number of building frames have been successfully designed by this method\(^1\). Because only limited information is available concerning the behavior of tall building frames, the current specification limits the application of such method to single and two story frames\(^2\). The promise which the plastic method has shown in the design of low buildings has encouraged further research into the plastic strength of multi-story building frames. Such a research program is currently underway at the Fritz Engineering Laboratory of Lehigh University.

In discussing the behavior of multi-story frames, it is convenient to classify the frames into two categories: braced frames and unbraced frames. In the following, the structural action of these two types of frames will be separately described.

The deformed configuration of a braced three-story, three-bay frame is illustrated in Fig. 1a. If the bracing members of the frame are sufficiently strong to resist the horizontal forces, the lateral deflection of the ends of the columns will generally be very small, and, for purposes of analysis, the columns may be assumed to be fixed against horizontal movement at the floor levels. To determine the strength of a column, such as column AB in Fig. 1a, it is possible to isolate the column segment and its neighboring members from the frame and treat this beam-and-column subassemblage as a unit. Figure 1b shows the subassemblage representing the structural action of column AB. In this subassemblage, the restraints offered by the beam members which are not framed directly into the columns
are approximated by springs at the far ends of the adjacent beams and columns. It has been shown that these restraints do not influence significantly the behavior of the column in question\(^3\). In performing an engineering analysis the restraining effect at the far ends can be estimated or even neglected. Consequently, the strength of column AB can be readily analyzed when the behavior of the beams and columns connected to its upper and lower ends is known. The analysis of the beam-and-column subassemblage can therefore be made by considering the simplified arrangement shown in Fig. 1c. The restraining effects of the neighboring beams and columns are represented by two rotational restraints, each having a known characteristic.

The behavior of an unbraced multi-story frame differs considerably from that of a braced frame. This can be seen by comparing the deformation of the two types of frames under load. Figure 2a shows the deflected configuration of the frame in Fig. 1a when the bracing members are removed. The frame will deflect laterally upon application of the external load, and the upper end of each column will translate through a certain distance with respect to the lower end. This causes additional moments in the columns. The magnitude of these moments depends on the amount of translation and on the intensity of axial forces in the columns.

The analysis of a column subjected to translation can also be made by isolating its neighboring members from the structure and forming a beam-and-column subassemblage. A subassemblage that would represent the behavior of column AB of the frame in Fig. 2a is illustrated in Fig. 2b. In this subassemblage the resistance to sway offered by the members which are not directly framed into the column is represented by lateral springs. The characteristics of the springs depend on the lateral rigidity of the entire
frame. In studying the behavior of column AB in the subassemblage, the rotational restraints offered by the adjacent beams and columns are replaced by springs as was done in the previous case. Thus, a simplified subassemblage as shown in Fig. 2c is obtained. This subassemblage may be thought of as a restrained column with joint translation.

This report treats the analysis of beam-and-column subassemblages loaded by external forces (horizontal and vertical) and bending moments. Both the sway and nonsway types of subassemblages will be studied. To simplify the discussion, the solutions developed will be for the subassemblages shown in Figs. 1c and 2c. However, subassemblages comprised of real beams and columns will also be considered after the development made on these simple subassemblages is completed.

The reasons for the study of subassemblages are: 1) The analysis of an entire multi-story, multi-bay frame is almost prohibitive if stability and/or deflection effects are a predominant consideration, and 2) Subassemblages with conservative assumptions with regard to end conditions can be used in the analysis and design of individual members and of groups of members.

The types of subassemblages which adequately describe the action of a portion of a frame and the effect of various types of loading on the choice of subassemblages will be studied in a subsequent report.
1.1 PREVIOUS RESEARCH

Numerous theoretical and experimental studies have been made on the behavior of beam-and-column subassemblages in the elastic and elastic-plastic range. These studies deal primarily with the determination of the load carrying capacity of such structures when translation of the joints is prevented (nonsway case). A comprehensive survey of the state of knowledge up to 1959 can be found in Ref. 4. In 1960, a systematic approach to restrained columns without sway was developed by Ojalvo who emphasized the use of nomographs in solving such problems. The same problem has also been investigated by Collins and Sidebottom using a computer analysis. Ojalvo's work was further extended by Levi to include a general treatment of the various problems. He also developed criteria for stability and adequacy for a general subassemblage. Recently, an experimental program was conducted at the Fritz Engineering Laboratory to check the theoretical solutions obtained by Ojalvo and by Levi.

In contrast to the extensive work done on nonsway subassemblages, only few attempts have been made to study the behavior of subassemblages with sway. A very simple case has been analyzed by Oxford and by Knothe using an approximate method originally proposed by Merchant. In 1962, Levi extended his method of solution to take into account the effect of sidesway and developed a general procedure for analyzing sway subassemblages. Subsequently, experiments on subassemblages with sidesway were performed.

This report summarizes the studies made by Levi on beam-and-column subassemblages as reviewed above. Examples will be given to illustrate the application of the procedures developed for the various problems.
1.2 SCOPE OF DEVELOPMENT

The beam-and-column subassemblages studied in this report are summarized in Fig. 3. To simplify the presentation, the two types of subassemblages; namely, the sway and nonsway subassemblages, will be treated as separate problems. In all the cases, the following are assumed to be known: the length and the cross-section of the column, the properties of the restraining members, and the magnitude of the axial force.* Additional quantities are specified in certain cases. The purpose of the analysis is to determine either the ultimate moment-carrying capacity or, for the sway subassemblages, the maximum value of the horizontal force. In almost all the procedures developed herein, graphical techniques are used to obtain the solutions.

First, a general method for analyzing subassemblages which are prevented from sidesway is presented. The method is an adaptation of the restrained column theory developed previously and is applicable to any type of nonsway subassemblages. This will be followed by a number of special methods which are developed for the following cases:

1. Subassemblages with symmetrical loading and restraint
2. Subassemblages with antisymmetrical loading
3. Subassemblages with one end free of external moment
4. Subassemblages with one end fixed against rotation
5. Beam-columns continuous over several supports.

The articles where the various methods are presented are listed in Fig. 3.

* The material properties of the members are always assumed to be given.
Next, the analysis of subassemblages which are permitted to sway is treated in detail. Three additional effects, which do not occur in nonsway subassemblages, are included in the analysis; namely, the effects of the sway deflection, the horizontal force and the lateral restraint. Methods will be developed for the general subassemblages for the following two cases: (1) When the lower joint moment and the horizontal force are specified, determine the maximum value of the upper joint moment which can be carried by a given subassemblage, and (2) When the two joint moments are known, determine the maximum horizontal force. The general methods are subsequently simplified and applied to the following four types of subassemblages:

1. Subassemblages loaded by joint moments only and free to sway in the lateral direction.
2. Subassemblages loaded by joint moments and restrained against sidesway by a specified lateral restraint.
3. Subassemblages pinned at one end and subjected to a horizontal force.
4. Subassemblages pinned at one end and subjected to a combination of horizontal force and external moment.

These subassemblages were selected to illustrate separately the three effects associated with sway subassemblages as mentioned above. The properties of the subassemblages are shown in Fig. 3.

A separate study on the sidesway buckling of subassemblages is also included in this report. An analytical criterion is first presented for subassemblages with one end pinned. The criterion is later generalized and a graphical method applicable to general subassemblages is developed.
The various procedures presented herein are essentially graphical methods and depend on the availability of certain types of charts. These charts have been developed in an earlier study and their use will be described briefly in the later development. A few sample charts which will be used in solving the illustrating problems given in this report are included in Appendices A, B and C. In preparing the charts, it has been assumed that the subassemblages are made of material which behaves in an elastic and perfectly plastic manner* and that the individual members contain cooling residual stresses of known magnitude and distribution.** More detailed discussion concerning the charts can be found in Ref. 13.

Throughout this study, the subassemblages are assumed to be sufficiently braced to prevent out-of-plane deformation. Failure is therefore due to either excessive bending or buckling in the plane of the structures.

* The material constants used in preparing the charts are: yield stress \( \sigma_y = 33 \) ksi and Young's modulus \( E = 30,000 \) ksi.
** A linearly varying symmetrical residual stress pattern with a maximum compressive stress of \( 0.3\sigma_y \) occurring at the flange tips is adopted. This pattern has been used in previous studies on beam-columns and restrained columns12,13,14.
2. ANALYSIS OF SUBASSEMBLAGES WITHOUT SIDESWAY

2.1 GENERAL DEFINITION AND SIGN CONVENTION

This article is concerned with the analysis of the carrying capacity of subassemblages prevented from sidesway. A typical subassemblage to be analyzed is shown in Fig. 4. The carrying capacity of such a structure is measured by the amount of external moments at the upper and lower joints, \((M_e)_U\) and \((M_e)_L\), together with the axial thrust which the structure is able to sustain. The column in the subassemblage is assisted in resisting the joint moments by rotational springs with known characteristics. The moment-rotation characteristics of the springs can be either linear (restraining members remain elastic) or nonlinear (restraining members are partially plastic). As shown in Fig. 4, the springs react to the joint rotations \(\theta_U\) and \(\theta_L\) with restraining moments \((M_r)_U\) and \((M_r)_L\), while the column carries the end moments \(M_U\) and \(M_L\) which are compatible with the joint rotations.

The sign convention for moments and end rotations adopted in the analysis is as follows: The external joint moments, \((M_e)_U\) and \((M_e)_L\) and the moments \(M_U\) and \(M_L\) acting at the ends of the column are positive when clockwise, and the restraining moments offered by the springs, \((M_r)_U\) and \((M_r)_L\), are positive when counterclockwise. The rotations \(\theta_U\) and \(\theta_L\) are positive when the tangent to the deformed curve of the column has rotated clockwise from its original direction. Thus, all quantities shown in Fig. 4 are positive.
2.2 METHOD OF ANALYSIS

From the above description it is evident that the following variables will effect the behavior of a subassemblage:

1. The height of the subassemblage, \( h \), or the slenderness ratio of its column \( h/r \).
2. The axial force \( P \), or the ratio of the axial force to the yield force of the column section, \( P/P_y \).
3. The external moments \( (M_e)_U \) and \( (M_e)_L \).
4. The restraining moments \( (M_r)_U \) and \( (M_r)_L \).
5. The column moments \( M_U \) and \( M_L \).
6. The end rotations \( \theta_U \) and \( \theta_L \).

The external moments, restraining moments and column moments are related by the joint equilibrium conditions

\[
(M_e)_U = (M_r)_U + M_U \quad \text{(1a)}
\]
\[
(M_e)_L = (M_r)_L + M_L \quad \text{(1b)}
\]

Compatibility requires that the restraint and the column rotate through the same angle at a joint. When the characteristics of the restraints are known, the restraining moments can be expressed as functions of the end rotations

\[
(M_r)_U = F_U (\theta_U, \text{properties of restraining members}) \quad \text{(2a)}
\]
\[
(M_r)_L = F_L (\theta_L, \text{properties of restraining members}) \quad \text{(2b)}
\]

* These moments can also be expressed nondimensionally as the ratios of their values to the yield moment of the column section \( M_y \), for example, \( (M_e)_U/M_y \).
In the previous studies on beam-columns, it has been found that the following functional relationships exist among the variables $M_U$, $M_L$, $Q_U$, and $Q_L$:

\[
\frac{M_U}{M_L} = q_L = f_L(M_L, \theta_L) \tag{3a}
\]

\[
\theta_U = g_L(q_L M_L, \frac{1}{q_L}) \tag{3b}
\]

in which the symbol $q_L$ represents the ratio of the moment of the upper end to that of the lower end and is taken to be positive when the two moments act in the same sense.

Equations 1, 2 and 3 make up six independent equations relating the variables. When $h/r$, $P/P_y$, $(M_e)_L$ and the restraining functions are given, there are seven unknowns to be determined. These unknowns are $(M_e)_U$, $(M_T)_U$, $(M_T)_L$, $M_U$, $M_L$, $\theta_U$ and $\theta_L$. Since only six equations are available, the system is indeterminate to the first degree.

However, by assuming a value of $\theta_L$, the equations can be solved for the other unknowns. If the analysis is repeated for several values of $\theta_L$, a curve relating $(M_e)_U$ and $\theta_L$ can be constructed. The maximum point on this curve corresponds to the desired ultimate value of $(M_e)_U$.

* An alternate set of the functional relationships is

\[
\frac{M_L}{M_U} = q_U = f_U(M_U, \theta_U) \nonumber
\]

\[
\theta_L = g_U (q_U M_U, \frac{1}{q_U}) \nonumber
\]

in which $q_U = M_L/M_U = 1/q_L$. 

---
The solution of six simultaneous equations for each assumed $\theta_L$ can be avoided by using the charts contained in Appendix A. Each chart relates the variables $M_L$, $\theta_L$ and $q_L$ or, equivalently, $M_U$, $\theta_U$ and $q_U$ for a given combination of $P/P_y$ and $h/r$. The procedure for determining $(M_e)_U$ for an assumed $\theta_L$ is as follows: Corresponding to an assumed $\theta_L$, the restraining moment $(M_r)_L$ is first computed from the known restraining function of the lower joint; the moment applied to the lower end of the column can then be determined from Eq. 1b, $M = (M_e)_L - (M_r)_L$. With $M_L$ and $\theta_L$ known, the moment ratio of the lower end, $q_L$, can be found from the chart in Appendix A, corresponding to the given combination of $P/P_y$ and $h/r$. By definition, the moment of the upper end is given by $M = q_L M$ and the corresponding moment ratio is $q_U = 1/q_L$. The rotation $\theta_U$ can be found by reentering the same chart with the values of $M_U$ and $q_U$ just determined. When $\theta_U$ is known, the restraining moment $(M_r)_U$ can be computed from the given restraining function of the upper joint. The external joint moment $(M_e)_U$ is finally determined, according to Eq. 1a, as the sum of $M_U$ and $(M_r)_U$. The following example will illustrate the application of the method.

Illustrative Example 1

PROBLEM: Given a subassemblage (see the inset in Fig. 5) whose column member has a slenderness ratio of 40 and is subjected to a constant axial force $P = 0.6P_y$ and a constant external moment $(M_e)_L = -0.5M_y$ (counterclockwise moment). The restraining functions are given by $(M_r)_L = 25 M_y \theta_L$ and $(M_r)_U = 12.5 M_y \theta_U$. It is assumed that the restraints will reach their limiting capacities when $(M_r)_L = 0.5 M_y$ and $(M_r)_U = 0.3 M_y$. 
Find the relationship between the external moment at the upper joint, $(M_e)_U$, and the rotation at the lower joint, $\theta_L$, and the maximum value of $(M_e)_U$.

**SOLUTION:** First, values of $\theta_L$ are chosen and tabulated in Column 1 of Table 1. The restraining moment corresponding to each $\theta_L$ is computed from the lower restraining function and is listed in Column 2. By subtracting the restraining moment from the external joint moment, the column moment at the lower end is obtained. This is given in Column 4. From the chart in Appendix A (for the case $P = 0.6P$ and $h = 40r$) the moment ratio $q_L$ is determined and recorded in Column 5. The negative sign of $q_L$ indicates that the moment at the upper end is clockwise. The moment ratio of the upper end, $q_U$, is computed by taking the inverse of $q_L$ and is given in Column 6. Multiplying the lower column moment by $q_L$ gives the moment at the upper end. The result is listed in Column 7.

With $M_U$ and $q_U$, the same chart is entered to find $\theta_U$ (Column 8) from which the restraining moment at the upper end is determined (Column 9). Adding Columns 7 and 9 yields the external joint moment which is in equilibrium with the given external moment at the lower end and the assumed $\theta_L$. The values of $(M_e)_U$ and $\theta_U$ listed in Columns 7 and 10 are all positive, indicating clockwise sense.

The results obtained in Table 1 are plotted as a $(M_e)_U$ vs. $\theta_U$ curve in Fig. 5. The maximum moment of the subassemblage corresponds to the peak of the curve and is found to be 0.72 M.
2.3 STABILITY CRITERION

Figure 5 shows that corresponding to a specified value of the external joint moment there are two end rotations, each representing a possible equilibrium configuration of the subassembly. One configuration represents a stable equilibrium (point A) and the other an unstable equilibrium (point C). The peak of the curve (point B) determines the limit of stability, since an increase in \( \theta_L \) would require a decrease in \( M_e \). At this limit, then,

\[
\frac{d(M_e)}{d\theta_L} = 0
\]  

(4)

Also, since \( M_e \) is constant

\[
\frac{d(M_e)}{d\theta_L} = 0
\]  

(5)

Substituting the expressions for \( (M_e)_U \) and \( (M_e)_L \) from Eqs. 1 into Eqs. 4 and 5 results in

\[
\frac{d(M_e)}{d\theta_L} = \frac{d(M_r)}{d\theta_L} + \frac{dM}{d\theta_L} = 0
\]  

(6a)

\[
\frac{d(M_e)}{d\theta_L} = \frac{d(M_r)}{d\theta_L} + \frac{dM}{d\theta_L} = 0
\]  

(6b)

The restraining moments \( (M_r)_U \) and \( (M_r)_L \) are functions of \( \theta_U \) and \( \theta_L \), respectively, and the column moments \( M_U \) and \( M_L \) are each functions of both \( \theta_U \) and \( \theta_L \). (See Eqs. 2 and 3). In the above example, however, it was observed that once a value of \( \theta_L \) is chosen, the rotation \( \theta_U \), the column moment \( M_U \) and the restraining moment \((M_r)_U \) of the upper end can be
uniquely determined. Therefore $\theta_L$ is the only independent variable.

Upon application of the chain rule of partial differentiation, Eqs. 6 become

$$\frac{d(M_L)}{d\theta_U} \frac{d\theta_U}{d\theta_L} + \frac{\partial M_U}{\partial \theta_L} \frac{d\theta_U}{d\theta_L} + \frac{\partial M_U}{\partial \theta_U} = 0 \quad (7a)$$

$$\frac{d(M_L)}{d\theta_L} + \frac{\partial M_L}{\partial \theta_U} \frac{d\theta_U}{d\theta_L} + \frac{\partial M_L}{\partial \theta_L} = 0 \quad (7b)$$

Eliminating $d\theta_U$ from the above equations leads to

$$\left(\frac{d(M_L)}{d\theta_U} + \frac{\partial M_U}{\partial \theta_U}\right) \left(\frac{d(M_L)}{d\theta_L} + \frac{\partial M_L}{\partial \theta_L}\right) - \frac{\partial M_U}{\partial \theta_L} \frac{\partial M_L}{\partial \theta_U} = 0 \quad (8)$$

Equation 8 defines the limit of stability for the general case of a nonsway subassemblage.

A simpler method of checking stability is developed in the next article.

2.4 CRITERION OF ADEQUACY

A subassemblage is considered adequate if it can resist the applied loads in a stable equilibrium configuration. It will be shown that a subassemblage is adequate if it is possible to find a deformed shape in equilibrium with some loads which are equal to or greater than the applied loads. For the case of constant $P/P_y$ and constant $(M_e)_L$, the criterion can be expressed as follows: If there exists at least one $\theta_L$ such that equilibrium and compatibility result in

$$(M_L)_L + M_L = (M_e)_L \quad (9a)$$
and
\[(M_\text{L})_U + M_U \geq (M_\text{e})_U\]  \hspace{1cm} (9b)

then the subassemblage is adequate.

When \((M_\text{e})_L\) is constant and a lower joint rotation \(\theta_L\) has been chosen, equilibrium at the lower joint and the resulting deformed shape determine \(\theta_U\), which fixes the value of \((M_\text{L})_U\) and \(M_U\) necessary for equilibrium. If the sum of \((M_\text{L})_U\) and \(M_U\) is equal to or greater than the applied external moment, then the column is adequate.

A proof of the criterion for adequacy is given in Appendix D.

2.5 ANALYSIS OF SPECIAL CASES

Under special conditions of loading and restraint, the behavior of a subassemblage can be described by a reduced number of variables. The study of these cases is of practical interest because they usually represent extreme conditions of geometry and/or loading of the general case.

(1) Subassemblages with Symmetrical Loading and Restraint

When a subassemblage with equal restraining beams is subjected to equal but opposite external moments, its column will be bent into a symmetrical configuration (See Fig. 6a). The moments and the rotations at the two ends of the column will be equal, but of opposite sense. The end moment ratio, \(q\), is therefore equal to -1. This special case has been solved in Ref. 5, and a design method along with a concept of adequacy has been developed in Ref. 15. The case is treated here for completeness.
In this case, only one equation for joint equilibrium is required

\[ M_e = M_r + M \]  

(10)

and only one restraining function needs to be defined. If the magnitude of the axial force and the slenderness ratio of the column are specified, the moment \( M_e \) varies only with the end rotation; and the relationship can be determined very simply by a graphical method.

A subassemblage bent into a symmetrical configuration is adequate when a rotation \( \Theta \) can be found such that

\[ M_r + M \geq M_e \]  

(11)

Inequality 11 insures that the moment-carrying capacity of the structure is equal to or greater than the applied moment.

The types of problems that can be studied for a symmetrical subassemblage are: 1) Given a restraining function and the magnitude of the axial force in the column, find the maximum external moment which the system can support, and 2) Given a restraining function and the magnitude of the axial force, check if the system is adequate to carry a specified amount of external moment.

**Illustrative Example 2**

**PROBLEM:** A subassemblage whose column length \( h \) is equal to 30\( r \) is deformed symmetrical by two joint moments of equal magnitude. The compressive force in the column is equal to 0.8\( P_y \); and the restraining function is defined by \( M_r = 12.5 M_y \theta \). It is required to determine: 1) the moment carrying capacity of the structure, and 2) the adequacy of the system to support an external joint moment of \( M_e = 0.2 M_y \), using the criterion for adequacy stated by Eq. 11.
SOLUTION: The determination of the ultimate moment of the subassemblage is illustrated in Fig. 7. The moment-rotation characteristic of the column is defined by curve OAB in Fig. 7a*, and that of the spring is represented by a straight line with a slope of -12.5. The latter is shown inverted in Fig. 7a. For a given external moment \( M_e \) equilibrium is represented by the point of intersection of the two curves (point A in the figure). Since the slope of the spring curve is constant, the maximum value of the external moment is the one which will result in tangency of the two curves. That is

\[
\frac{dM}{d\theta} = -\frac{dM_r}{d\theta}
\]

When the spring moment is added to the column moment, a curve relating the external moment to the end rotation is obtained. This curve is shown in Fig. 7b. The limit of stability, which is represented by point B, occurs for the same rotation as the point of tangency of Fig. 7a. The ultimate joint moment of the system is found to be 0.285 \( M_y \).

If it is only necessary to check the adequacy of the system to support a given moment, the analysis can be made very simply without resorting to the graphical technique. In the second part of the problem, if a value of \( \theta \) can be found which will result in a pair of \( M_r \) and \( M \) values satisfying Eq. 11, then the system is adequate. A trial value of \( \theta = 0.002 \) rad. is assumed, and the corresponding restraining moment is \( M_r = 12.5 \times 0.002 \text{ M}_y = 0.025 \text{ M}_y \). From the moment-rotation curve of the column, the end moment \( M \) for the assumed \( \theta \) is found to be \( M = 0.1 \text{ M}_y \). The total resisting moment is therefore equal to \( M_r + M = 0.125 \text{ M}_y \). This moment is less than the applied

* The moment-rotation curve of the column is taken from the second chart in Appendix B corresponding to \( P/P_y = 0.8 \) and \( h/r = 30 \).
moment $M = 0.2M_e$. A $\theta$ equal to 0.004 is next assumed, and restraining moment and column moment are found to be $0.05M_y$ and $0.155M_y$, respectively. The sum of these moments is $0.205M_y$, which is greater than the applied moment. The subassemblage is therefore adequate.

(2) **Subassemblages With Antisymmetrical Loading**

If a subassemblage with equal restraining beams is subjected to equal external moments $M_e$ applied in the same sense, its deformed shape will be antisymmetrical, as shown in Fig. 6b. The magnitude of the end moments and end rotations will again be equal. The method described previously for the analysis of symmetrical subassemblages can also be used in this case. The only modification is that the moment-rotation curves of the columns should be determined for the double curvature configuration, instead of the symmetrical configuration. The required curves for the column can be obtained from the chart in Appendix B (for the case of a column subjected to one end moment), using an effective slenderness ratio equal to half of the actual value.

For columns with slenderness ratios less than 40, previous studies have shown that the maximum moment will occur at the end of the members. Therefore the ultimate moment that can be carried by the column may be assumed to be equal to the plastic moment of the section reduced for the effect of axial force (usually denoted by $M_{pc}$).

(3) **Subassemblages With One End Free of Moment**

Another special case occurs when a subassemblage has no restraining beam and is free of applied moment at one end. This is shown in Fig. 6c. In analyzing such a structure, the required moment-rotation curve of the
column can also be obtained from the chart contained in Appendix B (for the case of a column subjected to one end moment). Extensive charts covering a wide range of $P/P_y$ and $h/r$ are available in Ref. 12. This case has been solved by Ojalvo in an earlier study. The analysis can readily be made by following the procedure described previously for the symmetrical case. If the slenderness ratio of the column is less than 30, its moment carrying capacity will be close to the plastic moment $M_{pc}$ of the section 12. The maximum joint moment that can be carried by the subassemblage is therefore equal to the sum of the plastic moment of the column and the maximum restraining moment that can be provided by the beam.

(4) **Subassemblages With One End Fixed**

When one end of a subassemblage is fixed against rotation (Fig. 6d), it is again possible to simplify the analysis by using the charts developed specially for this case. From these charts the moment vs. rotation relationships of columns which are fixed at one end can be obtained. A sample chart for $P/P_y = 0.4, 0.6$ and $0.8$, with slenderness ratio $h/r$ equal to 30, is included in Appendix B.

The case can be solved very simply after the required moment-rotation curve for the column is constructed. The method outlined in (1) may be followed. For a subassemblage with column slenderness ratio less than about 40, the maximum external moment of the system can again be determined as the sum of $M_{pc}$ and the limiting restraining moment.
(5) **Subassemblages That Can Be Solved by Statics**

For certain restraint conditions, it is possible to analyze and design subassemblages without recourse to compatibility. It has already been pointed out that when a column, with h/r less than a certain value, is pinned at one end or is bent in double curvature, the maximum moment will often occur at one or both ends of the member. Columns fixed at one end fall also into this category, as they are also bent in double curvature. When such columns are restrained by beams having known moment resisting capacities (see the spring curves in Fig. 8), the ultimate joint moment is equal to the sum of the ultimate moments of the beams and that of the column (Fig. 8a)

\[
(M_e)_{\text{max}} = (M_r)_{\text{max}} + M_{\text{ax}}
\]

(12)

For such cases, the lower bound theorem of plastic theory holds. When the restraining beams behave in an elastic and perfectly plastic manner, but the column has a peak beyond which it must unload, the ultimate strength of the subassemblage is again determined by Eq. 12, if the rotation corresponding to yielding of the restraining beams is smaller than the rotation corresponding to the maximum column moment (Fig. 8b).

\[
\theta_p \leq (\theta)_{\text{max}}
\]

(13)

However, the joint can not rotate further than \(\theta_{\text{max}}\) before unloading of the structure takes place.

If the restriction imposed by Eq. 13 is met, the ultimate external moment of a subassemblage is also the sum of the maximum moments of the restraining beam and column.
2.6 COLUMNS CONTINUOUS OVER SEVERAL SUPPORTS

The procedure described in Art. 2.2 can be adapted to analyze continuous columns subjected to axial forces and external moments. Figure 9 shows a column which is fixed at one end and is continuous over five supports. The distances between the supports are designated by $h_1, h_2, \ldots, h_5$. Joint moments and axial forces are applied at the levels of the supports. Each joint is numbered and moment, axial force and rotation at a joint are distinguished by a subscript. Thus, at Joint 1, the external moment is $M_1$, the rotation is $\theta_1$, and the total axial loads is $\sum_{i=1}^{5} P_i$. The column moments are $M_{10}$ and $M_{12}$ at Joint 1. Joint equilibrium requires that:

$$M_1 = M_{1,i-1} + M_{1,i+1}$$

(14)

According to the sign convention adopted in Art. 2.1, the moments $M_1, M_3$ and $M_5$, and the corresponding rotations $\theta_1, \theta_3$ and $\theta_5$ are positive. The other moments and rotations shown in Fig. 9 are all negative.

The problem to be analyzed is the following: If the member size of the column and the segment lengths $h_1$ through $h_5$ are given, and when all the vertical forces $P_1, \ldots, P_5$ and the moments $M_1, M_2, M_3$ and $M_4$ are specified, determine the relationship between $M_5$ and $\theta_5$ and the maximum value of $M_1$ which the column can sustain. The analysis can proceed from the lowest segment by assuming a value of $\theta_1$ and determining the corresponding column moment $M_{12}$ from the chart giving the moment-rotation relationship of fixed end column segments (Appendix B). The equilibrium condition at Joint 1 requires that the lower moment for the segment $h_2$ be given by $M_{12} = M_1 - M_{10}$. With $M_{12}$ and $\theta_1$ known, the upper column moment $M_{21}$ and
Joint rotation $\theta_2$ can be determined by the procedure outlined in Art. 2.2. Charts of the type given in Appendix A are used for this purpose. The above analysis is repeated for the other three segments until a pair of $M_5$ and $\theta_5$ is found. This yields one equilibrium configuration. Additional equilibrium configurations can be found by assuming different values of $\theta_1$ and repeating the above procedure. When the various pairs of $M_5$ and $\theta_5$ (each pair corresponding to one equilibrium configuration) are plotted, a curve defining the relationship between $M_5$ and $\theta_5$ is obtained. From this curve the maximum value of $M_5$ can be determined. The following example will illustrate the application of the method.

**Illustrative Example 3**

**PROBLEM:** Consider the continuous column shown in Fig. 10. It has three equal segments, each having a length of 30r. An axial force $P = 0.6P_y$ acts at the top of the column, and two external moments $M_1 = 0.6M_y$ and $M_2 = -0.5M_y$ are applied at Supports 1 and 2. Determine the relationship between $M_3$ and $\theta_3$ and the maximum value of $M_3$.

**SOLUTION:** The first step in the solution is to select several values of $\theta_1$; each corresponds to a possible equilibrium configuration and will result in a pair of $M_3$ and $\theta_3$ values. Subsequent computations for the various values of $\theta_1$ can be performed in a tabular form as shown in Table 2. For each $\theta_1$, $M_{10}$ is found from the chart for fixed end columns given in Appendix B. Since $M_1$ is given, $M_{12}$ can be computed by subtracting $M_{10}$ from $M_1$. The moment ratio $q_{12}$ is obtained by entering the appropriate chart of Appendix A with $M_{12}$ and $\theta_1$. The moment $M_{12}$ is computed by multiplying $M_{12}$ by $q_{12}$, and the moment ratio $q_{21}$ is simply the inverse of...
With $q_{1}$ and $M_{1}$ known, the same chart in Appendix A can be entered to determine $\theta_{2}$. The above procedure is continued for the top segment and the calculations finally yield values of $M_{32}(=M_{3})$ and $\theta_{3}$ which are compatible with the initial rotation $\theta_{1}$ and in equilibrium with the applied moments and axial force. The four pairs of $M_{3}$ and $\theta_{3}$ obtained in Table 2, corresponding to the four selected values of $\theta_{1}$, are plotted in Fig. 11. From this figure the maximum value of $M_{3}$ is found to be 0.50 $M_{Y}$. 
3. ANALYSIS OF SUBASSEMBLAGES WITH SIDESWAY

3.1 GENERAL DEFINITION AND SIGN CONVENTION

If the upper joint of a subassemblage is allowed to translate with respect to the lower joint, the subassemblage will generally be deformed into a swayed configuration when subjected to external loads. A typical example of a sway subassemblage is illustrated in Fig. 12a. The subassemblage is loaded by external joint moments \( (M_e)_U \) and \( (M_e)_L \) and by a vertical force acting directly on the top of the column. In addition, a horizontal force \( Q \) is applied at the level of the upper joint. The column in the structure is restrained at its two ends by beams (simulated by springs) which react to joint rotations \( \Theta_U \) and \( \Theta_L \) with moments \( (M_r)_U \) and \( (M_r)_L \). At the upper end of the column a horizontal spring providing lateral restraint against sidesway is attached. The spring has a stiffness \( k \) and reacts with a horizontal force \( k \Delta \) to a lateral displacement \( \Delta \).

It is assumed that the characteristics of both the rotational and lateral springs are known in a given problem.

As in the case of a nonsway subassemblage, the external moment applied at each joint is carried partly by the column and partly by the rotational restraint. The manner in which the joint moment is distributed depends on the moment-rotation characteristics of the column and the spring. In addition to the externally applied moments, there are moments developed in the subassemblage due to the lateral displacement \( \Delta \) and the horizontal shear \( V \) which is equal to the difference between \( Q \) and \( k \Delta \). These moments are also carried jointly by the column and the rotational
restraints. It is important that the effect of these moments be properly taken into account in the analysis of a sway subassemblage.

For a given subassemblage subjected to a specified axial force $P$, the following two problems may be investigated: 1) When the horizontal force $Q$ and one of the two external moments are given, determine the maximum value of the other moment which can be safely carried by the structure, and 2) when both moments are given, determine the maximum value of $Q$. Solutions to both problems will be presented in Arts. 3.2 and 3.3.

The variables involved in the analysis of a subassemblage with sidesway include the ten variables listed in Art. 2.2 plus the following three: the lateral deflection $\Delta$, the horizontal force $Q$ and the spring stiffness $k$. Thus, the total number of variables is thirteen. The ten variables defined previously for nonsway subassemblages have the same physical significance in analyzing subassemblages with sidesway. However, in a sway subassemblage, the rotations $\theta_U$ and $\theta_L$ are interpreted slightly differently. By definition $\theta_U$ and $\theta_L$ are the angles between the vertical (corresponding to the undeformed position of the column) and the tangent of the deflected shape at the two ends. In the nonsway case, the vertical coincides with the chord of the deflected column. This is no longer true if the top of the column moves horizontally with respect to the bottom. The joint rotations $\theta_U$ and $\theta_L$ to be used in the subsequent analysis of sway subassemblages are as indicated in Fig. 12a.

The three new variables introduced above represent the effects of lateral deflection and horizontal force. These variables can be expressed nondimensionally as $\Delta/h$, $Q/P$ and $k \Delta/P$. The ratio $\Delta/h$ can be seen to
represent the rotation of the chord with respect to the vertical.

The sign convention for bending moments and joint rotations is the same as that adopted in the previous discussions on nonsway subassemblages, that is, clockwise moments acting on joints or column ends and clockwise end rotations (measured from the vertical) are considered positive. A horizontal force is positive if it produces a clockwise moment about the lower joint. A chord rotation is positive when the chord of the deformed column has rotated clockwise from the vertical position. According to this sign convention, all quantities shown in Fig. 12a, except \((M_e)_U\), \((M_e)_L\), and \(k \Delta\), are positive.

3.2 METHOD OF ANALYSIS

As stated above the total number of variables involved in the analysis of subassemblages with sidesway is thirteen. The equations that can be used to relate these variables are two joint equilibrium equations

\[
\begin{align*}
(M_e)_U &= (M_r)_U + M_U \\
(M_e)_L &= (M_r)_L + M_L
\end{align*}
\]

(1a) (1b)

and two equations defining the restraining characteristics of the rotational springs

\[
\begin{align*}
(M_r)_U &= F_U \theta_U, \text{ properties of restraining members} \\
(M_r)_L &= F_L \theta_L, \text{ properties of restraining members}
\end{align*}
\]

(2a) (2b)

In addition the shear equilibrium of the column requires that

\[ M_U + M_L + P \Delta + Vh = 0 \]

(15)
in which $V$ represents the shear in the column and is equal to $Q - k \triangle$. Equation 15 can be derived by considering the equilibrium of the column shown in Fig. 12b*.

In analyzing a given subassemblage, some of the thirteen variables are usually given. In the following discussion, it will be assumed that the slenderness ratio of the column $h/r$ and the magnitude of the vertical force $P$ (or the ratio $P/P_y$) are the given quantities and that the characteristics of the rotational and lateral springs are known. The functional relations of Eq. 2a and 2b are specified when the properties of the rotational springs are given. Similarly, if the characteristic of the lateral spring is known, the spring stiffness $k$ can be considered as a known quantity. Thus, the number of variables is reduced by five, and the remaining variables are $\left(M_e\right)_U$, $\left(M_e\right)_L$, $M_U$, $M_L$, $\theta_U$, $\theta_L$, $\Delta/h$ and $Q$.

Some of the above eight variables will also be specified in a given problem. In the first type of problem stated in Art. 3.1, the horizontal force $Q$ and one of the two external moments, $(M_e)_U$ or $(M_e)_L$, are given. The known variables in the second type of problem are $(M_e)_U$ and $(M_e)_L$. So the total number of variables becomes six. As will be seen in the later development, the proposed method of analysis requires that an initial joint rotation (either $\theta_U$ or $\theta_L$) be assumed and subsequent analysis be made to determine the value of either the unknown moment or the unknown horizontal force consistent with the assumed joint rotation. Thus, in using this procedure, only five variables need to be determined: $(M_e)_U$ (if $(M_e)_U$ is the known moment) or $Q$, $M_U$, $M_L$, $\theta_U$ (if $\theta_L$ is the assumed rotation) and $\Delta/h$.

* Equation 15 is written for a set of positive end moments $M_U$ and $M_L$ (clockwise), although the moments shown in Fig. 12b are negative.
The two joint equilibrium conditions (Eqs. 1a and 1b) and the shear equilibrium condition (Eq. 15) constitute three basic equations that can be used to solve the five unknowns. Two additional relationships are necessary if a complete solution to the problem is sought. The charts contained in Appendix C have been prepared to provide the needed relationships.

Before discussing the use of these charts, it is necessary to examine the deformation behavior of an individual column subject to sway. In Fig. 12b is shown the deformed configuration of the column member isolated from the subassemblage of Fig. 12a. The joint rotations \( \Theta_U \) and \( \Theta_L \) and the chord rotation \( \Delta/h \) are the same as those illustrated in Fig. 12a. Two additional angles \( \gamma_U \) and \( \gamma_L \) relating, respectively, \( \Delta/h \) to \( \Theta_U \), and \( \Delta/h \) to \( \Theta_L \), are also shown. These angles represent the rotations of the tangents from the chord and are positive if the tangents rotate clockwise from the chord*. The column moments \( M_U \) and \( M_L \) are shown in their negative sense, being consistent with the applied external joint moments. The shear force \( V \) and the vertical force \( P \) form a resultant force acting along the direction of the thrust line. Three types of angles \( \alpha, \beta \) and \( \tau \) may be associated with this line. \( \alpha_U \) and \( \alpha_L \) are the angles measured from the thrust line to the tangents at the two ends, and \( \beta \) is the angle measured from the same line to the chord. The angle \( \tau \) defines the direction of the thrust line and is given by \( \tau = \tan^{-1} V/P \). It has been shown that in practical cases this angle may be assumed to be equal to \( V/P \), because \( P \) is usually much greater than \( V \). All angles are positive when they represent clockwise rotations from the thrust line.

* \( \gamma_U \) and \( \gamma_L \) as shown in Fig. 12b are therefore negative.
Using the definitions and sign convention stated above, the angles $\alpha_U$ and $\alpha_L$ can be expressed in terms of $\theta_U$, $\theta_L$ and $\gamma$.

$$\alpha_U = \gamma + \theta_U$$

$$\alpha_L = \gamma + \theta_L$$

(16a)

(16b)

and the angle $\beta$ is given by

$$\beta = \gamma + \Delta/h$$

(17)

Similarly, the rotations $\gamma_U$ and $\gamma_L$ can be determined from $\theta_U$, $\theta_L$ and $\Delta/h$ by the relationships

$$\gamma_U = \theta_U - \Delta/h$$

(18a)

and

$$\gamma_L = \theta_L - \Delta/h$$

(18b)

The charts presented in Appendix C may be used to determine the angle $\beta$ of a column, when the angle $\alpha$ and the moment $M$ at one end are specified. They may also be used to fund the angle $\alpha$ at one end of a column, when the angle $\beta$ and the moment $M$ at the same end are known. As will be shown in the later discussions, the use of these charts makes the solution to a sway subassemblage possible. The steps which were followed in preparing the charts have been described in detail in an earlier report. The report also includes examples illustrating the procedure of using the charts in column analysis.

The method developed for analyzing subassemblies in which the lower external moment $(M_e)_L$ and the horizontal force $Q$ are specified and the maximum value of $(M_e)_U$ is to be determined is summarized as follows.
1. Select a value of $\Theta$ and compute the corresponding restraining moment $(M_r)_L/M_y$ from the known moment-rotation characteristic of the lower restraining beam. The moment $M_L$ of the column can then be determined from Eq. 1b, $M_L/M_y = (M_e)_L/M_y - (M_r)_L/M_y$.

2. Assume a trial value for $\Delta$ and determine the ratio $\Delta/h$ which is equal to the chord rotation of the column*. The shear force $V$ can then be determined as the difference between $Q$ and $k\Delta$ (see Fig. 12b). Since the axial force $P$ is specified in a given problem, the angle $\tau$ can be computed from $\tau = V/P$. The sum of $\tau$ and $Q$ determines the angle $\alpha_L$ (Eq. 16b).

3. With $\alpha_L$ and $M_L/M_y$, determine the angle $\beta$ from the appropriate chart in Appendix C. The chord angle $\Delta/h$ can then be calculated from Eq. 17, $\Delta/h = \beta - \tau$. This value should be compared with that assumed in Step 2. If the calculated $\Delta/h$ does not agree with the assumed, a new $\Delta$ (or $\Delta/h$) should be tried. The process is repeated until agreement is found.

4. Knowing the deflection $\Delta$ and the shear $V$, the moments $P \Delta$ and $Vh$ can be determined. The following convenient relationship may be used for this purpose

$$\frac{P \Delta + Vh}{M_y} = \frac{P}{P_y} \cdot \frac{h}{r} \cdot \frac{d}{2r} \cdot \beta$$

in which $d$ is the depth of the column section. The derivation of this equation is given in Ref. 13.

* It is assumed here that in a given problem not only the slenderness ratio $h/r$ of the column member is known, but also the actual column length $h$.

** In the examples that follow the axial force is specified as the ratio of $P$ to $P_y$. The actual axial force can be determined by multiplying this ratio by $P_y$. 
5. From the shear equilibrium condition of Eq. 15, the moment at the upper end of the column can be determined, \( M_{U/M} = M_{L/M} - (P \Delta + Vh)/M_y \).

6. With \( M_{U/M} \) and \( \beta \), determine the angle \( \alpha_U \) from the chart used previously in Step 3. The joint rotation \( \theta_U \) is then obtained from Eq. 16a, \( \theta_U = \alpha_U - \tau \).

7. When \( \theta_U \) is known, the restraining moment \( (M_r)_U \) can be determined from the moment rotation relationship of the upper restraint. The external joint moment is then obtained by applying Eq. 1a, \( (M_e)_{U/y} = (M_r)_U/M_y + M_U/M_y \). This is the external moment consistent with the initial lower joint rotation \( \theta_L \).

8. The above steps are repeated for several selected value of \( \theta_L \). The results are then plotted in the form of a \( (M_e)_{U/y} \) vs. \( \theta_L \) (or \( \Delta/h \)) curve from which the maximum value of \( (M_e)_{U/y} \) can be determined.

To analyze the case in which the two external moments \( (M_e)_U \) and \( (M_e)_L \) are specified and the maximum value of \( Q \) is to be determined, the procedure described above has to be modified. In such a case, the shear force \( V \) of the column is not known, and, consequently, the angle \( \tau \) cannot be determined. This complicates considerably the analysis and makes the above method very difficult to apply. A modified procedure for solving subassemblies with unspecified horizontal force is therefore developed.

The first step in this procedure, which is the same as that in the previous method, consists of selecting a value of \( \theta_L \) and determining the corresponding column moment \( M_L \). Since the horizontal force \( Q \) is not known, the angle \( \alpha_L \) cannot be computed from Eq. 16b. It is therefore not
possible to determine directly the angle $\beta$ by using the appropriate chart. However, if a value of $\beta$ is assumed, the corresponding value of $\alpha_L$ can be obtained from the chart. Whether the assumed $\beta$ is correct or not may be checked by the following procedure: Since $\alpha_L$ is now known, the angle $\tau$ can be determined from $\tau = \alpha_L - \theta_L$ (Eq. 16b). Consequently, the rotation $\Delta/h$ is also known (from Eq. 17, $\Delta/h = \beta - \tau$). When the assumed value of $\beta$ is substituted into the right hand side of Eq. 19, the bending moments due to $\Delta$ and $V$ are determined. By knowing $\Delta$ and $(P\Delta + Vh)/M_y$, the shear force $V$ can be calculated and a new value of $\tau (\tau = V/P)$ is obtained. If the assumed $\beta$ is correct, the angle $\tau$ just determined should agree with the $\tau$ value used previously. Otherwise, a new $\beta$ value is assumed and the procedure is repeated until an agreement between the two $\tau$ values is found.

After the correct value of $\beta$ corresponding to the selected value of $\theta_L$ is found, the shear force $V$ and, in turn, the horizontal force $Q$ ($Q = V + k\Delta$) can be determined, using the procedure just described. The moment at the upper end of the column is determined from the shear equilibrium condition of Eq. 15,

$$\frac{M_U}{M_y} = -\left(\frac{M_L}{M_y} + \frac{P\Delta + Vh}{M_y}\right) = -\left(\frac{M_L}{M_y} + \frac{P}{y} \frac{h}{r} \frac{d}{2r} \beta\right)$$

(20)

with $M_U/M_y$ and $\beta$ known, the chart in Appendix C is used to find $\alpha_U$; and, consequently, the upper joint rotation can be determined, $\theta_U = \alpha_U - \tau$. The known moment-rotation characteristic of the upper spring determines the restraining moment $(M_r)_U/M_y$ corresponding to the $\theta_U$ just found. The sum of $(M_r)_U/M_y$ and $M_U/M_y$ gives the external moment $(M_e)_U$. When this moment and the given $(M_e)_L$ together with the horizontal force $Q$ found in
the analysis are applied to the subassemblage, the rotation $\theta_L$ produced at the lower joint will be the rotation which was selected at the beginning of the analysis.

Recall that the upper joint moment is a known quantity in the problem; the external moment determined from the final step of the above analysis should therefore be compared with this moment. If the computed external moment differs from the given moment, a new $\theta_L$ should be selected and the above analysis is repeated. A correct value of $\theta_L$ is found when the upper joint moment obtained from the analysis agrees with the given moment. Accordingly, the horizontal force $Q$ consistent with this $\theta_L$ is the correct horizontal force which should be applied to the subassemblage.

If all the above steps are repeated for different values of $\theta_L$, several pairs of $\theta_L$ and $Q$ values may be obtained. The maximum value of $Q$ can be found graphically by plotting the results in the form of a $Q$ vs. $\theta_L$ (or $\triangle/h$) curve and determining the peak of the curve. Numerical examples illustrating this procedure are given in Art. 3.3.

The method of analysis outlined previously for subassemblages with specified $(M_e)_L$ and $Q$ is now illustrated by the following example.

Illustrative Example 4

PROBLEM: A subassemblage whose column member has a slenderness ratio of 30 is subjected simultaneously to external joint moments $(M_e)_U$ and $(M_e)_L$ and a horizontal force $Q$. In addition, a vertical force $P = 0.6 P_y$ is applied on the top of the column. The lower external moment and the horizontal force are given: $(M_e)_L = -0.2 M_y$ and $Q = 0.002 P$. The
characteristics of the rotational springs are defined by \( (M)\_r^U = 12.5 \Theta^U M \) and \( (M)\_r^L = 25 \Theta^L M \). For simplicity, it is assumed that no lateral spring is attached to the structure \( (k = 0) \). It is required to determine the relationship between \( (M^e)\_U \) and \( \Delta/h \) and the maximum value of \( (M^e)\_U \).

**SOLUTION:** The step-by-step calculations made for this problem are summarized in Table 3. Each horizontal row in the table records the solution for a selected value of \( \Theta^L \) (given in Column 1). The nondimensional restraining moment \( (M)\_r^L /M \_y \) corresponding to a given \( \Theta^L \) is obtained by multiplying the rotation by 25 (characteristic of the lower spring) and is recorded in Column 2. The nondimensional column moment \( M^L /M \_y \) given in Column 3 is the difference between \( (M^e)\_L /M \_y \) and \( (M)\_r^L /M \_y \). Since there is no lateral spring attached to the subassemblage, the shear in the column is equal to the applied horizontal force, that is, \( V = Q \). The angle \( \tau \) can therefore be determined directly. By definition, it is given by \( \tau = V/P = 0.002\* \). The algebraic sum of \( \tau \) and \( \Theta^L \) gives the angle \( \alpha^L \) (Column 4). The chart in Appendix C corresponding to \( P/P_y = 0.6 \) and \( h/r = 30 \) is entered to find the value of \( \beta \). The chord rotation \( \Delta/h \) which determines the magnitude and direction of the sway deflection of the subassemblage is then obtained from the relation \( \Delta/h = \beta - \tau \). The values of \( \beta \) and \( \Delta/h \) are listed in Columns 5 and 6, respectively.

The next step in the solution is to determine the moments caused by the sidesway deflection and the shear force \( V \). From Eq. 19, it is known that the sum of these moments (nondimensionalized by dividing the sum of

* If a lateral spring is present, the shear force in the column can not be determined without knowing the deflection \( \Delta \). In such a case the trial-and-error procedure described in Steps 2 and 3 should be followed.
When the value of \( \beta \) in Column 5 is multiplied by 20.7, the product gives directly the required moment sum (Column 7). Knowing the magnitude of \( M_L/M_y \) and \( (P\Delta + Vh)/M_y \), the upper column moment can be determined by applying Eq. 20. This is done by taking the algebraic sum of the values in Columns 3 and 7 and changing its sign. The result is given in Column 8.

The angle \( \alpha_U \) at the upper end of the column is obtained by entering the same chart in Appendix C with \( M_L/M_y \) and \( \beta \). The joint rotation \( \theta_U \) is then determined from the relationship \( \theta_U = \alpha_U - \tau \). The values of \( \alpha_U \) and \( \theta_U \) are given, respectively, in Columns 9 and 10.

When \( \theta_U \) is known, the restraining moment \( (M_e U)/M_y \) at the upper joint can be obtained by multiplying this angle by 12.5 (characteristic of the upper spring). This is recorded in Column 11. The algebraic sum of the values in Columns 8 and 11 gives the external joint moment \( (M_e U)/M_y \) (listed in Column 12).

In this example a wide range of \( \theta_L \) values has been selected. The results show that the subassemblage can sway in both directions, depending on the magnitude of the upper joint moment \( (M_e U)/M_y \). This can be seen more clearly from Fig. 13 in which the upper joint moment \( (M_e U)/M_y \) (Column 12) is plotted against the chord rotation \( \Delta/h \) (Column 6). When \( (M_e U)/M_y \) is

*The ratio \( d/2r \) is taken to be 1.15 in this example. This is the value for an 8 WF 31 section. However, for most of the available WF shapes, this ratio is nearly constant.
less than 0.125, the subassemblage always sways to the left. The structure remains vertical when the upper joint moment \( (M_e)_u/M_y \) equals 0.125. In this case the secondary moment \( P \Delta \) is zero.

Figure 13 shows that the maximum external moment which can be applied to the upper joint of the subassemblage is 0.295 \( M_y \). Also shown in the figure is a \( (M_e)_u/M_y \) vs. \( \Delta/h \) curve for the same structure when no external horizontal force is applied*. The maximum value of \( (M_e)_u \) for this case is found to be 0.35 \( M_y \). A comparison of these results indicates that the presence of the horizontal force causes considerable reduction in the moment-carrying capacity of the subassemblage.

3.3 ANALYSIS OF SPECIAL CASES

(1) Subassemblages Loaded Without Horizontal Force

The analysis of a subassemblage subjected to external moment only is somewhat simpler than that shown above for the general case. If the lateral spring is also absent, the analysis can be simplified even further**. (See Fig. 14a). In this case the angle \( \tau \) is equal to zero, because \( V = 0 \). Therefore \( \alpha_L = \Theta_L \), \( \alpha_U = \Theta_U \) and \( \beta = \Delta/h \). The secondary moment due to the lateral displacement \( \Delta \) is again determined by Eq. 19, which is simplified to become

\[
\frac{P\Delta}{M_y} = \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \beta
\]

When the moments \( M_L/M_y \) and \( P\Delta/M_y \) are known, the shear equilibrium condition of Eq. 20 can be used directly to compute the moment \( M_{u y} \). The following example will illustrate the analysis of this special case.

* The analysis made for this case is given in Art. 3.3
** The next section will discuss the analysis of subassemblages with a specified lateral spring.
Illustrative Example 5

The subassemblage to be analyzed is the one used in Example 4, the only difference being that in this example Q is assumed to be zero. The solution is presented in a tabular form in Table 4. For each selected value of $\theta_L$ (Column 1), eight steps are required to determine the corresponding value of $(M_{e}^U)^y/M_y$ (Column 9). As in the previous example, the computations also yield the column moments, $M^U/M_y$ and $M^L/M_y$, the joint rotation $\theta^U$, and the chord rotation $\Delta/h$. In Fig. 15 the resulting moments, $M^U/M_y$, $M^L/M_y$, and $(M_{e}^U)^y/M_y$, are plotted against the chord rotation $\Delta/h$, which is actually a measure of the lateral displacement. The solid line shows the relationship between the external moment $(M_{e}^U)^y/M_y$ and the chord rotation $\Delta/h$ of the subassemblage. This curve was also given in Fig. 13 as a dashed line.

The sway character of the subassemblage may be studied by examining the two curves in Fig. 15, one showing the relationship between $M^U/M_y$ and $\Delta/h$, and the other $M^L/M_y$ and $\Delta/h$. The subassemblage tends to sway to the left when the upper column moment $M^U/M_y$ is larger than the lower column moment $M^L/M_y$. When $M^L/M_y$ is larger than $M^U/M_y$, the structure sways to the right. There is no sway when the two column moments are equal, but acting in the opposite sense. This fact is illustrated by the intersection of the two dashed curves on the ordinate. The external moment at the upper joint corresponding to no sway is equal to 0.167 $M$. If the two external moments $(M_{e}^U)^y/M_y$ and $(M_{e}^L)^y/M_y$ are allowed to increase simultaneously (instead

* The curve for $M^L/M_y$ is plotted with the sign of the moment changed. This is done for reason of convenience. It would therefore be simpler, for purpose of discussion, to refer to both $M^U/M_y$ and $M^L/M_y$ by their absolute values.
of one being fixed), the subassemblage will not sway when a constant ratio of $(M_e)^U/(M_e)^L = 0.167/0.200 = 0.835$ is maintained between the two moments.

From the results given in Table 4 the deformed configurations of the column for various values of $(M_e)^U/M_y$ can be determined. Four such configurations are sketched in Fig. 15. For low values of $(M_e)^U/M_y$, the column is bent into a double curvature by two positive end moments. The two end rotations, $\theta_U$ and $\theta_L$, and the chord rotation $\Delta / h$ are all negative. When $(M_e)^U/M_y = 1.41$, the upper joint rotation becomes zero, but the chord rotation is still negative. The lower column moment has changed sign and is now positive. When $(M_e)^U/M_y$ is increased to 0.2, the lower joint rotation becomes zero, and the upper joint rotation and the chord rotation are both positive. The subassemblage now sways to the right. Any further increase in the upper joint moment will produce a positive rotation at the lower end.

Criterion of Adequacy

The adequacy of a subassemblage subject to sidesway due to unequal external moments is insured if:

1. A $\Delta / h$ can be found such that the sum of the column moment and the restraining moment is equal to or greater than the applied joint moment

   \[(M_r)^U + M_u \geq (M_e)^U\]  \hspace{1cm} (22)

2. Another $\Delta / h$ can be found such that the sum of the column moment and the restraining moment is equal to or less than the applied joint moment,

   \[(M_r)^U + M_u \leq (M_e)^U\]  \hspace{1cm} (23)
The second condition is necessary to insure that the upper external moment is not smaller than the smallest possible value which a subassemblage subjected to a given \( M_e \) can sustain.

The proof of Inequalities 22 and 23 is analogous to the proof given in Appendix D for subassemblages which can not sway.

(2) **Subassemblages Having Specified Lateral Restraint**

Another special case which can be studied analytically is the case in which a subassemblage is loaded with external joint moments and is restrained by a lateral spring with a stiffness \( k = P/h \) (Fig. 14b). The reason for choosing such a stiffness is that the resisting force offered by the spring nullifies completely the effect of secondary moment, \( P \Delta \), on the strength of the subassemblage. For every sidesway deflection \( \Delta \), the spring reacts with a force \( P \Delta /h \). The resultant moment due to \( P \) and \( P \Delta /h \) about the lower joint is

\[
P \Delta - \frac{P \Delta}{h} h = 0
\]

Since there is no external horizontal force applied to the structure, Eq. 15 can be reduced to

\[
M_U + M_L = 0
\]

or \( M_u = - M_L \). Therefore, the insertion of a lateral spring having \( k = P/h \) insures that the column member will always be bent in a single curvature.

Because of the reduced number of variables involved, the analysis of a subassemblage of this type is considerably simpler than that of a general sway subassemblage. In this case, once the end rotation \( \theta_L \) is selected and the corresponding column moment \( M_L \) is determined, the
rotation $\theta_U$, the moment $M_U (M_U = - M_L)$ and the chord rotation $\Delta/h$ can be readily found without resorting to the charts in Appendix C. Instead, the simpler charts presented in Appendix B may be used.

Since it is known in the present problem that $M_U = - M_L$, the rotation $\gamma_U$ (angle between the chord and the tangent at the upper end) is equal to $-\gamma_L$. Equations 18a and 18b can therefore be combined to yield

$$\theta_U = \theta_L + 2\gamma$$

in which $\gamma$ is the positive angle between the chord and the tangent at the upper end (shown as $\gamma_U$ in Fig. 14b). The angle $\gamma$ corresponding to a given end moment $M_U$ can be found from the appropriate chart in Appendix B (for the case of columns bent in a symmetrical single curvature). After the value of $\gamma$ is found, Eq. 24 can be used to compute the angle $\theta_U$.

With $\theta_U$ known, the external moment $(M_e)$ is readily determined by applying the equilibrium condition of the upper joint. The chord rotation corresponding to the known value of $\theta_U$ is given by Eq. 18a

$$\Delta/h = \theta_U - \gamma$$

Thus, the amount of sidesway is also determined.

**Illustrative Example 6**

Consider the case when a lateral spring having $k = P/h$ is added to the subassemblage of Example 5. Since the insertion of such a spring has the effect of nullifying the secondary moment $P \Delta$, the carrying capacity of the subassemblage will be considerably increased. The step-by-step calculations made for this example are given in Table 5, and the results are summarized graphically in Fig. 16. In solving this example, the moment versus rotation curve of a column with $h/r = 30$ and $P/P_y = 0.6$
and subjected to two equal, but opposite, end moments is used. This curve can be found in the chart of Appendix B. The useful range of the curve terminates at a maximum rotation of 0.0285 radian which corresponds to the initiation of local buckling of the column section*. The calculations given in Table 5 therefore end when the value of \( \gamma \) has reached this limiting value.

The resulting external moment at the upper joint \( (M_e)_U / M_y \) and the corresponding chord rotation \( \Delta/h \) of the subassemblage are plotted as a solid line in Fig. 16. The maximum external moment that can be applied to the subassemblage before the occurrence of local buckling in the column is found to be 0.985 \( M_y \). Also shown in Fig. 16 is the result obtained for the same subassemblage but with the upper rotational spring removed (\( M_r = 0 \)). The maximum external moment for this case is equal to 0.39 \( M_y \). A comparison of the two results shows that the presence of the rotational restraint increases greatly the moment-carrying capacity of the structure.

The increase in strength derived from the lateral spring can be seen by comparing the result of this example with that of Example 5. Such a comparison is shown in Fig. 17. The maximum moment which can be carried by the subassemblage with the specified lateral spring is almost three times of that which can be carried by the same structure but without the lateral spring. The great difference between the two maximum moments indicates the importance of considering the effect of secondary moment in the analysis of subassemblies permitted to sway.

* See Ref. 16 for details.
(3) **Subassemblages Pinned at One End**

In many practical situations it will be found useful to have solutions to subassemblages pinned at one end. Since the procedure for analyzing such subassemblages is considerably simpler than that described for the general case, it is possible to study the effect of variation of horizontal force on the sway displacement of the structures. A study of this type is very time consuming for the more general subassemblage. The first structure to be analyzed is a subassemblage pinned at the lower joint and subjected to a horizontal force \( Q \) applied at the level of the upper joint (Fig. 14c).

In this case, the equilibrium condition for the upper joint becomes

\[
M_r + M = 0
\]

or \( M_r = -M \). When the rotation \( \theta \) of this joint is selected in the solution and the restraining moment \( M \) corresponding to this rotation is computed from the known restraining function, the moment \( M \) at the top of the column is determined. The rotation \( \gamma \) of the column (measured from the chord to the tangent) corresponding to this moment may be obtained from the appropriate chart in Appendix B (for the case of columns bent by one end moment). The compatibility condition of Eq. 18a can then be used to compute the chord rotation

\[
\Delta/h = \theta - \gamma
\]

The shear equilibrium condition of Eq. 15 is reduced to

\[
M + P\Delta + Qh = 0
\]

or equivalently,

\[
\frac{Qh}{M_y} = -\left(\frac{M_y}{M_y} + \frac{P\Delta}{M_y}\right) = -\left(\frac{M_y}{M_y} + \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \frac{\Delta}{h}\right)
\]

(25)
The last equation is derived by applying Eq. 21. Equation 25 determines the horizontal force $Q$ which is consistent with the selected joint rotation $\Theta$.

**Illustrative Example 7**

**PROBLEM:** Determine the relationship between $Q$ (or $Qh/M_y$) and $\Delta/h$ of the subassemblage shown in Fig. 14c, assuming $h = 40r$ and $P = 0.6 P_y$.

The structure will be studied for the following four restraining functions: $M_r = 0$ (pinned end), $M_r = 100 \Theta M_y$, $M_r = 200 \Theta M_y$ and $M_r = \infty$ (fixed end).

**SOLUTION:** The computations for the cases with $M_r = 100 \Theta M_y$ and $200 \Theta M_y$ are tabulated in Table 6 and the results are plotted in Fig. 18. The procedure outlined above has been followed closely in the solution. The case with $M_r = 0$ is equivalent to a pinned end column free to sway at the top. In this case, $M = 0$

consequently,

$$Qh = -P \Delta$$

or

$$\frac{Qh}{M_y} = -\frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \frac{\Delta}{h} = -27.6 \frac{\Delta}{h}$$

This shows that a negative horizontal force (acting toward the left) is required in order to keep the column displaced in the positive direction. The above equation is plotted as a straight line passing through the origin in Fig. 18. In the other extreme case in which $M_r = \infty$, the joint rotation $\Theta$ is always zero, and the chord rotation $\Delta/h$ equals $-\gamma$. The horizontal force is again given by

$$\frac{Qh}{M_y} = \left( \frac{M}{M_y} + \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \frac{\Delta}{h} \right) = \left( \frac{M}{M_y} - \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \gamma \right)$$
When a value of \( \gamma \) is assumed, the moment \( M/M_y \) can be found from the appropriate chart contained in Appendix B. The above equation may be used to determine the horizontal force \( Q \) consistent with the assumed \( \gamma \).

The results of this example show clearly the significant effect of the rotational restraint. The subassemblage is unstable, if the restraint is not present. As the stiffness of the restraint increases, the ultimate strength of the structure increases accordingly. The horizontal force reaches the absolute maximum when the restraint becomes infinitely stiff. For this example the maximum value of \( Q \) is equal to 0.226 \( M_y/h \).

Figure 18 also shows that the horizontal force \( Q \) changes its sign as the rotational stiffness of the restraint increases from 0 to 100 \( M_y \). There is a value of the stiffness for which the subassemblage can carry no horizontal force \( (Q = 0) \). The structure tends to buckle laterally under the applied axial force. The required stiffness can therefore be determined by a buckling analysis and is found to be 44 \( M_y \). A detailed description of the buckling analysis is presented in Art. 3.4.

The Effect of External Joint Moment

It is now of interest to study the effect of an external moment \( M_e \) on the load-deformation behavior of the subassemblage. The deformed configuration of the subassemblage is shown in Fig. 14d. The structure may be analyzed by the procedure described above and illustrated in Example 7. The effect of the external moment is taken into account in the joint equilibrium equation which is written as

\[
M_e = M_r + M
\]
or

\[ M = M - M_e \]

Therefore, the moment at the upper end of the column is equal to the difference between \( M_e \) and \( M_r \).

In Fig. 19 three curves showing the relationship between the horizontal force \( Qh/M_y \) and the chord rotation \( \Delta/h \) of the subassemblage considered in the previous example are plotted. The restraining function of the spring is assumed to be \( M_r = 100 \theta M_y \). The curves are computed for three values of the external moment: \( M_e = -0.2 M_y \), \( M_e = 0 \) and \( M_e = 0.2 M_y \). The significant concept illustrated by the curves is that the external moment will either help the subassemblage resist more horizontal force or reduce the subassemblage's load-carrying capacity, depending on the sense of the moment. If \( M_e \) is counterclockwise, it acts as a restraining moment; and if \( M_e \) is clockwise, it hinders the resistance to horizontal force.

3.4 BUCKLING OF SUBASSEMBLAGES

The term "buckling" in this report refers to the existence of more than one position of equilibrium for a subassemblage subjected to a given set of loads. The phenomenon is frequently described in the literature as the bifurcation of equilibrium positions, and is a typical failure mode for subassemblies which are loaded by vertical forces and are permitted to sway. In such a case, there exists a value of the vertical load \( P \) (called buckling or critical load) for which the structure can
either remain straight or buckle into an adjacent swayed configuration. When this occurs, the structure is said to have failed by sidesway buckling.

In this section, procedures for the determination of the buckling load of subassemblages are developed. An analytical expression will first be derived for the case of a subassemblage pinned at one end. This expression results in the solution to a frame buckling problem which has been investigated previously by conventional methods. However, the expression derived herein is in a form which can be readily generalized to inelastic structures. Finally, a method using the charts in Appendix C will be developed for the more general case of a subassemblage restrained at both ends. This method is again valid for the elastic as well as the inelastic range of structural action.

(1) Subassemblages Pinned at One End

Consider the subassemblage shown in Fig. 20a. The lower end of its column is pinned, while the upper end is restrained by a rotational spring. The structure is permitted to translate in the lateral direction. If the applied load $P$ is below the critical value, the subassemblage remains in a straight position, and no bending moment exists in any section of the column. When $P$ is equal to the critical value, there exists an adjacent swayed configuration which is in equilibrium with $P$ and the additional moments $\delta M$ and $\delta M_r$ developed due to the sway displacement. (See Fig. 20b). The swayed configuration is characterized by the joint rotation $\delta \theta$, the displacement $\delta \Delta$ and the rotation between the chord and the tangent, $\delta y$. First order expressions for $\delta M$ and $\delta M_r$ are:
These relationships are justified because both $\delta \theta$ and $\delta \gamma$ are considered to be infinitesimally small quantities. The joint equilibrium condition requires

$$\delta M_r + \delta M = 0$$

or, when the above relationships are substituted,

$$\frac{dM_r}{d\theta} \delta \theta + \frac{dM}{d\gamma} \delta \gamma = 0$$

The shear equilibrium of the column member yields

$$\delta M + P \delta \Delta = 0$$

or

$$\frac{dM}{d\gamma} \delta \gamma + P \delta \Delta = 0$$

From compatibility considerations, the quantities $\delta \theta$, $\delta \Delta$ and $\delta \gamma$ are related by the following equation

$$\delta \gamma = \delta \theta - \frac{\delta \Delta}{h}$$

Substituting Eq. 30 into Eqs. 28 and 29 leads to the following set of equations:

$$\left( \frac{dM_r}{d\theta} + \frac{dM}{d\gamma} \right) \delta \theta - \frac{dM}{d\gamma} \frac{\delta \Delta}{h} = 0$$

(31)

$$\frac{dM}{d\gamma} \delta \theta + (P_h - \frac{dM}{d\gamma}) \frac{\delta \Delta}{h} = 0$$

(32)
Equations 31 and 32 are two linear homogeneous equations in the two unknowns $\delta \theta$ and $\delta \Delta/h$. As usual, a nontrivial solution is possible if and only if the determinant of the coefficients vanishes; this represents the characteristic equation of the problem. The vanishing of the determinant corresponds to

$$
\left( \frac{dM}{d\theta} + \frac{dM}{d\gamma} \right) (Ph - \frac{dM}{d\gamma}) + \left( \frac{dM}{d\gamma} \right)^2 = 0
$$

(33)

After some algebraic manipulations, Eq. 33 can be simplified to become

$$
\frac{1}{dM_r} \frac{dM}{d\theta} + \frac{1}{dM} \frac{dM}{d\gamma} = \frac{1}{Ph}
$$

(34)

This is the basic expression which may be used to determine the critical load of the subassemblage.

If the response of the column and the restraint to the moments $dM$ and $dM_r$ is linear, the derivatives in Eq. 34 can be easily evaluated as the slopes of the $M$ vs. $\gamma$ and $M_r$ vs. $\theta$ lines. The solution of Eq. 34 gives the elastic buckling load of the structure. In subassemblies whose buckling load is higher than the load which causes initial yielding, or whose restraining member has nonlinear properties, the relationships between $M$ and $\gamma$ and $M_r$ and $\theta$ are no longer linear. The derivatives $dM/d\gamma$ and $dM_r/d\theta$ are the slopes of the moment-rotation curves at $\gamma = 0$ and $\theta = 0$, provided that the initial position of the subassemblage is vertical and straight*. If the initial position is vertical but the subassemblage is bent by an external joint moment, the derivatives must be evaluated at points corresponding to $\gamma = \gamma_i$ and $\theta = \theta_i$. The quantities $\gamma_i$ and $\theta_i$ are the initial or prebuckling rotations of the structure**.

* The critical load determined by using these derivatives will be the tangent modulus load of inelastic buckling.

** If the chord of the column member remains straight, the rotation $\gamma_i$ is equal to $\theta_i$. 
The use of the critical condition of Eq. 34 may be illustrated by analyzing the elastic buckling of a simple subassemblage whose restraining member has a linear characteristic. The restraining function is defined by

\[ M_r = C \Theta \]  

(35)
in which C is a constant. The derivative of \( M_r \) with respect to \( \Theta \) is therefore equal to

\[ \frac{dM_r}{d\Theta} = C \]  

(36)
The moment vs. rotation relationship of the column has been derived in Ref. 17 and is given by

\[ M = \frac{Ph \tan \lambda h}{\tan \lambda h - \lambda h} \gamma \]  

(37)
in which \( \lambda \) is defined as

\[ \lambda = \sqrt{\frac{P}{EI}} \]  

(38)
The derivative of \( M \) with respect to \( \gamma \) is

\[ \frac{dM}{d\gamma} = \frac{Ph \tan \lambda h}{\tan \lambda h - \lambda h} \]  

(39)
When Eqs. 36 and 39 are substituted into Eq. 34, the following equation for determining the critical load is obtained

\[ \frac{1}{C} + \frac{\tan \lambda h - \lambda h}{Ph \tan \lambda h} = \frac{1}{Ph} \]  

(40)
This equation can be simplified and rearranged to become

\[ \lambda h \tan \lambda h = \frac{Ch}{EI} \]  

(41)
The solution of this simple transcendental equation gives the critical load expressed in terms of \( E, I, h, \) and \( C. \)
Equation 41 can be shown to be also the solution for the sidesway buckling of a pinned base portal frame. In such a case, the cross beam of the frame is considered to be the restraining member and its moment vs. rotation relationship is given by \( M_r = \frac{6EI_b}{L_b} \gamma \), in which \( I_b \) and \( L_b \) are, respectively, the moment of inertia and the length of the beam. The constant \( C \) is therefore equal to \( \frac{6EI_b}{L_b} \). When this is substituted into Eq. 41, the buckling condition for the frame is obtained.

\[
\lambda h \tan \lambda h = \frac{6I_b h}{L_L} \tag{42}
\]

This equation checks with the solution given in Ref. 17 for the same problem.

(2) 

Buckling Analysis of General Subassemblages

The method developed above can also be applied to subassemblages having rotational restraints attached at both joints (Fig. 21a), but the number of variables involved in the analysis is increased considerably. In a general subassemblage, the existence of an adjacent configuration requires that a set of deformations \( \delta \Theta_U, \delta \Theta_L, \delta \Delta, \delta \gamma_U \) and \( \delta \gamma_L \) exists (Fig. 21b), and that equilibrium of the subassemblage be maintained with the load \( P \) equal to the critical load. The deformed configuration of the column member shows that the angles \( \delta \gamma_U \) and \( \delta \gamma_L \) are given in terms of \( \delta \Theta_U, \delta \Theta_L \) and \( \Delta/h \) by the following equations

\[
\delta \gamma_U = \delta \Theta_U + \frac{\delta \Delta}{h} \tag{43a}
\]

and

\[
\delta \gamma_L = \delta \Theta_L - \frac{\delta \Delta}{h} \tag{43b}
\]

Thus, the total number of the independent deformation variables is equal to three.
The restraining moments $\delta (M_r)_U$ and $\delta (M_r)_L$, developed in the restraints, and the column moments $\delta M_U$ and $\delta M_L$, should satisfy the joint equilibrium conditions:

$$\delta (M_r)_U + \delta M_U = 0$$ (44a)

and

$$\delta (M_r)_L + \delta M_L = 0$$ (44b)

In addition, the column moments $\delta M_U$ and $\delta M_L$ are related to the lateral displacement $\delta \Delta$ by the shear equilibrium equation

$$\delta M_U + \delta M_L + P \delta \Delta = 0$$ (45)

The moments $\delta (M_r)_U$, $\delta (M_r)_L$, $\delta M_U$ and $\delta M_L$ can be expressed in terms of the deformation variables $\delta \theta_U$, $\delta \theta_L$, $\delta \gamma_L$ as follows:

$$\delta (M_r)_U = \frac{d(M_r)_U}{d\theta_U} \delta \theta_U$$ (46a)

$$\delta (M_r)_L = \frac{d(M_r)_L}{d\theta_L} \delta \theta_L$$ (46b)

$$\delta M_U = \frac{\partial M_U}{\partial \gamma_U} \delta \gamma_U + \frac{\partial M_U}{\partial \gamma_L} \delta \gamma_L$$ (47a)*

and

$$\delta M_L = \frac{\partial M_L}{\partial \gamma_L} \delta \gamma_L + \frac{\partial M_L}{\partial \gamma_U} \delta \gamma_U$$ (47b)*

These are again the first order expressions of the moments. When these expressions are combined with Eqs. 44 and 45, a set of three simultaneous equations in the five unknowns $\delta \theta_U$, $\delta \theta_L$, $\delta \Delta/h$, $\delta \gamma_U$ and $\delta \gamma_L$ is obtained.

* In these expressions, the angles $\gamma_U$ and $\gamma_L$ are assumed to be the independent variables. Detailed discussion on the choice of the appropriate independent variables can be found in Ref. 13.
The unknowns $\delta \gamma_U$ and $\delta \gamma_L$ can be eliminated by the use of Eqs. 43a and 43b, and the resulting equations contain only three variables. Once again a nontrivial solution of the set of equations exists if the determinant of the coefficients is equal to zero. This leads directly the buckling condition of the general subassemblage from which the critical load can be determined.

In certain practical situations, it is sometimes desired to check whether or not a given subassemblage is stable under a certain load. Such a check can be made readily without knowing the actual critical load of the structure. A convenient method for checking the stability of portal frames which are permitted to sway has been developed in Ref. 18. The criterion for stability used in the method is as follows: Under a given set of applied loads, if an external horizontal force is required to cause sidesway, then the structure is stable. Buckling will occur when the structure can deform laterally without the application of any horizontal force. This criterion is also applicable to the general subassemblages considered here. The requirement for stability is that the sum of the column moments $\delta M_U$ and $\delta M_L$ corresponding to an assumed lateral deflection $\delta \Delta$ should be greater than the moment $P \delta \Delta$.

Expressed mathematically, the condition for stability is

$$\delta M_U + \delta M_L > P \delta \Delta$$

The assumed $\delta \Delta$ may be considered as the sway deflection produced by a certain external force acting in the direction of the deflection. In the following discussion, a modification to this criterion will be presented and a graphical method based on the modified criterion will be described. The new method makes use of the charts contained in Appendix C.
If a rotation $\delta \theta_L$ is introduced at the lower joint of the subassemblage of Fig. 21a, a sidesway configuration as shown in Fig. 21b results. This configuration is characterized by the deformations $\delta \theta_U$, $\delta \theta_L$, $\delta \Delta/h$, $\delta \gamma_U$, and $\delta \gamma_L$. These deformations must satisfy the compatibility relations of Eq. 43. Since there is no external moment applied at the lower joint, the sum of the moments $\delta (M_r)_L$ and $\delta M_L$ corresponding to the angle $\delta \theta_L$ should be zero. In addition, shear equilibrium of the column requires that

$$\delta M_U + \delta M_L + P \delta \Delta = 0$$

The subassemblage is stable and remains in its straight position under the load $P$, if the sum of the moments $\delta (M_r)_U$ and $\delta M_U$ is positive, that is

$$\delta (M_r)_U + \delta M_U > 0$$

The above inequality implies that the buckled configuration can not be maintained unless an external moment is applied at the joint.

Based on the above criterion a procedure for checking the stability of a subassemblage is developed. In using this procedure, the lower joint of the subassemblage is first rotated through an angle $\delta \theta_L$, and the corresponding restraining moment $\delta (M_r)_L$ is determined from the known restraining function. Equilibrium of the lower joint determines the column moment $\delta M_L (\delta M_L = - \delta (M_r)_L)$. With $\delta \theta_L$ and $\delta \theta_L$ known, the charts in Appendix C can be used to find the chord rotation $\delta \Delta/h$. The moment $P \delta \Delta$ corresponding to this chord rotation is then determined by applying Eq. 21. The moment $\delta M_U$ developed at the upper end of the column is found from the shear equilibrium condition, $\delta M_U = - \delta M_L - P \delta \Delta$. The same chart can

* The analysis is similar to that illustrated in Example 5. In using the charts in Appendix C, $\delta \theta_L$ should be identified as $\alpha_L$ and $\delta \Delta/h$ as $\beta$. 
now be used to find the upper joint rotation $\delta \theta_U$ corresponding to the values of $\delta \Delta/h$ and $\delta M_U$ just determined. With $\delta \theta_U$ known, the restraining moment $\delta (M_r)_U$ can be found from the upper restraining function. The external moment $\delta (M_e)_U$, which as to be applied to the upper joint in order to maintain the deformed shape (characterized by $\delta \theta_U$, $\delta \theta_L$ and $\delta \Delta/h$) is equal to the sum of $\delta (M_r)_U$ and $\delta M_U$. If $\delta (M_e)_U$ is found to be positive for a positive $\delta \Delta$, then the subassemblage is stable under the given load.

Illustrative Example 8

PROBLEM: A subassemblage whose column height $h$ is equal to 30$\pi$ is loaded by a constant vertical force $P = 0.8 P_y$. The restraining functions of the upper and lower springs are defined by $(M_r)_U = n M_y \theta_U$ and $(M_r)_L = n/2 M_y \theta_L$, respectively. The parameter which defines the stiffness of the restraints is considered to be a variable. It is required to determine the optimum value of $n$ so that the subassemblage will just buckle under the applied load $P$.

SOLUTION: A series of $n$ values is first selected and is listed in the first column of Table 7. For each value of $n$, the subassemblage is checked for its stability by using the procedure just developed. This check starts from an assumed rotation $\delta \theta_L$ (Column 2) and ends with an external joint moment $\delta (M_e)_U$ (Column 10). A positive $\delta (M_e)_U$ indicates that the value of $n$ is higher than $n_{opt}$; the structure is therefore safe to carry the given load. The optimum value of $n$ is reached when the external moment becomes zero. The results obtained in Table 7 are shown as a $n$ vs. $\delta (M_e)_U$ curve in Fig. 22. The intercept of this curve on the $n$-axis gives the value of $n_{opt}$.
4. **SUBASSEMBLAGES COMPRISED OF BEAMS AND COLUMNS**

The various methods developed above are for subassemblages in which the end restraint provided by the beams is represented by rotational springs. The columns in the subassemblages are subjected to compressive forces which remain constant throughout the application of the external joint moments. For subassemblages consisting of real beams and columns, the compressive force will generally change with the external moments because of the reactions developed at the ends of the beams. A simple example will illustrate this point.

Consider the subassemblage shown in Fig. 23a. The column in the subassemblage is restrained at the upper end by a rotational spring and is subjected to a constant vertical force \( P \) and a joint moment \( M_e \). In this case, the axial thrust in the column is not increased or decreased by the moment \( M_e \) and is always equal to \( P \). For a subassemblage whose column is restrained by an actual beam (Fig. 23b), the magnitude of the axial thrust depends upon the applied moment. Assuming that the moment carried by the beam is \( M_b \), the shear developed at the ends is equal to \( M_b /L \). This shear is subsequently transmitted to the column and causes an increase in the axial thrust.* The true axial thrust in the column is therefore equal to \( P + M_b /L \). Thus, the error involved in assuming a constant thrust is

\[
\text{Error} = \frac{M_b}{LP} \quad (48)
\]

* If the direction of the external moment is reversed, the shear force will cause a reduction in the axial thrust.
In many practical instances this error will be relatively small. However, it can be included in the procedures developed herein without much difficulty. The necessary steps for finding an equilibrium configuration for the subassemblage shown in Fig. 23b are as follows:

Corresponding to an assumed end rotation $\theta$, the beam moment is determined from the expression $M_b = \frac{3EI}{L} \cdot \theta$ in which the term $3EI/L$ may be identified as the stiffness of the restraining beam.* The axial thrust in the column is therefore equal to $P + \frac{3EI}{L^2} \cdot \theta$. Since both the end rotation and the axial thrust are now known, the column moment $M_c$ can be determined by entering the appropriate chart in Appendix B. The summation of $M_c$ and $M_b$ gives the external moment $M_e$ corresponding to the assumed $\theta$.

Thus, when necessary the effect of beam shear can be included in the analysis of subassemblies. Also if it is neglected the error can be easily estimated from Eq. 48.

* This is also the stiffness of the rotational spring of the structure shown in Fig. 23a.
5. SUMMARY AND CONCLUSIONS

It has been shown that the presence of bracing members in a multi-story frame affects greatly the behavior and strength of the structure. Multi-story frames are therefore classified into two categories: braced frames and unbraced frames. The analysis and design of the members in the two types of frames can be accomplished by partitioning the entire frame into subassemblies, each consisting of a column and several adjacent beams. In a braced frame, sidesway movement is prevented by bracing; and its subassemblies usually remain vertical and are called nonsway subassemblies. The deformation of a subassembly in an unbraced frame often involves sidesway. This type of subassembly is referred to as a sway subassembly.

Methods have been developed for analyzing both the sway and nonsway subassemblies. The various cases of general and special subassemblies that are treated in this report are summarized in Fig. 3. In the case of nonsway subassemblies, the analysis gives the moment versus joint rotation relationship in the elastic and inelastic range and also the maximum joint moment that can be applied to a given subassembly. The analysis of a general nonsway subassembly was discussed in Art. 22, and the procedure developed was illustrated by an example shown in Fig. 5. The result of this example revealed the possibility of establishing a general criterion which defines the limit of stability of a nonsway subassembly. Such a criterion was expressed mathematically by Eq. 8. It was pointed out that for practical applications the criterion for adequacy, as stated by Eqs. 9a and 9b, is more convenient and can be used to check whether a given
subassemblage is adequate for supporting the applied load. Four special types of subassemblages shown in Fig. 6, were studied in Art. 2.5 by the methods developed previously for analyzing restrained columns.\textsuperscript{5}

An additional problem which can be solved by applying the general method for nonsway subassemblages is the determination of the ultimate strength of beam-columns continuous over several supports. The structure is loaded simultaneously by axial forces and bending moments applied at the level of the supports (Fig. 9). The method of solution for this problem was outlined in Art. 2.6 and its application has been illustrated by an example (Fig. 10). The solution has its practical importance in the design of columns in braced frames.

The analysis of subassemblages permitted to sway was treated in detail in Arts. 3.2 and 3.3. In analyzing this type of subassemblage, it was found necessary to include three additional effects which do not occur in nonsway subassemblages. These are the effects of sway deflection, horizontal force and lateral restraint. General methods which may be used to determine the strength of a given subassemblage (either the maximum moment or the maximum horizontal force) were presented. The methods were later modified to become special methods for four types of subassemblages shown in Fig. 14. These methods were described in Art. 3.3 and applied to three example problems. Analysis of the results obtained from these examples led to the following conclusions:

1. The presence of a horizontal force acting in the direction of the sway displacement causes a reduction in the moment-carrying capacity of a subassemblage (Fig. 13).
2. For the case of a subassemblage loaded by two joint moments the direction of sidesway depends on the magnitude and the sense of the applied moments. It is possible that corresponding to certain combinations of the two moments the subassemblage may remain vertical (Fig. 15).

3. The secondary moment resulting from sway deflection reduces appreciably the ultimate strength of a subassemblage (Fig. 17). The reduction is particularly significant if high axial force is present in the column. This fact indicates the need of considering the effect of deformation in the analysis and design of unbraced building frames.

4. The maximum horizontal force that can be carried by a subassemblage increases as the stiffness of the rotational restraint increases, but approaches a limit when the restraint becomes infinitely stiff (Fig. 18).

A study on the sidesway buckling of subassemblages was made in Art. 3.4. Both analytical and numerical methods were developed for determining the critical load of two types of subassemblages. It was shown that the buckling strength of a general subassemblage also depends on the stiffness of the restraining beams.

The various procedures presented in this report for analyzing sway and nonsway subassemblages are being used in developing methods for the analysis and design of multi-story frames. Such methods will be described in future reports.
6. ACKNOWLEDGMENTS

The results reported herein were obtained from a research conducted in the Fritz Engineering Laboratory, Department of Civil Engineering, Lehigh University. Professor L. S. Beedle is director of the laboratory and Professor W. J. Eney is head of the department and the laboratory.

The work forms part of an investigation on "Welded Continuous Frames and Their Components." This project is sponsored by the Welding Research Council and the U. S. Navy Department. Funds are supplied by the American Institute of Steel Construction, American Iron and Steel Institute, Office of Naval Research, Bureau of Ships and the Bureau of Yards and Docks. Technical guidance for the project is provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council. Dr. T. R. Higgins is Chairman of the Lehigh Project Subcommittee.

The authors wish to express their appreciation to Dr. Morris Ojalvo for fruitful discussions during the early state of the development and to Dr. Theodore V. Galambos for a critical review of the manuscript.
7. APPENDICES

Appendix A
\( \frac{P}{P_y} h = 40 \)

\( p = 0.6 \)
\[
\frac{P}{P_y} = 0.8 \\
\frac{h}{r} = 30
\]
Appendix C
APPENDIX D PROOF OF CRITERION FOR ADEQUACY

The criterion for adequacy presented in Art. 2.4 for a non-sway subassemblage can be proven as follows:

1. Consider the curve of Fig. 5, which defines the relationship between $(M_e)_U$ and $\theta_L$ for a given subassemblage. On this curve, the following may be observed:
   a. Point B corresponds to the limit of stability
   b. Points to the left of point B are stable
   c. Points to the right of point B are unstable
   d. There is one stable point corresponding to each $(M_e)_U$ between the limit of stability and the abscissa
   e. There is one unstable point corresponding to each $(M_e)_U$ between the limit of stability and some lower limit (corresponding to local buckling or hinge formation)

2. An end rotation $\theta_L$ corresponding to an $(M_e)_U$ can be:
   a. At the limit of stability
   b. A stable configuration
   c. An unstable configuration

3. If $\theta_L$ is at the limit of stability (point B), then under the moment $[(M_e)_U]_B$, the subassemblage is in equilibrium and just stable. Therefore, it is adequate under any $(M_e)_U \leq [(M_e)_U]_B$. 
4. If $\theta$ is to the left of point B, say point A, the subassemblage is in equilibrium and stable under the moment $[(M_\theta)_{eU}]_A$. Therefore, it is adequate under any $(M_\theta)_{eU} \leq [(M_\theta)_{eU}]_A$.

5. If $\theta$ is to the right of point B, say point C, the subassemblage is in equilibrium but unstable under the moment $[(M_\theta)_{eU}]_C$. However, from 1-d and 1-e, if there is an unstable configuration corresponding to $[(M_\theta)_{eU}]_C$, there must also be one which is stable under the same moment $[(M_\theta)_{eU}]_C$. Therefore, an angle $\theta$ corresponding to an unstable configuration insures the existence of a smaller angle corresponding to a stable configuration which will resist the same unbalanced moment. As a result, the subassemblage is adequate under any $(M_\theta)_{eU} \leq [(M_\theta)_{eU}]_C$.

6. Since all the possible cases have been examined, it can be concluded that:

IF AN ANGLE $\theta$ IS FOUND WHICH RESULTS IN AN $(M_\theta)_{eU}$ EQUAL TO OR GREATER THAN THE EXISTING ONE, THE SUBASSEMBLAGE IS ADEQUATE TO RESIST THE APPLIED LOADS.
8. TABLES AND FIGURES
TABLE 1. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 1.

Computations for Curve of Fig. 5.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q_L (Rad.)</td>
<td>( \frac{M_r L}{M} )</td>
<td>( \frac{M_e L}{M} )</td>
<td>( \frac{M}{M} )</td>
<td>q_L</td>
<td>( \frac{1}{q_L} )</td>
<td>( \frac{M_U}{M} )</td>
<td>Q_U (Rad.)</td>
<td>( \frac{M_r U}{M} )</td>
<td>( \frac{M_e U}{M} )</td>
</tr>
<tr>
<td>1</td>
<td>-0.006</td>
<td>-0.150</td>
<td>-0.5</td>
<td>-0.350</td>
<td>-0.50</td>
<td>-2.00</td>
<td>0.175</td>
<td>0.004</td>
<td>0.050</td>
<td>0.225</td>
</tr>
<tr>
<td>2</td>
<td>-0.007</td>
<td>-0.175</td>
<td>-0.5</td>
<td>-0.325</td>
<td>-0.93</td>
<td>-1.08</td>
<td>0.302</td>
<td>0.007</td>
<td>0.087</td>
<td>0.389</td>
</tr>
<tr>
<td>3</td>
<td>-0.009</td>
<td>-0.225</td>
<td>-0.5</td>
<td>-0.275</td>
<td>-1.60</td>
<td>-0.63</td>
<td>0.440</td>
<td>0.012</td>
<td>0.150</td>
<td>0.590</td>
</tr>
<tr>
<td>4</td>
<td>-0.010</td>
<td>-0.250</td>
<td>-0.5</td>
<td>-0.250</td>
<td>-1.80</td>
<td>-0.56</td>
<td>0.450</td>
<td>0.015</td>
<td>0.190</td>
<td>0.640</td>
</tr>
<tr>
<td>5</td>
<td>-0.011</td>
<td>-0.275</td>
<td>-0.5</td>
<td>-0.225</td>
<td>-2.00</td>
<td>-0.50</td>
<td>0.450</td>
<td>0.019</td>
<td>0.238</td>
<td>0.688</td>
</tr>
<tr>
<td>6</td>
<td>-0.014</td>
<td>-0.350</td>
<td>-0.5</td>
<td>-0.150</td>
<td>-2.80</td>
<td>-0.36</td>
<td>0.420</td>
<td>0.030</td>
<td>0.300</td>
<td>0.720</td>
</tr>
<tr>
<td>7</td>
<td>-0.016</td>
<td>-0.400</td>
<td>-0.5</td>
<td>-0.100</td>
<td>-4.00</td>
<td>-0.25</td>
<td>0.400</td>
<td>0.036</td>
<td>0.300</td>
<td>0.700</td>
</tr>
<tr>
<td>8</td>
<td>-0.018</td>
<td>-0.450</td>
<td>-0.5</td>
<td>-0.050</td>
<td>-7.00</td>
<td>-0.14</td>
<td>0.350</td>
<td>0.039</td>
<td>0.300</td>
<td>0.650</td>
</tr>
</tbody>
</table>
TABLE 2. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 3.
Computation for Curve of Figure 11.

<table>
<thead>
<tr>
<th>Trial No.</th>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q_1$</td>
<td>0.0035</td>
<td>0.0037</td>
<td>0.0038</td>
<td>0.00382</td>
</tr>
<tr>
<td></td>
<td>$\frac{M_{10}}{M_y}$</td>
<td>0.40</td>
<td>0.408</td>
<td>0.41</td>
<td>0.411</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{M_1}{M_y}$</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>$\frac{M_{12}}{M_y}$</td>
<td>0.20</td>
<td>0.192</td>
<td>0.19</td>
<td>0.189</td>
</tr>
<tr>
<td>5</td>
<td>$q_{12}$</td>
<td>-1.125</td>
<td>-1.50</td>
<td>-1.70</td>
<td>-1.78</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{M_{21}}{M_y}$</td>
<td>-0.225</td>
<td>-0.288</td>
<td>-0.323</td>
<td>-0.336</td>
</tr>
<tr>
<td>7</td>
<td>$q_{21}$</td>
<td>-0.89</td>
<td>-0.67</td>
<td>-0.588</td>
<td>-0.562</td>
</tr>
<tr>
<td>8</td>
<td>$q_2$</td>
<td>-0.0036</td>
<td>-0.0043</td>
<td>-0.0047</td>
<td>-0.0048</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{M_2}{M_y}$</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.50</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{M_{23}}{M_y}$</td>
<td>-0.275</td>
<td>-0.212</td>
<td>-0.177</td>
<td>-0.164</td>
</tr>
<tr>
<td>11</td>
<td>$q_{23}$</td>
<td>-0.30</td>
<td>-1.60</td>
<td>-2.55</td>
<td>-3.0</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{M_{32}}{M_y}$</td>
<td>0.083</td>
<td>0.339</td>
<td>0.452</td>
<td>0.492</td>
</tr>
<tr>
<td>13</td>
<td>$q_{32}$</td>
<td>-3.33</td>
<td>-0.625</td>
<td>-0.392</td>
<td>-0.333</td>
</tr>
<tr>
<td>14</td>
<td>$q_3$</td>
<td>0.0022</td>
<td>0.0052</td>
<td>0.0080</td>
<td>0.0123</td>
</tr>
</tbody>
</table>
TABLE 3. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 4.
Computations for Curve of Figure 13.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>QL</td>
<td>$\frac{M}{M_y}$</td>
<td>$\frac{M}{M_y}$</td>
<td>$\alpha^*$</td>
<td>$\beta$</td>
<td>$\frac{P + Vh}{M_y}$</td>
<td>$\frac{M}{M_y}$</td>
<td>$\alpha^*$</td>
<td>$\theta$</td>
<td>$\frac{M}{M_y}$</td>
<td>$\frac{M}{M_y}$</td>
<td></td>
</tr>
<tr>
<td>-0.012</td>
<td>-0.300</td>
<td>0.100</td>
<td>-0.010</td>
<td>-0.0107</td>
<td>-0.0127</td>
<td>-0.2215</td>
<td>0.1215</td>
<td>-0.0097</td>
<td>-0.0117</td>
<td>-0.1463</td>
<td>-0.0248</td>
</tr>
<tr>
<td>-0.010</td>
<td>-0.250</td>
<td>0.050</td>
<td>-0.008</td>
<td>-0.0080</td>
<td>-0.0100</td>
<td>-0.1655</td>
<td>0.1155</td>
<td>-0.0068</td>
<td>-0.0088</td>
<td>-0.1100</td>
<td>0.0055</td>
</tr>
<tr>
<td>-0.008</td>
<td>-0.200</td>
<td>0</td>
<td>-0.006</td>
<td>-0.0054</td>
<td>-0.0074</td>
<td>-0.1118</td>
<td>0.1118</td>
<td>-0.0042</td>
<td>-0.0062</td>
<td>-0.0775</td>
<td>0.0343</td>
</tr>
<tr>
<td>-0.006</td>
<td>-0.150</td>
<td>-0.050</td>
<td>-0.004</td>
<td>-0.0028</td>
<td>-0.0048</td>
<td>-0.0580</td>
<td>0.1080</td>
<td>-0.0014</td>
<td>-0.0034</td>
<td>-0.0425</td>
<td>0.0655</td>
</tr>
<tr>
<td>-0.004</td>
<td>-0.100</td>
<td>-0.100</td>
<td>-0.002</td>
<td>-0.0003</td>
<td>-0.0023</td>
<td>-0.0062</td>
<td>0.1062</td>
<td>0.0014</td>
<td>0.0006</td>
<td>-0.0080</td>
<td>0.0982</td>
</tr>
<tr>
<td>-0.002</td>
<td>-0.050</td>
<td>-0.150</td>
<td>0</td>
<td>-0.0022</td>
<td>0.0002</td>
<td>0.0455</td>
<td>0.1045</td>
<td>0.0040</td>
<td>0.0020</td>
<td>0.0250</td>
<td>0.1295</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.200</td>
<td>0.002</td>
<td>0.0048</td>
<td>0.0028</td>
<td>0.0994</td>
<td>0.1006</td>
<td>0.0068</td>
<td>0.0048</td>
<td>0.0600</td>
<td>0.1606</td>
</tr>
<tr>
<td>0.002</td>
<td>0.050</td>
<td>-0.250</td>
<td>0.004</td>
<td>0.0075</td>
<td>0.0055</td>
<td>0.1553</td>
<td>0.0947</td>
<td>0.0047</td>
<td>0.0077</td>
<td>0.0963</td>
<td>0.1910</td>
</tr>
<tr>
<td>0.004</td>
<td>0.100</td>
<td>-0.300</td>
<td>0.006</td>
<td>0.0100</td>
<td>0.0080</td>
<td>0.2070</td>
<td>0.0930</td>
<td>0.0123</td>
<td>0.0103</td>
<td>0.1275</td>
<td>0.2205</td>
</tr>
<tr>
<td>0.006</td>
<td>0.150</td>
<td>-0.350</td>
<td>0.008</td>
<td>0.0128</td>
<td>0.0108</td>
<td>0.2650</td>
<td>0.0850</td>
<td>0.0153</td>
<td>0.0133</td>
<td>0.1650</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.008</td>
<td>0.200</td>
<td>-0.400</td>
<td>0.010</td>
<td>0.0160</td>
<td>0.0140</td>
<td>0.3312</td>
<td>0.0688</td>
<td>0.0185</td>
<td>0.0165</td>
<td>0.2063</td>
<td>0.2751</td>
</tr>
<tr>
<td>0.010</td>
<td>0.250</td>
<td>-0.450</td>
<td>0.012</td>
<td>0.0200</td>
<td>0.0180</td>
<td>0.4140</td>
<td>0.0360</td>
<td>0.0228</td>
<td>0.0208</td>
<td>0.2588</td>
<td>0.2948</td>
</tr>
</tbody>
</table>
TABLE 4. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 5.
Computation for Curves of Figure 15.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{M_e}{M_y}$</td>
<td>$\frac{M_L}{M_y}$</td>
<td>$\frac{P_d}{M_y}$</td>
<td>$\frac{M_U}{M_y}$</td>
<td>$\frac{P_c}{M_y}$</td>
<td>$\frac{M_{re}}{M_y}$</td>
<td>$\frac{M_{re}}{M_y}$</td>
<td>$\frac{M_{re}}{M_y}$</td>
</tr>
<tr>
<td>-0.012</td>
<td>-0.300</td>
<td>0.100</td>
<td>-0.0125</td>
<td>-0.2598</td>
<td>0.1598</td>
<td>-0.0110</td>
<td>-0.1375</td>
<td>0.0223</td>
</tr>
<tr>
<td>-0.010</td>
<td>-0.250</td>
<td>0.050</td>
<td>-0.0100</td>
<td>-0.2070</td>
<td>0.1570</td>
<td>-0.0085</td>
<td>-0.1063</td>
<td>0.0507</td>
</tr>
<tr>
<td>-0.008</td>
<td>-0.200</td>
<td>0.000</td>
<td>-0.0073</td>
<td>-0.1511</td>
<td>0.1511</td>
<td>-0.0055</td>
<td>-0.0688</td>
<td>0.0823</td>
</tr>
<tr>
<td>-0.006</td>
<td>-0.150</td>
<td>-0.050</td>
<td>-0.0046</td>
<td>-0.0952</td>
<td>0.1452</td>
<td>-0.0030</td>
<td>-0.0375</td>
<td>0.1077</td>
</tr>
<tr>
<td>-0.004</td>
<td>-0.100</td>
<td>-0.100</td>
<td>-0.0020</td>
<td>-0.0414</td>
<td>0.1414</td>
<td>0.000</td>
<td>0.1414</td>
<td>0.1414</td>
</tr>
<tr>
<td>-0.002</td>
<td>-0.050</td>
<td>-0.150</td>
<td>0.0005</td>
<td>0.0104</td>
<td>0.1396</td>
<td>0.0027</td>
<td>0.0338</td>
<td>0.1734</td>
</tr>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.200</td>
<td>0.0031</td>
<td>0.0642</td>
<td>0.1358</td>
<td>0.0052</td>
<td>0.0650</td>
<td>0.2008</td>
</tr>
<tr>
<td>0.002</td>
<td>0.050</td>
<td>-0.250</td>
<td>0.0055</td>
<td>0.1139</td>
<td>0.1361</td>
<td>0.0080</td>
<td>0.1000</td>
<td>0.2361</td>
</tr>
<tr>
<td>0.004</td>
<td>0.100</td>
<td>-0.300</td>
<td>0.0082</td>
<td>0.1697</td>
<td>0.1303</td>
<td>0.0110</td>
<td>0.1375</td>
<td>0.2678</td>
</tr>
<tr>
<td>0.006</td>
<td>0.150</td>
<td>-0.350</td>
<td>0.0110</td>
<td>0.2277</td>
<td>0.1223</td>
<td>0.0148</td>
<td>0.1850</td>
<td>0.3073</td>
</tr>
<tr>
<td>0.008</td>
<td>0.200</td>
<td>-0.400</td>
<td>0.0140</td>
<td>0.2898</td>
<td>0.1102</td>
<td>0.0180</td>
<td>0.2250</td>
<td>0.3352</td>
</tr>
<tr>
<td>0.010</td>
<td>0.250</td>
<td>-0.450</td>
<td>0.0180</td>
<td>0.3726</td>
<td>0.0774</td>
<td>0.0220</td>
<td>0.2750</td>
<td>0.3524</td>
</tr>
</tbody>
</table>
## TABLE 5. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 6.

Computation for Curve of Figure 16.

<table>
<thead>
<tr>
<th>$\Theta_L$</th>
<th>$\frac{(M_r)_L}{M_y}$</th>
<th>$\frac{(M_e)_L}{M_y}$</th>
<th>$\frac{M_L}{M_y} = - \frac{M_U}{M_y}$</th>
<th>$\gamma$</th>
<th>$\Theta_U$</th>
<th>$\frac{(M_r)_U}{M_y}$</th>
<th>$\frac{(M_e)_U}{M_y}$</th>
<th>$\Delta h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0060</td>
<td>-0.1500</td>
<td>-0.20</td>
<td>-0.0500</td>
<td>0.00680</td>
<td>-0.0044</td>
<td>-0.0550</td>
<td>-0.0505</td>
<td>-0.00520</td>
</tr>
<tr>
<td>-0.0040</td>
<td>-0.1000</td>
<td>-0.20</td>
<td>-0.1000</td>
<td>0.00145</td>
<td>-0.0099</td>
<td>-0.0125</td>
<td>0.0875</td>
<td>-0.00245</td>
</tr>
<tr>
<td>-0.0020</td>
<td>-0.0500</td>
<td>-0.20</td>
<td>-0.1500</td>
<td>0.00235</td>
<td>0.0027</td>
<td>0.0338</td>
<td>0.1838</td>
<td>0.00035</td>
</tr>
<tr>
<td>0.0020</td>
<td>0.0500</td>
<td>-0.20</td>
<td>-0.2500</td>
<td>0.00390</td>
<td>0.0098</td>
<td>0.1225</td>
<td>0.3725</td>
<td>0.00590</td>
</tr>
<tr>
<td>0.0040</td>
<td>0.1000</td>
<td>-0.20</td>
<td>-0.3000</td>
<td>0.00520</td>
<td>0.0144</td>
<td>0.1800</td>
<td>0.4800</td>
<td>0.00920</td>
</tr>
<tr>
<td>0.0060</td>
<td>0.1500</td>
<td>-0.20</td>
<td>-0.3500</td>
<td>0.00700</td>
<td>0.0200</td>
<td>0.2500</td>
<td>0.6000</td>
<td>0.01300</td>
</tr>
<tr>
<td>0.0070</td>
<td>0.1750</td>
<td>-0.20</td>
<td>-0.3750</td>
<td>0.00880</td>
<td>0.0246</td>
<td>0.3075</td>
<td>0.6825</td>
<td>0.01580</td>
</tr>
<tr>
<td>0.0075</td>
<td>0.1875</td>
<td>-0.20</td>
<td>-0.3875</td>
<td>0.01090</td>
<td>0.0293</td>
<td>0.3663</td>
<td>0.7538</td>
<td>0.01840</td>
</tr>
<tr>
<td>0.0075</td>
<td>0.1875</td>
<td>-0.20</td>
<td>-0.3875</td>
<td>0.01450</td>
<td>0.0365</td>
<td>0.4563</td>
<td>0.8438</td>
<td>0.02200</td>
</tr>
<tr>
<td>0.0070</td>
<td>0.1750</td>
<td>-0.20</td>
<td>-0.3750</td>
<td>0.01675</td>
<td>0.0405</td>
<td>0.5063</td>
<td>0.8813</td>
<td>0.02375</td>
</tr>
<tr>
<td>0.0060</td>
<td>0.1500</td>
<td>-0.20</td>
<td>-0.3500</td>
<td>0.01930</td>
<td>0.0446</td>
<td>0.5575</td>
<td>0.9075</td>
<td>0.02530</td>
</tr>
<tr>
<td>0.0040</td>
<td>0.1000</td>
<td>-0.20</td>
<td>-0.3000</td>
<td>0.02420</td>
<td>0.0524</td>
<td>0.6550</td>
<td>0.9550</td>
<td>0.02820</td>
</tr>
<tr>
<td>0.0020</td>
<td>0.0500</td>
<td>-0.20</td>
<td>-0.2500</td>
<td>0.02840</td>
<td>0.0588</td>
<td>0.7350</td>
<td>0.9850</td>
<td>0.03040</td>
</tr>
</tbody>
</table>
**TABLE 6. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 7.**

Computation for Curves of Figure 18.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>$\frac{M_x}{M_y}$</td>
<td>$\frac{M}{M_y}$</td>
<td>$\gamma$</td>
<td>$\frac{\Delta}{h}$</td>
<td>$\frac{P\Delta}{M_y}$</td>
<td>$\frac{Qh}{M_y}$</td>
</tr>
<tr>
<td>$M_r = 100 \Theta M_y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0020</td>
<td>0.20</td>
<td>-0.20</td>
<td>-0.0027</td>
<td>0.0047</td>
<td>0.1297</td>
<td>0.0703</td>
</tr>
<tr>
<td>0.0030</td>
<td>0.30</td>
<td>-0.30</td>
<td>-0.0043</td>
<td>0.0073</td>
<td>0.2015</td>
<td>0.0985</td>
</tr>
<tr>
<td>0.0040</td>
<td>0.40</td>
<td>-0.40</td>
<td>-0.0065</td>
<td>0.0105</td>
<td>0.2898</td>
<td>0.1102</td>
</tr>
<tr>
<td>0.0045</td>
<td>0.45</td>
<td>-0.45</td>
<td>-0.0082</td>
<td>0.0127</td>
<td>0.3505</td>
<td>0.0995</td>
</tr>
<tr>
<td>$M_r = 200 \Theta M_y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0010</td>
<td>0.20</td>
<td>-0.20</td>
<td>-0.0027</td>
<td>0.0037</td>
<td>0.1021</td>
<td>0.0979</td>
</tr>
<tr>
<td>0.0015</td>
<td>0.30</td>
<td>-0.30</td>
<td>-0.0043</td>
<td>0.0058</td>
<td>0.1601</td>
<td>0.1399</td>
</tr>
<tr>
<td>0.0020</td>
<td>0.40</td>
<td>-0.40</td>
<td>-0.0065</td>
<td>0.0085</td>
<td>0.2346</td>
<td>0.1654</td>
</tr>
<tr>
<td>0.00225</td>
<td>0.45</td>
<td>-0.45</td>
<td>-0.0082</td>
<td>0.0106</td>
<td>0.2925</td>
<td>0.1575</td>
</tr>
<tr>
<td>0.0023</td>
<td>0.46</td>
<td>-0.46</td>
<td>-0.0092</td>
<td>0.0115</td>
<td>0.3174</td>
<td>0.1426</td>
</tr>
</tbody>
</table>
### TABLE 7. SOLUTION OF ILLUSTRATIVE EXAMPLE NO. 8.

Computation for Curve of Figure 22.

<p>| | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>$\delta \theta_L$</td>
<td>$\frac{\delta (M_r)L}{M_y}$</td>
<td>$\frac{\delta M_L}{M_y}$</td>
<td>$\frac{\delta \Delta}{h}$</td>
<td>$\frac{P \delta \Delta}{h}$</td>
<td>$\frac{\delta M_U}{M_y}$</td>
<td>$\delta Q_U$</td>
<td>$\frac{\delta (M_r)U}{M_y}$</td>
<td>$\frac{\delta (M_e)U}{M_y}$</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.001</td>
<td>0.050</td>
<td>-0.050</td>
<td>0.00165</td>
<td>0.0456</td>
<td>0.0044</td>
<td>0.0020</td>
<td>0.0500</td>
<td>0.0544</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.001</td>
<td>0.040</td>
<td>-0.040</td>
<td>0.00150</td>
<td>0.0414</td>
<td>-0.0014</td>
<td>0.0018</td>
<td>0.0360</td>
<td>0.0346</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.001</td>
<td>0.030</td>
<td>-0.030</td>
<td>0.00132</td>
<td>0.0364</td>
<td>-0.0064</td>
<td>0.0015</td>
<td>0.0225</td>
<td>0.0161</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.001</td>
<td>0.020</td>
<td>-0.020</td>
<td>0.00116</td>
<td>0.0321</td>
<td>-0.0121</td>
<td>0.0012</td>
<td>0.0120</td>
<td>-0.0001</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.001</td>
<td>0.010</td>
<td>-0.010</td>
<td>0.00100</td>
<td>0.0276</td>
<td>-0.0176</td>
<td>0.0009</td>
<td>0.0045</td>
<td>-0.0131</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.001</td>
<td>0.001</td>
<td>-0.001</td>
<td>0.0009</td>
<td>0.0248</td>
<td>-0.0247</td>
<td>0.0001</td>
<td>0.0001</td>
<td>-0.0246</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 1 BRACED FRAME AND NONSWAY SUBASSEMBLAGE
FIG. 2 UNBRACED FRAME AND SWAY SUBASSEMBLAGE
FIG. 3 TYPES OF SUBASSEMBLAGES STUDIED IN THIS REPORT
FIG. 4 DEFORMED CONFIGURATION OF A SUBASSEMBLAGE PREVENTED FROM SWAY
FIG. 5 ANALYSIS OF A NONSWAY SUBASSEMBLAGE
(ILLUSTRATIVE EXAMPLE 1)
FIG. 6 SPECIAL CASES OF SUBASSEMBLAGES WITHOUT SIDESWAY
FIG. 7 ANALYSIS OF A SYMMETRICAL SUBASSEMBLAGE (ILLUSTRATIVE EXAMPLE 2)
FIG. 8a STATICAL SOLUTION OF A SUBASSEMBLAGE HAVING ROTATION CAPACITY

FIG. 8b STATICAL SOLUTION OF A SUBASSEMBLAGE HAVING NO ROTATION CAPACITY
FIG. 9 A TYPICAL CONTINUOUS COLUMN
FIG. 10 A COLUMN CONTINUOUS OVER THREE SUPPORTS
(ILLUSTRATIVE EXAMPLE 3)

\[ \frac{h}{r} = 30 \]

\[ M_3 = M_y \]

\[ M_2 = -0.5 M_y \]

\[ M_1 = 0.6 M_y \]

FIG. 11 MOMENT VS. ROTATION CURVE OF THE COLUMN SHOWN IN FIG. 10
FIG. 12a  DEFORMED CONFIGURATION OF
A SUBASSEMBLAGE PERMITTED TO SWAY

FIG. 12b  DEFORMED CONFIGURATION OF
A COLUMN IN A SWAY SUBASSEMBLAGE

Except \( M_U, M_L, \gamma_U \) and \( \gamma_L \)
all quantities as shown are positive
FIG. 13 ANALYSIS OF A SWAY SUBASSEMBLAGE WITH HORIZONTAL FORCE (ILLUSTRATIVE EXAMPLE 4)
FIG. 14 SPECIAL CASES OF SUBASSEMBLAGES WITH SIDESWAY
FIG. 15 ANALYSIS OF A SWAY SUBASSEMBLAGE WITHOUT HORIZONTAL FORCE (ILLUSTRATIVE EXAMPLE 5)
FIG. 16  ANALYSIS OF A SWAY SUBASSEMBLAGE WITH SPECIFIED LATERAL RESTRAINT (ILLUSTRATIVE EXAMPLE 6)

FIG. 17  COMPARISON OF MOMENT VS. SWAY RELATIONSHIPS OF A SUBASSEMBLAGE WITH AND WITHOUT LATERAL RESTRAINT
FIG. 18 ANALYSIS OF A SWAY SUBASSEMBLAGE WITH ONE END PINNED AND SUBJECTED TO A HORIZONTAL FORCE (ILLUSTRATIVE EXAMPLE 7)

FIG. 19 ANALYSIS OF A SWAY SUBASSEMBLAGE WITH ONE END PINNED AND SUBJECTED TO A HORIZONTAL FORCE AND AN EXTERNAL MOMENT
FIG. 20  BUCKLING OF A SUBASSEMBLAGE WITH ONE END PINNED

FIG. 21  BUCKLING OF A GENERAL SUBASSEMBLAGE
FIG. 22 BUCKLING ANALYSIS OF A GENERAL SUBASSEMBLAGE (ILLUSTRATIVE EXAMPLE 8)
FIG. 23a  SUBASSEMBLAGE COMPRISED OF ROTATIONAL SPRING AND COLUMN

FIG. 23b  SUBASSEMBLAGE COMPRISED OF BEAM AND COLUMN
9. **NOTATION**

\begin{align*}
C &= \text{Stiffness of restraining member (see Eq. 35)} \\
d &= \text{Depth of section} \\
E &= \text{Young's modulus} \\
F &= \text{Symbol representing restraining function} \\
h &= \text{Height of subassemblage} \\
I &= \text{Moment of inertia} \\
k &= \text{Stiffness of lateral spring} \\
L &= \text{Length of beam} \\
M_U &= \text{Moment at upper end of column} \\
M_L &= \text{Moment at lower end of column} \\
(M_e)_U &= \text{External moment applied at upper joint} \\
(M_e)_L &= \text{External moment applied at lower joint} \\
(M_r)_U &= \text{Restraining moment offered by upper spring} \\
(M_r)_L &= \text{Restraining moment offered by lower spring} \\
n &= \text{A parameter defining the stiffness of rotational restraint (see Example 8)} \\
P &= \text{Axial force in column} \\
P_y &= \text{Axial yield force of column cross section} \\
Q &= \text{Horizontal force} \\
q_U &= \frac{M_L}{M_U} \\
q_L &= \frac{M_U}{M_L} \\
r &= \text{Radius of gyration about strong axis} \\
V &= \text{Shear in column} \\
\alpha &= \text{Angle between thrust line and tangent at one end of column} \\
\beta &= \text{Angle between thrust line and chord of column}
\end{align*}
\( \gamma \) = Angle between tangent and chord of column
\( \Delta \) = Sway displacement
\( \delta \) = Symbol indicating small variation of certain quantity
\( \theta \) = Joint rotation = angle between vertical and tangent
\( \lambda \) = \( \sqrt{P/EI} \)
\( \sigma_y \) = Yield stress of material
\( \tau \) = Angle between thrust line and vertical
10. REFERENCES

1. Beedle, L. S.  

2. American Institute of Steel Construction  
SPECIFICATION FOR THE DESIGN, FABRICATION AND ERECTION OF STRUCTURAL STEEL FOR BUILDINGS, AISC, New York, 1961

PLASTIC DESIGN OF BRACED MULTI-STORY FRAMES, Fritz Engineering Laboratory Report 273.3, Lehigh University, 1961

4. Ojalvo, M.  
LITERATURE SURVEY ON THE ANALYSIS AND DESIGN OF RESTRAINED COLUMNS, Fritz Engineering Laboratory Report 278.1, Lehigh University, 1959

5. Ojalvo, M.  

6. Collins, W. H. and Sidebottom, O. M.  
INELASTIC BUCKLING OF COLUMNS WITH ELASTICALLY RESTRAINED ENDS, Department of Theoretical and Applied Mechanics Report 183, University of Illinois, 1961

7. Levi, V.  
PLASTIC DESIGN OF BRACED MULTI-STORY FRAMES, Ph.D. Dissertation, Lehigh University, 1962. (University Microfilms, Ann Arbor, Michigan)

TESTING TECHNIQUES FOR RESTRAINED COLUMNS, Fritz Engineering Laboratory Report 278.7, Lehigh University, 1963

DIE VERFAHREN ZUR STABILITÄTSBERECHNUNG STATISCH UNBESTIMMTER BIEGESTEIFER STAHLSTABWERKE, VERGLICHEN AN EINEM UNTERSUCHUNGSBEISPIEL, Der Stahlbau, Vol. 32, No. 2, 1963, p. 42

10. Knothe, K.  
VERGLEICHENDE DARSTELLUNG DER NÄHERUNGSMETHODEN ZUR BESTIMMUNG DER TRAGLÄSTE EINES BIEGESTEIFER STAHLSTABWERKES, Der Stahlbau, Vol. 32, No. 11, 1963, p. 330

11. Merchant, W.  
FRAME STABILITY IN THE PLASTIC RANGE, British Welding Journal, Vo. 3, No. 8, 1956, p. 366
12. Ojalvo, M. and Fukumoto, Y.
   NOMOGRAPHS FOR THE SOLUTION OF BEAM-COLUMN PROBLEMS, Welding Research Council Bulletin No. 78, 1962

   RESPONSE OF COLUMNS TO IN-PLANE LOADING, Fritz Engineering Laboratory Report 273.10, Lehigh University, 1963

   ULTIMATE STRENGTH TABLES FOR BEAM-COLUMNS, Welding Research Council Bulletin No. 78, 1962


16. Galambos, T. V. and Lay, M. G.
   END-MOMENT END-ROTATION CHARACTERISTICS FOR BEAM-COLUMNS, Fritz Engineering Laboratory Report 205A.35, Lehigh University, 1962

17. Timoshenko, S. P. and Gere, J. M.

18. Winter, G., Hsu, P. T., Koo, B. and Loh, M. H.