ANALYSIS OF RESTRAINED COLUMNS PERMITTED TO SWAY

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Fritz Engineering Laboratory Report No. 273.11b
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INTRODUCTION

In a rigidly jointed multi-story frame, which is not braced against joint translations, all columns can be considered as restrained columns permitted to sway. The behavior and strength of these columns directly affect those of the frame. Consider the unbraced three-story, three-bay frame shown in Fig. 1(a). Under the combined action of the gravity and lateral loads, the structure deforms to the sway configuration shown. All story levels translate horizontally, causing relative displacements of the ends of the column members (story sway). Each column in the frame is therefore subjected to three types of bending moments: 1) Moment caused by the gravity load due to rigid frame action, 2) Moment required to resist story shear, and 3) Moment resulting from displacing the axial force in the column through the story sway. The third type, often referred to as secondary moment, may become significant if the axial force or the story sway or both are relatively large.

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The behavior of a column in an unbraced frame may be studied by isolating the column and its neighboring members from the structure and forming a beam-to-column subassemblage. A subassemblage that represents the behavior of column AB of the frame in Fig. 1(a) is illustrated in Fig. 1(b). In this subassemblage the flexural restraints offered by the members that are not framed directly into the column are approximated by rotational springs, and the resistance to sway provided by these members is represented by lateral springs. In order to study the strength of column AB in the subassemblage, it is first necessary to develop a method for analyzing the restrained column shown in Fig. 1(c). In this column the rotational restraints offered by all the neighboring members at each joint are approximated by a single spring and the resistance to relative translation between A and B is represented by a lateral spring.

The reasons for the study of restrained columns and subassemblies are: 1) The analysis of an entire multi-story, multi-bay frame is almost prohibitive if stability and deflection effects are predominant considerations and 2) subassemblies can be used in the analysis and design of individual members and of member groups, when conservative assumptions are made for end conditions. A plastic design procedure for unbraced frames, based on the results presented in this paper, has already been developed and is presented elsewhere.

The analysis of restrained columns with sway has received little attention in the literature. A very simple column was analyzed by J. Oxfort and by K. Knothe and their results were compared with those obtained by applying an approximate method originally proposed by W. Merchant. The same structure was also studied by U. Vogel using an iterative procedure. An exploratory test on a sway column was conducted by M. G. Lay and T. V. Galambos.

This paper presents a numerical method of analysis for restrained columns having any type of rotational and lateral restraints and subjected to any combination of external moments and lateral force. The method is subsequently modified for simplified application to four special types of restrained columns. Column material is assumed to behave in an elastic and perfectly plastic manner (elasto-plastic).

In all the columns to be analyzed, the following data are assumed to be known: The length, the cross section, and the yield stress of the column, the properties of the rotational and lateral restraints, and the magnitude of the axial force. Additional

8 Vogel, U., "Die Traglastberechnung stahlerner Rahmentragwerke nach der Plastizitätstheorie II. Ordnung," Heft, Stahlbau-Verlag, Köln, 1965
quantities are specified when needed. The purpose of analysis is to determine the maximum joint moments or lateral force that can be carried by a given column.

Throughout this study, the columns are assumed to be sufficiently braced to prevent out-of-plane deformation. Failure is therefore the result of excessive bending in the plane of the applied moments.

Notation -- The symbols adopted for use in this paper are defined where they first appear and are listed alphabetically in the Appendix.

VARIABLES AND SIGN CONVENTION

The deformed configuration of a typical sway column is shown in Fig. 2(a). The column is loaded by two external joint moments, \( (M_e)_U \) and \( (M_e)_L \), and by a vertical force, \( P \), acting directly on the top of the column. In addition, a lateral force, \( Q \), is applied at the level of the upper joint. The column is restrained at its two ends by rotational restraints which react to the joint rotations, \( \theta_U \) and \( \theta_L \), with restraining moments, \( (M_r)_U \) and \( (M_r)_L \). At the upper end of the column a lateral restraint providing resistance against sidesway is attached. The lateral restraint has a stiffness \( k \) and reacts with a horizontal force \( k \Delta \) to a sway displacement \( \Delta \). It is assumed that the characteristics of both the rotational and lateral restraints are known in a given problem.
The external moment applied at each joint is carried partly by the column and partly by the rotational restraint. The manner in which the joint moment is distributed depends on the moment-rotation characteristics of the column and the restraint. In addition to the externally applied moments, there are moments developed in the column due to the lateral displacement $\Delta$ and due to the horizontal shear $V$ which is equal to the difference between $Q$ and $k\Delta$. These moments are also carried jointly by the column and the rotational restraints. It is important that the effect of these moments be properly taken into account in the analysis of a sway column.

For a given column subjected to a specified axial force $P$, the following two problems may be investigated: 1) When the horizontal force $Q$ and one of the two external moments are given, determine the maximum value of the other moment which can be safely carried by the structure, and 2) when both moments are given, determine the maximum value of $Q$. Both problems are considered in this paper.

The variables involved in the analysis of a general restrained column with sway are:

1. The height of the column, $h$, or the slenderness ratio, $h/r$;
2. The axial force, $P$, or the ratio of the axial force to the yield force of the cross section, $P/P_y$ (both $r$ and $P_y$ are cross-sectional properties);
3. the externally applied moments, \( (M_e)_U \) and \( (M_e)_L \);  
4. the lateral force, \( Q \);  
5. the properties of the rotational restraints;  
6. the restraining moments, \( (M_r)_U \) and \( (M_r)_L \);  
7. the column moments, \( M_U \) and \( M_L \);  
8. the joint rotations, \( \theta_U \) and \( \theta_L \);  
9. the lateral deflection, \( \Delta \); and  
10. the stiffness of the lateral restraint, \( k \).

As stated previously, the variables in 1, 2, 5, and 10 are always considered to be known for a given column. Thus, the total number of variables that must be determined or specified is ten. The lateral deflection \( \Delta \) can also be expressed nondimensionally as the ratio \( \Delta/h \) which represents the rotation of the chord with respect to the vertical.

The sign convention for moments, lateral force, and joint and chord rotations adopted in the analysis is as follows:

**Moments acting at a joint:** \( (M_e)_U \) and \( (M_e)_L \) are positive when clockwise, \( (M_r)_U \), \( (M_r)_L \), \( M_U \) and \( M_L \) are positive when counterclockwise (Fig. 2(b)).

**Moments and rotations at the ends of the column:** \( M_U \), \( M_L \), \( \theta_U \), \( \theta_L \), and \( \Delta/h \) are positive when clockwise (Figs. 2(a) and 2(b)).
Horizontal shear: $V$ is positive if it causes a clockwise moment about the lower joint.

According to this sign convention, all quantities shown in Figs. 2(a) and 2(b) are positive.

**METHOD OF ANALYSIS FOR GENERAL CASE**

As stated above the total number of variables involved in the analysis of a given column is ten. The equations that can be used to relate these variables are two joint equilibrium equations

$$\begin{align*}
(M_e)_U &= (M_r)_U + M_U \\
(M_e)_L &= (M_r)_L + M_L
\end{align*}$$

and two equations defining the restraining characteristics of the rotational restraints

$$\begin{align*}
(M_r)_U &= F_U (\theta)_U, \text{ properties of restraining members}) \\
(M_r)_L &= F_L (\theta)_U, \text{ properties of restraining members})
\end{align*}$$

In addition the shear equilibrium of the column requires that

$$M_U + M_L + P\Delta + Vh = 0$$
This equation can be derived by considering the equilibrium of the column shown in Fig. 2(b). Through Eqs. 2a and 2b, it is possible to determine the restraining moments, \( (M^r)_U \) and \( (M^r)_L \), when the joint rotations, \( \theta_U \) and \( \theta_L \), are known. Thus, the number of variables is reduced by two, and the remaining variables are \( (N^e)_U \), \( (N^e)_L \), \( Q \), \( M_U \), \( M_L \), \( \theta_U \), \( \theta_L \), and \( \Delta/h \).

Some of the above eight variables are usually specified in a given problem. In the first type of problem stated earlier, the lateral force, \( Q \), and one of the two external moments, \( (M^e)_U \) or \( (M^e)_L \), are given. The known variables in the second type of problem are \( (M^e)_U \) and \( (M^e)_L \). So the total number of variables becomes six. As will be seen in the later development, the proposed method of analysis requires that an initial joint rotation (either \( \theta_U \) or \( \theta_L \)) be assumed and subsequent analysis be made to determine the value of either the unknown moment or the unknown lateral force consistent with the assumed joint rotation. Thus, in using this procedure, only five variables need to be determined: \( (M^e)_U \) (if \( (M^e)_L \) is the known moment) or \( Q \), \( M_U \), \( M_L \), \( \theta_U \) (if \( \theta_L \) is the assumed rotation) and \( \Delta/h \).

The two joint equilibrium conditions (Eqs. 1a and 1b) and the shear equilibrium condition (Eq. 3) constitute three basic equations that can be used to solve for the five unknowns. Two additional relationships are necessary if a complete solution to the problem is to be obtained. A series of charts of special type has been developed to provide the needed relationships; Fig. 3 shows a sample chart.
Before discussing the use of the chart, it is necessary to examine the deformation of a column member in its swayed position. Fig. 2(c) shows the deformed column of Fig. 2(a) with all restraint notation removed for clarity. The joint rotations, $\theta_U$ and $\theta_L$, and the chord rotation, $\Delta/h$, are the same as those illustrated in Fig. 2(a). Two additional angles, $\gamma_U$ and $\gamma_L$, relating, respectively, $\Delta/h$ to $\theta_U$, and $\Delta/h$ to $\theta_L$, are also shown. These angles represent the rotations of the tangents from the chord and are positive if the tangents rotate clockwise from the chord. ($\gamma_U$ and $\gamma_L$ as shown in Fig. 2(c) are therefore negative.) The shear force $V$ and the vertical force $P$ form a resultant force acting along the direction of the thrust line.

Three types of angles, $\alpha$, $\beta$, and $\tau$, may be associated with this line. $\alpha_U$ and $\alpha_L$ are the angles measured from the thrust line to the tangents at the two ends, and $\beta$ is the angle measured from the same line to the chord. The angle $\tau$ defines the direction of the thrust line and is given by $\tau = \tan^{-1} V/P$. It has been shown that in practical cases this angle may be assumed to be equal to $V/P$, because $P$ is usually much greater than $V$. All angles are positive when they represent clockwise rotations from the thrust line.

Using the definitions and sign convention stated above, the angles $\alpha_U$ and $\alpha_L$ can be expressed in terms of $\theta_U$, $\theta_L$, and $\tau$.

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\[ \alpha_U = \tau + \theta_U \]  
\[ \alpha_L = \tau + \theta_L \]  

and the angle \( \beta \) is given by

\[ \beta = \tau + \frac{\Delta}{h} \]  

Similarly the rotations \( \gamma_U \) and \( \gamma_L \) can be determined from \( \theta_U, \theta_L, \) and \( \Delta/h \) by the relationships

\[ \gamma_U = \theta_U - \frac{\Delta}{h} \]  
\[ \gamma_L = \theta_L - \frac{\Delta}{h} \]

The type of chart shown in Fig. 3 may be used to determine the angle \( \beta \) of a column, when the angle \( \alpha \) and the moment \( M \) at one end are specified. It may also be used to find the angle \( \alpha \) at one end of a column, when the angle \( \beta \) and the moment \( M \) at the same end are known. The particular chart was prepared for a wide-flange column with \( h/r \) (strong-axis slenderness ratio) equal to 30 and loaded by an axial force equal to 0.6 \( P_y \). The bending moments are applied about the strong axis of the cross section. The steps which were followed in preparing the chart have been described in detail elsewhere.  

10 (The material constants used were a yield stress \( \sigma_y = 33 \text{ ksi} \) and Young's modulus \( E = 30,000 \text{ ksi} \).) Additional charts for a large number of \( P/P_y \) and \( h/r \) combinations are available in a separate booklet.  

The method developed for analyzing restrained columns with a specified lower external moment and a specified lateral force is summarized as follows:

1. Select a value of $\Theta_L$ and compute the corresponding restraining moment, $\langle M_T \rangle_L$, from the known moment-rotation characteristic of the lower restraint. The moment $M_L$ of the column can then be determined from Eq. 1b, $M_L = \langle M_e \rangle_L - \langle M_T \rangle_L$.

2. Assume a trial value for $\Delta$ and compute the chord rotation $\Delta/h$. The shear force $V$ can be determined as the difference between $Q$ and $k\Delta$. Since the axial force $P$ is specified in a given problem, the angle $\tau$ can be computed from $\tau = V/P$. The sum of $\tau$ and $\Theta_L$ determines the angle $\alpha_L$ (Eq. 4b).

3. With $\alpha_L$ and $M_L$, determine the angle $\beta$ from the chart. The chord rotation $\Delta/h$ can then be calculated from Eq. 5, $\Delta/h = \beta - \tau$. This value is compared with that assumed in Step 2. If the calculated $\Delta/h$ does not agree with the assumed, a new $\Delta$ (or $\Delta/h$) should be tried. The process is repeated until agreement is found.

4. Calculate the moments $P\Delta$ and $Vh$, using the $\Delta$ and $V$ values just determined. The following
convenient relationship may be used for this purpose.

\[ P\Delta + Vh = \frac{P}{P_\gamma} \frac{h}{r} d \frac{1}{2r} M_y \]  \hspace{1cm} (7)

in which \( d \) and \( M_y \) are, respectively, the depth and yield moment of the column section. The derivation of this equation is given elsewhere. \(^{10}\)

5. From the shear equilibrium condition of Eq. 3 the moment at the upper end of the column can be determined, \( M_U = -M_L - (P\Delta + Vh) \).

6. With \( M_U \) and \( \beta \), determine the angle \( \alpha_U \) from the chart. The joint rotation \( \Theta_U \) is then obtained from Eq. 4a, \( \Theta_U = \alpha_U - \tau \).

7. When \( \Theta_U \) is known, the restraining moment \( (M_e)_U \) can be determined from the moment-rotation relationship of the upper restraint. The external joint moment is then obtained by applying Eq. 1a,
\[ (M_e)_U = (M_r)_U + M_U \]  \hspace{1cm} This is the external moment consistent with the assumed lower end rotation \( \Theta_L \).

8. The above steps are repeated for several selected values of \( \Theta_L \). The results are then plotted in the form of a \((M_e)_U \) vs. \( \Theta_L \) (or \( \Delta/h \)) curve from which the maximum value of \((M_e)_U \) can be determined.
To analyze the case in which the two external moments \((M_e)_U\) and \((M_e)_L\) are specified and the maximum value of \(Q\) is to be determined, the procedure described above has to be modified. In such a case, the shear force \(V\) of the column is not known; consequently, the angle \(\tau\) cannot be determined. This complicates the analysis considerably and makes the above method very difficult to apply. A modified procedure for restrained columns with unspecified lateral force is therefore developed.

The first step in this procedure is the same as in the previous method; it consists of selecting a value of \(\theta_L\) and determining the corresponding column moment \(M_L\). Since the lateral force \(Q\) is not known, the angle \(\alpha_L\) cannot be computed from Eq. 4b. It is therefore not possible to determine directly the angle \(\beta\) by using the chart. However, if a value of \(\beta\) is assumed, the corresponding value of \(\alpha_L\) can be obtained from the chart. The assumed \(\beta\) value may be checked by the following procedure: Since \(\alpha_L\) is now known, the angle \(\tau\) can be determined from \(\tau = \alpha_L - \theta_L\) (Eq. 4b). Consequently the rotation \(\Delta/h\) is also known (from Eq. 5, \(\Delta/h = \beta - \tau\)). When the assumed value of \(\beta\) is substituted into the right hand side of Eq. 7, the bending moments due to \(\Delta\) and \(V\) are determined. By knowing \(\Delta\) and \(P\Delta + Vh\), the shear force \(V\) can be calculated and a new value of \(\tau (\tau = V/P)\) is obtained. If the assumed \(\beta\) is correct, the angle \(\tau\) just determined should agree with the \(\tau\) value used previously. Otherwise, a new \(\beta\) value is assumed and the procedure is repeated until an agreement between the two \(\tau\) values is found.
After the correct value of $\beta$ corresponding to the selected value of $\theta_L$ is found, the shear force $V$ and, in turn, the lateral force $Q$ ($Q = V + kA$) can be determined, using the procedure just described. The moment at the upper end of the column is determined from the shear equilibrium condition of Eq. 3,

$$M_U = -(M_L + P\Delta + \cdot VH) = -(M_L + \frac{P}{P} \cdot \frac{h}{r} \cdot \frac{d}{2r} \cdot \beta M_y)$$  \hspace{1cm} (8)

With $M_U$ and $\beta$ known, the chart can be used to find $\alpha_U$; and the upper joint rotation is then determined from Eq. 4a, $\theta_U = \alpha_U - \tau$.

The known moment-rotation characteristic of the upper restraint determines the restraining moment $(M_r)_U$ corresponding to the $\theta_U$ just found. The sum of $(M_r)_U$ and $M_U$ gives the external moment $(M_e)_U$. When this moment and the given $(M_e)_L$ together with the lateral force $Q$ found in the analysis are applied to the column, the rotation $\theta_L$ produced at the lower joint will be the rotation which was selected at the beginning of the analysis.

Recall that the upper joint moment is a known quantity in the problem; the external moment determined from the final step of the above analysis should therefore be compared with this moment. If the computed external moment differs from the given moment, a new $\theta_L$ should be selected and the above analysis is repeated. A correct value of $\theta_L$ is found when the upper joint moment obtained from the analysis agrees with the given moment. Accordingly, the lateral force $Q$ consistent with this $\theta_L$ is the correct lateral force which should be applied to the column.
If all the above steps are repeated for different values of $\theta_L$, several pairs of $\theta_L$ and $Q$ values may be obtained. The maximum value of $Q$ can be found graphically by plotting the results in the form of a $Q$ vs. $\theta_L$ (or $\Delta/h$) curve and determining the peak of the curve.

The method of analysis outlined previously for restrained columns with specified $(M_e)_L$ and $Q$ is illustrated by the following example.

**Illustrative Example 1**

**PROBLEM:** A restrained column shown in Fig. 4 with a slenderness ratio of 30 is subjected simultaneously to external joint moments $(M_e)_U$ and $(M_e)_L$ and a lateral force $Q$. In addition, an axial force $P = 0.6P_y$ is applied at the top of the column. The lower external moment and the lateral force are given: $(M_e)_L = -0.2M_y$ and $Q = 0.002P_y$. The characteristics of the rotational restraints are defined by $(M_r)_U = 12.5\theta_M y$ and $(M_r)_L = 25\theta_M y$. For simplicity, it is assumed that no lateral restraint is attached to the structure ($k = 0$). It is required to determine the relationship between $(M_e)_U$ and $\Delta/h$ and the maximum value of $(M_e)_U$.

**SOLUTION:** The step-by-step calculations made for this problem are summarized in Table 1. Each horizontal row in the table records the solution for a selected value of $\theta_L$ (given in Column 1). The nondimensional restraining moment $(M_r)_L/M_y$
corresponding to a given $\theta_L$ is obtained by multiplying the rotation by 25 (characteristic of the lower restraint) and is recorded in Column 2. The nondimensional column moment $M_L/M_y$ given in Column 3 is the difference between $(M_e)_L/M_y$ and $(M_r)_L/M_y$. Since there is no lateral restraint present, the shear in the column is equal to the applied lateral force, that is, $V = Q$. The angle $\tau$ can therefore be determined directly. By definition, it is given by $\tau = V/P = 0.002$. The algebraic sum of $\tau$ and $\theta_L$ gives the angle $\alpha_L$ (Column 4), and the angle $\beta$ is obtained by entering Fig. 3. The chord rotation $\Delta/h$ which determines the magnitude and direction of the sway deflection is then obtained from the relation $(\Delta/h) = \beta - \tau$. The values of $\beta$ and $\Delta/h$ are listed in Columns 5 and 6, respectively.

The next step in the solution is to determine the moments caused by the sidesway deflection $\Delta$ and the shear force $V$. From Eq. 7, it is known that the sum of these moments (nondimensionalized by dividing the sum by $M_y$) is equal to $P \frac{h}{r} \frac{d}{2r} \beta = (0.6)(30)(1.15)(0.002)20.7\beta$. (The ratio $d/2r$ is taken to be 1.15 which is the value for the 8WF31 section. However, for most of the available WF shapes, this ratio is nearly constant.) When the value of $\beta$ in Column 5 is multiplied by 20.7, the product gives directly the required moment sum (Column 7). Knowing the magnitude of $M_L/M_y$ and $(P_\Delta + Vh)/M_y$, the upper column moment, $M_u/M_y$, can be determined by applying Eq. 3 (Column 8).
The angle $\alpha_U$ at the upper end of the column is obtained by entering Fig. 3 with $M_U/M_y$ and $\beta$. The joint rotation $\theta_U$ is then determined from the relationship $\theta_U = \alpha_U - \tau$. The values of $\alpha_U$ and $\theta_U$ are given, in Columns 9 and 10, respectively.

When $\theta_U$ is known, the restraining moment $(M_r)_U/M_y$ at the upper joint can be obtained by multiplying this angle by 12.5 (characteristic of the upper restraint). This is recorded in Column 11. The algebraic sum of the values in Columns 8 and 11 gives the external joint moment $(M_e)_U/M_y$ listed in Column 12.

In this example a wide range of $\theta_L$ values has been selected. The results show that the column can sway in both directions, depending on the magnitude of the upper joint moment $(M_e)_U$. This can be seen more clearly from Fig. 4 in which the upper joint moment, $(M_e)_U/M_y$, is plotted against the chord rotation $\Delta/h$. When $(M_e)_U/M_y$ is less than 0.125, the column always sways to the left. The structure remains vertical when the upper joint moment equals 0.125 $M_y$. In this case, the secondary moment $P_A$ is zero.

Figure 4 shows that the largest external moment that can be applied to the upper joint of the subassemblage is 0.295 $M_y$. Also shown in the figure is a $(M_e)_U/M_y$ vs. $\Delta/h$ curve for the same structure when no lateral force is applied. (The analysis made for this case is given in the next section.) The largest external moment for this case is found to be 0.35 $M_y$. A comparison of these results
indicates that the presence of the lateral force causes considerable reduction in the moment-carrying capacity of the column.

ANALYSIS OF SPECIAL CASES

Restrained Columns Loaded without Lateral Force

The analysis of restrained columns subjected to external moments only is somewhat simpler than that of the general case. If the lateral restraint is also absent, the analysis can be simplified even further. (The next article will consider columns with specified lateral restraints.) The structure to be analyzed is shown in Fig. 5(a). For this column the angle $\tau$ is equal to zero, because $V = 0$. Therefore $\alpha'_{L} = 0$, $\alpha'_{U} = 0$, and $\beta = \Delta/h$. The secondary moment due to the lateral displacement $\Delta$ is again determined by Eq. 7, which is simplified to become

$$ P\Delta = \frac{P}{P_p} \frac{h}{2} \frac{d}{2} \beta M_y $$  \hspace{1cm} (9) 

When the moments $M_L$ and $P\Delta$ are known, the shear equilibrium condition of Eq. 3 can be used directly to compute the moment $M_U$. The following example will illustrate the analysis of this special case.

Illustrative Example 2

The column to be analyzed is the one used in Example 1, the only difference being that in this example $Q$ is assumed to be zero. The solution is presented in Table 2. For each selected value of $\theta_L$ (Column 1), eight steps are required to determine the corresponding value of $(M U_y) / M_y$ (Column 9). As in the previous example, the computations
also yield the column moments, \( M_U/M_y \) and \( M_L/M_y \), the joint rotation, \( \theta_U \), and the chord rotation, \( \Delta/h \). In Fig. 6 the resulting moments, \( M_U/M_y \), \( M_L/M_y \) and \( (M_e)_U/M_y \), are plotted against the chord rotation, \( \Delta/h \). The solid line shows the relationship between the moment \( (M_e)_U/M_y \) and the rotation \( \Delta/h \) of the column. This curve was also given in Fig. 4 as the dashed line.

The sway character of the column may be studied by examining the two dashed curves in Fig. 6, one showing the relationship between \( M_U/M_y \) and \( \Delta/h \), and the other \( M_L/M_y \) and \( \Delta/h \). (The curve for \( M_L/M_y \) is plotted with the sign of the moment changed. In the following discussion, both \( M_U/M_y \) will be referred to by their absolute values.) The column tends to sway to the left when the upper column moment, \( M_U/M_y \), is larger than the lower column moment, \( M_L/M_y \). When \( M_L/M_y \) is larger than \( M_U/M_y \), the structure sways to the right. There is no sway when the two column moments are equal, but acting in the opposite sense. This fact is illustrated by the intersection of the two dashed curves on the vertical axis. The clockwise moment at the upper joint corresponding to no sway is equal to 0.167 \( M_y \). If the two external moments, \( (M_e)_U/M_y \) and \( (M_e)_L/M_y \), are allowed to increase simultaneously (instead of one being fixed), the column will not sway when a constant ratio of \( (M_e)_U/(M_e)_L = 0.167/0.200 = 0.835 \) is maintained between the two moments.

From the results given in Table 2 the deformed configurations of the column for various values of \( (M_e)_U/M_y \) can be determined. Four such configurations are sketched in Fig. 6. For low values of \( (M_e)_U/M_y \),
the column is bent in double curvature by two clockwise end moments. The two end rotations, \( \theta_U \) and \( \theta_L \), and the chord rotation, \( \Delta/h \), are all negative. When \( (M_e)_U/M_y = 1.41 \), the upper joint rotation becomes zero, but the chord rotation is still negative. The lower column moment is now counterclockwise. When \( (M_e)_U/M_y \) is increased to 0.2, the lower joint rotation becomes zero, and the upper joint rotation and the chord rotation are both positive. The column now sways to the right. Any further increase in the upper joint moment will produce a positive rotation at the lower end.

**Restained Columns Having Specified Lateral Restraints**

Another special case which can be studied analytically is the case in which a column is loaded with external joint moments and is restrained by a lateral restraint with a stiffness \( k = P/h \) (Fig. 5b). The reason for choosing such a stiffness is that the resisting force offered by the restraint nullifies completely the effect of secondary moment \( P\Delta \) on the strength of the column. For every sidesway deflection \( \Delta \), the restraint reacts with a force \( P\Delta/h \). The resultant moment due to \( P \) and \( P\Delta/h \) about the lower joint is

\[
P\Delta - \frac{P\Delta}{h}h = 0
\]

Since there is no lateral force applied to the structure, Eq. 13 can be reduced to

\[
M_U + M_L = 0
\]
or \( M_U = -M_L \). Therefore, the presence of a lateral restraint having \( k = P/h \) assures that the column will always be bent in single curvature.

Because of the reduced number of variables involved, the analysis of a column of this type is considerably simpler than that of a general sway column. In this case, once the end rotation \( \theta_L \) is selected and the corresponding column moment \( M_L \) is determined, the rotation \( \theta_U \), the moment \( M_U (M_U = -M_L) \) and the chord rotation \( \Delta/h \) can be readily found without resorting to the type of chart shown in Fig. 3. It is only necessary to use the moment-rotation charts for beam-columns bent in symmetrical single curvature. These charts are available elsewhere.\(^{11,12}\)

Since it is known in the present problem that \( M_U = -M_L \), the rotation \( \gamma_U \) (angle between the chord and the tangent at the upper end) is equal to \( -\gamma_L \). Equations 6a and 6b can therefore be combined to yield

\[
\theta_U = \theta_L + 2\gamma
\]

in which \( \gamma \) is the positive angle between the chord and the tangent at the upper end (shown as \( \gamma_U \) in Fig. 5(b)). The angle \( \gamma \) corresponding to a given end moment \( M_U \) can be found from the available charts mentioned above. After the value of \( \gamma \) is found, Eq. 12 can be used to compute the angle \( \theta_U \).

With $\theta_U$ known, the external moment $(M_e)_U$ is readily determined by applying the equilibrium condition at the upper joint. The chord rotation corresponding to the known value of $\theta_U$ is given by Eq. 6a

$$\Delta/h = \theta_U - \gamma$$

Thus, the amount of side-way is also determined.

**Illustrative Example 3**

Consider the case when a lateral restraint with $k = P/h$ is added to the column of Example 2. Since the addition of such a restraint has the effect of nullifying the $P\Delta$ moment, the carrying capacity of the column will be considerably increased. The step-by-step calculations made for this example are given in Table 3, and the results are summarized graphically in Fig. 7. In solving this example, the moment-rotation curve of a column with $h/r = 30$ and $P/P_y = 0.6$ and subjected to two equal, but opposite, end moments is used. The useful range of that curve terminates at a maximum rotation of 0.0285 radian which corresponds to the initiation of local buckling of the column section. The calculations given in Table 3 therefore end when the value of $\gamma$ has reached this limiting value.

---

The resulting external moment at the upper joint, \( (M_e)_u/M_y \), and the corresponding chord rotation, \( \Delta/h \), of the column are plotted as the solid line in Fig. 7. The maximum external moment that can be applied to the column before the occurrence of local buckling in the column is found to be \( 0.985 \, M_y \). Also shown in Fig. 7 is the result obtained for the same column but with upper rotational restraint removed (\( M_r = 0 \)). The maximum external moment for this case is equal to \( 0.39 \, M_y \). A comparison of the two results shows that the presence of the rotational restraining increases greatly the moment-carrying capacity of the structure.

The increase in strength derived from the lateral restraint can be seen by comparing the result of this example with that of Example 2. Such a comparison is shown in Fig. 8. The maximum moment which can be carried by the column with the specified lateral restraint is almost three times that which can be carried by the same structure but without the restraint. The great difference between the two maximum moments indicates the importance of considering the effect of secondary moment in the analysis of columns permitted to sway.

**Restrained Columns With One End Pinned**

In many practical situations it will be found useful to have solutions to columns pinned at one end. Since the procedure for analysing such columns is considerably simpler than that described for the general case, it is possible to study the effect of variation of lateral force on the sway displacement of the structures. A study of this type is very time consuming for the
general restrained column. The first structure to be analyzed is a column pinned at the lower joint and subjected to a lateral force $Q$ applied at the level of the upper joint (Fig. 5(c)).

In this case, the equilibrium condition for the upper joint becomes

$$M_{r} + M = 0 \quad (14)$$

or $M_{r} = -M$. When the rotation $\theta$ of this joint is selected in the solution and the restraining moment $M_{r}$ corresponding to this rotation is computed from the known restraining function, the moment $M$ at the top of the column is determined. The rotation $\gamma$ of the column (measured from the chord to the tangent) corresponding to this moment may be obtained from the available moment-rotation chart for beam-columns bent by one end moment.\textsuperscript{11, 12} The compatibility condition of Eq. 6a can then be used to compute the chord rotation

$$\Delta/h = \theta - \gamma \quad (15)$$

The shear equilibrium condition of Eq. 3 is reduced to

$$M + P\Delta + Qh = 0 \quad (16)$$

or, equivalently,

$$Qh = -(M + P\Delta) = -(M + \frac{P}{P_{y}} \frac{h}{r} \frac{d}{2r} \beta M_{y}) \quad (17)$$

The last equation is derived by applying Eq. 9. Equation 17 determines the horizontal force $Q$ which is consistent with the selected joint rotation $\theta$. 
Illustrative Example 4

PROBLEM: Determine the relationship between \( Q \) (or \( Qh/M_y \)) and \( \Delta/h \) of the column shown in Fig. 5(c), assuming \( h = 40r \) and \( P = 0.6 \) \( P_y \). The structure will be studied for the following four restraining functions: \( M_x = 0 \) (pinned end), \( M_x = 100 \theta \) \( M_y \), \( M_x = 200 \theta \) \( M_y \) and \( M_x = \infty \) (fixed end).

SOLUTION: The computations for the cases with \( M_x = 100 \theta \) \( M_y \) and \( 200 \theta \) \( M_y \) are tabulated in Table 4 and the results are plotted in Fig. 9. The procedure outlined above has been followed closely in the solution. The case with \( M_x = 0 \) is equivalent to a pinned end column free to sway at the top. In this case

\[
M = 0
\]  \hspace{1cm} (18)

consequently,

\[
Qh = - P\Delta
\]  \hspace{1cm} (19)

or

\[
\frac{Qh}{M_y} = - \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \frac{\Delta}{h} = - 27.6 \frac{\Delta}{h}
\]  \hspace{1cm} (20)

This shows that a negative lateral force (acting toward the left) is required to keep the column displaced in the positive direction. The above equation is plotted as a straight line passing through the origin in Fig. 9. In the other extreme case in which \( M_x = \infty \), the joint rotation \( \theta \) is always zero, and the chord rotation \( \Delta/h \) equals \(- \gamma \). The lateral force is again given by
\[
\frac{Qh}{M_y} = -\frac{M}{M_y} + \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \Delta \right) = -\left(\frac{M}{M_y} - \frac{P}{P_y} \frac{h}{r} \frac{d}{2r} \gamma\right)
\]  \(21\)

When a value of \(\gamma\) is assumed, the moment \(M/M_y\) can be found from the appropriate moment-rotation chart available elsewhere.\(^{12}\) The above equation may be used to determine the lateral force \(Q\) consistent with the assumed \(\gamma\).

The results of this example show clearly the significant effect of the rotational restraint. The column is unstable if the restraint is not present. As the stiffness of the restraint increases, the ultimate strength increases accordingly. The lateral force reaches the absolute maximum when the restraint becomes infinitely stiff. For this example the maximum value of \(Q\) is equal to \(0.226 \, M_y/h\).

Figure 9 also shows that the lateral force \(Q\) changes its sign as the rotational stiffness of the restraint increases from 0 to 100 \(M_y\). There is a value of the stiffness for which the column can carry no lateral force \((Q = 0)\). The structure tends to buckle laterally under the applied axial force. The required stiffness can therefore be determined by a buckling analysis and is found to be 44 \(M_y\). A detailed description of the buckling analysis can be found elsewhere.\(^{14}\)

The Effect of External Joint Moment

It is now of interest to study the effect of an external moment \(M_e\) on the load-deformation behavior of the column. The

---

deformed configuration of the subassemblage is shown in Fig. 5(d). The structure may be analyzed by the procedure described above and illustrated in Example 4. The effect of the external moment is taken into account in the joint equilibrium equation which is written as

\[ M_e = M_r + M \quad (22) \]

or

\[ M = M_e - M_r \quad (23) \]

Therefore, the moment at the upper end of the column is equal to the difference between \( M_e \) and \( M_r \).

In Fig. 10 three curves showing the relationship between the lateral force \( \frac{Qh}{M_y} \) and the chord rotation \( \frac{\Delta}{h} \) of the column considered in the previous example are plotted. The restraining function is assumed to be \( M_r = 100 \frac{Q}{M_y} \). The curves are computed for three values of the external moment: \( M_e = -0.2 M_y \), \( M_e = 0 \) and \( M_e = 0.2 M_y \). The significant concept illustrated by the curves is that the external moment will either help the column resist more lateral force or reduce the structure's load-carrying capacity, depending on the sense of the moment. If \( M_e \) is counterclockwise, it acts as a restraining moment; and if \( M_e \) is clockwise, it hinders the resistance to lateral force.
SUMMARY AND CONCLUSIONS

Methods have been presented for the analysis of restrained columns that are permitted to sway in the plane of the applied loads. The columns are rotationally restrained at both ends and laterally restrained against sway at the upper joint. The loads applied to the columns are the axial force, \( P \), the lateral force, \( Q \), and the two joint moments, \( (M_e)^U \) and \( (M_e)^L \). In all the cases studied the force \( P \) and the moment \( (M_e)^L \) are assumed to be known.

The analysis gives either the relationship between the joint moment \( (M_e)^U \) and the sway displacement \( \Delta/h \) (if \( Q \) is specified) or the relationship between the lateral force \( Q \) and the displacement \( \Delta/h \) (if \( (M_e)^U \) is given). These relationships represent the behavior of the columns in the elastic and inelastic range and permit the determination of the maximum load-carrying capacities. The application of the methods was illustrated by the example shown in Fig. 4.

The general methods of analysis were simplified to become special methods for the four types of columns shown in Fig. 5. These methods were discussed in detail and their applications were illustrated by three examples. Analysis of the results obtained from all the examples led to the following conclusions:

1. The presence of a lateral force acting in the direction of the sway displacement causes a reduction in the moment-carrying capacity of a column (Fig. 4).
2. For the case of a column loaded by two joint moments the direction of sidesway depends on the magnitude and the sense of the applied moments. It is possible that for certain combinations of the two moments the column may remain vertical (Fig. 6).

3. The secondary moment resulting from sway deflections reduces appreciably the ultimate strength of a column (Fig. 8). The reduction is particularly significant if high axial force is present in the column. This fact indicates the need of considering the effect of deformation in the analysis and design of unbraced building frames.

4. The maximum lateral force that can be carried by a column increases as the stiffness of the rotational restraint increases, but approaches a limit when the restraint becomes infinitely stiff (Fig. 9).

The above conclusions have been incorporated in the development of a new design method for unbraced multi-story frames.4

ACKNOWLEDGMENTS

The results reported herein were obtained from a research conducted in the Fritz Engineering Laboratory, Department of Civil Engineering, Lehigh University, Bethlehem, Pa. L. S. Beedle is
director of the laboratory and acting head of the department.

The work represents part of an investigation on "Welded Continuous Frames and Their Components". This project is sponsored by the Welding Research Council and the U. S. Navy Department. Funds were supplied by the American Institute of Steel Construction, American Iron and Steel Institute, Office of Naval Research, Bureau of Ships and the Bureau of Yards and Docks. Technical guidance for the project is provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council. T. R. Higgins is Chairman of the Lehigh Project Subcommittee.

The writers wish to express their appreciation to T. V. Galambos, I. Hooper and E. R. Estes for reviewing the manuscript.

APPENDIX - NOTATION

The following symbols have been adopted for use in this paper:

\[ \begin{align*}
    d & = \text{Depth of section} \\
    E & = \text{Young's modulus} \\
    F & = \text{Symbol representing restraining function} \\
    h & = \text{Height of column} \\
    k & = \text{Stiffness of lateral restraint}
\end{align*} \]
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<th>Description</th>
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<td>$M_U$</td>
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<tr>
<td>$M_L$</td>
<td>Moment at lower end of column</td>
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<td>$M_{eU}$</td>
<td>External moment applied at upper joint</td>
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<tr>
<td>$M_{eL}$</td>
<td>External moment applied at lower joint</td>
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<td>$M_{rU}$</td>
<td>Restraining moment offered by upper restraint</td>
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<td>Horizontal force</td>
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### Table 1: Solution of Illustrative Example 1

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### TABLE 2 SOLUTION OF ILLUSTRATIVE EXAMPLE 2

RESTRAINED COLUMN SUBJECTED ONLY TO EXTERNAL MOMENTS (NO LATERAL FORCE)

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TABLE 3  SOLUTION OF ILLUSTRATIVE EXAMPLE 3

RESTRAINED COLUMN WITH SPECIFIED LATERAL RESTRAINT (NO LATERAL FORCE)

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<td>$\frac{M_e}{M_y}$</td>
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<td>$\gamma$</td>
<td>$Q_U$</td>
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**TABLE 4** SOLUTION OF ILLUSTRATIVE EXAMPLE 4

RESTRAINED COLUMN WITH ONE END PINNED AND Subjected TO LATERAL FORCE
FIG. 1 UNBRACED FRAME, SWAY SUBASSEMBLAGE AND RESTRAINED COLUMN
(a) Applied forces and moments.

(b) Forces and moments acting on joints and column.

(c) Angular references and rotations.

FIG. 2 A RESTRAINED COLUMN PERMITTED TO SWAY
FIG. 4 ANALYSIS OF A RESTRAINED COLUMN WITH SPECIFIED \((M_e)_L\) AND \(Q\) (EXAMPLE 1)
FIG. 5 SPECIAL TYPES OF RESTRAINED COLUMNS
FIG. 6  ANALYSIS OF A RESTRAINED COLUMN WITH NO LATERAL FORCE (EXAMPLE 2)
FIG. 7 ANALYSIS OF A RESTRAINED COLUMN WITH A SPECIFIED LATERAL RESTRAINT (EXAMPLE 3)

FIG. 8 COMPARISON OF MOMENT VS. SWAY RELATIONSHIPS OF A COLUMN WITH AND WITHOUT LATERAL RESTRAINT
FIG. 9 ANALYSIS OF A RESTRAINED COLUMN WITH ONE END PINNED AND SUBJECTED TO A LATERAL FORCE (EXAMPLE 4)

FIG. 10 ANALYSIS OF A RESTRAINED COLUMN WITH ONE END PINNED AND SUBJECTED TO A LATERAL FORCE AND AN EXTERNAL MOMENT