RESTRAINED COLUMNS

by

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ABSTRACT

This dissertation treats the problem of prismatic columns with non-translating ends. Transverse deflections are due to bending about one of the principal axes of the cross section. The columns are loaded only at their ends where rotational restraints may exist. These rotational restraints exert bending moments upon the column which tend to limit deformations and increase the carrying capacity of the column.

By considering the shapes (hereafter referred to as column deflection curves) which a compressed member would take if held in a bent configuration by axial loads applied to the ends, it is shown how solutions may be obtained for the carrying capacity of restrained columns. In the most general case discussed the compressive loads may be applied with equal or unequal end eccentricities and the rotational restraints may be equal or unequal functions of the end rotations. In these studies the stresses may exceed the proportional limit of the material.

By properly organizing the information supplied by the column deflection curves in nomographs, it is shown how a wide variety of restrained column solutions may be achieved graphically.

An "exact" elastic-plastic analysis is developed for symmetrical rigid frames loaded in a symmetrical manner. The analysis makes use of moment-rotation curves developed for columns. The frame analysis considers the non-linear effects of both the inelastic action and the deformations.
I. INTRODUCTION

I.1 THE COLUMN IN A CONTINUOUS FRAMEWORK

With the advent of welding there has been a steadily increasing trend towards the use of rigid joints to connect the beam and column elements of metal structures. The great advantage of this kind of construction lies in the fact that a weak element of the structure is restrained from failing prematurely by the stronger members to which it is rigidly connected. The collapse of a member in a continuous structure can only occur after it becomes impossible for the member to gain sufficient additional support from the neighboring elements to compensate for the increased load it is called upon to carry.

Unfortunately, the analysis for the failure load of even the simplest continuous structures often results in formidable problems. By and large these problems are connected with the behavior of the compression members of the structure. These members are usually compressed by loads that are carried to them by the beams. The beams, which are acted upon by transverse loads, flex and as a result they transmit to the columns at the earlier stages of loading both bending moments and axial loads. This causes the columns of the structure to bend from the very beginning of loading. As loading progresses the transverse deflections of the column are amplified by the secondary moments throughout the column length. These secondary moments, which
are referred to as deflection moments, are due to the compressive load of the column acting upon the transverse deflections. A stage in the loading may eventually be reached at which a column would collapse were it not for the restraining influence of the rest of the structure. At this time the rotations at the ends of the columns are prevented from increasing indefinitely by the beams and other members to which the columns are rigidly attached. These restraints increase the carrying capacity of the column so that additional loading of the structure is possible. Finally, at the collapse load (if over-all collapse is triggered by column failure), the restraints are no longer able to compensate for the increasing loads.

A complete analysis for the collapse load of a structure requires a knowledge of the structural behavior of its components. In the case of compressed members, this behavior is influenced to a marked extent by the deformations of the column and the non-linearity of the stress-strain relationship when the stresses exceed the proportional limit.

I.2 TYPES OF COLUMN FAILURES

A column may fail in a number of different ways and a complete investigation of column action must consider each of the following modes of deformation:

(1) Deformation of the column by bending about either one or both of the two principal axes of the cross section.
(2) Deformation of the column by twisting about the longitudinal axis passing through the shear center.

(3) Simultaneous twisting about a longitudinal axis and bending about both principal axes of the cross section.

In order to have bending about both principal axes of the cross section without twist about the longitudinal axes it is necessary that the thrust line of the column be parallel to the undeflected longitudinal axis and that it pass through the shear center.

When the shear center lies on one of the principal axis of the cross section, this axis together with a longitudinal axis passing through the centroid defines a plane. When the loads are applied in this plane the column will deform by bending about one of the principal axes of the cross section and each point on the column will move parallel to the plane in which the loads are applied. When the centroid and the shear center coincide there are two planes defined by the longitudinal centroidal axis and a principal axis containing the shear center. If the loads are applied in either one of these planes, the plane of loading and the plane of deformation will coincide.

Columns loaded in the manner described in the preceding paragraph may at a certain stage of the loading exhibit lateral-torsional buckling (deformation mode (3) described above). The factors that govern this tendency are the ratio of the bending stiffnesses about the principal axis, and the torsional stiffness of the column.
Deformation mode (2) associated with pure twist about a longitudinal axis of the column can only occur for a concentrically loaded column for which the centroid and shear center coincide. Such deformation occurs at the torsional buckling load of the column.

Deformation mode (3) will occur for all general cases of column loading not discussed above.

1.3 DEFINITION OF THE PROBLEMS TO BE STUDIED AND LIMITING ASSUMPTIONS

The present study will be concerned with the development of theory and procedures for the determination of the collapse load of columns which fail by excessive bending about one of the principal axes of the cross section. Restraining moments at the column ends may be linear functions of the end rotations of the column or they may vary in a non-linear manner with the end rotations. Although special attention will be given in the computations to rolled steel wide-flange columns bent about their major principal axis, the procedures will be applicable to any cross section and material. In the computations for the wide-flange sections a residual stress distribution will be assumed that has been found to be typical for as-milled wide-flange columns.

The column theory developed herein makes it possible to develop a procedure for the complete elasto-plastic analysis of simple frame structures subjected to symmetrical loading. This procedure which will be described, permits the computation of deformations and the
determination of the stiffnesses of the beam and column elements of
the structure at any stage of loading.

The study will be limited by the following assumptions:

(a) The columns are made of a ductile material which will sus-
tain the flexural strains without fracture.

(b) The columns have a constant cross sectional shape over
their entire length.

(c) The columns are initially straight.

(d) The deflections due to shear strain are small and can be
neglected.

(e) The entire transverse section of the column, originally
plane, remains plane and normal to the longitudinal fibers of the
column after bending.

(f) The deflections normal to the undeformed column axis are
small compared to the length of the column. The angle between the
tangent to the deflected column curve and the original direction of
the column is small so that its tangent and sine may be taken as
equal to the angle in radians. The cosine of this angle may be taken
as unity and the square of the tangent of this angle may be considered
to be a term of higher order than those considered in the analysis.

(g) The loads and restraints are applied only to the ends of
the column.

(h) The ends of the column are completely restrained from
translating.
(i) Strain reversal of material which is stressed beyond the proportional limit is neglected so that for each strain there corresponds but one stress as determined from a stress-strain curve of the virgin material. It will be shown subsequently that it is possible to consider residual stresses for a material such as steel under the assumption that it has an idealized elastic-plastic stress-strain diagram.

(j) Local instability (changes in the shape of the cross section) will be ignored.
II. THEORY AND APPROACHES
CURRENTLY AVAILABLE

II.1 BASIC CONCEPTS

The most general case of the beam column problem to be investigated is shown in Fig. 1. The compressive load $P$ is applied to each end with different eccentricities $e_A$ and $e_B$. In addition to the applied moments $P_e_A$ and $P_e_B$ there are restraining moments at the ends which depend upon the rotation characteristics of the supports. These moments are denoted by the symbols $f_A(\theta'_A)$ and $f_B(\theta'_B)$ to indicate that they are functions of the end rotations.

It is convenient in an analysis to assume that the column cross section and the material properties are specified. The remaining quantities which will determine the equilibrium configurations of the column are $f_A(\theta'_A)$, $f_B(\theta'_B)$, $P$, $e_A$, $e_B$, and $L$. The object of the analysis is to determine the relationship between the quantities enumerated at the limit of stable equilibrium (stability). To do this, all except one of the quantities are specified and equilibrium configurations are determined for various values of the open parameter. If the axial load $P$ is selected as the open parameter and increased gradually, the rotation $\theta'_A$ will vary with $P$ as shown in Fig. 2. The maximum value of $P$ for which an equilibrium configuration can be found is given by point A. If the column is further deformed as indicated by an increase in $\theta'_A$, the compressive force must be reduced to maintain equilibrium. Point A is therefore the demarcation
between stable and unstable equilibrium configurations of the beam column. This value of \( P \) together with the other quantities previously specified gives, therefore, one relationship of the type sought.

The example outlined above takes \( P \) for the open parameter. In other cases it might be more desirable to use \( L, e_A \), or even one of the restraining functions as the open parameter to obtain the desired relationship. This will lead to a curve similar to the one in Fig. 2 for the determination of the extremum value of the open parameter. Furthermore, since \( \theta_A' \) serves only to identify the equilibrium configuration corresponding to each value of the open parameter, it is possible to use other deformations of the bent column to serve the same purpose. It is thus possible to use the rotation at \( B \) or the maximum curvature in the beam column as the abscissa in Fig. 2. The maximum value of the open parameter will be the same no matter what deformation is chosen as the abscissa.

The consideration of a beam column problem in terms of the stability concepts outlined above is due to von Karman. In Ref. 2 he outlines an approach for the determination of the ultimate carrying capacity of unrestrained columns with equal applied end moments.

II.2 THE METHOD OF KARMAN AND CHWALLA

a. Moment-Curvature Relationships

All methods of determining the equilibrium configurations referred to in II.1 require that the moment-curvature relationship of the column be known. This relationship depends on the shape of
of the cross section, the average stress, and the material properties. A description of how an \( M-\Phi \) curve may be obtained for a given average stress is included in Section III.2a. With an \( M-\Phi \) diagram such as the one shown in Fig. 3 available, it is possible to determine the shape of a bent column by the numerical integration procedure described in Section III.2b.

**b. The Column Deflection Curve**

The column deflection curve (C.D.C.)\(^*\) is the shape that a column would take if it is held in a bent configuration by axial loads applied to the ends. Fig. 4 shows one complete wave of a C.D.C. The half wave length will be given by the Euler formula,

\[
\lambda = \pi \sqrt{\frac{EI}{P}}
\]

\[\text{ ..........(2.1)}\]

when the stresses do not exceed the proportional limit. In this case the \( M-\Phi \) relationship will be:

\[
y'' = \phi = \frac{M}{EI}
\]

\[\text{ ..........(2.2)}\]

and the differential equation for the C.D.C is,

\[
y'' = \frac{M}{EI} = -\frac{Py}{EI}
\]

\[\text{ ..........(2.3)}\]

For every value of \( \Theta_0 \) Eq. 2.3 may be integrated to give the shape of a C.D.C. All of these curves will be sine waves that differ only in their amplitude.

\* The abbreviation "C.D.C" will be used for column deflection curves.
As \( \Theta_o \) becomes larger, \( y_m \) also becomes larger until the stresses due to both compression and bending start to exceed the proportional-limit at and near points of greatest deflection on the C.D.C. The \( M-\Theta \) relationship at these sections is no longer given by Eq. 2.2 and it becomes necessary to go to a curve such as the one in Fig. 3 or to an analytical expression that takes into account the shape of the cross section, stress-strain diagram, and average stress. Numerical integration procedures for determining the shape of the C.D.C. under these conditions are outlined in Section III.2b. As \( \Theta_o \) increases, \( y_m \) increases and \( \Theta \) decreases.

It is apparent from the description of the C.D.C. that an infinite number of such curves are possible for a given column cross section, stress-strain diagram, and average compressive stress. Each of these may be conveniently identified by \( \Theta_o \) or \( y_m \).

Chwalla\(^{(4)}\) noticed that a real beam-column in equilibrium, with the end moments and compressive force applied to it, was really a segment of a C.D.C. This may be seen by a consideration of the beam-column of Fig. 5 and the series of C.D.C.s in Fig. 6. Let \( y_A = \frac{M_A}{P} \) and \( y_B = \frac{M_B}{P} \). On each of the C.D.C.s of Fig. 6 horizontal lines are drawn at the same distances \( y_A \) and \( y_B \) from the x-axis. The distance between the intercepts on the C.D.C.s are \( L_n \). In Fig. 7a a segment of a C.D.C. is drawn for which \( L_n \) equals the length of the column in Fig. 5. Figure 7b shows a free body diagram of the same segment rotated through the angle \( \alpha = \frac{y_A-y_B}{L} \). It is evident that the column segment of Fig. 7b and the beam column of Fig. 5 have the
same forces and moments acting upon them within the assumption of small deformations made in 1.2. Since the segment of Fig. 7b is in equilibrium it follows that it represents an equilibrium configuration of the column in Fig. 5.

From the von Karman concept of stability it follows that two equilibrium configurations are possible for every condition of loading up to the one corresponding to A in Fig. 2. It therefore follows that except for this condition of loading, two C.D.Cs. may be found to yield segments of length L in the manner described above. These two segments correspond to the two equilibrium configurations possible for the column of Fig. 5.

c. Column Solutions

To obtain solutions for unrestrained axially loaded columns with various end moments, Chwalla(5) took the length L as the open parameter and determined $L_n$ for several C.D.Cs. These were plotted against the $y_m$ of the corresponding C.D.C. to obtain a curve as in Fig. 8. Point A on this curve gives the required relationship at the limit of stability.

Chwalla also used these C.D.Cs. to obtain solutions for restrained columns with equal applied end moments and equal elastic rotational restraints.(5) Computations were made for an average compressive stress of 21,300 psi. The material was structural steel and the cross section was rectangular.
The restrained column solutions were obtained in the following manner: The restrained column is as shown in Fig. 9. The column under consideration extends from A to B. The moment at A or B is given by:

\[ M = -Pe + k\theta' \]  \hspace{1cm} \text{.........(2.4)}

The term k is the stiffness of the rotational restraints at the ends of column segment A-B due to the end spans. On a C.D.C. it is possible to determine \( L_n/2 \) by trial (Fig. 10) so that the internal moment \((P \cdot y_{A'})\) at A' is equal to the external moment given by Eq. 2,4. The angle \( \theta' \) in Eq. 2.4 is the slope at A' in Fig. 10. Several values of the open parameter \( L_n/2 \) are determined in this manner from several C.D.C's. When these are plotted against the corresponding values of \( y_m \) a curve similar to Fig. 8 is obtained giving the length at the limit of stable equilibrium.

II.3 THE METHOD OF BAKER, HORNE, AND RODERICK

a. Introductory Remarks

The essential difference between this method and the approach used by Chwalla(5) in solving a restrained column problem is in the choice of the open parameter. While Chwalla uses the length, the method of this section (method of Baker, et.al.) uses the average compressive stress as the open parameter. A second difference is in the way in which the equilibrium configurations of the bent column are determined. Von Karman and Chwalla used a numerical integration
procedure while the procedure of Ref. 7 expresses the $M$-$\phi$ relationships analytically and then proceeds to integrate them to arrive at the shape of the bent column.

### b. Mathematical Expressions for the $M$-$\phi$ Relationships

It has been previously stated that the $M$-$\phi$ diagram depends upon the shape of the cross section, the stress-strain diagram, and the average compressive stress. Obviously the stress-strain diagram and the cross section can not be too complicated if convenient analytical expressions are to be derived for the $M$-$\phi$ diagram. Mild grade structural steel of the type used in bridges and buildings has such an uncomplicated stress-strain diagram. It can be accurately represented in the region before strain hardening begins by two straight lines as shown in Fig. 11a or by the diagram in Fig. 11b which takes into account an upper and a lower yield level. Jezek found that the diagram of Fig. 11a gave substantially the same results when applied to unrestrained beam-column problems as those obtained with the use of an exact stress-strain diagram of a typical structural steel. (6)

A rectangular cross section and a stress-strain diagram like the one in Fig. 11a were used in the computations of Baker, Horne, and Roderick (7) in order to arrive at expressions relating moment to curvature. The equations as given in Ref. 7 and 9 must cover three cases of stress distribution over the cross section. [In Fig. 12a the stresses are entirely elastic and the $M$-$\phi$ relation is given by]
Eq. 2.2. Figure 12b shows a penetration of the yield stresses on the compression side. The equation in this case is:

\[ \phi = y'' = \frac{\sigma_y (P_y - P)}{E D P_y} \cdot \frac{1}{\xi_1^2} \] ...............(2.5)

where

\[ \xi_1^2 = \left[ \frac{3}{2} - \left( \frac{M}{b D^2} \right) \left( \frac{3}{4\sigma_y} \right) \left( \frac{P_y}{P_y - P} \right) \right]^2 \]

In Fig. 12c the curvature has become severe enough to cause compressive yielding on one side and tensile yielding on the other side. An elastic core remains between the two yield areas. The \( M-\phi \) relation in this case is:

\[ \phi = y'' = \frac{\sigma_y}{E D \xi_2} \] ...............(2.6)

where

\[ \xi_2 = \sqrt{\left\{ 1 - \left( I / P_y \right)^2 - \frac{M}{b D^2 \sigma_y} \right\}} \]

\( c. \) Procedures

In Ref. 7 a few solutions are obtained by this method for the case of equal applied end moments and equal elastic rotational restraints (Fig. 13a). The forces \( Q \) are assumed to be applied to the beams first and then an additional force \( F \) is added to the top of the column until the structure collapses. At several values of the load \( F \), the equilibrium shape of the column is determined.

Determining the equilibrium shape involves a trial and error
procedure. First a value for the slope at A and B is assumed. The axial load in the column and the bending moment at A can then be determined from the condition of elastic action in the beams. With the known slope and the computed bending moment at A, it is possible to determine the shape of the column. In the determination of the shape, Eqs. 2.5 and 2.6 are used for the appropriate stress distribution. If the slope at A is assumed correctly then the tangent at the midpoint of the bent column will be vertical. If this is not so then the process is repeated. The shape giving a vertical tangent at the midpoint represents an equilibrium shape for the load F.

After equilibrium shapes are determined for several values of F, a curve of F versus the center deflection of the column may be drawn. Point A on this curve (Fig. 13b) gives the ultimate load that can be superimposed on the given frame.
III. DEVELOPMENT OF GENERAL AND PARTICULAR PROCEDURES

III.1 INTRODUCTION

It is evident from the discussion in Chapter II that solutions to restrained column problems present considerable computational difficulties. The theoretical portion of the solution follows from the von Karman concept which considers that the limit of stability has been reached when an equilibrium configuration is no longer possible for the column. It is therefore necessary to fix all but one of the several variables that may be involved and to investigate several equilibrium configurations. Each will require a different value of the open parameter. The extreme value found in this manner gives the relation between the several variables (including the open parameter) at the limit of stability.

Of the variables involved in a restrained column problem, the following are suitable for use as the open parameter:

1. The average compressive stress.
2. The column length.
3. The eccentricity with which the applied load acts at either end.
4. The restraining moment at either end as a function of the rotation at the same end of the column.
It does not seem at all suitable to use either the shape of the column cross section or the shape of the stress-strain diagram as the open parameter.

In this connection it is noticed that the method of Karman and Chwalla \((2,4,5)\) uses the column length as the open parameter while in Ref. 8, Baker, Horne, and Roderick uses the average compressive stress. In Ref. 9, Horne again uses the compressive stress as the open parameter. In an approximate method developed by Bijlaard \((10,11)\) the average stress is the open parameter.

It is desirable to identify the several equilibrium configurations for the analysis. Since every column bends into a shape which is equivalent to the shape of a particular segment of a particular column deflection curve (Section II.2b), it is possible to use the column deflection curve (which in turn is identifiable by either its maximum deflection, maximum curvature, or slope at the point of zero deflection) as a means of identifying the shape of the column in equilibrium with the external loads. The particular segment of a column deflection curve which is equivalent to the bent column can be easily found since the moments at the ends of the column must equal the moments at the matching points of the column deflection curve. In what follows, a particular column deflection curve is identified by the compressive force (non-dimensionalized as \(P/P_y\)) and the slope at the point of zero deflection \(\Theta_o\).

The column deflection curve has been described in II.2b as the shape of a prismatic member held in a bent configuration by
purely axial loads applied to the ends. If the sum of the bending stress and the average stress is below the proportional limit, the half wave length is given by Eq. 2.1. For larger amplitudes of the wave, combined stresses due to axial force and bending moment will exceed the proportional limit at all sections where:

\[ y = \left( \sigma_{\text{p, f}} - \frac{P}{A} \right) \frac{S}{P} \quad \ldots \ldots (3.1) \]

In the equation above \( y \) is the deflection, \( \sigma_{\text{p, f}} \) is the stress at the proportional limit, \( S \) is the section modulus, \( A \) is the area, and \( P \) is the compressive load.

The solutions to be developed in this thesis depend upon the availability of numerous column deflection curves and the systematic interpretation of the information they contain. A detailed description of numerical integration procedures for the construction of the curves is therefore included in Section III.2. Procedures that depend upon the integration of algebraic terms to accomplish the same have not been used because of the excessive complications that are encountered even for the simplest cross sections and stress-strain diagrams (see Section II.3b). In contrast, the numerical integration procedures may be easily applied to any stress-strain diagram and cross section.

III.2 CONSTRUCTION OF THE COLUMN DEFLECTION CURVE BY NUMERICAL INTEGRATION

a. Determination of the M-\( \theta \) Diagrams

The virgin stress-strain diagram of the material considered is
shown diagrammatically in Fig. 14b. Above it in Fig. 14a the column cross section is drawn to scale so that its depth extends from a value of the strain $\varepsilon_2$ to a trial value of $\varepsilon_1$ on the diagram of Fig. 14b. Since the strain distribution across the section is assumed linear over the depth, the strain at any point of the cross section may be found by descending vertically to Fig. 14b and reading the value on the abscissa. The stress is given by the corresponding ordinate of the stress-strain diagram. In Fig. 14c a diagram of $\sigma$ vs $b$ (b is the width of the column cross section at the point under consideration) gives the compressive force per unit length of depth of the column cross section.

The diagram of Fig. 14c may be integrated by mechanical (planimeters) or numerical means to give the total axial load when the strains at the convex and concave sides are $\varepsilon_2$ and $\varepsilon_1$ respectively. If this load differs from the load for which the $M$-$\Phi$ diagram is desired, a new trial value of $\varepsilon_1$ is assumed while keeping $\varepsilon_2$ the same. Eventually this leads to the desired value of $P$.

When the proper value of $\varepsilon_1$ has been thus determined another integration is performed on Fig. 14c to determine the first moment of the $\sigma$-$b$ diagram about the axis of bending through the centroid of the cross section ($0$-$0'$ in Fig. 14a). This gives the bending moment for the given axial load and outer fiber strains $\varepsilon_1$ and $\varepsilon_2$. The corresponding curvature is $\frac{\varepsilon_1 - \varepsilon_2}{d}$.

By repeating the above procedure for several values of $\varepsilon_2$
a complete $M-\theta$ diagram may be obtained. In a similar manner $M-\theta$ curves are obtainable for other values of $P$. Figure 15 shows the general shape that these curves take.

The fact that only one moment is associated with each value of the curvature is a direct consequence of using a stress-strain diagram that gives but one value for the stress for each value of the strain.

A detailed procedure for computing $M-\theta$ diagrams of steel wide-flange sections with residual stresses is contained in Ref. 12.

b. Numerical Integration

An initial slope $\theta_0$ is chosen for the column deflection curve to be constructed. The axial load $P$, the cross section, and the stress-strain diagram have been specified so that the $M-\theta$ relationship is computable according to the method outlined in Section III.2a. The increment of length chosen for the integration will be designated by $\rho$ (Fig. 16). In the equations that follow it will be taken into consideration that the slopes of the column deflection curve are always small so that the cosines of the slope angles can be taken as equal to unity. The sines and tangents of these angles may also be taken as equal to the angles measured in radians.

Figure 16 shows how the deflection is obtained at the end of the first segment. The length of this segment is designated by $\rho_1$. Over this segment the radius of curvature is assumed constant and
equal to \(1/\phi_1\) where \(\phi_1\) is a mean curvature for the segment. Since curvatures are obtained from the appropriate \(M-\phi\) curve, it is necessary to first determine a mean value of the bending moment moment for the segment. For the first segment this was taken as \(P \cdot \theta_0 \cdot \rho_1 \cdot \frac{1}{2}\).

The vertical deflection at the end of the first segment is:

\[
y_1 = \rho_1 \theta_0 - \frac{\rho_1^2 \phi_1}{2}
\]

.........(3.2)

The slope at the end of the first segment is:

\[
\theta_1 = \theta_0 - \rho_1 \phi_1
\]

.........(3.3)

The slope and deflection at the end of the second segment are determined in a similar manner. The mean moment for this segment is taken as \(P \left[y_1 + \theta_1 \cdot \rho_2 \cdot \frac{1}{2}\right]\). The \(M-\phi\) curve gives the mean value of the curvature for this segment which is designated as \(\phi_2\).

The deflection at the end of the second segment is:

\[
y_2 = y_1 + \theta_1 \rho_2 - \frac{\rho_2^2 \phi_2}{2}
\]

.........(3.4)

The slope at the end of the second segment is:

\[
\theta_2 = \theta_1 - \rho_2 \phi_2
\]

.........(3.5)

In the manner described above the deflections and slopes are obtained at numerous discrete points along the column deflection curve. The procedure described in the previous two paragraphs is
continued until the slope $\theta_n$ becomes equal to zero. This corresponds to the quarter wave point of the column deflection curve. The entire C.D.C. may be reproduced from only a quarter wave length because of the symmetrical nature of the wave. The bending moment at any point on the column deflection curve is equal to the product of the compressive load $P$ and the deflection $y$ of that point.

Appendix VII.4 shows a table set up for the computation of a C.D.C. Columns 1, 2, and 3 of any row give the station, deflection, and slope of the tangent to the C.D.C. Column 4 when added to column 1 gives column 5, the average deflection of the next segment. Column 6 is obtained from column 5 by multiplying the axial load and dividing by $M_y$. Column 7 is obtained by going into the $M-\theta$ curves with the value from column 6. Multiplying column 7 by $\theta_y$ gives the average curvature of the segment (column 8). Column 9 is the deviation from the tangent (Fig. 16). Column 10 gives the additional displacement at the end of the next segment had the curve followed the tangent. The actual deflection at the end of the next segment ($y_{n+1}$) is found in the succeeding row and is equal to the sum of the values in columns 2 and 10 of the preceding row minus the value in column 9 of the preceding row. Column 11 is the product of the segment length and the average curvature appearing in column 8. It is the change in slope over the segment. The slope at the next station appears in column 2 of the next row and is determined by subtracting column 11 of the preceding row from the slope as given in column 2 of the preceding row. This process is continued until the slope of the tangent becomes horizontal at the quarter wave point.
The accuracy with which the column deflection curve is determined depends largely on the length of the increments used for the numerical integration. To obtain a measure of the errors involved from this source, comparisons were made with the results from analytical determinations of the column deflection curve. The cross section used for this purpose was the 8WF31 rolled steel section. The modulus of elasticity was taken as 30,000 ksi. Segment lengths were four times the length of the radius of gyration. Since the only analytical solution available is not applicable for stresses above the proportional limit, the values of P/A and θ₀ were chosen to insure elastic action. The axis of bending was the major axis of the cross section. The theoretical value for the half wave length is given by Eq. 2.1 and the theoretical shape of the column deflection curve for elastic action is a sine wave. It follows that the maximum deflection of the wave is given by:

\[ y_m = \int_{0}^{\pi} \frac{\theta_0}{\pi} \]  

\[ ........(3.6) \]

Table I gives a comparison for the amplitudes and half wave lengths of the column deflection curves:

<table>
<thead>
<tr>
<th>P/A (psi)</th>
<th>θ₀ (radians)</th>
<th>L/r Numerical Integration</th>
<th>Analytical</th>
<th>y_m Numerical Integration</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>3960</td>
<td>0.01</td>
<td>273.22</td>
<td>273.44</td>
<td>3.0185</td>
<td>3.0202</td>
</tr>
<tr>
<td>3960</td>
<td>0.02</td>
<td>273.28</td>
<td>273.44</td>
<td>6.0378</td>
<td>6.0405</td>
</tr>
<tr>
<td>6600</td>
<td>0.02</td>
<td>211.70</td>
<td>211.81</td>
<td>4.6757</td>
<td>4.6790</td>
</tr>
<tr>
<td>9900</td>
<td>0.02</td>
<td>172.80</td>
<td>172.94</td>
<td>3.8152</td>
<td>3.8203</td>
</tr>
</tbody>
</table>
The errors in the length vary between 0.05% and 0.08% while the
errors in the amplitudes vary from 0.04% to 0.13%. Errors in this
range are quite insignificant and are smaller than those introduced
by the assumptions of Section 1.2.

III.3 SOLUTION FOR THE GENERAL CASE OF RESTRAINED COLUMNS*

Figure 1 shows the most general case of a restrained column.
Let is be assumed that the following quantities are specified. P, L,
f_A(θ'), f_B(θ'), e_A/e_B, the stress-strain diagram, and the cross
section of the column. It is required to determine the maximum
value e_A may have for which an equilibrium configuration of the
column can be found.

Several column deflection curves available for the given axial
load and column section will be examined to determine the location
of the segment of length L which makes e_A/e_B equal to the prescribed
value. Figure 17 illustrates the procedure for the location of the
correct segment on a column deflection curve. End A is located at
a trial distance x_A from the end of a column deflection curve and
end B is found on the curve at a horizontal distance L from A. The
angles θ_A and θ_B are known at any section of the column deflection
curve from the numerical integration used to determine the curve.
The angles θ_A', θ_B' and the eccentricities e_A, e_B are given by the
following equations:

* The material in this section appears in Ref. 15 and is in essentially
the same form. It is included here for completeness and continuity.
\[ \theta'_A = \theta_A - \alpha \quad \ldots \ldots (3.6) \]
\[ \theta'_B = \theta_B + \alpha \quad \ldots \ldots (3.7) \]
\[ \alpha = \frac{y_A - y_B}{L} \quad \ldots \ldots (3.8) \]
\[ e_A = f\left( \theta'_A \right) - Py_A \frac{A}{A'} \quad \ldots \ldots (3.9) \]
\[ e_B = f\left( \theta'_B \right) - Py_B \frac{B}{B'} \quad \ldots \ldots (3.10) \]
\[ e_A/e_B = f\left( \theta'_A \right) - Py_A \frac{A}{A'} \frac{B}{B'} \quad \ldots \ldots (3.11) \]

Values for \( \theta'_A, \theta'_B, \alpha, y_A, y_B, e_A, \) and \( e_B \) are positive when they are as shown in Figs. 17 and 1. The bending is positive when the upper fibers of the column in Fig. 17 are in tension.

Equations 3.9 and 3.10 stem from the requirement that at A and B the external moment must equal the internal moment. The internal moments at A and B are \( P_yA \) and \( P_yB \) respectively as can be verified from the column deflection curve (Fig. 17). By plotting \( e_A/e_B \) vs. \( x_A \) as in (Fig. 18) it is possible to arrive graphically at a value of \( x_A \) which locates the segment of the column deflection curve that represents an equilibrium shape for the given column when the ratio of eccentricities is specified.

For the segment yielding the correct ratio of \( e_A/e_B \) the value of \( e_A \) is determined from Eq. 3.9. Several values of \( e_A \) are computed
in this manner from several column deflection curves. These values are plotted as in Fig. 19 against the values of $\theta_p$ for the corresponding column deflection curves. The horizontal tangent gives the maximum value of $e_A$ consistent with an equilibrium configuration. This value together with the other parameters that were initially specified is the solution to the stability problem. Appendix 3 contains an illustrative problem of this kind.

III.4 SOLUTIONS FOR COLUMNS WITH ONE END PINNED*

The special case shown in Fig. 20a is well suited to rapid solution by means of nomographs. The pin end of the column must always correspond to the origin of the column deflection curve (Fig. 20b). The end rotation $\theta_B'$ at the restrained end of the column of length $L$ is given by:

$$\theta_B' = \theta' = \theta - y/L \quad \cdots \cdots \cdots (3.12)$$

The internal moment at the restrained end of the column is $P \cdot y$ (Fig. 20b) while the external moment is:

$$M = f_B(\theta_B') - P \cdot e_B \quad \cdots \cdots \cdots (3.13)$$

The way in which the nomograph is constructed may best be understood by a consideration of Fig. 21. It shows how the information from a single column deflection curve is organized. In the upper section the internal moment at a point $B'$ is plotted against the value of

* The material appearing in this section appears in Ref. 15 and is in essentially the same form. It is included here for completeness and continuity.
\[ \theta' \] at that point. In the lower section the horizontal distance \[ L \] from the origin of the column deflection curve is plotted against \[ \theta' \]. Coordinates of typical points \[ B' \] are shown in the upper and lower portions of the nomograph. Each column deflection curve gives a pair of curves on the nomograph. A nomograph for an 8WF31 steel column is shown in Fig. 22. The M-\[ \theta \] diagram used in the construction of the column deflection curves was taken from the diagrams developed in Ref. 12. The stress-strain diagram used in Ref. 12 is similar to Fig. 11a with \[ \sigma_y \] equal to 33 ksi and \[ E \] equal to 30,000 ksi. A cooling residual stress pattern as shown in Fig. 23 was also assumed to be present. While the nomograph is strictly applicable to the 8WF31 section only, it has been non-dimensionalized so that use can be made of it for all wide-flange rolled steel sections normally used as columns. When the nomograph is used for sections other than the 8WF31, the error will be small because the distribution of the areas for all wide-flange column sections about the neutral axis is similar. In fact the results based upon an 8WF31 section nearly always will give conservative values for the strength of a column because this section has one of the more unfavorable thrust-moment-curvature relationships of the wide-flange column sections rolled. It should be noted that all numerical results given in the appendices are strictly applicable only to the 8WF31 section described above.

The maximum length is the most convenient open parameter in achieving a solution by nomographs for a restrained pin ended column (Fig. 20a). For an equilibrium configuration of the column,
the internal moment at the restrained end must equal the external moment due to the eccentrically applied load and the restraining moment.

\[ M_y = f_B(\theta_B') - P e_B \tag{3.14} \]

The right hand side of Eq. 3.14 is plotted as a line on the upper portion of the nomograph (Fig. 24). Since in general \( f_B(0) \) is equal to zero (the restraining moment disappears when end B has a zero rotation) the line will intersect the moment axis at \( M/M_y \) equal to \(-P e_B/M_y\). With increasing values of the rotation at B, the line tends upward since the restraining moment grows with the rotation. Wherever this line intersects one of the upper nomograph curves, Eq. 3.14 is satisfied and an equilibrium configuration has been found \( (\theta_B' = \theta') \). By carrying the intersections to the lower portion of the diagram to the corresponding curves, the equilibrium lengths for each configuration are determined. The largest \( L/r \) found gives the solution. Several illustrative problems of this nature are considered in Appendix 3b.

When the moment restraint at end B disappears the problem reduces to one of a pin ended column with a moment applied at one end. In this case it is convenient to use the maximum value of the end moment at B \( (P e_B) \) as the open parameter. Figure 25 shows how this is accomplished. A horizontal line in the lower portion of the nomograph is drawn at the specified slenderness ratio. Intersections with the curves of the nomograph are carried up to the corresponding
nomograph curves in the upper portion. By connecting these points in the upper portion one obtains the relationship between the end moment and end rotation at B for each equilibrium configuration. This is known as the M-Θ' curve. It gives the relationship up to and beyond the ultimate capacity of the beam column. Appendix 2 gives such curves for several values of P/Ey and L/r.

Another approach to the restrained case would determine the minimum restraint for stability. This approach is only convenient when fB is a linear function of ΘB. Figure 26 shows the construction. A straight line is drawn tangent to the M-Θ' curve of the column and intersecting the vertical axis at -P·eB. The slope of this line gives the minimum stiffness of the end restraint necessary for stability.

If eB were the open parameter, the maximum value can be obtained by drawing a line parallel to the restraint curve and tangent to the M-Θ' curve of the column (Fig. 27). Its intersection with the vertical axis determines the maximum value of eB.

Finally it is possible to use the load P as the open parameter as Horne did in Ref. 9. It will be recalled (Section II.1) that if the compressive load is plotted against a suitable deformation function that a curve such as the one in Fig. 2 is obtained and that for each P smaller than the P at the limit of stability two values of the deformation function will be found. For the present case let the deformation function be ΘB'. Since L, eB, and fB(Θ) are defined, the line representing the external moment will intersect
the M-θ' curve at two points when P is smaller than the load P at the limit of stability (Fig. 28). By going to nomographs for successively higher values of P/Py and repeating the process it is possible to arrive at a condition wherein the line of external moment at end B will just be tangent to the M-θ' curve as in Fig. 27. This is the value of P at the limit of stability. There is but one equilibrium configuration for this value of P. This method is more cumbersome than the others but it may prove useful if it is desired to compare theory with the usual column test results where the eccentricity is kept constant and the load is gradually increased.

III.5 SOLUTIONS FOR COLUMNS SYMMETRICALLY LOADED AND SYMMETRICALLY RESTRAINED* 

Nomographs from which a wide variety of solutions may be obtained are possible for the symmetric case. In addition the same nomographs will provide solutions for the general case of unrestrained columns. For symmetric cases it will be noticed that the center of the column will always correspond to a point of maximum amplitude of a column deflection curve.

Figure 29a shows a symmetrically loaded and symmetrically restrained column while Fig. 29b shows the column deflection curve that includes the column of Fig. 29a. The segment of the column deflection curve A'-B' corresponds to the restrained column A-B. At point B or A in Fig. 29a the equality of internal and external moments requires that:

---

*The material appearing in this section appears in Ref. 15 and is in essentially the same form. It is included here for completeness and continuity.
\[ P.y = f(\theta') - P.e \]  
\[ \ldots \ldots \text{(3.15)} \]

Because of the symmetry, the end rotation of the column will be equal to the slope \( \theta \) at \( A' \) or \( B' \) on the column deflection curve.

Figure 30 shows how the information from one column deflection curve is plotted for the nomograph. In the lower portion the vertical axis gives the half length of the column. The complete nomograph (diagrammatically shown in Fig. 31) for a particular value of \( P/Py \) is used in the same manner as a nomograph described in Section III.5. It is therefore possible to use \( L, f(\theta), e, \) or \( P \) as the open parameter for the solution of a symmetrically restrained column. Nomographs for several values of \( P/Py \) and for the 8WF31 steel section described in Section III.5 are given in Appendix 1. Appendix 2 contains \( M-\theta \) diagrams for several values of \( P/Py \). Appendix 3c contains an illustrative problem of the kind considered in this section.

III.6 **SOLUTIONS FOR THE GENERAL CASE OF UNRESTRAINED COLUMNS**

The nomographs described in Section III.5 may also be used to give a rapid graphical solution for the case of an unrestrained column with different end moments. Figure 32 shows such a column. The maximum length is to be determined at the limit of stability. The quantities \( M_A, M_B, \) and \( P \) are the specified parameters in this study.
Before proceeding with the details of the solution, it is well to consider a particular column deflection curve and to determine the length (or lengths) consistent with $M_A$, $M_B$, and $P$. In Fig. 33a it is seen that three equilibrium lengths $L$ may be obtained (lengths $L_3$, $L_2$, and $L_1$). Since $M_B$ is the numerically smaller of the two end moments, it follows that $L_3$ is larger than the half wave length of the column deflection curve. Cases of this sort always represent unstable equilibrium configurations. This is so because the deflections would be predominantly in the direction in which the smaller of the two end moments tends to force the column (Fig. 33b). This would result in an unwinding tendency of the column and consequently $L_3$ will not enter into consideration for the maximum value of $L$ consistent with equilibrium. Conversely, $L_2$ does not tend to unwind and therefore represents another equilibrium configuration (Fig. 33c) that must be considered along with $L_1$. Since $L_2$ is always equal to or larger than $L_1$ for each C.D.C., it follows that $L_2$ is the only length of interest for each column deflection curve. The above statement follows from the approach which consists of examining all possible stable equilibrium configurations with the object of determining the one that yields the extremum value for the open parameter (Section II.1).

The length $L_2$ is seen to be equal to $L_A$ plus $L_B'$ from Fig. 33a. On the partial nomograph of Fig. 34a the value of $L_A$, $L_B$, $L_A'$ and $L_B'$ are shown for a particular column deflection curve. The determination of the maximum value of $L$ at the limit of stability involves the determination of $L_2$ for several column deflection curves and then
plotting them against the corresponding values of $\theta_0$ as shown in Fig. 34b. The required value of $L$ is the point of horizontal tangency of the curve. Appendix 3d contains an illustrative problem of the kind considered in this section.

### III.7 SOLUTIONS FOR COLUMNS WITH ONE END FIXED

This special case which is shown in Fig. 35 may also be treated by nomographs. It will be noticed that the actual deformed shape of the loaded column must again be a segment of a column deflection curve. A line drawn from $A$ to $B$ in Fig. 35 must be tangent to the deflected column at the fixed end $B$. This means that the segments of a column deflection curve that are admissible as possible deflected column shapes must be such that the secant line joining the ends is tangent to the column deflection curve at least at one extremity. Several admissible segments are indicated on a column deflection curve in Fig. 36.

Nomographs for different values of $P/P_y$ will follow the same pattern as the nomographs for the column with one end pinned. The lengths $L_1, L_2, L_3 \ldots$ will be plotted against $\theta_1', \theta_2', \theta_3', \ldots$ in the lower portion while in the upper portion the moments at $A_1', A_2', A_3', \ldots$ are plotted against the $\theta'$ rotations. Figure 37 shows a typical pair of curves obtained in this manner for a particular $\theta_0$ and $P$ value.

At end $A$ of the column of Fig. 35, equilibrium requires:
The use of the nomograph is based upon Eq. 3.16 and follows all the procedures outlined for the pin-ended column in Section III.4.

III.8 VERIFICATION OF THE THEORY

There are no test results available at present for restrained steel wide-flange columns bent about the major axis. It is therefore necessary to turn for the present to the limited verification that is available from tests of unrestrained beam columns.

In Ref. 14 the results of a theoretical study of unrestrained beam columns is presented. The method of this reference (Ref. 14) gives the maximum end moments that can be applied to compressed columns. The type of failure assumed is identical to the type assumed in this dissertation, that is, failure by excessive bending about a single axis. Reference 14 gives extensive results which can be compared with the cases reported in Appendix VII.2. The tops of the curves in Appendix VII.2 give values which are in practically all cases identical to the results given in Ref. 14. Since Ref. 14 finds that its theoretical results are in good agreement with test results, it follows that the present theory gives good results at least for this limited application of the theory.

A further verification is shown in Fig. 38 wherein an experimental moment-rotation curve obtained at the Fritz Engineering Institute is compared with the theoretical curve.
Laboratory is compared with theoretical curves obtained by the methods developed in this dissertation. It should be noted that whereas the theory of Ref. 14 gives only the curve up to the maximum moment capacity of the compressed column, the present theory can give the curve beyond this maximum value of the moment.

111.9 APPLICATION OF THE THEORY TO DESIGN PROCEDURES

It has been pointed out in Section 111.1 that a column in a continuous framework does not behave as an isolated member. Although it has been generally assumed in this chapter that the rotational restraints are known, the restraints in an actual structure cannot be readily determined. For this reason it is felt that a design method suitable for the practicing engineer must sacrifice exactness and to some extent economy of material in order to achieve a method suitable for rapid computations.

At the present time a method is being developed at the Fritz Laboratory that takes advantage of the economy due to the restraints the beams impose on the columns. The method is based upon moment-rotation curves such as the ones in Appendix VII.2 which indicate how far the ends of a member can rotate and still sustain the compressive load. As the ends of the column rotate it is recognized that the moments a column must sustain may be substantially reduced.
IV. THE ELASTIC-PLASTIC ANALYSIS OF SIMPLE RIGID FRAMES

IV.1 INTRODUCTION

One of the assumptions in conventional analyses of indeterminate structures is that equilibrium is formulated on the basis of the undeformed structure. When the internal moments, shear, and axial loads are to be determined at some section within the structure, a cut is assumed to pass through the section in question so that it separates the structure into two parts. The equilibrium equations for internal and external forces on one of the parts may then be used to obtain the internal forces at the cut section. When formulating the equilibrium equations in a conventional analysis the approximation is made that the structure has not deformed. In order that all of the traditional methods such as slope-deflection, moment distribution, column analogy, and the method of consistent deformations be applicable, it is first necessary to make the foregoing simplification. Unfortunately, the determination of the buckling load for a rigid frame structure with primary moments can only proceed from a consideration of the deformed structure and a more exact solution is therefore necessary for this type of problem.\(^{(3)}\)

Another requirement of the methods used in conventional analyses of indeterminate structures is that the stress be linearly proportional to strain. When this is not the case the deformations no longer are proportional to the loads and it is no longer possible
to use the principle of superposition upon which all of the classical methods of indeterminate analysis are based.

The purpose of this chapter is to develop an exact theory for certain symmetrical structures loaded in a symmetrical manner. This theory is made practicable by the systematic investigation of columns as outlined in Chapter III. The method proposes to consider both the exact shape of the loaded structure and non-linearity of the stress-strain relation. For any magnitude of the loading it will be possible to determine the bending stiffnesses of the beam and column elements of the structure. These frame properties are required for a rigorous solution to the sidesway frame buckling problem in the plastic range (3).

Some examples of structures that may be analyzed by the methods of this chapter are shown in Fig. 39. The frame of Fig. 39a may be analyzed for the cases of fixed or pin ends and for either a uniform load, concentrated loads symmetrically placed, or for the simultaneous application of uniform and symmetrically placed concentrated loads (see Fig. 39a). The present analysis assumes that the deformations are symmetric. Figure 39b shows a symmetrical box frame that may be analyzed in a similar manner provided the loads are as indicated in the figure.

Rather than treat each combination of structure and loading as a separate case, only one case will be discussed. The principles behind the analysis are the same for all combinations of symmetrical structure and loading. The minor changes to accommodate different
cases will be obvious after a thorough knowledge of the nomographs explained in Chapter III is attained. The case that will receive special attention is shown in Fig. 40.

IV.2 ANALYSIS OF A SYMMETRICAL PIN ENDED FRAME WITH A UNIFORM LOAD

The final deformed shape of the structure under consideration is shown in Fig. 41. The loading consists of the specified distributed load \( w \). Members are prismatic with the cross sectional properties specified. It is assumed that nomographs as described in Section III.4 are available for the column with one end pinned for the actual column section used in the frame. It is required to determine the horizontal thrust \( H \), the transverse displacement of any point on the frame, and the bending moment, thrust, and shear at any section of the frame. The rotational stiffness at \( B \) for the column member \( BA \), and the rotational stiffness at \( B \) and \( B' \) for the beam member \( BB' \) are also required. The last two requirements were included because they are necessary for the computation of the sidesway buckling load for the frame. It will be shown that the column stiffnesses are directly obtainable from the nomograph mentioned above. The beam stiffnesses may be determined once the state of stress is known for the beam element.

In Fig. 41 a value of \( \theta' \) and a value of the bending moment at point \( B \) of the loaded frame may be assumed. If the assumed values are correct, then a numerical integration procedure starting from \( B \) may be employed to develop the shape of the column \( BA \). When starting the
integration at B it is noticed that the shear is $M_B/L_1$ and that the compressive load is $WL_2/2$. At A the integration (similar to the integration procedure described in III.2) will yield a zero transverse deflection. It will be recalled from the discussion of III.4 and Fig. 25 that there are an infinite number of combinations of $\theta'$ and $M_B$ that will satisfy the requirement of zero deflection at A. The combinations that are possible solutions (those that yield zero deflection at A) appear as an $M-\theta'$ curve for the column of length $L_1$ (Fig. 25). The point on the $M-\theta'$ curve representing the solution is determined only after a consideration of an additional compatibility condition for the beam element.

The value of $M_B$ and $\theta'$ representing the solution must also satisfy the requirement that an integration starting from B to establish the beam shape must give a slope of zero at the center of the span. Again it will be shown that a curve defines the infinite combinations of $M_B$ and $\theta'$ which will yield zero slope at C. The intersection of this $M_B-\theta'$ curve for the beam with the $M_B-\theta'$ curve for the column will give the values of $M_B$ and $\theta'$ representing the solution to the problem.

In the determination of the $M_B-\theta'$ curve for the beam it will be assumed that the small axial load component in the beam has negligible effect on the structural behavior of the member. Actually it is possible to take this small compressive load $H$, into account by introducing an element of trial and error into the analysis, but
the multiplication of the effort in achieving a more accurate solution is not justified. The variation in the approach to achieve this end would involve assuming a value of H beforehand and performing successive trials until the H obtained from the solution is equal to the H assumed at the outset.

Figure 42 shows half of a beam with a uniform load $w$ applied to it. Point C corresponds to the center of member B-B' of the frame in Fig. 41. The shear is zero and the tangent to the beam is horizontal at C. A moment $M_C$ is assumed at C and the shape of the beam due to the distributed load $w$ may be obtained by a numerical integration in the positive direction of $x$. It will be noticed that the bending moment at any section is $M_x = M_C - \frac{wx^2}{2}$. Corresponding to each $M_x$ there is a value of $\phi_x$ given by the $M-\phi$ curve of the cross section for a zero value of the average compressive stress. Suitably small increments of length are used in the integration for the beam shape so that accurate values for the slope and the deflection are obtained at numerous points along the beam. The integration is similar to the integration procedure described in Section III.2b for members with axial loads. The only difference is that the moment at any point on the column deflection curve is $P.y$ while for the beam it is $M_C - \frac{wx^2}{2}$. The shape of the beam is developed until the slope again becomes horizontal.

Figure 43a shows several deflected beam shapes constructed for the same cross section and distributed load. The differing shapes are due to the different values assumed for $M_C$ in each case. At a
distance from point C equal to the half span length of the frame under investigation, the bending moment \( M(x = L_2/2) \) and the slope \( \Theta_B' \) are determined for each beam. These are plotted in Fig. 43b and a smooth curve is drawn connecting the points. This curve is the \( M_B-\Theta' \) curve for the beam. All combinations of \( M \) and \( \Theta' \) given by points on this curve will give a horizontal slope at C.

Finally if the \( M_B-\Theta' \) curves for the column and beam are plotted on one graph as in Fig. 44, the intersection of the curves gives the one pair of values, \( M_B \) and \( \Theta_B' \), which satisfies the condition of zero deflection at A and zero slope at C.

Line "a" in Fig. 44 is drawn tangent to the \( M-\Theta' \) curve for the column at the point where the two curves intersect. The slope of this line represents the rotational stiffness (on the basis of no unloading) of member BA when the frame is loaded with a distributed load of magnitude w. The foregoing statement follows from the fact that the stiffness is, \( k_{BA} = \frac{\Delta N_B}{\Delta \Theta_B'} \) as \( \Delta M_B \) tends to zero and that this ratio is the slope of the \( M_B-\Theta' \) curve for the column. The stiffness considering unloading is equal to the initial slope of the \( M_B-\Theta' \) curve since a reduction of \( M_B \) involves a reduction of strain and stress intensity throughout the column according to the stress-strain law, \( \sigma = E \epsilon \).

The stiffnesses for the beam member may be obtained once the exact elastic-plastic state of this member has been determined. If
the material is an ideal elastic-plastic material and it is desired to determine the stiffnesses $K_{BB}$ and $K_{B'B}$ on the basis of no unloading, one must consider as effective only that portion of the beam that is still in the elastic state. If the stiffness considering unloading is desired, then in addition to the elastic portions, those plasticized portions must be considered effective in which unloading occurs during a small deviation of the structure from the equilibrium position.
V. SUMMARY

It has been stated in Chapter One that the present study would be concerned with the development of theory and procedures for the determination of the collapse load of columns which fail by excessive bending about one of the principal axis of the cross section. The original results of the study are summarized below:

1. An exact solution has been obtained for a column with unequal end eccentricities of the compressive load and unequal rotational restraints at the ends of the column. It has thus been demonstrated that solutions may be obtained for any prismatic column where the rotational restraints may be considered as continuous functions of the end rotations.

2. All restrained columns of the type considered in this study may be considered as special cases of the general restrained column described under item 1. Nomographs for rapid solutions have been developed for the following special cases.

   (a) The column with one end pinned and the other end restrained. The compressive load is applied with an eccentricity at the restrained end (Fig. 20a). The solution includes the special case when the restraint disappears.

   (b) The column with the compressive load applied with equal eccentricities at the ends and with equal rotational end restraints (Fig. 29a). The solution includes the special case when the restraints disappear.

3. Nomographic methods have been developed for solving the
following column problems.

(a) The general case of unrestrained columns (Fig. 32).

(b) The column with one end completely fixed against rotation and with the other end partially restrained. The compressive load is applied eccentrically to the partially restrained end (Fig. 35).

4. In Appendix VII.2 moment-rotation curves are given for several beam-columns. These are of importance because they show the rotation capacity, an item of considerable importance in the field of plastic design.

5. An exact method of structural analysis making use of the moment rotation curves of Appendix VII.2 has been presented. It permits the analysis of simple symmetrical frames loaded in a symmetrical manner. The method considers non-linear effects due to both plasticity and the deformations of the structure. The method is of importance because it gives the stiffnesses of the members of the frame at any stage of the loading. This information is essential to the computation of the sidesway buckling loads of these structures in the plastic range.

6. Some experimental verification of the theory presented herein has been presented. The experimental evidence is limited to the comparison of theoretically computed moment-rotation curves for beam-columns with results from a beam-column test. The test results are in good agreement with the theory. For the present no tests of restrained wide-flange steel columns
have been performed so that general experimental verification is lacking.
VI. NOMENCLATURE

A  - Cross sectional area of a column (in.$^2$)
P  - Compressive load (lbs.)
E  - Young's Modulus (psi)
P_y - $\sigma_y A$ (lbs.)
I  - Moment of inertia about axis of bending (in.$^4$)
M  - Bending moment (lb.-in.)
M_p - Plastic bending moment of a cross section (lb.-in.)
S  - Section modulus of a cross section (in.$^3$)
M_y  - Bending moment of a cross section at initial yield
H  - Horizontal thrust at support of a rigid frame (lbs.)
D  - Half depth of a column of rectangular cross section (in.)
F,Q - Force (lbs.)
L,L_n - Column lengths, span lengths of a beam in a rigid frame (in.)
$M_n^a$ - Mean bending moment for a segment of a column deflection curve (lb.in.)
e_A,e_B - Eccentricities with which a load is applied to a structure (in.)
\l - Half wave length of a column deflection curve (in.)
k  - Stiffness of a rotational restraint (in. lbs./radian)
b  - Width of a column cross section (in.)
d  - Depth of a column cross section (in.)
r  - Radius of gyration about axis of bending (in.)
w  - Distributed load (lbs./in.)
y_m - Maximum amplitude of a column deflection curve (in.)
\gamma,\gamma_A,\gamma_B,\gamma_n - Deflections of points on a column deflection curve (in.)
$x_A$ - A distance giving the location of a segment on a column deflection curve (in.)

$x$ - A distance measured from the center of a beam (in.)

$s$ - Length of restraining span (in.)

$L_A, L'_A, L_B, L'_B$ - Partial lengths of a column deflection curve (in.)

$\theta_1, \theta_2, \theta_n$ - Slope at end of segment $n$ of a column deflection curve (in./in.)

$\theta_A, \theta_B$ - Slope of column deflection curve at $A, B$ (in./in.)

$\theta_{o1}, \theta_{o2}, \ldots \theta_{on}$ - Initial slopes of column deflection curves (in./in.)

$\theta, \theta'_A, \theta'_B$ - Rotations at end of column (in./in.)

$\phi$ - Curvature (radians/in.)

$\phi_a$ - Mean curvature (radians/in.)

$\phi_n$ - Mean curvature for a segment of a column deflection curve (radians/in.)

$\alpha = \frac{y_A - y_B}{L}$ (in./in.)

$c_1, c_2$ - Numerical values $0 \leq \xi \leq 1$

$\delta$ - Transverse deflection on a bent column (in.)

$\Delta$ - Symbol denoting increment

$\Delta \theta$ - Change in slope over length in a column deflection curve (radians)

$\sigma$ - Normal unit stress (psi)

$\sigma_Y$ - Yield stress (psi)

$\sigma_{pl}$ - Stress of proportional limit (psi)

$\varepsilon, \varepsilon_1, \varepsilon_2$ - Unit strains (in./in.)
\( \tau_{rc} \) - Maximum residual compressive stress (psi)

\( p \) - Increment of column length (in.)

\( f_A(\theta), f_B(\theta) \) - Functions giving restraining moments at column ends (lb.in./radian)
VII.1 NOMOGRAPHS
VII. 2 MOMENT-ROTATION CURVES FOR

UNRESTRAINED COLUMNS
\[
\frac{M}{M_y} = 0.2 \\
\sigma_{RC} = 0.3 \sigma_y \\
\sigma_y = 33 \text{ ksi}
\]
\[ \frac{M}{M_y} \]

\[ \frac{M}{M_y} = \begin{cases} 
\frac{P}{P_y} & \text{if } \sigma_{RC} = 0.3 \sigma_y \\
0 & \text{if } \sigma_y = 33 \text{ ksi}
\end{cases} \]
\[ \frac{M}{M_y} \]

\[ \frac{P}{P_y} = 0.12 \]

\[ \sigma_{RC} = 0.3 \sigma_y \]

\[ \sigma_y = 33 \text{ ksi} \]
\[
\frac{M}{M_y} = 0.3 \quad \frac{P}{P_y} = 0.3 \quad \sigma_{RC} = 0.3 \sigma_y \quad \sigma_y = 33 \text{ ksi}
\]
\[
\frac{M}{M_y} \quad \frac{P}{P_y} = 0.4 \quad \sigma_{RC} = 0.3 \sigma_y \\
\sigma_y = 33 \text{ ksi}
\]
\[ \frac{M}{M_y} \]

- \( \frac{L}{r} = 40 \)
- \( \frac{L}{r} = 60 \)
- \( \frac{L}{r} = 80 \)
- \( \frac{L}{r} = 100 \)

\[ \sigma_{RC} = 0.3 \sigma_y \]
\[ \sigma_y = 33 \text{ ksi} \]
VII 3a. ILLUSTRATIVE PROBLEM - GENERAL CASE

Given: (1) The rolled steel wide-flange column A-B of Fig. 45 with restraining spans A'-A and B-B'

(2) \( P/P_y = 0.3 \)

(3) \( \phi_{r_c} = 0.3 \phi_y \)

(4) \( e_A/e_B = 2.0 \)

(5) \( M_p = 1.11 M_y \)

(6) \( d/r = 2.3 \) (near constant for all standard rolled steel column sections)

(7) \( E = 30 \times 10^3 \text{ k.s.i.}, \phi_y = 33 \text{ k.s.i.} \)

Required: Maximum \( e_A \)

Solution: The restraining functions at A and B are approximated according to simple plastic theory, that is, the restraining beams are assumed to remain entirely elastic up to the formation of a plastic hinge.

The restraining function at A is therefore given by:

\[
f_A(\theta'_A) = \begin{cases} 
54.5 M_y \theta'_A & , \quad \theta'_A \leq 0.0204 \\
M_p & , \quad \theta'_A > 0.0204 
\end{cases}
\]

The restraining function at B is given by:

\[
f_B(\theta'_B) = \begin{cases} 
109 M_y \theta'_B & , \quad \theta'_B \leq 0.0102 \\
M_p & , \quad \theta'_B > 0.0102 
\end{cases}
\]
The various column deflection curves for $P/P_y = 0.3$ are examined to determine the location of the segment on each of them which is equivalent to the column shown in Fig. 45. When $x_A$ (see Figs. 17 and 18) is chosen correctly Eq. 3.11 is satisfied. After several trials $x_A$ is determined for each column deflection curve. Equation 3.9 is then used to determine the value of $e_A$ for each of the equilibrium conditions.

The values of $x_A$ and $e_A$ determined in this manner are given in the table below.

<table>
<thead>
<tr>
<th>Column Deflection Curve ($\Theta_o$)</th>
<th>$x_A$ Satisfying Eq. 3.11 (See Figs. 17 and 18)</th>
<th>$e_A$ (Eq. 3.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>99.8 r</td>
<td>1.33 r</td>
</tr>
<tr>
<td>0.02</td>
<td>99.8 r</td>
<td>2.66 r</td>
</tr>
<tr>
<td>0.03</td>
<td>98.9 r</td>
<td>4.03 r</td>
</tr>
<tr>
<td>0.035</td>
<td>96.7 r</td>
<td>4.75 r</td>
</tr>
<tr>
<td>0.04</td>
<td>91.0 r</td>
<td>5.32 r</td>
</tr>
<tr>
<td>0.05</td>
<td>62.8 r</td>
<td>5.65 r</td>
</tr>
<tr>
<td>0.06</td>
<td>32.0 r</td>
<td>5.01 r</td>
</tr>
</tbody>
</table>

A plot of $e_A$ versus $\Theta_o$ (Fig. 45 shows that 5.65 r is the maximum value of the eccentricity at $A$.

The maximum value of $e_A$ is reached when the column is deformed so that it is very nearly equivalent to the segment of the C.D.C. who's $\Theta_o$ is 0.05 radians. The effect of the restraint at $A$ is to reduce the effective eccentricity of the load on the column from 5.65 r to 2.44 r.
The reduction corresponds to the plastic moment that is developed in member A'-A. The restraint at B reduces the effective eccentricity on the column at this end by the same amount so that the moment in the column is actually opposite in sense to the moment that the load would tend to induce. The column moment at this end is equivalent to the moment due to the axial load applied with an eccentricity of -0.38 r.

The bending moment in the column at A amounts to 96% of the moment capacity of the section modified for the axial thrust of 0.3 $P_y$. It is thus seen that in this case the restraints are sufficient to increase the carrying capacity of a column with a slenderness ratio of 80 to nearly the capacity of an eccentrically compressed member of negligible length.

VII. 3b. ILLUSTRATIVE PROBLEM - ONE END PINNED

Given: (1) The rolled steel wide-flange column A-B of Fig. 46 with restraint span B-B'.

(2) $P/P_y = 0.3$
(3) $\n_{rc} = 0.3 \n_y$
(4) $M_p = 1.11 M_y$
(5) $e_B = 2.88 r$
(6) $d/r = 2.3$
(7) $E = 30 \times 10^3$ k.s.i., $\n_y = 33$ k.s.i.
Required: The maximum length $L$, of span $A-B$. Consider the cases when $s/r = 0, 56.5, 113$ and $226$.

Solution: The restraining function at $A$ (in accordance with the approximation of VII.3a) is given by:

$$f(\theta_B') = \begin{cases} 
3080 \frac{r}{s} M_y \theta_B', & \theta_B' \leq 0.00036 \frac{s}{r} \\
M_p, & \theta_B' > 0.00036 \frac{s}{r}
\end{cases}$$

The external moment, $f(\theta_B') - P e_B$ must be equal to the internal moment of the column at $B$. The equality when non-dimensionalized is:

$$\frac{M}{M_y} = 3080 \frac{r}{s} \theta_B' - 1.0$$

when $\theta_B' \leq 0.00036 \frac{s}{r}$ and

$$\frac{M}{M_y} = 1.11 - 1.0 = 0.11$$

when $\theta_B' > 0.00036 \frac{s}{r}$.

The non-dimensionalized function $\frac{f(\theta_B') - P e_B}{M_y} = \frac{M}{M_y}$ is plotted in the upper portion of the appropriate nomograph of Appendix 1. The intersections with the $\frac{M}{M_y} - \theta'$ curves of the $\theta_1^1, \theta_2^2, \theta_3^3, \ldots, \theta_n^n$ column deflection curves are carried down to give lengths $L_1/r, L_2/r, L_3/r, \ldots, L_n/r$ representing equilibrium configurations as explained in section III.4. The maximum value of $L/r$ for each case
considered is indicated by an arrow in the table below.

<table>
<thead>
<tr>
<th>s/r</th>
<th>θ₀</th>
<th>L/r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>251.4</td>
</tr>
<tr>
<td>56.5</td>
<td>0.020</td>
<td>173</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>185</td>
</tr>
<tr>
<td></td>
<td>0.035</td>
<td>181</td>
</tr>
<tr>
<td>113</td>
<td>0.030</td>
<td>161</td>
</tr>
<tr>
<td></td>
<td>0.035</td>
<td>166</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>168</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>142</td>
</tr>
<tr>
<td>226</td>
<td>0.035</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>131</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>118</td>
</tr>
</tbody>
</table>

A plot of s/r versus maximum L/r is shown in Fig. 46.

VII.3c. ILLUSTRATIVE PROBLEM - SYMMETRICALLY RESTRAINED AND SYMMETRICALLY LOADED COLUMN

Given: (1) The rolled steel wide-flange column A-B of Fig. 47 with symmetrical restraining spans A'-A and B-B'.

(2) \( P/P_y = 0.3 \)
(3) \( \nabla_{rc} = 0.3 \nabla_y \)
(4) \( M_p = 1.11 M_y \)
(5) \( e = e_A = e_B = 2.88 r \)
(6) \( d/r = 2.3 \)
(7) \( E = 30,000 \text{ k.s.i.}, \quad \nabla_y = 33 \text{ k.s.i.} \)

Required: The maximum length \( L \), of span A-B. Consider the case when \( s/r = 0, 56.5, 113, \) and 226.
Solution: The restraining function at A and B (in accordance with the same approximation of VII.3a) are given by:

\[
f(\theta_B) = f(\theta_A) = f(\theta) = \begin{cases} 
3080 \frac{r}{s} M_\theta & , \quad \theta \leq 0.00036 \frac{s}{r} \\
M_p & , \quad \theta > 0.00036 \frac{s}{r}
\end{cases}
\]

The external moments at A and B (f(\theta) - P_e) acting on column A-B must equal the internal moments at these points. In non-dimensional terms:

\[
\frac{M}{M_y} = 3080 \frac{r}{s} \quad \theta - 1.0 \quad (a)
\]

when \( \theta \leq 0.00036 \frac{s}{r} \) and

\[
\frac{M}{M_y} = 0.11 \quad \text{when} \quad \theta > 0.00036 \frac{s}{r} \quad (b)
\]

The expression given by (a) and (b) is plotted in the upper portion of the appropriate nomograph of Appendix 1 and the intersections with the M-\( \theta \) curves of the \( \theta_o, \theta_o^1, \ldots, \theta_o^n \) column deflection curves are carried down to the lower portion of the nomograph as explained in III.5 to give lengths \( L_o^1/2r, L_o^2/2r, \ldots, L_o^n/2r \) representing equilibrium configurations of the column. In the table below the maximum value of L/r for each value of s/r is indicated by an arrow.
### VII.3d. ILLUSTRATIVE PROBLEM - UNRESTRAINED COLUMN

**Given:**

1. The pin ended rolled steel, wide-flange column A-B of Fig. 48 with unequal end moments.

2. \( \frac{P}{P_y} = 0.3 \)

3. \( \frac{\nabla^*_{rc}}{\nabla^*_{y}} = 0.3 \)

4. \( M_p = 1.11 M_y \)

5. \( \frac{M_A}{M_y} = +0.6 \)

6. \( \frac{M_B}{M_y} = -0.3 \)

**Required:** The maximum length of column A-B consistent with equilibrium.

<table>
<thead>
<tr>
<th>( s/r )</th>
<th>( \theta_0 )</th>
<th>( L/2r )</th>
<th>( L/r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>173</td>
<td>346</td>
</tr>
<tr>
<td>56.5</td>
<td>0.01</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>95</td>
<td>190</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>96</td>
<td>192</td>
</tr>
<tr>
<td></td>
<td>0.035</td>
<td>94</td>
<td>188</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>90</td>
<td>180</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>73</td>
<td>146</td>
</tr>
<tr>
<td>113</td>
<td>0.035</td>
<td>82.5</td>
<td>165</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>87.5</td>
<td>175</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>74.0</td>
<td>148</td>
</tr>
<tr>
<td>226</td>
<td>0.040</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>45</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>0.060</td>
<td>41</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>0.070</td>
<td>38.5</td>
<td>77</td>
</tr>
<tr>
<td></td>
<td>0.080</td>
<td>31.5</td>
<td>63</td>
</tr>
</tbody>
</table>
Solution: In accordance with the procedure outlined in III.6, $L_A$ and $L_B' \text{ lengths (see Fig. 33a) are obtained from several}

column deflection curves at $P/P_y = 0.3$. These are obtained from the symmetrical case nomograph of VII.1 where $P=0.3P_y$. A horizontal line is drawn in the upper portion of the diagram for $M/M_y = +0.6$. The intersections with the $M-\Theta$ curves are carried down to the lower portion of the nomo-

graph to give the distances $L_A$ on each column deflection curve. Similarly the intersections with the horizontal line $M/M_y = -0.3$ are carried down to give the $L_B'$ distance.

In the following table the values of $L_A$ and $L_B'$ are added for each column deflection curve to give the length of column A-B. An arrow indicates the maximum value of $L$ consistent with equilibrium.

<table>
<thead>
<tr>
<th>$\Theta_0$</th>
<th>$L_A/r$</th>
<th>$L_B'/r$</th>
<th>$L = L_A + L_B'/r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>0.02</td>
<td>136</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>0.025</td>
<td>123</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>0.030</td>
<td>117</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>0.035</td>
<td>111</td>
<td>23</td>
<td>134</td>
</tr>
<tr>
<td>0.040</td>
<td>104.5</td>
<td>32</td>
<td>136.5</td>
</tr>
<tr>
<td>0.050</td>
<td>85</td>
<td>30</td>
<td>115</td>
</tr>
</tbody>
</table>
VII.4 \textbf{TYPICAL NUMERICAL INTEGRATION COMPUTATIONS}

Section - 8WF31, Major Axis Bending

\( P = 0.3 \, P_y \)

\( \theta_0 = 0.05 \) radians

\( \frac{M}{M_y} = 0.3 \, P_y \cdot y \cdot \frac{\tau_y}{S} = 0.09976y \)

\( \frac{\phi}{\phi_y} \) determined from \( M-\phi \) diagrams of Reference 12

\( \phi_a = \phi_y \cdot \phi_y = \frac{\tau_y}{E \cdot Ar^2} \cdot \phi_y = 275.10^{-6} \cdot \phi_y \)

\( \rho^2/2 \cdot \phi_a = \) Deviation of \( y_n \) from tangent drawn at \( y_{n-1} \)

\( \Delta \theta = \rho \phi_a = \) Change in slope over length increment \( \rho \)
<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y_n</td>
<td>( \theta_n )</td>
<td>( \rho/2\theta_n )</td>
<td>( y_n^{\text{ave}} )</td>
<td>( M/M_y^{\text{ave}} )</td>
<td>( \rho/\rho_y^{\text{ave}} )</td>
<td>( \theta_a \cdot 10^{-9} )</td>
<td>( \rho^2/2\rho_a )</td>
<td>( \rho\theta_n )</td>
<td>( \Delta \theta )</td>
</tr>
<tr>
<td>0 r</td>
<td>0</td>
<td>0.050000</td>
<td>0.3470</td>
<td>0.3470</td>
<td>0.346</td>
<td>0.346</td>
<td>9.520</td>
<td>0.0009</td>
<td>0.6940</td>
<td>0.00013</td>
</tr>
<tr>
<td>4 r</td>
<td>0.6931</td>
<td>0.04987</td>
<td>3.46</td>
<td>3.46</td>
<td>1.0392</td>
<td>1.037</td>
<td>8.059</td>
<td>0.0007</td>
<td>0.6922</td>
<td>0.00040</td>
</tr>
<tr>
<td>8 r</td>
<td>1.3826</td>
<td>0.04947</td>
<td>3.433</td>
<td>3.433</td>
<td>1.7259</td>
<td>1.712</td>
<td>47.350</td>
<td>0.0046</td>
<td>0.6866</td>
<td>0.00066</td>
</tr>
<tr>
<td>12 r</td>
<td>2.0646</td>
<td>0.04881</td>
<td>3.387</td>
<td>3.4033</td>
<td>2.4003</td>
<td>2.398</td>
<td>65.931</td>
<td>0.0064</td>
<td>0.6775</td>
<td>0.00092</td>
</tr>
<tr>
<td>16 r</td>
<td>2.7357</td>
<td>0.04789</td>
<td>3.324</td>
<td>3.0681</td>
<td>3.061</td>
<td>3.061</td>
<td>84.169</td>
<td>0.0081</td>
<td>0.6647</td>
<td>0.00117</td>
</tr>
<tr>
<td>20 r</td>
<td>3.3923</td>
<td>0.04672</td>
<td>3.242</td>
<td>3.7165</td>
<td>3.708</td>
<td>3.708</td>
<td>101.959</td>
<td>0.0098</td>
<td>0.6485</td>
<td>0.00142</td>
</tr>
<tr>
<td>24 r</td>
<td>4.0310</td>
<td>0.04530</td>
<td>3.144</td>
<td>4.3434</td>
<td>4.335</td>
<td>4.335</td>
<td>119.213</td>
<td>0.0115</td>
<td>0.6288</td>
<td>0.00165</td>
</tr>
<tr>
<td>28 r</td>
<td>4.6483</td>
<td>0.04354</td>
<td>3.029</td>
<td>4.9512</td>
<td>4.939</td>
<td>4.960</td>
<td>136.400</td>
<td>0.0131</td>
<td>0.6059</td>
<td>0.00189</td>
</tr>
<tr>
<td>32 r</td>
<td>5.2411</td>
<td>0.04176</td>
<td>2.898</td>
<td>5.5309</td>
<td>5.518</td>
<td>5.620</td>
<td>154.550</td>
<td>0.0149</td>
<td>0.5796</td>
<td>0.00215</td>
</tr>
<tr>
<td>36 r</td>
<td>5.8058</td>
<td>0.03961</td>
<td>2.749</td>
<td>6.0807</td>
<td>6.066</td>
<td>6.275</td>
<td>172.563</td>
<td>0.0166</td>
<td>0.5498</td>
<td>0.00240</td>
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<tr>
<td>40 r</td>
<td>6.3390</td>
<td>0.03721</td>
<td>2.582</td>
<td>6.5972</td>
<td>6.581</td>
<td>7.055</td>
<td>194.013</td>
<td>0.0187</td>
<td>0.5165</td>
<td>0.00269</td>
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<tr>
<td>44 r</td>
<td>6.8368</td>
<td>0.03452</td>
<td>2.396</td>
<td>7.0764</td>
<td>7.059</td>
<td>8.160</td>
<td>224.400</td>
<td>0.0216</td>
<td>0.4791</td>
<td>0.00311</td>
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<tr>
<td>48 r</td>
<td>7.2943</td>
<td>0.03141</td>
<td>2.180</td>
<td>7.5123</td>
<td>7.494</td>
<td>9.460</td>
<td>260.150</td>
<td>0.0251</td>
<td>0.4360</td>
<td>0.00361</td>
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<td>7.7052</td>
<td>0.02780</td>
<td>1.929</td>
<td>7.8981</td>
<td>7.879</td>
<td>1.2170</td>
<td>334.675</td>
<td>0.0322</td>
<td>0.3859</td>
<td>0.00465</td>
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<tr>
<td>56 r</td>
<td>8.0589</td>
<td>0.02315</td>
<td>1.607</td>
<td>8.2196</td>
<td>8.200</td>
<td>1.6100</td>
<td>442.750</td>
<td>0.0426</td>
<td>0.3213</td>
<td>0.00615</td>
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<tr>
<td>60 r</td>
<td>8.3376</td>
<td>0.01700</td>
<td>1.180</td>
<td>8.4556</td>
<td>8.435</td>
<td>2.1500</td>
<td>591.250</td>
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<td>0.2360</td>
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<tr>
<td>64 r</td>
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<td>8.5166</td>
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</tbody>
</table>

\( \rho^* = 12.5r \), \( \rho/2 = 6.25r \), \( \rho = 3.60r \)

\[ \begin{array}{cccccccc}
67.60 r & 8.5713 & -0.00004 \\
0.0549 & 8.5715 & 0.8551 & 2.5700 & 706.750 & 0.0552 & 1.0999 & 0.00883 \\
\end{array} \]

\[ \Sigma \text{Col.(9)} - \Sigma \text{Col.(8)} = 8.5713 \]

Summations for checking

\[ \begin{array}{cccc}
3.410 & 8.9123 & 0.0004 \\
\end{array} \]

* This \( \rho \) is smaller than \( 4r \), since at \( x = 68r \), \( \theta \) becomes negative.

The value of the last \( \rho(3.60r) \) is obtained by trial, so that at \( x = 67.60r \), \( \theta \approx 0 \).
Fig. 4

Fig. 5
Fig. 8

Fig. 9

Fig. 10
Fig. 11a

Fig. 11b

Fig. 12a

Fig. 12b

Fig. 12c
Fig. 14a  
(Cross Section)

Fig. 14b  
(Stress-Strain Diagram)

Fig. 14c  
(LOAD Diagram)

Fig. 15

\[ M = \begin{cases} 0.4P_y & \text{for} \quad \theta \end{cases} \]
Initial Tangent

Column Deflection Curve

\[ \theta_1 = \theta_0 - \rho_1 \phi_1 \]

\[ y_1 = \rho_1 \theta_0 - \rho_1 \phi_1 / 2 \]

Fig. 16
Fig. 20a

Fig. 20b
Positive Moment

Negative Moment

Fig. 21
Fig. 22
Fig. 23
Fig. 24
Fig. 25
Fig. 26

Fig. 27
Fig. 28

Fig. 29a

Fig. 29b
Fig. 30
Fig. 31
Fig. 32

Fig. 33a

Fig. 33b

Fig. 33c
Fig. 34a

Fig. 34b
Section - 8WF31

$\theta_0 = 0.04$

$P/P_y = 0.3$

Fig. 37
Fig. 38

Experimental Points

Theoretical

Experimental

8WF13 in Major Axis Bending

\[ \frac{L}{r} = 111 \]

\[ \frac{P}{P_y} = 0.3 \]

\[ \frac{P}{P_y} = 0.33 \]

\[ \frac{P}{P_y} = 0.4 \]
Concentric Loads
Symmetrically Placed

Uniformly Distributed Load

Pin Ends

Simple Frames

Fig. 39 a

Concentrated Loads
Symmetrically Placed

Uniformly Distributed Load

Fig. 39 b

Box Section

Uniformly Distributed Load

Concentrated Loads
Symmetrically Placed
Fig. 40

Fig. 41

Fig. 42
Fig. 44

- M-θ' Curve for Beam
- M_B, θ'_B
- M-θ' Curve for Column

Fig. 44
$e_A \text{ Maximum} = 5.65 r$
$e_B = 2.88r$

$P = 0.3Py$

$\frac{L}{r}$

$\frac{s}{r}$

Fig. 46
Fig. 47

\[ P = 0.3 \, P_y \]

\[ e = 2.88 \, r \]

Fig. 48

\[ M_A = 0.6 \, M_y \]

\[ M_B = -0.3 \, M_y \]

\[ P = 0.3 \, P_y \]
IX. REFERENCES


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X. V I T A

The author was born in New York City on March 4, 1924, the son of Fanny and Nessim Ojalvo. He completed his secondary school training at Stuyvesant High School in New York City.

He studied Civil Engineering at the City College of New York and the Rensselaer Polytechnic Institute in Troy, New York. He received a Bachelor of Civil Engineering Degree from R.P.I. in 1944 and a Master of Civil Engineering from the same institute in 1952.

The author has taught as a tutor at the City College of New York from 1947 to 1949, as an instructor of Civil Engineering at R.P.I. from 1949 to 1951, and as an Assistant Professor at Princeton University from 1951 to 1958. In 1958 he accepted a position as instructor at Lehigh University and simultaneously began his studies for a Ph.D. degree at Lehigh.

The author is married to the former Anita Bedein and they are the parents of Lynne, Joseph, Howard, and Isobel Ojalvo.