ELASTIC, ELASTIC-PLASTIC AND PLASTIC BUCKLING
OF PLATES WITH RESIDUAL STRESSES

by

Yukio Ueda

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Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

(Date)  Lambert Tall
Professor in Charge

Accepted, (Date)

Special Committee directing the doctoral work of Mr. Yukio Ueda

Professor Alexis Ostapenko
Chairman

Professor Voris V. Latshaw

Professor Fazil Erdogan

Professor Cornie L. Hulsbos

Professor William J. Eney

Professor Lambert Tall
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Welded built-up members are being used more frequently in steel construction due to economy, convenience and aesthetics. The residual stresses produced in the members as a result of the welding play an important role in the buckling strength of the members.

This dissertation presents the results of an investigation into the elastic, elastic-plastic and plastic buckling of steel plates containing residual stresses. Particular attention is given to the local buckling of built-up columns of box-shaped cross section.

The material of the members is steel with a stress-strain relationship assumed to be elastic perfectly plastic, and with a Poisson's ratio of 0.3 in the elastic range and 0.5 in the plastic range.

The analysis of the behavior of the plate material in the plastic range was based on both the secant modulus deformation theory and the flow theory. The theorem of minimum potential energy was applied to solve the buckling problem. A simplified residual stress distribution was used in the analysis.

Analytical solutions were obtained for the elastic, elastic-plastic and plastic buckling of a plate simply supported at the loading edges with the other edges:

- 1 -
a) elastically restrained
b) simply supported
c) fixed.

Numerical examples of the analytical solution were presented for the study on the strength of local buckling of square built-up columns, that is, case (a) above. This study showed that the first term of the series of the assumed deflection function,
\[ w = a \cos \frac{\pi}{2} \frac{x}{b} \sin N \pi \frac{z}{L} \]
was sufficient to investigate the elastic, elastic-plastic and plastic buckling of the plate with residual stresses.

An experimental study was performed on two short columns to check the theory for the square built-up column of the numerical examples. Good agreement was obtained with the results of the numerical calculation for elastic buckling and for elastic-plastic buckling, based on the secant modulus deformation theory. The experiments also showed that the ultimate load was very close to the critical buckling load for the elastic-plastic buckling, but that the post buckling strength was large for the elastic buckling.
1. INTRODUCTION

This dissertation presents the results of an analytical and experimental investigation into the elastic, elastic-plastic and plastic buckling of steel plates with residual stresses. Particular attention is given to the local buckling of built-up columns of box-shaped cross section.

Welded built-up members are being used more frequently in steel construction due to economy, convenience and aesthetics. These members contain residual stresses due to welding, but it is only recently that research on the behavior of members under load has shown that the residual stress distribution inherent in the cross section plays a major role in their strength.

When a plate containing residual stresses is subjected to thrust, it will behave elastically until the thrust reaches a certain value which causes yielding at some point in the plate. Under a thrust less than this value, the plate may buckle elastically.

When the thrust exceeds this value, some parts of the plate start to yield due to the presence of compressive residual stresses resulting from welding. Thereafter, the plate consists of elastic and plastic parts and the buckling of the plate is called elastic-plastic buckling. The theory of elasticity is no longer applicable.
to the plastic parts. As the thrust increases, the plastic zone in the plate will widen and the method of analysis becomes complicated for the elastic-plastic buckling of the plate.

Two types of plastic theory, the deformation theory and the flow theory, will be employed to analyze the behavior of the plastic zone.

When the thrust reaches the value which makes the plate completely plastic, the plate cannot carry any more load for the case when the material is assumed to be of elastic perfectly plastic material. The buckling at this magnitude of the thrust is called plastic buckling.

It had been believed that residual stresses do not affect elastic buckling of members, but this is only true for the case of column buckling of the Euler type. Residual stresses do influence the elastic buckling of members whose strength is a function of the product of the stress and the exponent of the lever arm when this exponent is other than unity or zero. This occurs with torsional buckling of columns and the local buckling of plates.

Prior to the investigation of this dissertation, the influence of residual stresses on the buckling strength of the plates has not been studied except for some particular cases of elastic buckling.
In the analysis of this study, the theorem of minimum potential energy has been applied to solve the buckling problem for which the residual stress distribution has been simplified.

Analytical solutions were obtained for the elastic, elastic-plastic, and plastic buckling of a plate with residual stresses, in the following three cases when the plate is simply supported at the loading edges, and at the other edges is:

a) elastically restrained
b) simply supported
c) fixed.

The solution is numerically illustrated for the local buckling of a built-up column of a square cross section. This is an application of the solution to case (b). The numerical calculation was carried out by a digital computer, the L.G.P.30 at Lehigh University.

Two short columns of square cross section were tested to check the numerical computation for the elastic and elastic-plastic buckling cases.

The new contributions of this dissertation are the theoretical and experimental investigation of the elastic, elastic-plastic, and plastic buckling of a plate containing residual stresses, with particular attention paid to the local buckling of box-shaped columns.
2. LITERATURE SURVEY

A thin plate may fail due to lateral deflection when the plate is subjected to compressive forces, shearing forces or their combination in its plane along the sides. The differential equation of such plates, originally derived by Saint Venant (1), has the following form:

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] + h \left[ \sigma_x \frac{\partial^2 w}{\partial x^2} + 2 \tau_{xy} \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} \right] = 0 \quad (2.1)
\]

where

- \( D \) = flexural rigidity of the plate = \( \frac{E \ h^3}{12 \ (1- \nu^2)} \)
- \( w \) = deflection of the plate
- \( \sigma_x, \sigma_y \) = normal stress components in the cartesian coordinates
- \( \tau_{xy} \) = shearing stress in the cartesian coordinates
- \( E \) = Young's modulus
- \( h \) = thickness of plate
- \( \nu \) = Poisson's ratio

In 1891, G. H. Bryan (2)(3) investigated theoretically the phenomenon of buckling of rectangular plates which were simply-supported on all edges and acted upon on two opposite sides by a
uniformly distributed compressive force in the plane of the plate. He applied the energy criterion of stability to the buckling.

More than fifteen years later, S. Timoshenko, H. Reissner, K. Sezawa, H. Wagner, and G. I. Taylor treated problems concerning the buckling of rectangular plates fixed at edges and also under various boundary conditions. In particular, Timoshenko investigated extensively the stability of plates with various conditions of support under compressive forces, shearing forces or their combination.

An attempt to extend the theory of plate stability into the inelastic range was made by F. Bleich by considering the plate which has a reduced modulus of elasticity in the direction of the loading, but which retains the Young's modulus in the direction perpendicular to the loading. The equilibrium equation in this case becomes:

\[ D \left( \alpha_t \frac{\partial^4 w}{\partial x^4} + 2 \sqrt{\alpha_t} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + h \sigma_x \frac{\partial^2 w}{\partial x^2} = 0 \]  

(2.2)

where

\[ \alpha_t = \frac{E_t}{E} \]

\[ E_t = \text{tangent modulus} \]
The middle term in the parentheses is associated with the distortion of a square element of the plate due to the twisting moment acting on that element. Taking into consideration the limiting values 1 and \( \alpha_t \), the coefficient of the middle term was assumed arbitrarily as \( \sqrt{\alpha_t} \).

On the other hand, E. Chwalla, M. Ros and A. Eichenger assumed that the plate is isotropic in the plastic range and hence that the modulus of elasticity is the same in both directions. The corresponding equilibrium equation becomes:

\[
D \alpha_r \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + h \sigma_x \frac{\partial^2 w}{\partial x^2} = 0
\]

where

\[
\alpha_r = \frac{E_r}{E}
\]

\[
E_r = \text{reduced modulus}
\]

From this equation the critical buckling stress is \( \alpha_r \) times that in the elastic range.

As the theory of plasticity developed after 1940, new theories of plastic buckling were presented.

There were two main currents in this development, one based on Hencky's deformation theory and the other on Prandtl-Reuss flow theory. After 1940, Bijlaard and Ilyushin.
independently applied the deformation stress-strain relationship to
the plate buckling problem and obtained a solution. The deforma-
tion theory was further developed by Stowell\(^{(17)(18)}\) who modified
it based on Shanley's concept which is explained below. The
improved Stowell theory showed a good agreement with experimental
results. In contrast with the deformation theory, Handelman-Prager
in 1948 presented the buckling theory of plates\(^{(20)}\) in the plastic
range, based on the flow theory. The flow theory, which is more
complete than the deformation theory from the viewpoint of mathe-
matical theory of plasticity, showed no resemblance to the experi-
mental results.\(^{(21)(22)(23)}\) Pearson\(^{(24)}\) improved the flow theory by
using Shanley's concept.

According to Shanley's concept of the tangent modulus
theory, inelastic buckling of uniformly loaded members will occur
at the tangent modulus load. It is assumed that there is no strain
reversal in any part of the member at the instant of buckling, and
that buckling of the member proceeds simultaneously with increase
in load. This improved theory still gave much higher critical
values when compared to experimental results.

Onat and Drucker\(^{(25)}\) investigated the influence of initial
imperfections on torsional buckling of a simplified model of a uni-
form cruciform cross section and showed that the flow theory leads
to a reasonable correlation with experiment when unavoidable initial
imperfections are taken into account. However, their conclusions
have no application to the general study of plate buckling.

In 1955, Yamamoto \(^{(26)}\) presented the theory of plastic buckling of plates with consideration of the effect of an initial imperfection. The theory predicts, approximately, the tangent modulus load and correlates reasonably with test results.

In the following year, Thurlimann and Haaijer \(^{(27)}\) developed the plastic buckling theory of plates, based on the flow theory. Taking into account the initial imperfections, the four independent instantaneous flexure and shear moduli of an orthotropic plate were determined from the test results of the material under consideration. This theory gave good correlation with experimental results.

When plates are used as members of structures, such as built-up columns, shells of ship structures, plate girders and so on, the structures are quite often fabricated by welding. Residual stresses due to welding have presented problems concerning their influence on the strength of welded structures. Before the strength of welded structures could be studied, it was necessary to know the behavior of materials due to welding.

Before 1936, the analytical and experimental investigation of the welded joint had presumed an elastic behavior throughout the complete process of welding. Boulton and Lance Martin \(^{(28)}\) and Rosenthal \(^{(29)}\) simultaneously and independently showed, both analytically and experimentally, that welding induced plastic deformation
of material in and near the weld, and that the residual stresses resulting after cooling were due to these plastic deformations. Further analytical and experimental investigations were made by Fujita (30) and Tall (31). Experimental work was carried out by many other investigators. Okerblom (41) studied the deformations of welded metal structures and presented a theoretical method for the calculation of welding deformations.

In 1960 Okerblom presented a paper (42) concerning the influence of residual stresses on the stability of welded structures and structural members based on experimental results. His paper showed the possibility of elastic buckling of plate elements in the structure due to welding.

In the same year Yoshiki and others (43) investigated analytically the influence of residual stresses on the elastic buckling of center welded plates with the aid of integral equations and showed that the residual stresses could influence the elastic buckling strength of a plate, particularly in certain cases of residual stress distribution.

References 42 and 43 are apparently the only two papers concerned with elastic buckling of plates with residual stresses. There is no research on plate buckling other than in the elastic range.
3. PLATE BUCKLING

3.1 Stress-Strain Relationship in the Elastic and the Plastic Ranges

When the deformation and stresses of plates are analyzed, the relationship between stress and strain must be defined both in the elastic and plastic ranges of the material. The most fundamental relation of stress and strain is that obtained from a coupon test in uniaxial tension or compression. Figure 3.1(a) shows a typical stress-strain relationship of a strain hardening material and Fig. 3.1(b) presents an idealized stress-strain relationship (elastic perfectly plastic) for steel, used in the analysis of this dissertation.

In the elastic range the well established theory of elasticity is based on the following relationship between stress and strain \( (44)(45) \);
\[ \varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu (\sigma_y + \sigma_z) \right) \]
\[ \varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu (\sigma_z + \sigma_x) \right) \]
\[ \varepsilon_z = \frac{1}{E} \left( \sigma_z - \nu (\sigma_x + \sigma_y) \right) \]

\[ \gamma_{yz} = \frac{2 (1+\nu)}{E} \tau_{yz} \]
\[ \gamma_{zx} = \frac{2 (1+\nu)}{E} \tau_{zx} \]
\[ \gamma_{xy} = \frac{2 (1+\nu)}{E} \tau_{xy} \]

where

\[ \varepsilon_x, \varepsilon_y, \varepsilon_z = \text{normal strain components in the cartesian coordinates} \]
\[ \gamma_{yz}, \gamma_{zx}, \gamma_{xy} = \text{shearing strain components in the cartesian coordinates} \]
\[ \sigma_x, \sigma_y, \sigma_z = \text{normal stress components in the cartesian coordinates} \]
\[ \tau_{yz}, \tau_{zx}, \tau_{xy} = \text{shearing stress components in the cartesian coordinates} \]

When the problem under consideration is one of plane stress, such as is the case with thin plates, this relationship becomes simpler since \( \sigma_z, \tau_{yz} \) and \( \tau_{zx} \) are zero.
In the plastic range the analysis of the plate material may be based on either the deformation theory or the flow theory.

The deformation theory\(^{(12)}\) assumes a one-to-one correspondence between stress and strain in the plastic range when the material is under load. The flow theory\(^{(13)(14)}\) on the other hand, assumes a one-to-one correspondence between the rate of change of stress and the rate of change of strain.

The important basic difference between these two theories lies in the fact that the stress-strain relationship is independent of the loading history in the deformation theory. In the flow theory, the strain depends upon the manner in which the state of stress is obtained. Although it appears logical that the loading history must play a role, test results\(^{(21)(22)(23)}\) have shown that only the deformation theory gives good results.

The secant modulus deformation theory assumes that the material is isotropic in the plastic range and that, the intensities of stress and strain are defined by the square root of the second invariant of the stress and strain tensors with constant factors
that is for the state of plane stress.

\[
\sigma_i = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau^2}
\]

\[
\varepsilon_i = \frac{2}{\sqrt{3}} \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_x \varepsilon_y + \tau^2/4}
\]

where

- \( \sigma_i \) = intensity of stress
- \( \varepsilon_i \) = intensity of strain

Under proportional loading, the assumed isotropy of the material requires the relationship between stress and strain to be of such a form that

\[
\frac{\sigma_x - \nu \sigma_y}{\varepsilon_x} = \frac{\sigma_y - \nu \sigma_x}{\varepsilon_y} = \frac{\tau_{xy}}{2(1+\nu) \varepsilon_s} = \frac{\sigma_i}{\varepsilon_i} = E_s
\]

where

- \( E_s \) = secant modulus

Accordingly, the stress-strain relationship during the process of loading is:

\[
\varepsilon_x = \frac{1}{E_s} \left[ \sigma_x - \nu \sigma_y \right]
\]

\[
\varepsilon_y = \frac{1}{E_s} \left[ \sigma_y - \nu \sigma_x \right]
\]

\[
\gamma_{xy} = \frac{2(1+\nu)}{E_s} \tau_{xy}
\]
For unloading, the material is assumed to behave completely elastically and the relationship between stress and strain may be defined as follows:

\[
\begin{align*}
\frac{d\varepsilon_x}{\lambda} &= \frac{1}{E} \left[ d\sigma_x - \nu d\sigma_y \right] \\
\frac{d\varepsilon_y}{\lambda} &= \frac{1}{E} \left[ d\sigma_y - \nu d\sigma_x \right] \\
\frac{d\tau_{xy}}{\lambda} &= \frac{2(1+\nu)}{E} d\tau_{xy}
\end{align*}
\]  

(3.6)

where the relationship is given in the form of a variation to eliminate the effect of a permanent set.

Based on the flow theory, the stress-strain relationship for loading may be defined by the following equations\(^{(20)}\)

\[
\begin{align*}
\dot{\varepsilon}_x &= \frac{1}{E} \left[ \lambda \dot{\sigma}_x - (\nu + \frac{\lambda - 1}{2}) \dot{\sigma}_y \right] \\
\dot{\varepsilon}_y &= \frac{1}{E} \left[ -(\nu + \frac{\lambda - 1}{2}) \dot{\sigma}_x + \frac{\lambda + 3}{4} \dot{\sigma}_y \right] \\
\dot{\varepsilon}_z &= \frac{1}{E} \left[ -(\nu + \frac{\lambda - 1}{2}) \dot{\sigma}_x - (\nu - \frac{\lambda - 1}{4} \dot{\sigma}_y \right] \\
\ddot{\gamma}_{xy} &= \frac{2(1+\nu)}{E} \ddot{\tau}_{xy}
\end{align*}
\]

(3.7)

where

\[
\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z = \text{rate of change of strain components in the cartesian coordinates with respect to the independent parameter}
\]

\[
\dot{\sigma}_x, \dot{\sigma}_y, \dot{\sigma}_z = \text{rate of change of stress components in the cartesian coordinates with respect to the independent parameter}
\]

\[
\lambda = \frac{E}{E_t}
\]
The equations for unloading are:

\[
\begin{align*}
\dot{\varepsilon}_x &= \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) \\
\dot{\varepsilon}_y &= \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) \\
\dot{\gamma}_{xy} &= \frac{2 (1 + \nu)}{E} \dot{\tau}
\end{align*}
\]  

(3.8)

3.2 Potential Energy in the Plate

The strain energy per unit volume which is stored during the deformation of the plate may be expressed by:

\[
dW(x,y,z) = \int_0 \dot{\varepsilon}_x \sigma_x \, d\varepsilon_x + \int_0 \dot{\varepsilon}_y \sigma_y \, d\varepsilon_y + \int_0 \dot{\gamma}_{xy} \tau_{xy} \, d\gamma_{xy}
\]

(3.9)

where

\[W = \text{strain energy stored in the plate.}\]

This is an equation valid both in the elastic range and in the plastic range. The actual evaluation of the strain energy will be performed by introducing the stress-strain relationships presented in Section 3.1 and by integrating \(dW\) over the entire volume of the plate.

The total potential energy is obtained in the form of a summation of the strain energy stored in the plate, \(W\), and the work
done by the external forces acting on the plate, $U$, with an arbitrary
additive constant depending on the reference position. The value of
this arbitrary constant may be selected by making the potential energy
equation to zero for a suitable reference position. In stability
problems, it is usually convenient to take the loaded state prior to
buckling as such a reference position.

The potential energy at buckling, with the reference position
as that prior to buckling, may be expressed for the following cases:
(a) in the elastic region, and (b) in the plastic region, where both
the deformation and the flow theories are used.

(a) For the elastic part of the plate, the energy equation may
be shown to be (1)

$$V = \int \frac{D}{2} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2\nu \left( \frac{\partial^2 w}{\partial y^2} \right) \left( \frac{\partial^2 w}{\partial y \partial x} \right) + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx \, dy$$

$$- \int \left[ \frac{h}{2} \left( \sigma_x \left( \frac{\partial w}{\partial x} \right)^2 + 2\tau \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) + \sigma_y \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] dx \, dy \quad (3.10)$$

where

$$V = \text{potential energy of the plate.}$$

(b) In contrast with the above expression for the elastic
part, the energy equations for the plastic part of the plate are
presented below in two different forms. One is based on the secant
modulus deformation theory and the other on the flow theory,
Stowell improved Ilyushin's deformation theory by basing it on the Shanley concept that no strain reversal occurs in any part of the member under a uniform compressive load. Equation 3.11 was derived in this study, and resembles Stowell's equation. The difference between Stowell's equation and Eq. 3.11 is in the coefficients $C_3'$ and $C_3''$. In Stowell's equation $C_3' = C_3'' = 1 - \frac{3}{4} \frac{\tau_0 + 2 \tau^2}{\sigma_0^2}$.

$$V = \iint_D \left[ \frac{C_1'}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 - \frac{C_2'}{2} \left( \frac{\partial w}{\partial x} \right)^2 \left( \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{C_3'}{2} \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx \, dy$$

$$- \iint_D \left[ \frac{h}{2} \left( \frac{\sigma_x}{\partial x} \right)^2 + 2\tau \left( \frac{\partial w}{\partial x} \right)^2 \left( \frac{\partial^2 w}{\partial y \partial x} \right) + \sigma_y \left( \frac{\partial w}{\partial y} \right)^2 \right] dx \, dy$$

(3.11)

where

$C_1' = 1 - \frac{\sigma_x^2}{\sigma_0^2} (1 - \nu^2) \kappa$

$C_2' = \frac{4 \tau_{xy}}{\sigma_0^2} (1 - \nu^2) \kappa$

$C_3' = 2 \left[ (1 - \nu) - \frac{\tau_{xy}}{\sigma_0^2} (1 - \nu^2) \kappa \right]$

$C_3'' = 2 \left[ \nu - \frac{\sigma_x \tau_{xy}}{\sigma_0^2} (1 - \nu^2) \kappa \right]$

$C_4' = \frac{4 \tau_{xy}}{\sigma_0^2} (1 - \nu^2) \kappa$

$C_5' = 1 - \frac{\sigma_y^2}{\sigma_0^2} (1 - \nu^2) \kappa$

$\kappa = 1 - \frac{E_t}{E_s}$

$D_d$ = flexural rigidity of plate in the plastic range, based on the deformation theory, $= E_s \cdot h^3/12(1-\nu^2)$
When the integral of Eq. 3.11 is applied to the elastic zone, the coefficients change and $E_t = E_{sec} = E$ so that the Eq. 3.11 coincides with Eq. 3.10.

Pearson\(^{(24)}\) introduced Shanley's concept into Handelman-Prager's equations for plastic buckling of plates\(^{(20)}\) which were based on the flow theory. The following energy equation was derived in this study by using Pearson's considerations and the Handelman-Prager equations.

\[
\begin{align*}
V = & \iint \left\{ \frac{D_f}{2} \left[ C_1 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + C'_3 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + C''_3 \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] + C_5 \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right\} dx \, dy \\
& - \iint \left\{ \frac{h}{2} \left[ \sigma_x \left( \frac{\partial w}{\partial x} \right)^2 + \sigma_y \left( \frac{\partial w}{\partial y} \right)^2 + \sigma_{xy} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \right\} dx \, dy 
\end{align*}
\]

(3.12)

where

\[
\begin{align*}
C_1 &= \frac{(1-\nu^2)(\lambda + 3)}{(5-4\nu)\lambda - (1-2\nu)^2} \\
C'_3 &= 2(1-\nu) \\
C''_3 &= \frac{4(1-\nu^2)(2\nu + \lambda - 1)}{(5-4\nu)\lambda - (1-2\nu)^2} \\
C_5 &= \frac{4\lambda(1-\nu^2)}{(5-4\nu)\lambda - (1-2\nu)^2} \\
D_f &= \text{flexural rigidity of plate in the plastic range, based on the flow theory,} = E_h^3 / 12(1-\nu^2). 
\end{align*}
\]
Equation 3.12 may also be reduced to Eq. 3.10 for the elastic zone, since $E_t = E$.

The energy equations are easily handled in elastic-plastic problems by properly adjusting the coefficients of the strain energy for the elastic and plastic parts of the plate.

3.3 Theorem of Minimum Potential Energy and Equilibrium Differential Equation

From the theory of elasticity the theorem of minimum potential energy may be stated as

"Of all displacements satisfying the given boundary conditions those which satisfy the equilibrium equation make the potential energy an absolute minimum."

Conversely, when the potential energy of the body is a minimum the body is in a state of equilibrium.

This is valid only for an elastic body, but not necessarily obeying Hooke's law. This fact implies that as long as the body is not subject to unloading in the plastic range this theorem is in effect. In this study the two theories used for solving the buckling problem are the secant modulus deformation theory and the flow theory; both are modified by Shanley's concept. Consequently it is assumed that there is no place in the plate which is subject to
strain reversal at the instant of buckling and the theorem of minimum potential energy can be applied.

The fact that the theorem of minimum potential energy does lead to equilibrium differential equations in the plastic range (when no unloading occurs) was proved by comparing these equations with the equilibrium differential equations obtained from consideration of the equilibrium of an element of the body. This was done for both plasticity theories in this dissertation.

The differential equations of equilibrium of the plate in the plastic range are shown as follows for both plasticity theories:

(a) secant modulus deformation theory

\[
\begin{align*}
C_1 \frac{\partial^4 w}{\partial x^4} - C_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + C_3 \frac{\partial^4 w}{\partial x \partial y^3} - C_4 \frac{\partial^4 w}{\partial x \partial y} + C_5 \frac{\partial^4 w}{\partial y^4} &= -\frac{h}{D_4} \left[ \sigma_x \left( \frac{\partial^2 w}{\partial x^2} \right) + 2 \tau \left( \frac{\partial^2 w}{\partial x \partial y} \right) + \sigma_y \left( \frac{\partial^2 w}{\partial y^2} \right) \right] \\
\end{align*}
\]

(3.13)

where

\[
\begin{align*}
C_1 &= C_1 \quad \text{in Eq. (3.11)} \\
C_2 &= C_2 \\
C_3 &= C_3' + C_3'' \\
C_4 &= C_4 \\
C_5 &= C_5
\end{align*}
\]
(b) flow theory

\[ C_1 \frac{\partial^4 w}{\partial x^4} + 2C_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + C_5 \frac{\partial^4 w}{\partial y^4} = -\frac{h}{D_f} \left[ \sigma_x \left( \frac{\partial^2 w}{\partial x^2} \right) + 2\tau \left( \frac{\partial^2 w}{\partial x \partial y} \right) + \sigma_y \left( \frac{\partial^2 w}{\partial y^2} \right) \right] \]  \[(3.14)\]

where

\[ C_1 = c_1 \quad \text{in Eq. (3.12)} \]
\[ C_3 = c_3' + c_3'' \quad " \]
\[ C_5 = c_5 \quad " \]

The characteristic values of the above differential equations (3.13 and 3.14) give the exact values of buckling of strength.

In general, a calculation using this exact method for solution of the differential equations is much more involved than the energy method. If the solution of the differential equation is difficult to obtain, the energy method can be used as a powerful tool to solve the problem to sufficient accuracy for engineering purposes.

3.4 Introduction of Residual Stresses into the Plate Equation

In the case of an elastic isotropic plate of a constant thickness, h, which is subjected to edge thrust in the x direction
the equilibrium equation may be expressed as:\(^{(1)}\)

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - \sigma_x h \frac{\partial^2 w}{\partial x^2} \tag{3.15}
\]

Next, consider the edge thrust being applied in two steps. The stresses corresponding to these two load increments are \(\sigma_{x,y}^{(1)}\) and \(\sigma_{x,y}^{(2)}\) (Fig. 3.2). At the end of the first load increment, the differential equation becomes:

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - (\sigma_{x,y}^{(1)}) h \frac{\partial^2 w}{\partial x^2} \tag{3.16}
\]

where

\(\sigma_{x,y}^{(1)}\) = normal stress in x direction due to the first loading, varying with respect to the y ordinate.

Keeping the first load increment constant and applying the second load increment, the differential equation becomes:

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - (\sigma_{x,y}^{(1)} + \sigma_{x,y}^{(2)}) h \frac{\partial^2 w}{\partial x^2} \tag{3.17}
\]

where

\(\sigma_{x,y}^{(2)}\) = normal stress in x direction due to the second loading, varying with respect to the y ordinate.

The same concept may also be applied to a plate which consists of several parts, a, b, ..., Each of these parts is homogeneous.
in itself, but different from each other. Each part is subject to a stress from the first load increment. The following simultaneous differential equations hold true for the elastic isotropic plate when the second load increment is uniformly applied to the whole plate.

\[
D_a \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - \left[ \sigma_a^{(1)} + E_a \varepsilon_g^{(2)} \right] h_a \frac{\partial^2 w}{\partial x^2}
\]

\[
D_b \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - \left[ \sigma_b^{(1)} + E_b \varepsilon_g^{(2)} \right] h_b \frac{\partial^2 w}{\partial x^2}
\]

\[
D_m \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = - \left[ \sigma_m^{(1)} + E_m \varepsilon_g^{(2)} \right] h_m \frac{\partial^2 w}{\partial x^2}
\]

where

\( D_a, D_b, \ldots D_m \) = bending rigidity of plate in the domains of \( a, b, \ldots \)

\( \sigma_a, \sigma_b, \ldots \sigma_m \) = normal stresses in the domains of \( a, b, \ldots \) due to the first loading

\( E_a, E_b, \ldots E_m \) = Young's Modulus in the domains of \( a, b, \ldots \)

\( h_a, h_b, \ldots h_m \) = thickness of plate in the domains of \( a, b, \ldots \)

\( \varepsilon_g \) = increment of strain due to second loading

The solutions to Eqs. 3.18 must satisfy the proper boundary conditions along the conjunction of each part (including the edges of the plates).

The boundary condition may be expressed as:
Introducing the boundary conditions into the solutions of the differential equations \( n \) simultaneous homogeneous equations are obtained for \( w \) from Eq. 3.19. Therefore, a non-trivial solution will be obtained only when the coefficient determinant is zero. The characteristic value of \( \varepsilon^m \) corresponds to the critical buckling strain of a plate with a system of initial loadings.

The procedure of load increments may be illustrated for the energy method in a manner similar to that just completed for the differential equation.

The corresponding coefficient determinant to the simultaneous differential equation, Eq. 3.19, may be obtained by applying the Ritz Method to the following energy equation, which is Eq. 3.10 modified by taking into account the load increments.

\[
V = \iint \frac{D_m}{2} \left\{ \frac{\partial^2 w}{\partial x^2} + 2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} + 2\nu \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \right\} \, dx \, dy
- \iint \frac{h_m}{2} \left( \sigma_{m}^{\varepsilon} + E_m \varepsilon^{\varepsilon} \right) \left( \frac{\partial w}{\partial x} \right)^2 \, dx \, dy
\]  

(3.20)
In Eq. 3.20, $D_m, E_m, h_m, \sigma_m^{(1)}$, and $\nu$ are known for each domain and $\varepsilon^{(2)}$ is unknown. The integration must be carried out separately in each domain. The extreme value of the total potential energy will be determined from the homogeneous equations, that is,

$$\frac{\partial V}{\partial \alpha_p} = 0 \quad (p = 1, 2, \ldots, n.) \quad (3.21)$$

where $\alpha_p$ is the coefficient of a sequence of functions which are assumed as the deflection, the functions satisfy the end conditions and the sequence of families of the functions are relatively complete. The determinant of Eq. 3.21 with respect to $\alpha_p$ corresponds to the determinant from the simultaneous differential equations Eq. 3.18.

If the assumed shape of the deflection surface is the exact one, the final solution is the same as the one which is obtained from the solution of the differential equation.

When the first load increment is the residual stress distribution in the plate, the methods explained in this section will determine the critical buckling strain for a plate containing residual stresses. Moreover, the principle of this method of solution is applicable to both the elastic-plastic and the plastic cases within the limits to which the theorem of minimum potential energy is valid, that is, no unloading. When $\sigma^{(1)}$ is regarded as a residual stress, an additional condition is imposed in that the distribution of $\sigma^{(1)}$.
must be in equilibrium across the plate width.

The above discussion was concerned with the problem of the buckling of a plate with a system of residual stresses in one direction, \( \sigma_{x,y}^{(0)} \).

In practice it is quite possible to have additional residual stresses, that is, in the other direction, such as \( \sigma_{y,x}^{(0)} \) and/or \( \tau_{xy}^{(0)} \).

The same procedure as used above for the introduction of residual stresses \( \sigma_{x,y}^{(0)} \) into the equilibrium differential equation may be directly used for the other systems of residual stresses, \( \sigma_{y,x}^{(1)} \) and/or \( \tau_{xy}^{(1)} \). The differential equation of a plate with residual stresses, \( \sigma_{x,y}^{(1)} \), \( \sigma_{y,x}^{(1)} \) and \( \tau_{xy}^{(1)} \) under thrust in the direction of the x axis may be expressed by Eq. 3.22, which corresponds to Eq. 3.18.

\[
D_m \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = -h_m \left( \left( \sigma_{x,y}^{(1)} + E_m \varepsilon_{y}^{(2)} \right) \frac{\partial^2 w}{\partial x^2} + 2 \tau_{xy}^{(1)} \frac{\partial^2 w}{\partial x \partial y} + \sigma_{y,x}^{(1)} \frac{\partial^2 w}{\partial y^2} \right) \tag{3.22}
\]

Similarly, the energy equation in this case may be shown to be:
\[ V = \int \int \frac{D_m}{2} \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2\nu \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right\} \, dx \, dy \]

\[ -\int \int \frac{h_m}{2} \left\{ \left( \sigma_{xy}^{(0)} + E \epsilon_{xy}^{(0)} \right) \left( \frac{\partial w}{\partial x} \right)^2 + \tau_{xy} \left( \frac{\partial w}{\partial y} \right) \left( \frac{\partial w}{\partial x} \right) + \sigma_{yy} \left( \frac{\partial w}{\partial y} \right)^2 \right\} \, dx \, dy \]  

(3.23)

The Eqs. 3.22 and 3.23 are most general forms of Eqs. 3.17 and 3.20 and are applicable to a problem of plate buckling with combined type of residual stresses. All these equations are for the elastic domain, but similar equations may be derived for the plastic domain, by taking into account the yield condition.
4. **ANALYSIS OF BUCKLING STRENGTH OF A PLATE WITH RESIDUAL STRESSES**

4.1 **General Approach**

In general, the buckling problem is handled mathematically as a boundary value problem. From these equations characteristic values are obtained, that is, critical loads. The energy method for the stability problem leads to the so-called minimum values which are obtained from the calculus of variation.

The exact solution of the differential equations of this dissertation is very difficult to obtain. For this reason the problem was solved by the energy method.

In Chapter 3 the stress-strain relationship and the strain energy equations were presented for the elastic and plastic ranges. Adopting a proper stress-strain relationship for each domain of the plate, and assuming a suitable series of deflection functions which are relatively complete, the potential energy in the plate may be evaluated in the elastic or in the plastic parts taking into account the effect of residual stresses in the plate. This potential energy for the plate will be minimized with the aid of the Ritz method,
which gives the equilibrium condition according to the theorem of minimum potential energy. This results in a set of simultaneous homogeneous equations. The non-trivial solution of this set of simultaneous equations is obtained only if the coefficient determinant is equal to zero. The roots of such a determinant correspond to the boundary-value problem characteristic values of which the lowest is the desired buckling load.

In the analysis, the material is presumed to be steel and assumed to be elastic perfectly plastic and incompressible.

4.2 Residual Stress Distribution

Steel structures fabricated by welding contain residual stress due to plastic deformations set up by the temperature gradient induced at welding.

Two residual stress patterns may be regarded as typical for welded plates and for shapes fabricated from plates by welding. One is due to an edge weld, and the other due to a center weld.
Built-up box sections, stiffened plates, webs of I-shapes and channel sections, all correspond to the first residual stress distribution, and flanges of I sections to the second distribution.

The buckling strength of these plate elements may be investigated in the same way.

The main purpose of this dissertation is to investigate the local buckling of built-up box sections. For this reason, this study was directed to the buckling of plates with residual stresses due to edge welds.

The residual stress patterns of welded plates have been studied both theoretically and experimentally. The residual stress pattern in this study will be of the form shown in Fig. 4.1, which corresponds to the pattern obtained in experimental work.

It is advantageous to simplify the residual stress distribution for the analysis of the buckling strength of plates. Such simplifications may be made in the form of straight, parabolic or sinusoidal lines, as shown in Fig. 4.2.

The curve may be chosen from the viewpoint of the accuracy of the simplified curve and the simplicity of handling in the analysis.

The residual stress distribution assumed must satisfy the requirement that the equilibrium of residual stresses in the plate
must be maintained and that the stress at any point may not exceed
the yield point of the material. Usually, the curve is assumed to
be symmetric with respect to its center line because of symmetry
of plate and of welding conditions.

The residual stress distribution is normally the same for
similar plates welded under similar conditions. But the distribu-
tion is not always the same for different plate sizes and differ-
ent welding conditions. For example, Fig. 4.2(c) is a good approxi-
mation for small widths. With increasing width, the simplified
distribution of Fig. 4.2(e) is a good approximation with its con-
stant residual stress distribution; it is appropriate in most prac-
tical cases. The residual stress distribution along the length is
the same except in the vicinity of the end portion, approximately
the dimension of the width.

The assumption that the distribution of residual stress does
not vary along the plate infers that shearing stress due to residual
stress can not exist in the plate. This is shown in Appendix A.

The residual stress distribution shown in Fig. 4.3(a) was
chosen as a pattern appropriate for this study.

By adjusting appropriate parameters this distribution of
residual stress can be readily reduced to other patterns such as
those shown in Fig. 4.3(b), Fig. 4.3(c) and Fig. 4.3(d).
Referring to Fig. 4.3(a) when \((b - b_2)\) is assumed to be zero, the distribution changes to that of Fig. 4.3(b). When \((b_2 - b_0) = 0\) Fig. 4.3(c) results, and when \((b - b_2) = 0, b_0 = 0\), the distribution is transformed to the pattern of Fig. 4.3(d). In particular, the distribution of Fig. 4.3(c) is taken as a reasonable simplification for plates with large b/t ratios.

Strictly speaking, the structural steel plate is neither homogeneous nor isotropic. This is particularly true when the stress is near the yield point. But it is customary to regard the material as homogeneous and even isotropic in the elastic range since the resulting error is rather small.

When the material is subjected to heat treatment due to welding, the material properties change and among the mechanical properties the change in the yield point is pronounced in the vicinity of weld due to the introduction of new metal, electrode, and due to the heat input. In built-up box sections, these changes are limited to small areas around the weld and the change there has only a slight influence on the buckling strength of the plate because the weld and the heat affected zone are at, and close to, the boundary of plate element. The plastification of the material starts at the middle and progresses to the edges under the compressive load. The stress at the heat-affected zone will remain tensile even with compressive stresses acting on the plate due to external forces in the case of elastic and elastic-plastic buckling of plates.
The elastic-plastic and plastic buckling of a plate with residual stresses may occur earlier or later than the buckling of a plate free of residual stresses, depending upon the type of residual stress. Early buckling is caused by early plastification when the residual stress is compressive in the middle of the plate. Likewise, later buckling occurs when the residual stress is tensile in the middle of the plate. Therefore, a low estimation of the yield point at and in the vicinity of the weld is presumed to have very little effect on the prediction of buckling strength of the plate since the weld is located at the supporting edge.

In this study, the yield point of the material is assumed to have the same value over the entire plate in both tension and compression and equal to the average value of the yield point of the parent material.

4.3 **Relationship Between Stress and Strain, and Secant and Tangent Moduli Under Load**

When a plate free of residual stresses is subjected to a thrust on opposite edges, the stress and the strain in the plate are distributed uniformly. The relationship between them is specified by Eqs. 3.2, 3.5 and 3.7 in the elastic and plastic ranges. The secant modulus was defined as $E_s = \frac{\sigma_l}{\varepsilon'_l}$ (Eq. 3.4), and the tangent modulus is zero in the plastic range because the material is assumed to be perfectly plastic.
For the case where residual stresses exist in the plate, the stress and strain in the plate are no longer uniform, although their relationship at any point is still governed by Eqs. 3.2, 3.5 and 3.7.

The basic residual stress distribution used for computation is given in Fig. 4.3(a). When edge thrust is applied, the average increment of strain, $\Delta \varepsilon$ is superimposed upon the residual strain and the actual strain at any point is $\varepsilon_x = \frac{\sigma_r}{E} - \Delta \varepsilon$. Therefore, the stress at the point is $\sigma_x = \left( \frac{\sigma_r}{E} - \Delta \varepsilon \right) \varepsilon$ in the elastic range and $\sigma_x = -\sigma_y$ at and above the yield strain.

For elastic buckling, the critical stress may be expressed by the product of Young's modulus and the critical strain; for plastic buckling, the critical stress can not exceed the yield stress in compression. Elastic-plastic buckling has a complicated relationship between the strain and the growth of the plastic zone. The following two equations give this relationship:

\[
\frac{\sigma_{cr}}{\sigma_Y} = \frac{\sigma_c}{\sigma_Y} - \frac{1}{2} \left( \frac{b_0 + b_1}{b} \right) \left[ \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_c}{\sigma_Y} - 1 \right] \tag{4.1}
\]

\[
\frac{b_j}{b} = \frac{b_0}{b} + \left( \frac{b_2}{b} - \frac{b_0}{b} \right) \frac{\left[ \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_c}{\sigma_Y} - 1 \right]}{\left[ \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_c}{\sigma_Y} \right]} \tag{4.2}
\]
where

\[
\sigma_c = \varepsilon c E = \sigma_{cr} \text{ for elastic buckling only}
\]

\[
\sigma_{cr} = \text{average critical normal stress}
\]

\[
\sigma_r = \text{magnitude of maximum compressive residual stress in the assumed pattern}
\]

\[
\sigma_{rz} = \text{magnitude of maximum tensile residual stress in the assumed pattern}
\]

\[
\sigma_y = \text{yield stress in tension or compression}
\]

When a plate is subjected to an edge thrust, the states of stress and strain are defined according to the magnitude of the thrust and of the residual stress. These relationships are illustrated in Fig. 4.4. Figure 4.4(a) is the basic residual stresses distribution and the corresponding strains are evaluated by dividing the stresses by \(E\) (Young's Modulus). The relationship between stress and strain is listed in Table 4.1.

When the thrust increases the strain by \(\Delta \varepsilon\) then, depending on the magnitude of \(\Delta \varepsilon\), the plate may be in an elastic, elastic-plastic or plastic state. Figure 4.4(b) shows stress and strain distribution in the case of elastic buckling and Table 4.1 presents their relationship. When some part of the plate reaches the yield point, the plate will be composed of elastic parts and a plastic part. Figure 4.4(c) shows the stress and strain distribution. Table 4.1 gives the relationship between stress, strain and secant modulus. Similarly, Fig. 4.4(d) and Table 4.1 give information on stress, strain and secant modulus of the plate for plastic buckling.
It should be mentioned here that the residual strain at and in the vicinity of the weld is generally higher than the tensile yield strain of the parent material, but the stress there cannot be greater than the tensile yield point. As the edge thrust causes additional compressive strains in the plate, the element of material where the residual strain is greater than the tensile yield strain behaves according to the elastic law since there is an unloading of tension.

4.4 Local Buckling of Built-Up Columns

Among the problems of buckling of plates with residual stresses of this investigation, special attention was paid to the study of the local buckling of built-up columns. Built-up columns may be classified into two groups with respect to the shape of the cross section, that is, closed sections or open sections.

The method of solution of this kind of problem is quite similar for both sections except for the boundary conditions at the free edges.

The study in this dissertation is limited to the case of the closed section.

A closed column is composed of several walls each of which consists of a flat plate. That is, the local buckling strength of this kind of columns is reduced to the problem of buckling of plates
connected at their edges. The local buckling of a column is solved under the following assumptions:

1. The column does not buckle before local buckling of the walls occurs.
2. The deflection at the conjunction of each plate is zero.
3. The deflection and bending moment at both loading ends are zero.
4. The angle between two adjacent plates does not change.
5. The wave length of buckling is identical on each wall, and there is no phase lag between the walls.

Then, making reference to Fig. 4.5 the above-mentioned assumptions may be expressed in the form of equations:

Assumption 2 may be written as

\[ w_i = 0 \quad \text{at} \quad y_i = \pm \frac{b}{2} \quad (4.3) \]

where

\[ w_i = \text{deflection of plate on the side } i \]
\[ y_i = \text{y-axis of cartesian coordinate on a plate on the side } i \]
\[ \frac{b}{2} = \text{half width of plate element on the side } i \]

Assumption 4 may be written as
\[ \begin{align*} \theta_i &= \theta_{i+1} \\ M_i &= M_{i+1} \\ \theta_i &= \theta_{i-1} \\ M_i &= M_{i-1} \end{align*} \]  
\[ y_i = \omega \]
\[ y_i = -b \]

where

\( \theta_i \) = angle of rotation at edge of plate \( i \)

\( M_i \) = bending moment per unit length of section of plate perpendicular to \( x \) axis

The assumptions 3 and 5 suggest the following equation for the deflection function:

\[ w_i = f_i(y_i) \sin N \pi \frac{L}{L} \]  
\[ (4.5) \]

where

\( N \) = number of half waves in the direction of the \( x \) axis

\( L \) = entire length of column

\( f_i(y) \) = deflection function in the direction of the \( x \) axis

Special attention should be paid to assumption 3 concerning the influence of the aspect ratio of the plate elements upon the critical strain. According to assumption 3, no deflection and no bending moment at the loading edges are assumed, although it
may not be true in actual cases, and this infers that the longitudi-

dinal deflection may be expressed in the form of \( \sin N \pi \frac{N}{L} \). \( (4.4) \)

Studies\(^{(1)(45)}\) on the elastic buckling of flat plates have shown that, for the plate with an aspect ratio of more than 4, the buckling strength is almost identical in both of the following two cases, simply supported or fixed at the loading edges, regardless of edge conditions. From this, it is quite rational to estimate that the walls of a column will buckle at the ratio of \( \ell/b \), which gives the lowest critical value, that is, the case of a plate simply supported at the loading edges, regardless of the conditions of the other edges. The local buckling of a built-up column may be predicted from this point of view. Even though assumption 3 may not be correct in actual cases, it gives a good prediction for the critical strain because the problem under study is concerned with the local buckling of built-up columns and the aspect ratio of plate elements is more than four in most practical cases.

If the solution, \( \omega \), of the differential equation containing the effect of residual stresses is obtained for each wall of the column, it will have four integration constants. The boundary conditions, previously mentioned, give as many homogeneous equations as there are integration constants in the solution of deflection. These equations have no independent coefficients, which are not zero simultaneously. To get the solution for this group of equations, the coefficient determinant of the equations must be zero. In general, it is not easy to find a solution for a boundary
value problem except in simple cases. In this study it was not possible to obtain the particular solution for the differential equation except for the case where the residual stress distribution is that of Fig. 4.3(c). In this case, the exact solution can be obtained under the basic assumptions, but, the actual calculation is still tedious and lengthy when compared with the energy method. To obviate these difficulties, the energy method was chosen for this study.

When a column consists of $s$ number of walls, it has $s$ number of corners. Each corner supplies four boundary conditions, $w = 0$, $M_i = M_{i+1}$ and $\theta_i = \theta_{i+1}$, that is, there are $4s$ boundary conditions.

In the energy method, if a deflection function is assumed in a form which has $m$ independent coefficients for each plate, $m$ independent constants need to be determined. The required number of conditions are supplied by the process of minimization of the potential energy, $V$, with the aid of the Ritz Method,

$$\frac{\partial V}{\partial a_i} = 0 \quad i = 1, 2, \ldots, m \cdot s \quad (4.6)$$

Enough conditions then exist to analyze the problem.

4.5 Analytical Solutions for Buckling Strength of Plates with Residual Stress

Analytical solutions are obtained for the elastic, elastic-plastic and plastic buckling of a plate with residual stresses when
the plate is simply supported at the loading edges and at the other edges is:

a) elastically restrained
b) simply supported
c) fixed.

From these results, the local buckling strength of a built-up column of rectangular cross section will be obtained. Case (a) corresponds to a rectangular cross section, case (b) to a square cross section and case (c) to the limiting case. All these cases are shown in Fig. 4.6.

4.5.1. Plate Elastically Restrained

- Rectangular Cross Section -

This type of cross section consists of two different pairs of plates. (Fig. 4.7)

The following additional assumptions may be added to those given in Section 4.4.

1. Material properties of all plates are the same: yield point, Young's modulus, Poisson's ratio, both in the elastic and plastic ranges.

2. Each pair of parallel plates are of the same size.

3. Each pair of parallel plates has the same residual stress distribution.
4. The residual stress distribution in both pairs of plates is similar in shape to each other (Fig. 4.7).

Assumption 4 is reasonable for practical cases since the size of plates do not differ so much as to produce different patterns of residual stresses. Moreover, this assumption leads to the fact that the plastification starts in both pairs of plates at the same time, leading to a certain degree of simplification in the computations.

The following considerations were made in the assumption of a deflection function for the theoretical study. In the elastic buckling of a plate without residual stresses, a plate simply supported at four edges buckles in a sinusoidal form under edge thrust. On the other hand, in the case of a plate simply supported at the loading edges and fixes at sides, the deflection function,

$$w = \left[ \cos \frac{\pi}{b} y - 1 \right] \sin \frac{N \pi}{L} \frac{x}{L}$$

as used by H. L. Cox (48) is not a true deflection but a very good approximation. Each element of a rectangular box column is regarded as a plate which is supported simply at the loading edges and with various degrees of fixity at the other edges. Consequently, it is reasonable to assume that the plate will buckle into a form between those expressed by the deflection functions in the cases of (b) and (c). A more accurate result is obtained by taking a series of the deflection functions in the y direction instead of a first one term. Although the symmetry of deflection was tacitly assumed in the above
discussion, it is quite rational because the structure and its residual stress distribution are considered to be symmetric with reference to their center lines.

A combination of two series of sinusoidal functions was chosen as the deflection function for each plate element of a rectangular box column, as shown in Eq. 4.7.

\[
\begin{align*}
\omega_1 &= \left[ \sum a_{m} \cos \left( \frac{2m-1}{2} \pi \frac{y_1}{b} \right) + \sum c_{n} \left\{ \cos \left( n \pi \frac{y_1}{b} \right) - 1 \right\} \right] \sin n \pi \frac{x}{L} \\
\omega_2 &= \left[ \sum a_{2m} \cos \left( \frac{2m-1}{2} \pi \frac{y_2}{b} \right) + \sum c_{2n} \left\{ \cos \left( n \pi \frac{y_2}{b} \right) - 1 \right\} \right] \sin n \pi \frac{x}{L}
\end{align*}
\]

(4.7)

where

\[
\begin{align*}
a_{m}, a_{2m}, c_{n}, c_{2n} &= \text{coefficients of deflection functions} \\
m, n &= \text{positive integers}
\end{align*}
\]

These deflections must satisfy the boundary conditions enumerated in Section 4.4. The symmetry of the structure and the assumed deflection reduces the number of the boundary conditions, so that the boundary conditions at only one corner are sufficient for the solution.

These deflection functions satisfy the condition that the deflection vanishes at the edges, that is

\[
\begin{align*}
\omega_1 &= 0 \quad \text{at} \quad y_1 = \pm \frac{b}{2} \\
\omega_2 &= 0 \quad \text{at} \quad y_2 = \pm \frac{b}{2}
\end{align*}
\]

(4.8)
The second condition is that the change of slope of both plates at their conjunction is the same,

\[ \theta_1 \omega = \theta_2 \omega \quad y_1 = \pm b \quad y_2 = \mp b \] (4.9)

or, rewriting

\[ \left( \frac{\partial \omega^1}{\partial y_1} \right) = \left( \frac{\partial \omega^2}{\partial y_2} \right) \quad y_1 = \pm b \quad y_2 = \mp b \] (4.10)

Substituting \( \omega^1 \) and \( \omega^2 \) into Eq. 4.10, the relation of the coefficients may be obtained.

\[ \sum a_{zm} = \alpha \sum a_{im} \] (4.11)

where

\[ \alpha = -\frac{b}{\omega} \]

The bending moment at the corner of both plates must coincide with each other. This condition infers that,

\[ M^1 = M^2 \quad y_1 = \pm b \quad y_2 = \mp b \] (4.12)

The bending moment in the plate may be expressed in the form of
\[
M = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \tag{4.13}
\]

At the edge, \( \omega = \frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0 \) and consequently the bending moment takes the following form: \( M = -D \left( \frac{\partial^2 w}{\partial y^2} \right) \), and the boundary condition is

\[
D \left( \frac{\partial^2 w_1}{\partial y_1^2} \right)_{y_1 = \pm b} = D \left( \frac{\partial^2 w_2}{\partial y_2^2} \right)_{y_2 = \mp b} \tag{4.14}
\]

where \( D_1 \) and \( D_2 \) are the respective bending rigidities of the plates, and, from Section 3.3,

\[
D = \frac{E h^3}{12 (1 - \nu^2)} \quad \text{in the elastic range}
\]

\[
D = \frac{E_s h^3}{12 (1 - \nu^2)} = \frac{E_s h^3}{12 \nu} \quad \text{in the plastic range (deformation theory)}
\]

\[
D = \frac{E h^3}{12 (1 - \nu^2)} = \frac{E h^3}{12 
u} \quad \text{in the plastic range (flow theory)}
\]

The boundary condition Eq. 4.14 gives the relationship between the coefficients in the expression for \( w_1 \) and \( w_2 \) as

\[
\sum c_{2n} = \beta \sum c_{1n} \tag{4.16}
\]

where

\[
\beta = \left( \frac{D_1}{D_2} \right) \left( \frac{b_2}{b_1} \right)^2
\]
The coefficients of the deflection functions were determined so as to satisfy all boundary conditions. Substituting the relationship obtained above (Eqs. 4.11 and 4.16) into Eq. 4.7 the final assumed deflection may be shown to be

\[ w_1 = \left[ \sum a_m \cos \left( \frac{2m-1}{2} \pi \frac{y}{b} \right) + \sum c_n \left\{ \cos \left( n \pi \frac{y}{b} \right) - 1 \right\} \right] \sin N \pi \frac{x}{L} \]

\[ w_2 = \left[ \alpha \sum a_m \cos \left( \frac{2m-1}{2} \pi \frac{y}{b} \right) + \beta \sum c_n \left\{ \cos \left( n \pi \frac{y}{b} \right) - 1 \right\} \right] \sin N \pi \frac{x}{L} \]  \hspace{1cm} (4.17)

The first term in the brackets of the equation corresponds to the deflection of a plate simply supported at all four edges and the second term is associated with that for a plate simply supported at the loading edges and fixed at the other edges.

Reasonably accurate results are obtained by taking only the first term of each series \((m = 1, n = 1)\) in Eq. 4.17. Then the assumed deflection becomes:

\[ w_1 = \left\{ \alpha \cos \left( \frac{\pi}{2} \frac{y}{b} \right) + \beta \left\{ \cos \left( \frac{\pi}{2} \frac{y}{b} \right) - 1 \right\} \right\} \sin N \pi \frac{x}{L} \]

\[ w_2 = \left\{ \alpha \cos \left( \frac{\pi}{2} \frac{y}{b} \right) + \beta \left\{ \cos \left( \frac{\pi}{2} \frac{y}{b} \right) - 1 \right\} \right\} \sin N \pi \frac{x}{L} \]  \hspace{1cm} (4.18)

Introducing the above equations into the expression for the energy integral and carrying out the integration in each part, the elastic parts and the plastic part, and taking into consideration the different stresses and the secant moduli, the total potential energy will be obtained. \( V = V_1 + V_2 \) (where \( V_1 \) and \( V_2 \) are the potential energy...
in each plate).

Using the Ritz method and minimizing the potential energy, the partial differentiation with respect to the coefficients \( a \) and \( c \) leads to the following homogeneous equations:

\[
\frac{\partial V}{\partial a} = 0 \quad \text{and} \quad \frac{\partial V}{\partial c} = 0
\]

(4.19)

with the result that

\[
\begin{aligned}
a \ F_{11} + c \ F_{12} &= 0 \\
\ F_{21} + c \ F_{22} &= 0
\end{aligned}
\]

Each component in the above equation, \( F_{11}, F_{12}, F_{21}, \) and \( F_{22} \), is listed in Appendix B in the sequence of elastic, elastic-plastic (deformation theory), elastic-plastic (flow theory), and plastic buckling (deformation theory) cases. Consequently, the requirement that the coefficient determinant of the above equations is zero gives the stability condition

\[
F_{11} \ F_{22} - F_{12} \ F_{21} = 0
\]

(4.21)

from which the critical value can be computed.
4.5.2. **Plate Simply Supported**

- Square Cross Section -

The square cross section is the most common for columns in practical use. In dealing with this problem it was assumed that the four plates which compose a square column are completely identical to each other in material properties, size of plates and the distribution of the residual stress.

The complete symmetry of the structure and of the residual stress distribution render the analysis comparatively simple. The solution for this case will be obtained as a limiting case of the previous problem. The assumed deflection function must satisfy the same boundary conditions as before. The symmetry of this structure leads to the fact that only one plate simply supported at all the edges need be investigated.

Choosing only terms associated with the deflection of a simply supported plate in Eq. 4.17 and, taking \( m = 2 \) the following equation is obtained, which satisfies all the boundary conditions.

\[
\psi = \left[ a_1 \cos \left( \frac{\pi y}{2b} \right) + a_2 \cos \left( \frac{3\pi y}{2b} \right) \right] \sin N\pi \frac{x}{L} \quad (4.22)
\]

Following exactly the same procedure as in the previous section, the final results were obtained, with
\[
\begin{align*}
\alpha_1 F_{11} + \alpha_2 F_{12} &= 0 \\
\alpha_1 F_{21} + \alpha_2 F_{22} &= 0
\end{align*}
\] (4.23)

and

\[
F_{11} F_{zz} - F_{12} F_{z1} = 0	ag{4.24}
\]

which gives the critical strain. \( F_{11}, F_{22} \) and \( F_{12} (= F_{21}) \) are presented in Appendix B.

The following equation gives the first approximation for the critical buckling strain:

\[
F_{11} = 0
\] (4.25)

This corresponds to the limiting case of the previous problem and also to the case where deflection is assumed as:

\[
\omega = a \cos \left( \frac{\pi y}{b} \right) \sin N \pi \frac{x}{L}
\] (4.26)

When there is no residual stresses in the plate this deflection is the exact one, according to the explanation of Section 4.5.1.

4.5.3 Plate Fixed

- Limiting Case

Another limiting case of the rectangular section is that which corresponds to a pair of opposite plates which have infinite
bending rigidity.

The boundary conditions in this case are fundamentally the same as case (a) of Section 4.5, except the condition that \( \theta_1 = \theta_2 = 0 \) at the edges. This leads to \( q_1 = 0 \) in the deflection function.

The second term in the brackets of the deflection Eq. 4.17 (the case of rectangular cross section) fulfills these boundary conditions. That is

\[
\omega = \sum_{n=1}^{\infty} c \left[ \cos \left( n \pi \frac{y}{b} \right) - 1 \right] \sin n \pi \frac{x}{L} \tag{4.27}
\]

and for \( n = 2 \),

\[
F_{11} F_{22} - F_{12} F_{21} = 0 \tag{4.28}
\]

which gives the critical strain.

The components of the determinant, \( F_{11}, F_{22}, F_{12} (= F_{21}) \), are shown in Appendix B.

The first approximation uses the deflection,

\[
\omega = c \left[ \cos \left( \pi \frac{y}{b} \right) - 1 \right] \sin n \pi \frac{x}{L} \tag{4.29}
\]

which is identical to that used by H. L. Cox\(^{(48)}\) for the elastic buckling without residual stresses.
As explained in the previous section, $F_{11} = 0$ gives the first approximation for this solution, and is the same as for the limiting case of rectangular sections.
5. Numerical Illustrations for Local Buckling of Built-Up Columns of Square Cross Section

5.1 General Method of Calculation

Among the many types of built-up columns, box columns of square cross section are the most practical and economical (31) because the cross section is closed, and is symmetrical about both major axes with the material distributed away from its centroid. These facts result in advantages such as greater torsional rigidity and greater radius of gyration in both directions.

From these points of view, and taking into account the utility of the results, the built-up column with square cross section was selected for the numerical studies.

When the plate sizes and the distribution of residual stress are specified, the critical stress or strain of the elastic, elastic-plastic and plastic buckling may be obtained from Eqs. 4.24 and 4.25. Rewriting

\[ F_{ll} = 0 \]  \hspace{1cm} (5.1)

or

\[ F_{ll} F_{22} - F_{12} F_{21} = 0 \]  \hspace{1cm} (5.2)
In order to save time the numerical calculation was carried out by a digital computer, the LGP 30 at Lehigh University. The program was prepared by the author according to the "Act III" code for this computer.

Poisson's ratio, \( \nu \) is assumed to be 0.3 in the elastic range and 0.5 in the plastic range. It is commonly accepted that the average value of Poisson's ratio for steel is 0.3 in the elastic range. Some sample calculations in this study showed that a variation of Poisson's ratio within 5% results in a negligible difference in the critical b/t ratio. Therefore it is quite reasonable to take 0.3 for Poisson's ratio for steel in the elastic range regardless of the kind of steel. In the plastic range, this ratio was assumed 0.5 because of the assumption that the material is incompressible in the plastic range.

In the process of finding the critical strain, some difficulties were encountered in the case of elastic-plastic buckling, in particular, the difficulty of arranging the equation for obtaining \( \varepsilon_c \). The manner of obtaining \( \varepsilon_c \) directly, results in a trial and error method.

The equations for buckling may be solved in two ways: (a) the trial and error method for determining \( \varepsilon_c \) for a given (b/t) ratio, and (b) the direct method which gives the critical (b/t) ratio for a particular \( \varepsilon_c \). The trial and error method would normally not be used and would only be used when determining buckling
loads for a plate of a particular geometry. For this study, the direct method was used, since it entails the least computation for a general study.

The following sections will be devoted to the numerical analysis of the elastic, elastic-plastic and plastic buckling of a simply supported plate, an element of a built-up column, taking into account the influence of residual stress, and the results were summarized in Fig. 5.1 to Fig. 5.11.

5.2 Elastic Buckling

The first computation carried out was a comparison of the accuracy obtained by taking the first term and the first two terms of the equations for deflection, Eqs. 4.26 and 4.22 respectively. The sample computation indicated that the use of the first term is sufficiently accurate to analyze the elastic buckling problem.

The critical strain may be computed from Eq. 5.1, $F_{11} = 0$, which may be arranged more explicitly in its final form as:

$$\frac{\varepsilon_c}{\varepsilon_Y} = \frac{\sigma_c}{\sigma_Y} = \left( \frac{\sigma_{cr}}{\sigma_Y} \right)_0 - R \quad (5.3)$$

where

$$\left( \frac{\sigma_{cr}}{\sigma_Y} \right)_0 = \text{ratio of critical stress to yield point, elastic buckling, without residual stresses}$$

$R = \text{reduction of buckling strength due to residual stresses} = f \left( \frac{\sigma_a}{\sigma_Y}, \frac{\sigma_z}{\sigma_Y}, \frac{b_a}{b}, \frac{b_z}{b} \right)$
Equation 5.3 shows that the influence of residual stresses may be evaluated from the residual stress distribution independently of the critical stresses.

The value of \( \left( \frac{\sigma_{er}}{\sigma_Y} \right)_o \), the elastic buckling without residual stress, may be taken from reference books, since it has been solved completely by others.

In this study the shape and distribution of residual stresses were investigated with respect to their influence on R, the reduction of buckling strength. When the values of \( \sigma_{r_1} \) and \( \sigma_{r_2} \) are specified, the widths of compressive residual stress and tensile residual stress may be determined from the equilibrium condition.

The effect of the residual stress distribution on the reduction, R, was studied. The residual stress distribution is defined by \( \mu \) in Fig. 5.1. Keeping \( \sigma_{r_1} \) and \( \sigma_{r_2} \) constant, R was evaluated for varying \( \mu \). The result is shown in Fig. 5.1. For smaller values of \( \mu \), R becomes less. The most severe reduction is for the distribution when \( \mu \) is unity, that is, the residual stress distribution is of rectangular shape.

The relationship between the reduction of buckling strength and the amount of compressive residual stresses is presented in Fig. 5.2. The curves for R vs. \( \sigma_{r_1} / \sigma_Y \) start to deviate from the straight line, \( R = \sigma_{r_1} / \sigma_Y \) at approximately \( \sigma_{r_1} / \sigma_Y = 0.15 \).
The deviation may be described with very good accuracy by a second order parabola. An approximate equation for \( R \) is proposed in the form,

\[
R = \left( \frac{\sigma_{\text{cr}}}{\sigma_Y} \right) - k \left( \frac{\sigma_{\text{rl}}}{\sigma_Y} \right)^2
\]  
(5.4)

where

\[
k = \text{the deviation of the curve from the line}
\]

\[
R = \frac{\sigma_{\text{rl}}}{\sigma_Y} \quad \text{at} \quad \frac{\sigma_{\text{rl}}}{\sigma_Y} = 1.0
\]

Thus, the elastic buckling strength of a plate simply supported at all edges may be predicted by the following equations,

\[
\frac{\sigma_c}{\sigma_Y} = \left( \frac{\sigma_{\text{cr}}}{\sigma_Y} \right) - R
\]  
(5.3)

\[
\frac{\sigma_c}{\sigma_Y} = \left( \frac{\sigma_{\text{cr}}}{\sigma_Y} \right) - \left( \frac{\sigma_{\text{rl}}}{\sigma_Y} \right) + k \left( \frac{\sigma_{\text{rl}}}{\sigma_Y} \right)^2 \quad \text{for} \quad \frac{\sigma_{\text{rl}}}{\sigma_Y} > 0.15
\]  
(5.5)

\[
\frac{\sigma_c}{\sigma_Y} = \left( \frac{\sigma_{\text{cr}}}{\sigma_Y} \right) - \left( \frac{\sigma_{\text{rl}}}{\sigma_Y} \right) \quad \text{for} \quad \frac{\sigma_{\text{rl}}}{\sigma_Y} < 0.15
\]  
(5.6)

where

\[
k = R \quad \text{for} \quad \frac{\sigma_{\text{rl}}}{\sigma_Y} = 1.0
\]

The factor \( \ell/b \) influences the buckling strength of a plate, and at a certain value of \( \ell/b \) the most critical strain may be obtained. However, in the elastic buckling, the influence
of residual stresses on the buckling strength of a plate is independent of the critical load, as shown by Eq. 5.3. The factor \((\xi/b)\) is contained only in the first term of Eq. 5.3 but not in the second term. The first term gives the elastic buckling strength for a plate without residual stress which is a minimum for \((\xi/b) = 1.0\).

The results of numerical calculations are summarized in Fig. 5.3. The results corresponding to elastic buckling are shown by solid lines according to the values of \(\sigma_{r1}/\sigma_y = 1/8, 1/4, 1/2\) and \(3/4\). The last three curves for \(\sigma_{r1}/\sigma_y = 1/4, 1/2, 3/4\) intersect the abscissa. This interesting fact shows the possibility of the buckling of a plate without any external load and explains the reason why plates can be distorted solely due to welding.

Figure 5.5 was drawn for the critical strain ratio \((\varepsilon_c/\varepsilon_y)\) vs.\((b/t)\) ratio. Again, solid curves depict the cases of elastic buckling.

The curves calculated in this section are concerned with elastic buckling and hence the critical stress and strain relationship reduces to \(\sigma_{cr} = \varepsilon_c\). Therefore the shape of the curves for elastic buckling is the same in both figures.

5.3 Elastic-Plastic Buckling

In this section, numerical computation for the elastic-plastic buckling case was carried out based on the theoretical
preparation utilizing the deformation theory and the flow theory. Exactly the same process of calculation was used for both the computations, based on the deformation theory and on the flow theory.

There are two ways to consider how the residual stresses influence the buckling strength of the plate: one is consider the change of stress distribution of the external load and the other is to consider the plastification of the material. In the elastic buckling the first factor of the change of the stress distribution affects the buckling strength of the plate, and in the elastic-plastic buckling both factors together influence the buckling strength of the plates.

As in the preceding section, a comparison was made of the accuracy obtained by taking the first and the first two terms of the deflection equation. The computation showed that the value for \( \sigma_{cr}/\sigma_Y \) as obtained by the deflection equations, differed only by 3% in the worst case. It was judged that the use of the first term is accurate enough to carry out a comprehensive numerical computation with due consideration to the economy of the computer time. (For instance, using only the first term, the running time of the computer is only one fourth of the time spent for the computation using the first two terms.)

Even when only the first term of the deflection equation was used, the computation did not become very much simpler than was the case for elastic buckling. In that case, the influence of
residual stresses on the buckling strength of a plate was separated from the original buckling strength of the plate without residual stresses. In the case of elastic-plastic buckling it is impossible to separate these two factors in the equation. Consequently, the influence of residual stresses may change depending on the critical strain. The main reason why such a difficulty arises is that, as the external thrust increases, the most compressed fiber in the plate reaches its yield point and the resulting plastic zone spreads out according to the amount of the critical load.

In contrast with the buckling of a single plate, the computation for the local buckling of a box column normally requires the determination of the critical \((\ell/b)\) ratios which give the minimum critical stress.

For each residual stress pattern, various values of \((\ell/b)\) were chosen for a given value of critical strain. From the results of the computation, the curves \((L/B)\) vs. critical \((b/t)\) were drawn and the most critical \((\ell/b)\) was determined corresponding to the minimum critical strain. This is illustrated in Fig. 5.7 to Fig. 5.11. For example, in Fig. 5.9 the lowest critical stress corresponds to an \((\ell/b)\) ratio between 0.7 and 0.8 in the case of the elastic-plastic buckling of a plate with residual stress. The results are shown for the deformation theory; the results for the flow theory is not shown.
From the results of the computation, it was shown that, for practical cases, an approximate value of \((\ell/b)_{cr} = 0.8\) may be assumed to determine the local buckling of a square box column. The approximation, however, results in negligible error, as shown by this study.

Figures 5.3, 5.4, 5.5 and 5.6 summarize the computation results for the elastic-plastic local buckling of the box column using the deformation theory and the flow theory. Figures 5.3 and 5.4 show the ratio of the average critical stress to the yield point vs. the \((b/t)\) ratio, and Figs. 5.5 and 5.6 show the ratio of the average critical strain to the yield strain vs. the \(b/t\) ratio. In the elastic buckling case, the critical stress is calculated from the critical strain multiplied by the Young's modulus. But in the elastic-plastic buckling this relationship is no longer applicable and the critical stress must be calculated from Eqs. 4.1 and 4.2. Consequently, the shape of the curve changes.

As expected, the flow theory gives higher critical stresses than does the deformation theory. The elastic-plastic buckling curves based on the flow theory lie very close to the boundary of the elastic and the elastic-plastic buckling regions.

The curves which represent the buckling strength of a plate vs. \((b/t)\) ratio are hyperboli for elastic buckling, but are almost straight lines for elastic-plastic buckling. At the transition from the elastic buckling to the elastic-plastic buckling, there is
a discontinuity in the curve in Figs. 5.3, 5.4, 5.5 and 5.6. This is due to the sudden plastification of the material in the plate.

5.4 Plastic Buckling

This section presents the results of the computation for the plastic buckling of a plate. The plate material is assumed to be homogeneous and elastic perfectly plastic. When the entire plate reaches the yield point, the plate can no longer carry any additional load, although the strains may increase. Consequently, the critical strain may be investigated but the critical stress of the buckled plate can no longer be studied. For this reason the result of numerical analysis does not appear in Figs. 5.3 and 5.4 which are drawn with respect to $\frac{\sigma_{cr}}{\sigma_Y}$ and $b/t$. The tangent modulus of the elastic perfectly plastic material is zero in the plastic range independent of the strain. On the other hand, the secant modulus is affected by the magnitude of plastic strain.

As far as the flow theory is concerned, the complete plastification of the plate may be delayed by the existence of residual stresses, but after the whole plate has reached the yield point, the plate may behave completely plastically in the same manner as if the plate had not been subjected to any residual stresses before. The evaluation of the critical strain can be made by the flow theory in the same way as for material free of...
residual stresses but not for material which is elastic perfectly plastic. While the residual stresses do not play any role in the flow theory for the plastic buckling, they do influence the deformation theory because the secant modulus can define the relationship between strain and stress in the plastic range for elastic perfectly plastic material.

The result of the numerical calculation according to the deformation theory is shown in Fig. 5.5. The plastic buckling curves go above $2 \varepsilon_Y$ because the original tensile residual strain is assumed as $\sigma_Y/E$ and hence requires twice the compressive yield strain to reach the yield point of the material. The calculation was carried out for the critical strain up to $6 \varepsilon_Y$. Computation for a larger $\varepsilon_c/\varepsilon_Y$ is of no significance because the irregularity of material greatly overshadows the effect of residual stresses at the higher plastic strain.

In elastic buckling, the equation for the critical strain consists of two terms, the original buckling strain of a plate free of residual stress, and the effect of residual strains. This separation simplifies the investigation of the effect of residual stresses on plate buckling. In plastic buckling, on the other hand, this kind of advantage can not be taken and the procedure for the study of the influence of residual stresses is similar to that for elastic-plastic buckling.
The numerical computation for the critical \((b/t)\) ratio was carried out for four cases where the ratio of the residual stresses to yield point, \((\sigma_r/\sigma_y)\), equal to 0, 1/8, 1/2 and 1. For other values of the ratio, the corresponding values of the critical \((\varepsilon_c/\varepsilon_Y)\) ratio may be obtained simply by interpolation.

At the critical strain of \(2\varepsilon_Y\) there is a discontinuity, which is the transition from the elastic-plastic buckling to the plastic buckling of the plate (Fig. 5.5). The discontinuity of the curves is due to the sudden plastification of the elastic part remaining in the plate.

When the plate is not subject to any residual stresses, the plastic buckling of the plate occurs at \(1/\sqrt{2}\) of the \((\varepsilon/b)\) ratio which gives the lowest critical strain among any other values of \((\varepsilon/b)\). For the plate with residual stresses, the corresponding critical value of \((\varepsilon/b)\) is approximately 0.7, which is approximately the same as \(1/\sqrt{2}\). This fact suggests that the existence of residual stresses in the plate affects the critical strain of the plastic buckling of the plate, but not the wave length of buckling. (Figs. 5.7, 5.9 and 5.11)

The buckling strength of a plate in the strain-hardening range was not computed since the effect of residual stresses on the plate buckling in such a high strain is presumably negligible.
5.5 **Summary of Numerical Calculations**

The numerical analysis of the elastic, elastic-plastic and plastic buckling of the plate with residual stresses has been carried out for the case where the plate is simply supported at all edges. In the plastic range, the analysis was based both on the secant modulus deformation theory and on the flow theory.

From the numerical analysis, the following important information was obtained:

1. The approximation of using only the first term of the deflection equation gives an answer which is very close to the exact solution, and is accurate enough to analyze the buckling problem. This fact is important especially in the elastic-plastic buckling of the plate because of the complications arising from the fact that the plate has both elastic and plastic parts and a sudden change of material properties exists.

2. In the case of the elastic buckling of the plate with residual stresses, the influence of the residual stresses on the buckling strength of the plate is independent of the critical stress, and can be evaluated from the residual stress distribution, according to Eq. 5.3,
The possibility that the plate with residual stresses may buckle without any external load was demonstrated. This fact explains the reason why a plate can distort only due to welding.

The ratio of \( \ell/b \) which gives the minimum critical strain is 1.0 for elastic buckling, 0.7 to 0.8 for elastic-plastic buckling and 0.7 for plastic buckling. In the case of the elastic-plastic buckling, the ratio of \( \ell/b \) is closer to 0.7 for the plate with a wide distribution of compressive residual stress and closer to 0.8 for the plate with a narrow distribution of compressive residual stress.

An interesting fact concerning the ratio \( \ell/b \) in elastic-plastic buckling is that the smaller the elastic part in the plate, the closer is the ratio \( \ell/b \) to 0.7; conversely, the larger the elastic part the closer is the ratio \( \ell/b \) to 1.0.

In the case of elastic-plastic buckling of the plate, the analysis based on the flow theory gives a much higher critical strain than the one based

\[
\frac{\varepsilon_c}{\varepsilon_Y} = \frac{\sigma_c}{\sigma_Y} = \left( \frac{\sigma_{cr}}{\sigma_Y} \right)_0 - R
\]
on the deformation theory.

(6) Only the deformation theory was applied for the plastic buckling of the plate with residual stresses, since the flow theory is not applicable for the elastic perfectly plastic material in this case.

(7) A plate containing residual stresses will not buckle until the critical stress reaches the yield point, if the (b/t) ratio of the plate is less than

- a) $1.17 \sqrt{E/\sigma_y}$ based on the deformation theory

- b) $1.83 \sqrt{E/\sigma_y}$ based on the flow theory regardless of magnitude of the residual stresses, and less than

- c) $1.90 \sqrt{E/\sigma_y}$ for the plate free of residual stresses.
6. EXPERIMENTAL INVESTIGATION

6.1 Introduction

Experiments were planned to verify the theory for elastic buckling and elastic-plastic buckling of plate elements in built-up square columns of A7 steel.

Making reference to the results of the numerical calculation of Chapter 5, as summarized in Figs. 5.3 and 5.4, (b/t) ratios for the specimens were selected to be 45 and 65. These ratios were designed to produce elastic and elastic-plastic buckling respectively. The dimensions of the specimens are listed in Table 6.1. The experiments consisted of tensile coupon tests, residual stress measurements and plate buckling tests. The test columns fabricated from plates with b/t ratios of 45 and 65 were designated as S.1 and S.2 respectively.

The plate buckling tests were conducted on the short columns to simulate local buckling of columns without the occurrence of column buckling.

6.2 Fabrication of Test Specimens

The test specimens were cut from two long columns which were
fabricated from plates of 1/4" thickness. The fabrication of the columns was carried out according to standard practice, that is,

a) the plates were sheared to size,

b) the edges of the plates were not prepared for welding,

c) the plates were first assembled into the square shape and tackwelded,

d) the final weld was made by an automatic welding machine with a speed of welding of 19 in./minute and with average values of 27 volt and 325 amp. of current,

e) the sequence of welding the corners of the columns was such that the second weld was placed at the corner diagonally opposite the first weld in order to obtain a similar residual stress distribution for the four sides of the column.

f) the reinforcement of the weld was about 1/8", although 1/16" had been specified.

6.3 Testing

6.3.1. Residual Stress Measurement

The sectioning method\(^{(53)}\) was used for the residual stress measurement of the column cross sections. The cross sections
measured were at a distance greater than the cross sectional dimension from the cut edges so that the residual stress distribution in the plates were not disturbed. The gage length was ten inches. The measurement was performed using a Whittemore gage.

6.3.2. Tension Coupon Test

Certain sectioned pieces used for the residual stress measurement were tested as tension coupons. The coupons were taken from the center of the plate elements and from the edges for tests on the parent material and welded portions of the plates, respectively.

The static yield stress was obtained with the strain rate equal to zero and was used in the numerical analysis of the problem as the yield point of the material.

6.3.3. Plate Buckling Test

The two short columns S1 and S2 were tested with slenderness ratios of 11 and 12 respectively, so that column buckling of the specimens was prevented. The plate elements of these two specimens had aspect ratios of 4.5 and 5.0 respectively, with corresponding ratios of the column length to half-wave length of buckling of approximately 5.0 for both specimens. These column lengths were sufficient to eliminate the edge effect on the residual stress.
distributions in the middle part of the plates and on the local buckling strength of the columns. The square cross section was chosen to check the numerical analysis of simply supported plates described in Chapter 5.

For this test, an 800-kip Universal mechanical screw-type testing machine was used. The strain of the columns under the load was measured from an average reading of four SR-4 strain gages mounted at the four corners at mid-height of the columns.

The transverse deflection of the plates was measured at four cross sections around the mid-height of column S1 and at six cross sections for column S2, on two opposite sides. The deflection was measured by a 1/10,000 dial gage fixed to a frame held manually. The reference positions for the deflection measurement were located at the edges of the plate. This simple apparatus was accurate with the maximum deviation of several measurements of the deflection at the same position being within 3/10,000 inches.

6.4 Test Results and Discussion

6.4.1. Residual Stress Measurement

The average experimental residual stress distributions in the cross section of the column are shown in Fig. 6.1 for columns S1 and S2 respectively. These curves were obtained from the average experimental values for eight half-plate widths. Figure 6.1 also
shows the simplified residual stress patterns for the parameter
\( \mu = 1/2 \) used in the analysis. The simplified distribution for
the parameter \( \mu = 1/2 \) is a good approximation for the experi-
mental one in both cases.

For the local buckling load prediction, using Fig. 6.1,
the compressive residual stress, \( \sigma_{r1} \) is 12.5 ksi for column S1
and 10.5 ksi for column S2. The tensile residual stress, \( \sigma_{r2} \) was
taken as the yield point of the parent material as determined from
the results of the tensile coupon tests; that is 39.0 ksi for
column S1 and 38.5 for column S2. Hence, the ratios of \( \sigma_{r1}/\sigma_y \)
used were 0.320 for column S1 and 0.273 for column S2.

6.4.2. Tension Coupon Test

The results of the tension coupon tests are listed in Table
6.1.

The yield point of the weld metal is somewhat higher than
that of the parent material. This fact was neglected in the com-
putations as mentioned in Section 4.2.

6.4.3. Plate Buckling Test

The relationships between load and strain, load and trans-
verse deflection, and load and the square of the transverse deflec-
tion are shown in Figs. 6.2, 6.3 and 6.4.
The relationship between load and deflection did not indicate a bifurcation load, (Fig. 6.3), presumably due to the existence of initial imperfections and an eccentrical loading. The delta-squared, \( (\delta^2) \), method\(^{(52)} \) was used to determine the critical buckling load of the plate element. The results of this method are shown in Fig. 6.4 and the critical loads obtained were 340 kips and 261 kips respectively for columns S1 and S2. (The corresponding critical stresses, \( \sigma_{cr} \), were 30.2 ksi and 16.0 ksi, with the ratio \( (\sigma_{cr}/\sigma_Y) \) as 0.775 and 0.415. These results are compared with the theoretical predictions in Fig. 6.5. According to the residual stress measurements, the value of \( \sigma_{ri}/\sigma_Y \) is 0.320 for column S1 and 0.273 for column S2. Hence the corresponding predicted critical ratios of \( \sigma_{ri}/\sigma_Y \) are 0.795 and 3.90 respectively.

Comparison was made between the experimental and the theoretical results. For column S2, (elastic buckling), the theoretical prediction was a little low, while for S1, (elastic-plastic buckling), it was a little high when based on the deformation theory and was very high when based on the flow theory. However, except for the flow theory, the difference is very small and it can be concluded that the experimental results correlated well with the theoretical prediction.

The ultimate loads attained in the tests were 357 kips for column S1 and 337 kips for column S2. (The corresponding ultimate
stresses, \( \sigma_u \), were 31.7 ksi and 20.7 ksi, with the ratio \( \sigma_u / \sigma_y \) as 0.813 and 0.538.

The post buckling strength of the plate element of the columns was approximately 5\% and 30\% of the buckling strength for columns S1 and S2 respectively. The post buckling strength above the apparent buckling load for column S1 was 17 kips with the stress difference \( \sigma_u - \sigma_{cr} = 1.5 \text{ ksi} \), the ratio \( (\sigma_u - \sigma_{cr}) / \sigma_y = 0.038 \), and the ratio \( (\sigma_u - \sigma_c) / \sigma_{cr} = 0.050 \). The corresponding figures for column S2 were 76 kips, 4.7 ksi, 0.123 and 0.294.

The results of these two pilot tests have indicated that considerable post buckling strength may be expected for elastic buckling of the plates, although not for elastic-plastic buckling.
7. SUMMARY AND CONCLUSIONS

This dissertation presents the results of an investigation into the elastic, elastic-plastic and plastic buckling of plates containing residual stresses. Particular attention was paid to the local buckling of plate elements of built-up columns of box-shaped cross sections.

In the theoretical analysis, the pattern of the residual stress distribution was simplified and the theorem of minimum potential energy was employed with the restriction that there is no reversal of strain at any point in the plastified material. The plastic part of the plate was analyzed by plastic theories, the secant modulus deformation theory and the flow theory, both of which were modified according to the Shanley concept of column buckling.

The new contributions of this dissertation are as follows:

a) The method of analysis presented is believed to be the first approximate solution for the elastic-plastic and plastic buckling of a plate with residual stresses.

b) Analytical solutions are presented for the elastic, elastic-plastic and plastic buckling of a plate with
residual stresses when the plate is simply supported at the loading edges and at the other edges is:

i ) elastically restrained

ii ) simply supported

iii ) fixed.

3) The result of numerical computations for the analytical solution to the local buckling of a welded built-up square column is presented for elastic, elastic-plastic and plastic buckling. The results are presented of a pilot experimental study which verified the theoretical analysis. The experimental study showed the relationship between the buckling strength and the ultimate strength of a plate element of the column.

The following conclusions may be drawn from the previous chapters:

1. Numerical Analysis

   The analytical solutions presented in this dissertation are believed to give a good prediction for local buckling strength of built-up columns since good correlation was obtained between the analytical and experimental studies.

   a) The approximation of using only the first term of the deflection equation gives an answer which is
very close to the exact solution, and is accurate enough for analysis of the buckling problem. (Sections 5.2 to 5.4).

b) In the case of the elastic buckling of the plate with residual stresses, the influence of the residual stresses on the buckling strength of the plate is independent of the critical stress, and can be evaluated from the residual stress distribution, according to Eqs. 5.3 and 5.6. (Section 5.2)

c) The possibility that the plate with residual stresses may buckle without any external load was demonstrated. This fact explains the reason why a plate can distort only due to welding. (Section 5.2, Figs. 5.3 and 5.4)

d) The ratio of $\ell/b$, which gives the minimum critical strain is 1.0 for elastic buckling, 0.7 to 0.8 for elastic-plastic buckling and 0.7 for plastic buckling. (Section 5.2 to 5.4, Figs. 5.7 to 5.11)

e) In the case of elastic-plastic buckling of the plate, the analysis based on the flow theory gives a much higher critical strain than the one based on the deformation theory. Only the deformation theory was applied for the plastic buckling of the plate with residual stresses, since the flow theory is not
applicable for the elastic-plastic material in this case. (Sections 5.3 and 5.4, Figs. 5.3 to 5.6)

f) A plate containing residual stresses will not buckle until the critical stress reaches the yield point, if the (b/t) ratio of the plate is less than the aspects of the ratio:

i) \(1.17\sqrt{E/\sigma_y}\) based on the deformation theory

ii) \(1.83\sqrt{E/\sigma_y}\) based on the flow theory regardless of magnitude of the residual stresses, and less than

iii) \(1.90\sqrt{E/\sigma_y}\) for the plate free of residual stresses.

(Sections 5.2 and 5.3, Figs. 5.3 and 5.4)

2. **Experimental Study**

a) The experiments verified the validity of the theoretical analysis for the elastic and elastic-plastic buckling of a plate containing residual stresses.

For elastic-plastic buckling, the theory based on the secant modulus deformation theory gave good correlation with the experimental results, but the theory based on the flow theory did not. (Section 6.4.3, Fig. 6.5)
b) The (L/l) ratios of the plate elements of the built-up columns were approximately 1.00 for the elastic buckling and 0.75 for elastic-plastic buckling. These values were the same as predicted by the theory. (Sections 5.2, 5.3 and 6.4.3)

c) Although considerable post buckling strength occurred for elastic buckling of the plate, this was not the case for elastic-plastic buckling. (Section 6.4.3, Fig. 6.2)

3. Recommendations for Future Research

The results of this dissertation have indicated that future work should be directed along the following lines:

1) Computation for the numerical analyses for a plate simply supported at the loading edges and at the other edges
   a) fixes
   b) elastically supported.

   The theoretical results for these cases have been presented in this dissertation.

2) Further experimental investigations for a wider range of b/t in all the different cases.
3) Application of the theory in this dissertation to other cases; such as the local buckling of
   a) a center welded plate
   b) the flange of open shapes
with consideration of the effect of residual stresses. This would be accomplished by following exactly the same procedure as in the dissertation except that a suitable deflection shape must be adopted for the case under consideration.

4) Extension of the theory to the case of a plate with combined residual stresses.

5) Investigation into the ultimate strength of plates with residual stresses.
8. N O M E N C L A T U R E

\[ a_0, a_1, a_2 \quad = \quad \text{Coefficients of deflection functions} \]
\[ a_{1m}, a_{2m} \quad = \quad \text{Coefficients of deflection functions} \]
\[ B \quad = \quad \text{Width of plate} \]
\[ b \quad = \quad \text{Half width of plate} \]
\[ b_o \quad = \quad \text{Half width of horizontal compressive residual stress distribution in an assumed pattern} \]
\[ b_1 \quad = \quad \text{Half width of plastic zone} \]
\[ b-b_2 \quad = \quad \text{Width of maximum tensile residual stress distribution} \]
\[ b(i) \quad = \quad \text{Half width of plate element on side } i \]
\[ c_{1m}, c_{2m} \quad = \quad \text{Coefficients of deflection functions} \]
\[ D \quad = \quad \text{Flexural rigidity of a plate} = \frac{E h^3}{12 (1 - \nu^2)} \]
\[ D_i \quad = \quad \text{Flexural rigidity of plate on side } i \]
\[ D_d \quad = \quad \text{Flexural rigidity of plate in the plastic range, based on the deformation theory} \]
\[ D_f \quad = \quad \text{Flexural rigidity of plate in the plastic range, based on the flow theory} \]
\[ D_a, D_b \quad = \quad \text{Flexural rigidity of plate in the domains } a, b, c, \ldots \]
$D_{x,y}$ = Flexural rigidity of plate varying with respect to $x$ and $y$ coordinates

$E$ = Modulus of elasticity

$E_a, E_b, ...$ = Modulus of elasticity in the domains, $a, b, c, ...$

$E_r$ = Reduced modulus

$E_s$ = Secant modulus

$E_t$ = Tangent modulus

$F_{11}, F_{12}, F_{21}, F_{22}$ = Components of the coefficient determinant of the stability equation

$g_1, g_2, ...$ = Functions expressing boundary conditions

$h$ = Thickness of plate

$h_a, h_b, ...$ = Thickness of plate in the domains, $a, b, ...$

$h_i$ = Thickness of plate on side $i$

$h_{x,y}$ = Thickness of plate varying with respect to $x$ and $y$ ordinates

$k$ = Reduction of buckling strength, $R$, for $\frac{\sigma_{r1}}{\sigma_Y} = 1$

$L$ = Entire length of column or plate

$2l$ = Half wave length of buckling of plate

$(\ell/b)_{cr}$ = $(\ell/b)$ ratio giving minimum critical strain of buckling of plate
\( M_i \) = Bending moment per unit length of section of plate about x axis on side i

\( m \) = An integer

\( N \) = Number of buckling modes in the direction of x axis

\( n \) = A number

\( R \) = Magnitude of reduction of elastic buckling strength due to residual stresses

\( r \) = Radius of gyration of cross section of columns

\( s \) = Number of walls composing a built-up column

\( 2t \) = Thickness of a plate

\( V \) = Potential energy of plate

\( V_i \) = Potential energy of plate on the side i

\( W \) = Strain energy stored in plate

\( w \) = Deflection of plate

\( w_i \) = Deflection of plate on the side i

\( w' \) = Rate of change of deflection of plate

\( x, y, z \) = Cartesian coordinates
\( y_i \) = y-axis of cartesian coordinate on a plate on the side \( i \)

\( \alpha \) = A parameter, \( \alpha = \frac{\omega}{b_0} \)

\( \alpha_r \) = The ratio \( E_r/E \)

\( \alpha_t \) = The ratio \( E_t/E \)

\( \beta \) = A parameter, \( \beta = \left( \frac{D_1}{D_2} \right) \left( \frac{b_0}{b} \right)^2 \)

\( \gamma \) = Shearing strain

\( \gamma_{x_1}, \gamma_{x_2}, \gamma_{x_3} \) = Shearing strain components in the cartesian coordinates

\( \dot{\gamma} \) = Rate of change of shearing strain

\( \omega \) = Normal strain due to second loading

\( \varepsilon_c \) = Critical normal strain

\( \varepsilon_r \) = Residual strain

\( \varepsilon_{r_1} \) = Magnitude of maximum compressive residual strain in the assumed pattern

\( \varepsilon_{r_2} \) = Magnitude of maximum tensile residual strain in the assumed pattern

\( \varepsilon_i \) = Intensity of strain

\( \varepsilon_x, \varepsilon_y, \varepsilon_z \) = Normal strain components in the cartesian coordinates
\( \varepsilon_y \) = Yield strain in tension or compression

\( \dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z \) = Rate of change of strain components in the cartesian coordinates

\( \Delta b \) = Width of tensile residual stress is rectangular distribution, \( \Delta b = b - b_2 \)

\( \Delta \varepsilon \) = Increment of normal strain

\( \Theta_i \) = Angle of rotation at edge of plate \( i \)

\( \kappa \) = \( 1 - E_t/E_s \)

\( \mu \) = A parameter, when \( \mu \Delta b \) is width of tensile residual stress distribution in the assumed pattern

\( \nu \) = Poisson's ratio

\( \nu_i \) = Poisson's ratio for the plate \( i \)

\( \lambda \) = \( E/E_t = 1/\alpha_t \) (ratio)

\( \sigma \) = Normal stress

\( \sigma_a, \sigma_b, \ldots \) = Normal stress due to first loading in the domains \( a, b, \ldots \)

\( \sigma_c \) = Computational parameter, \( \sigma_c = \varepsilon_c E \)

\( \sigma_{cr} \) = Average critical normal stress

\( \sigma_i \) = Intensity of stress
\( \sigma_r \) = Residual stress

\( \sigma_{r1} \) = Magnitude of maximum compressive residual stress in the assumed pattern

\( \sigma_{r2} \) = Magnitude of maximum tensile residual stress in the assumed pattern

\( \sigma_x, \sigma_y, \sigma_z \) = Normal stress components in the cartesian coordinates

\( \sigma_{x, y} \) = Rate of change of stress components in the cartesian coordinates

\( \sigma_{x, y}^{(1)} \) = Normal stress in the x-direction varying with respect to y ordinate

\( \sigma_{x, y}^{(2)} \) = Normal stress in x-direction, varying with respect to y ordinate, due to first loading

\( \sigma_{y, x}^{(1)} \) = Normal stress in x-direction, varying with respect to y ordinate, due to second loading

\( \sigma_{y, x} \) = Normal stress due to first loading in y direction varying with respect to x ordinate

\( \sigma_{y} \) = Yield stress in tension or compression

\( \tau_{xy} \) = Shearing stress component in y-direction

\( \tau_{xy}^{(1)} \) = Shearing stress component in y-direction, due to first loading

\( \tau_{yz}, \tau_{xz}, \tau_{xy} \) = Shearing stress components in the cartesian coordinates
9. APPENDICES
APPENDIX A

PROOF FOR $\sigma_y^{(1)}$ AND $\tau_{xy}^{(1)}$ EQUAL TO ZERO

IN THE ASSUMED RESIDUAL STRESS DISTRIBUTION

When the residual stress $\sigma_x^{(1)}$ in the cross section of the plate varies in the direction of width and is constant along its length, the other components of residual stresses, $\sigma_y^{(0)}$ and $\tau_{xy}^{(0)}$ cannot exist over the plate.

Proof:

Boundary conditions

a) $\sigma_y^{(1)}$ is zero along the edges parallel to the x-axis,

b) $\tau_{xy}^{(1)}$ vanishes at the centerline due to the symmetry of the plate and of the residual stress distribution.

The equilibrium equation for plane stress

$$\frac{\partial \sigma_y^{(0)}}{\partial x} + \frac{\partial \tau_{xy}^{(0)}}{\partial y} = 0 \quad (a)$$

$$\frac{\partial \sigma_y^{(0)}}{\partial y} + \frac{\partial \tau_{xy}^{(0)}}{\partial x} = 0 \quad (b)$$

as $\sigma_x^{(0)}$ is only a function of $y$,
Substituting this expression into Eq. (a),
\[
\frac{\partial \tau_{xy}^{(i)}}{\partial y} = 0
\]
then
\[
\tau_{xy}^{(i)} = g(x)
\]

Taking into consideration the symmetry of the plate and the residual stress distribution, \( \tau_{xy} \) vanishes on the centerline, that is,
\[
\tau_{xy}^{(i)} \bigg|_{y=0} = 0
\]
With this condition, \( g(x) \) which is a function of \( x \) only, must disappear at any value of \( y \) and consequently
\[
\tau_{xy}^{(i)} = g(x) \equiv 0
\]
With the above result, Eq. (b) yields
\[
\frac{\partial \sigma_y^{(i)}}{\partial y} = 0
\]
Then
\[
\sigma_y^{(i)} = h(x)
\]

From boundary condition (a),
\[
\sigma_y^{(i)} \bigg|_{y=b} = \sigma_y^{(i)} \bigg|_{y=b} = 0
\]
Accordingly, \( \sigma_y^{(i)} \) is constant and its value is zero.
\[
\sigma_y^{(i)} = 0
\]
As a result, \( \sigma_y^{(i)} \) and \( \tau_{xy}^{(i)} \) disappear at any point in the plate under the above mentioned boundary conditions.
APPENDIX B

ANALYTICAL SOLUTIONS FOR BUCKLING STRENGTH OF PLATES WITH RESIDUAL STRESSES

Analytical solutions are presented for elastic, elastic-plastic and plastic buckling of a plate which is simply supported at the loading edges and at other edges is:

a) elastically restrained
b) simply supported
c) fixed.

B.1. General Notation

\[
\begin{align*}
\Gamma_0 & = \frac{b_0}{b} \\
\Gamma_1 & = \frac{b_1}{b} \\
\Gamma_2 & = \frac{b_2}{b} \\
\Gamma_3 & = \frac{b}{\ell} \\
\jmath \Gamma_3 & = \frac{\ell}{b} \\
\Gamma_4 & = \frac{t}{b}
\end{align*}
\]
\[
\Gamma_5 = \frac{h_2}{h_1} \left( \frac{\sigma_r b_2 + \sigma_{r2} b_2}{\sigma_Y b} \right) \frac{b_2 - b_0}{b}
\]

\[
\Gamma_6 = \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_{r2}}{\sigma_Y} \frac{b_2 - b_0}{b}
\]

\[
k_1 = \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_{r2}}{\sigma_Y}
\]

\[
k_2 = \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_{r2}}{\sigma_Y} \frac{b_2 - b_0}{b}
\]

\[
k_3 = \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_{r2}}{\sigma_Y} \frac{b_2 - b_0}{b}
\]

\[
k_4 = \frac{\sigma_{r1}}{\sigma_Y} - \frac{\sigma_{r2}}{\sigma_Y}
\]

\[
u = \frac{32}{12} \left( \frac{1}{N\pi} \right)^2 \left( \frac{E}{\sigma_Y} \right) \left( \frac{h}{b} \right)^2 \frac{1}{1 - \nu^2}
\]

\[
u_0 = \frac{32}{9} \left( \frac{1}{N\pi} \right)^2 \left( \frac{E}{\sigma_Y} \right) \left( \frac{h}{b} \right)^2
\]

\[
u_1 = \frac{32}{9} \left( \frac{1}{N\pi} \right)^2 \left( \frac{E}{\sigma_Y} \right) \left( \frac{h}{b} \right)^2 \frac{1}{\sigma_{r1}/\sigma_Y + \sigma_{r2}/\sigma_Y}
\]
\[ u_2 = \frac{32}{q} \left( \left( \frac{1}{N \pi} \right) \left( \frac{E}{\sigma_Y} \right) \left( \frac{b}{h} \right) \right)^2 \frac{b_2 - b_0}{\sigma_Y} \left( \frac{\sigma_{L1}}{\sigma_Y} \right) + \frac{\sigma_{L2}}{\sigma_Y} \]

\[ u_3 = \frac{32}{q} \left( \left( \frac{1}{N \pi} \right) \left( \frac{E}{\sigma_Y} \right) \left( \frac{b}{h} \right) \right)^2 \frac{1}{\sigma_Y} \left( \frac{\sigma_{L1}}{\sigma_Y} \right) - \frac{\sigma_{L2}}{\sigma_Y} \]

\[ S_k^m = \cos m \pi \Gamma_6 \cdot \log \frac{\Gamma_k - \Gamma_0}{\Gamma_6 - \Gamma_0} - \left( m \pi \right) \frac{1}{1} \sin m \pi \left( \left( \Gamma_k - \Gamma_0 \right) - \left( \Gamma_6 - \Gamma_0 \right) \right) \]

\[ - \left( m \pi \right)^2 \frac{1}{2} \cos m \pi \left( \left( \Gamma_k - \Gamma_0 \right) - \left( \Gamma_6 - \Gamma_0 \right) \right) \]

\[ + \left( m \pi \right)^3 \frac{1}{3} \sin m \pi \left( \left( \Gamma_k - \Gamma_0 \right) - \left( \Gamma_6 - \Gamma_0 \right) \right) \]

\[ + \left( m \pi \right)^4 \frac{1}{4} \cos m \pi \left( \left( \Gamma_k - \Gamma_0 \right) - \left( \Gamma_6 - \Gamma_0 \right) \right) \]

\[ \text{where } k = 1 \text{ or } 2, \quad m = \frac{1}{2}, 1, \frac{3}{2}, 2, 3 \text{ or } 4. \]

For elastic region \( i = E \) and \( \omega = 1 \)

For plastic region \( i = p \) and \( \omega = \frac{1}{4} \)

For the side 1 of column \( j = 1 \)

For the side 2 of column \( j = 2 \).
B.2. Analytical Solutions for Elastically Restrained Plate

B.2.1. Additional Notation

\[ j C_{11} = \pi^4 \left( \omega \cdot j \Gamma_3^4 + \frac{1}{2} j \Gamma_3^2 + \frac{1}{16} \right) \]
\[ j C_{12} = \pi^4 \left( \omega \cdot j \Gamma_3^4 - \frac{1}{2} (1 - 2\nu) j \Gamma_3^2 + \frac{1}{16} \right) \]
\[ j C_{13} = \pi^4 \left( -\omega \cdot j \Gamma_3^4 + \frac{1}{4} (4 - \nu) j \Gamma_3^2 + \frac{1}{4} \right) \]
\[ j C_{14} = \pi^4 \left( \omega \cdot j \Gamma_3^4 - \frac{1}{4} (4 - 9\nu) j \Gamma_3^2 + \frac{1}{4} \right) \]
\[ j C_{21} = j C_{13} \]
\[ j C_{22} = j C_{14} \]
\[ j C_{23} = \pi^4 \left( 3 \omega \cdot j \Gamma_3^4 + 2 j \Gamma_3^2 + 1 \right) \]
\[ j C_{24} = -4 \pi^4 \left( \omega \cdot j \Gamma_3^4 + \nu j \Gamma_3^2 \right) \]
\[ j C_{25} = \pi^4 \left( \omega \cdot j \Gamma_3^4 - 2 (1 - 2\nu) j \Gamma_3^2 + 1 \right) \]

B.2.2. Analytical Solutions

The analytical solution is given in the form:

\[ F_{11} F_{22} - F_{12} F_{21} = 0 \]
where

\[ F_i = (\Gamma_i^2)(\nu_{110}) + (\nu_{111}) \]
\[ F_{12} = F_{21} = (\Gamma_4^2)(\nu_{120}) + (\nu_{121}) \]
\[ F_{22} = (\Gamma_4^2)(\nu_{220}) + (\nu_{221}) \]

B.2.2.a. **Elastic Buckling**

\[ (\nu_{110}) = u \ (t_{130}) \]
\[ (\nu_{120}) = u \ (t_{230}) \]
\[ (\nu_{220}) = u \ (t_{430}) \]
\[ (\nu_{111}) = -k_1 (t_{140}) - k_2 (t_{151}) + k_3 (t_{152}) + k_4 (t_{160}) \]
\[ (\nu_{121}) = -k_1 (t_{240}) - k_2 (t_{251}) + k_3 (t_{252}) + k_4 (t_{260}) \]
\[ (\nu_{221}) = -k_1 (t_{440}) - k_2 (t_{451}) + k_3 (t_{452}) + k_4 (t_{460}) \]

\[ (t_{130}) = \left( C_{ii}^E - \frac{\alpha^3}{\beta} C_{ii}^E \right) \]
\[ (t_{140}) = (1 - \alpha^3 \Gamma_5^e) \left( \Gamma_0 - \frac{1}{\pi \mu} \sin \pi \Gamma_0 \right) \]
\[ (t_{151}) = (1 - \alpha^3 \Gamma_5^e) \left( (\Gamma_2 - \Gamma_0^e) + \frac{1}{\pi \mu} (\sin \pi \Gamma_2 - \sin \pi \Gamma_0^e) \right) \]
\((t_{152}) = (1 - \alpha^2) \Gamma_5 \left\{ \frac{1}{2} (\Gamma_z^2 - \Gamma_0^2) + \frac{1}{\pi} (\Gamma_z \sin \pi \Gamma_z - \Gamma_0 \sin \pi \Gamma_0) \right. \\
\left. + (\frac{1}{\pi})^2 (\cos \pi \Gamma_z - \cos \pi \Gamma_0) \right\} \\
(t_{160}) = (1 - \alpha^2) \Gamma_5 \left( (\Gamma_z - 1) + (\frac{1}{\pi}) \sin \pi \Gamma_z \right) \\
(t_{230}) = \left( C_1^E - C_{13}^E \right) \left( \frac{2}{\pi} \right) - \left[ \left( C_{14}^E - C_{14}^E \right) \left( \frac{2}{3 \pi} \right) \\
(t_{240}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ - \frac{2}{\pi} \sin \pi \Gamma_0 - \frac{2}{3 \pi} \sin \frac{3 \pi}{2} \Gamma_0 \right\} \\
(t_{251}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ \frac{2}{\pi} (\sin \frac{\pi}{2} \Gamma_z - \sin \frac{\pi}{2} \Gamma_0) + \frac{2}{3 \pi} (\sin \frac{3 \pi}{2} \Gamma_z - \sin \frac{3 \pi}{2} \Gamma_0) \right\} \\
(t_{252}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ \frac{2}{\pi} (\Gamma_z \sin \frac{\pi}{2} \Gamma_z - \Gamma_0 \sin \frac{\pi}{2} \Gamma_0) + \frac{2}{3 \pi} (\cos \frac{\pi}{2} \Gamma_z - \cos \frac{\pi}{2} \Gamma_0) \\
+ \frac{2}{3 \pi} (\Gamma_z \sin \frac{3 \pi}{2} \Gamma_z - \Gamma_0 \sin \frac{3 \pi}{2} \Gamma_0) + (\frac{2}{3 \pi})^2 (\cos \frac{3 \pi}{2} \Gamma_z - \cos \frac{3 \pi}{2} \Gamma_0) \right\} \\
(t_{260}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ \frac{2}{\pi} (1 + \sin \frac{\pi}{2} \Gamma_z) + \frac{2}{3 \pi} (1 + \sin \frac{3 \pi}{2} \Gamma_z) \right\} \\
(t_{430}) = \left( C_2^E + \frac{\beta^3}{\alpha^4} \right) \left( C_{25}^E \right) \\
(t_{440}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ 3 \Gamma_0 - \frac{4}{\pi} \sin \pi \Gamma_0 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_0 \right\} \\
(t_{451}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ 3 (\Gamma_z - \Gamma_0) - \frac{4}{\pi} (\sin \pi \Gamma_z - \sin \pi \Gamma_0) \\
+ \frac{1}{2 \pi} (\sin 2 \pi \Gamma_z - \sin 2 \pi \Gamma_0) \right\} \\
(t_{452}) = (1 - \alpha^2 \beta) \Gamma_5 \left\{ \frac{3}{2} (\Gamma_z^2 - \Gamma_0^2) - \frac{4}{\pi} (\Gamma_z \sin \pi \Gamma_z - \Gamma_0 \sin \pi \Gamma_0) \\
- (\frac{2}{\pi})^2 (\cos \pi \Gamma_z - \cos \pi \Gamma_0) + \frac{1}{2 \pi} (\Gamma_z \sin 2 \pi \Gamma_z - \Gamma_0 \sin 2 \pi \Gamma_0) \\
+ (\frac{1}{2 \pi})^2 (\cos 2 \pi \Gamma_z - \cos 2 \pi \Gamma_0) \right\} \\
(t_{460}) = (1 - \alpha^2 \beta) \Gamma_5 \left( (\Gamma_z - 1) + \frac{1}{\pi} \sin \pi \Gamma_z + \frac{1}{2 \pi} \sin 2 \pi \Gamma_z \right)
B.2.2.b. Elastic-plastic Buckling

(Based on the deformation theory)

\[ (u_{110}) = u_1(t_{110}) - u_2(t_{120}) + u(t_{130}) \]

\[ (u_{120}) = u_1(t_{120}) - u_2(t_{122}) + u(t_{123}) \]

\[ (u_{220}) = u_1(t_{410}) - u_2(t_{420}) + u(t_{430}) \]

\[ (u_{111}) = -(t_{140}) - k_2(t_{151}) + k_3(t_{152}) + k_4(t_{160}) \]

\[ (u_{121}) = -(t_{240}) - k_2(t_{251}) + k_3(t_{252}) + k_4(t_{260}) \]

\[ (u_{221}) = -(t_{440}) - k_2(t_{451}) + k_3(t_{452}) + k_4(t_{460}) \]

\[ (t_{110}) = \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] \Gamma_0 + \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] (\frac{1}{2\pi} \sin \pi \Gamma_0) \]

\[ (t_{120}) = \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] \log \frac{\Gamma_0 - \Gamma_6}{\Gamma_6 - \Gamma_0} + \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] S_1 \]

\[ (t_{130}) = \left[ C_1^E - \frac{\alpha}{\beta} C_2^E \right] (1 - \Gamma_0) - \left[ C_1^E - \frac{\alpha}{\beta} C_2^E \right] (\frac{1}{2\pi} \sin \pi \Gamma_1) \]

\[ (t_{140}) = (1 - \frac{3}{5}) \left( \Gamma_1 - \frac{1}{2\pi} \sin \pi \Gamma_1 \right) \]

\[ (t_{151}) = (1 - \frac{3}{5}) \left( \Gamma_2 - \frac{1}{2\pi} \sin \pi \Gamma_2 \right) \]

\[ (t_{152}) = (1 - \frac{3}{5}) \left( \Gamma_2 - \frac{1}{2\pi} \sin \pi \Gamma_2 \right) \]

\[ (t_{221}) = (1 - \frac{3}{5}) \left( \Gamma_2 - \frac{1}{2\pi} \sin \pi \Gamma_2 \right) \]

\[ (t_{210}) = \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_0 + \left[ C_1^p - \frac{\alpha}{\beta} C_2^p \right] \frac{2}{\pi} \sin \frac{3\pi}{2} \Gamma_0 \]
(t 220) = \left( C_{13} - \frac{P_{13}}{2} \right) S_{13}^1 + \left( C_{13} - \frac{P_{13}}{2} \right) S_{13}^1

(t 230) = \left( C_{15} - \frac{E_{15}}{2} \right) \frac{2}{\pi} (1 - \sin \frac{\pi}{2} \Gamma) - \left( C_{14} - \frac{E_{14}}{2} \right) \frac{2}{3 \pi} (1 + \sin \frac{3\pi}{2} \Gamma)

(t 240) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ -\frac{2}{\pi} \sin \frac{\pi}{2} \Gamma + \frac{2}{3 \pi} \sin \frac{3\pi}{2} \Gamma \right]

(t 250) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ \frac{2}{\pi} (\sin \frac{\pi}{2} \Gamma - \sin \frac{3\pi}{2} \Gamma) + \frac{2}{3 \pi} (\sin \frac{3\pi}{2} \Gamma - \sin \frac{9\pi}{2} \Gamma) \right]

(t 252) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ \frac{2}{\pi} (\Gamma - \Gamma' - \Gamma'') \right] \left( \sin \frac{\pi}{2} \Gamma + \frac{2}{3 \pi} \sin \frac{3\pi}{2} \Gamma \right)

(t 260) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ -\frac{2}{\pi} \sin \frac{\pi}{2} \Gamma + \frac{2}{3 \pi} (1 + \sin \frac{3\pi}{2} \Gamma) \right]

(t 410) = \left( C_{23} - \frac{P_{23}}{2} C_{23} \right) \sin \pi \Gamma_0 + \left( C_{24} + \frac{P_{24}}{2} C_{24} \right) \sin \pi \Gamma_0 + \left( C_{25} + \frac{P_{25}}{2} C_{25} \right) \sin 2\pi \Gamma_0

(t 420) = \left( C_{25} - \frac{P_{25}}{2} C_{25} \right) \log \frac{\Gamma - \Gamma'}{\Gamma - \Gamma''} + \left( C_{24} + \frac{P_{24}}{2} C_{24} \right) S_{11}^1 + \left( C_{25} + \frac{P_{25}}{2} C_{25} \right) S_{11}^1

(t 430) = \left( C_{23} + \frac{P_{23}}{2} C_{23} \right) (1 - \Gamma) - \left( C_{24} + \frac{P_{24}}{2} C_{24} \right) \frac{1}{\pi} \sin \pi \Gamma_1

(t 440) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ 3 \Gamma_1 - \frac{4}{\pi} \sin \pi \Gamma_1 + \frac{1}{2 \pi} \sin 2\pi \Gamma_1 \right]

(t 451) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ 3 \Gamma_1 - \frac{4}{\pi} (\sin \pi \Gamma_1 - \sin \pi \Gamma_1) + \frac{1}{2 \pi} (\sin 2\pi \Gamma_1 - \sin 2\pi \Gamma_1) \right]

(t 452) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ \frac{3}{2} \left( \Gamma_1 - \Gamma' \right)^2 - \frac{4}{\pi} (\Gamma_1 \sin \pi \Gamma_2 - \Gamma_1 \sin \pi \Gamma_1)

- \frac{2}{\pi} \left( \cos \pi \Gamma_1 - \cos \pi \Gamma_1 \right) \right] + \frac{1}{2 \pi} \left( \Gamma_1 \sin 2\pi \Gamma_2 - \Gamma_1 \sin 2\pi \Gamma_1 \right)

(t 460) = (1 - \alpha^2 \beta \Gamma_5^5) \left[ (\Gamma_2 - 1) + \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2 \pi} \sin 2\pi \Gamma_2 \right]
B.2.2.c. Elastic-plastic Buckling

(Based on the flow theory)

\[
\begin{align*}
(u_{110}) &= u_0 (t_{110}) + u(t_{130}) \\
(u_{120}) &= u_0 (t_{210}) + u(t_{230}) \\
(u_{220}) &= u_0 (t_{410}) + u(t_{430}) \\
(u_{111}) &= -(t_{140}) - k_2 (t_{151}) + k_3 (t_{152}) + k_4 (t_{160}) \\
(u_{121}) &= -(t_{240}) - k_2 (t_{251}) + k_3 (t_{252}) + k_4 (t_{260}) \\
(u_{221}) &= -(t_{440}) - k_2 (t_{451}) + k_3 (t_{452}) + k_4 (t_{460}) \\
(t_{110}) &= \left[ C_{1}^{p} - \alpha_{12}^{p} C_{12}^{p} \right] \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \left( \frac{1}{\pi} \sin \pi \Gamma_1 \right) \\
(t_{130}) &= \left[ C_{1}^{p} C_{13}^{p} \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \frac{2}{3\pi} \sin \frac{3\pi}{2} \Gamma_1 \\
(t_{210}) &= \left[ C_{1}^{p} C_{13}^{p} \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \frac{2}{3\pi} \sin \frac{3\pi}{2} \Gamma_1 \\
(t_{230}) &= \left[ C_{1}^{p} C_{13}^{p} \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \frac{2}{3\pi} \sin \frac{3\pi}{2} \Gamma_1 \\
(t_{410}) &= \left[ C_{1}^{p} C_{13}^{p} \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \frac{2}{3\pi} \sin \frac{3\pi}{2} \Gamma_1 \\
(t_{430}) &= \left[ C_{1}^{p} C_{13}^{p} \right] \frac{2}{\pi} \sin \frac{\pi}{2} \Gamma_1 + \left[ C_{12}^{p} C_{12}^{p} \right] \frac{2}{3\pi} \sin \frac{3\pi}{2} \Gamma_1 \\
(t_{140}), (t_{151}), (t_{152}), (t_{160}), (t_{240}), (t_{251}), (t_{252}), (t_{260}), (t_{440}), \\
(t_{451}), (t_{452}) and (t_{460}) are the same as in Section B.2.2.b.
B.2.2.d. Plastic Buckling

(Based on the deformation theory)

\[(\nu_{110}) = u_1(t_{110}) - u_2(t_{120}) + u_3(t_{130})\]
\[(\nu_{120}) = u_1(t_{210}) - u_2(t_{220}) + u_3(t_{130})\]
\[(\nu_{220}) = u_1(t_{410}) - u_2(t_{240}) + u_3(t_{430})\]
\[(\nu_{111}) = (t_{140})\]
\[(\nu_{121}) = (t_{240})\]
\[(\nu_{222}) = (t_{440})\]
\[(t_{110}) = \left(\frac{c_p}{c_{11}^p - \frac{\alpha}{\beta} c_{11}}\right) \Gamma_0 + \left(\frac{c_p}{c_{12}^p - \frac{\alpha}{\beta} c_{12}}\right) \frac{1}{2} \pi \sin \pi \Gamma_0\]
\[(t_{120}) = \left(\frac{c_p}{c_{11}^p - \frac{\alpha}{\beta} c_{11}}\right) \log \frac{\Gamma_0 - \Gamma_0}{\Gamma_0} + \left(\frac{c_p}{c_{12}^p - \frac{\alpha}{\beta} c_{12}}\right) S^2\]
\[(t_{130}) = \left(\frac{c_p}{c_{11}^p + \frac{\alpha}{\beta} c_{11}}\right) (1 - \Gamma_2) - \left(\frac{c_p}{c_{12}^p - \frac{\alpha}{\beta} c_{12}}\right) \frac{1}{2} \pi \sin \pi \Gamma_2\]
\[(t_{140}) = (1 - \frac{3}{3} \Gamma_5)\]
\[(t_{210}) = \left(\frac{c_p}{c_{11}^p - \frac{\alpha}{\beta} c_{11}}\right) \frac{2}{3} \pi \sin \frac{\pi}{2} \Gamma_0 + \left(\frac{c_p}{c_{14}^p - \frac{\alpha}{\beta} c_{14}}\right) \frac{2}{3} \pi \sin \frac{3 \pi}{2} \Gamma_0\]
\[(t_{220}) = \left(\frac{c_p}{c_{11}^p - \frac{\alpha}{\beta} c_{11}}\right) S^2 + \left(\frac{c_p}{c_{14}^p - \frac{\alpha}{\beta} c_{14}}\right) S^2\]
\[(t_{230}) = \left(\frac{c_p}{c_{11}^p - \frac{\alpha}{\beta} c_{11}}\right) \frac{2}{3} \pi (1 - \sin \frac{\pi}{2} \Gamma_2) - \left(\frac{c_p}{c_{14}^p - \frac{\alpha}{\beta} c_{14}}\right) \frac{2}{3} \pi (1 + \sin \frac{3 \pi}{2} \Gamma_2)\]
\[(t_{240}) = (1 - \frac{2}{2} \beta \Gamma_5)\]
\[(t_{410}) = \left(\frac{c_p}{c_{13}^p + \frac{\beta}{\alpha} c_{23}}\right) \Gamma_0 + \left(\frac{c_p}{c_{124}^p + \frac{\beta}{\alpha} c_{24}}\right) \sin \pi \Gamma_0 + \left(\frac{c_p}{c_{125}^p + \frac{\beta}{\alpha} c_{25}}\right) \sin 2 \pi \Gamma_0\]
$$(t_{420}) = \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] \log \left(\frac{\gamma_{z} - \gamma_{z}}{\gamma_{z} - \gamma_{z}}\right) + \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] s^{2} + \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] s^{2}$$

$$(t_{430}) = \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] \left(1 - \gamma_{z}\right) - \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] \frac{1}{\pi^{2}} \sin \pi \gamma_{z}
- \left[c_{m}^{p} + \frac{\beta^{3}}{\alpha} c_{z}^{p}\right] \frac{1}{2 \pi^{2}} \sin 2 \pi \gamma_{z}$$

$$(t_{440}) = 3 \left(1 - \alpha \beta^{2} \gamma_{z}^{2}\right)$$

B.3. **Analytical Solutions for Simply Supported Plates**

B.3.1. **Additional Notation**

$$c_{i}^{11} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} + 2 \gamma_{3}^{2} + 1\right)$$

$$c_{i}^{12} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} - 2(1 - 2 \nu) \gamma_{3}^{2} + 1\right)$$

$$c_{i}^{13} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} + (6 + 4 \nu) \gamma_{3}^{2} + 9\right)$$

$$c_{i}^{14} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} - 2(3 - 8 \nu) \gamma_{3}^{2} + 9\right)$$

$$c_{i}^{15} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} + 18 \gamma_{3}^{2} + 9\right)$$

$$c_{i}^{16} = \frac{\pi^{4}}{32} \left(\omega \gamma_{3}^{4} + 18 \nu \gamma_{3}^{2} + 9\right)$$
B.3.2. **Analytical Solutions**

The analytical solution is given in the form:

\[ F_{ll} = 0 \]

or

\[ F_{ll} F_{z2} - F_{ll} F_{z1} = 0 \]

where

\[ F_{ll} = (\Gamma_4^2) (\nu_{110}) + (\nu_{111}) \]

\[ F_{12} - F_{21} = (\Gamma_4^2) (\nu_{120}) + (\nu_{121}) \]

\[ F_{22} = (\Gamma_4^2) (\nu_{220}) + (\nu_{221}) \]

B.3.2.a. **Elastic Buckling**

\[ (\nu_{110}) = u \ (t_{130}) \]

\[ (\nu_{120}) = 0 \]

\[ (\nu_{220}) = u \ (t_{430}) \]

\[ (\nu_{111}) = -k_1(t_{140}) - k_2(t_{151}) + k_3(t_{152}) + k_4(t_{160}) \]

\[ (\nu_{121}) = -k_1(t_{240}) - k_2(t_{251}) + k_3(t_{252}) + k_4(t_{260}) \]

\[ (\nu_{221}) = -k_1(t_{440}) - k_2(t_{451}) + k_3(t_{452}) + k_4(t_{460}) \]
\begin{align*}
(\tau_{130}) &= c_{13}^E \\
(\tau_{140}) &= \Gamma_0 + \frac{1}{\pi} \sin \pi \Gamma_0 \\
(\tau_{151}) &= \Gamma_2 - \Gamma_0 + \frac{1}{2\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right) \\
(\tau_{152}) &= \frac{1}{2} \left( \Gamma_2^2 - \Gamma_0^2 \right) + \frac{1}{2\pi} \left( \Gamma_2 \sin \pi \Gamma_2 - \Gamma_0 \sin \pi \Gamma_0 \right) \\
&\hspace{1cm} + \left( \frac{1}{\pi} \right)^2 \left( \cos \pi \Gamma_2 - \cos \pi \Gamma_0 \right) \\
(\tau_{160}) &= \Gamma_2 - 1 + \frac{1}{2\pi} \sin \pi \Gamma_2 \\
(\tau_{240}) &= \frac{1}{\pi} \sin \pi \Gamma_0 + \frac{1}{2\pi} \sin \pi \Gamma_0 \\
(\tau_{251}) &= \frac{1}{\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right) + \frac{1}{2\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right) \\
(\tau_{252}) &= \frac{1}{2\pi} \left( \Gamma_2 \sin \pi \Gamma_2 - \Gamma_0 \sin \pi \Gamma_0 \right) \\
(\tau_{260}) &= \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2\pi} \sin \pi \Gamma_2 \\
(\tau_{380}) &= c_{15}^E \\
(\tau_{440}) &= \Gamma_0 + \frac{1}{3\pi} \sin \pi \Gamma_0 \\
(\tau_{451}) &= \Gamma_2 - \Gamma_0 + \frac{1}{3\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right) \\
(\tau_{452}) &= \frac{1}{2} \left( \Gamma_2^2 - \Gamma_0^2 \right) + \frac{1}{3\pi} \left( \Gamma_2 \sin \pi \Gamma_2 - \Gamma_0 \sin \pi \Gamma_0 \right) \\
&\hspace{1cm} + \left( \frac{1}{3\pi} \right)^2 \left( \cos \pi \Gamma_2 - \cos \pi \Gamma_0 \right) \\
(\tau_{460}) &= \Gamma_2 - 1 + \frac{1}{3\pi} \sin \pi \Gamma_2 \\
\end{align*}
B.3.2.b, Elastic-plastic Buckling

(Based on the deformation theory)

\[ (v_{110}) = u_1(t_{110}) - u_2(t_{120}) + u(t_{130}) \]

\[ (v_{210}) = u_1(t_{210}) - u_2(t_{220}) + u(t_{230}) \]

\[ (v_{220}) = u_1(t_{410}) - u_2(t_{420}) + u(t_{430}) \]

\[ (v_{111}) = -u_1(t_{140}) - k_2(t_{151}) + k_3(t_{152}) + k_4(t_{160}) \]

\[ (v_{121}) = -u_1(t_{240}) - k_2(t_{251}) + k_3(t_{252}) + k_4(t_{260}) \]

\[ (v_{221}) = -u_1(t_{440}) - k_2(t_{451}) + k_3(t_{452}) + k_4(t_{460}) \]

\[ (t_{110}) = c_{11}^p \Gamma_0^o + \frac{1}{\pi} c_{12}^p \sin \pi \Gamma_0^o \]

\[ (t_{120}) = c_{11}^p \log \left( \frac{\Gamma_6^o - \Gamma_0^o}{\Gamma_6^o - \Gamma_1^o} \right) + c_{12}^p S_1^i \]

\[ (t_{130}) = c_{11}^p (1 - \Gamma_6^i) - \frac{1}{\pi} c_{12}^p \sin \pi \Gamma_1^i \]

\[ (t_{140}) = \Gamma_1^o + \frac{1}{\pi} \sin \pi \Gamma_1^o \]

\[ (t_{151}) = (\Gamma_2^o - \Gamma_1^i) + \frac{1}{\pi} \left[ \sin \pi \Gamma_2^o - \sin \pi \Gamma_1^i \right] \]

\[ (t_{152}) = \frac{1}{2} (\Gamma_2^o - \Gamma_1^i)^2 + \frac{1}{\pi} \left[ \Gamma_2^o \sin \pi \Gamma_2^o - \Gamma_1^i \sin \pi \Gamma_1^i \right] + \frac{1}{(\pi)^2} \left[ \cos \pi \Gamma_2^o - \cos \pi \Gamma_1^i \right] \]

\[ (t_{160}) = \Gamma_2^o - 1 + \frac{1}{\pi} \sin \pi \Gamma_2^o \]

\[ (t_{210}) = \frac{1}{\pi} c_{13}^p \sin \pi \Gamma_0^o + \frac{1}{\pi} c_{14}^p \sin 2\pi \Gamma_0^o \]

\[ (t_{220}) = c_{13}^p S_1^o + c_{14}^p S_1^i \]
\[(t_{230}) = -\frac{1}{\pi} c_{13} E \sin \pi \Gamma_1 - \frac{1}{2 \pi} c_{14} E \sin 2 \pi \Gamma_1\]

\[(t_{240}) = \frac{1}{\pi} \sin \pi \Gamma_1 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_1\]

\[(t_{251}) = \frac{1}{\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_1 \right) + \frac{1}{2 \pi} \left( \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_1 \right)\]

\[(t_{252}) = \frac{1}{\pi} \left( \Gamma_2 \sin \pi \Gamma_2 - \Gamma_1 \sin \pi \Gamma_1 \right) + \frac{1}{2 \pi} \left( \Gamma_2 \sin 2 \pi \Gamma_2 - \Gamma_1 \sin 2 \pi \Gamma_1 \right)\]

\[(t_{260}) = \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_2\]

\[(t_{410}) = c_{15}^P + \frac{1}{3 \pi} c_{16}^P \sin 3 \pi \Gamma_0\]

\[(t_{420}) = c_{15}^P \log \frac{\Gamma_1 - \Gamma_6}{\Gamma_1 - \Gamma_6} + c_{16}^P \frac{s_1}{3}\]

\[(t_{430}) = c_{15}^E (1 - \Gamma_1) - \frac{1}{3 \pi} c_{16}^E \sin 3 \pi \Gamma_1\]

\[(t_{440}) = \Gamma_1 + \frac{1}{3 \pi} \sin 3 \pi \Gamma_1\]

\[(t_{451}) = \Gamma_2 - \Gamma_1 + \frac{1}{3 \pi} \left( \sin 3 \pi \Gamma_2 - \sin 3 \pi \Gamma_1 \right)\]

\[(t_{452}) = \frac{1}{2} \left( \Gamma_2^2 - \Gamma_1^2 \right) + \frac{1}{3 \pi} \left( \Gamma_2 \sin 3 \pi \Gamma_2 - \Gamma_1 \sin 3 \pi \Gamma_1 \right)\]

\[+ \left( \frac{1}{3 \pi} \right)^2 \left( \cos 3 \pi \Gamma_2 - \cos 3 \pi \Gamma_1 \right)\]

\[(t_{460}) = \Gamma_2 - 1 + \frac{1}{3 \pi} \sin 3 \pi \Gamma_2\]
B.3.2.c. Elastic-plastic Buckling

(Based on the flow theory)

\[(v_{110}) = u_0(t_{110}) + u(t_{130})\]
\[(v_{120}) = u_0(t_{210}) + u(t_{230})\]
\[(v_{220}) = u_0(t_{410}) + u(t_{430})\]

\[(v_{111}) = -(t_{140}) - k_2(t_{151}) + k_3(t_{152}) + k_4(t_{160})\]
\[(v_{121}) = -(t_{240}) - k_2(t_{251}) + k_3(t_{252}) + k_4(t_{260})\]
\[(v_{221}) = -(t_{440}) - k_2(t_{451}) + k_3(t_{452}) + k_4(t_{460})\]

\[(t_{110}) = C_{11} \gamma_1 + \frac{1}{\pi C_{12}} \sin \pi \gamma_1\]
\[(t_{130}) = C_{12} \left(1 - \gamma_1\right) - \frac{1}{\pi C_{12}} \sin \pi \gamma_1\]
\[(t_{210}) = \frac{1}{\pi C_{13}} \sin \pi \gamma_1 + \frac{1}{2\pi C_{14}} \sin 2\pi \gamma_1\]
\[(t_{230}) = -\frac{1}{\pi C_{13}} \sin \pi \gamma_1 + \frac{1}{2\pi C_{14}} \sin 2\pi \gamma_1\]
\[(t_{410}) = C_{15} \gamma_1 + \frac{1}{3\pi C_{16}} \sin 3\pi \gamma_1\]
\[(t_{430}) = C_{15} \left(1 - \gamma_1\right) - \frac{1}{3\pi C_{16}} \sin 3\pi \gamma_1\]

\[(t_{140}) (t_{151}) (t_{152}) (t_{160}) (t_{240}) (t_{251}) (t_{252}) (t_{260}) (t_{440})\]

\[(t_{451}) (t_{452})\] and \[(t_{460})\] are the same as in Section B.3.2.b.
B.3.2.d. Plastic Buckling

(Based on the deformation theory)

\[(v_{110}) = u_1(t_{110}) - u_2(t_{120}) + u_3(t_{130})\]

\[(v_{120}) = u_1(t_{120}) - u_2(t_{220}) + u_3(t_{230})\]

\[(v_{220}) = u_1(t_{410}) - u_2(t_{420}) + u_3(t_{430})\]

\[(v_{111}) = -1\]

\[(v_{121}) = 0\]

\[(v_{221}) = -1\]

\[(t_{110}) = \frac{c_1^p}{\Gamma_0} + \frac{1}{\Gamma_0} c_{12}^p \sin \pi \Gamma_0\]

\[(t_{120}) = c_{11}^p \log \frac{\Gamma_0 - \Gamma_6^2}{\Gamma_0} + c_{12}^p S_1\]

\[(t_{130}) = c_{11}^p (1 - \Gamma_0^2) - \frac{1}{\Gamma_0} c_{12}^p \sin \pi \Gamma_2\]

\[(t_{210}) = \frac{1}{\Gamma_0} c_{13}^p \sin \pi \Gamma_0 + \frac{1}{\Gamma_0} c_{14}^p \sin 2\pi \Gamma_0\]

\[(t_{220}) = c_{13}^p S_1^2 + c_{14}^p S_2^2\]

\[(t_{230}) = \frac{1}{\Gamma_0} c_{13}^p \sin \pi \Gamma_2 + \frac{1}{2\pi} c_{14}^p \sin 2\pi \Gamma_2\]

\[(t_{410}) = c_{15}^p \Gamma_0 + \frac{1}{3\pi} c_{16}^p \sin 3\pi \Gamma_0\]

\[(t_{420}) = c_{15}^p \log \frac{\Gamma_0 - \Gamma_6^2}{\Gamma_0} + c_{16}^p S_3^2\]

\[(t_{430}) = c_{15}^p (1 - \Gamma_0^2) - \frac{1}{3\pi} c_{16}^p \sin 3\pi \Gamma_2\]
B.4. **Analytical Solutions for Fixed Plates**

B.4.1 **Additional Notation**

\[ c_{11} = \pi^4 \left[ 3 \omega \Gamma_3^4 + 2 \Gamma_3^2 + 1 \right] \]
\[ c_{12} = 4 \pi^4 \left( \omega \Gamma_3^4 + \nu \Gamma_3^2 \right) \]
\[ c_{13} = \pi^4 \left( \omega \Gamma_3^4 - 2(1 - 2\nu) \Gamma_3^2 + 1 \right) \]
\[ c_{14} = -2 \pi^4 \omega \Gamma_3^4 \]
\[ c_{15} = \pi^4 \left[ -\omega \Gamma_3^4 + (4 - \nu) \Gamma_3^2 + 4 \right] \]
\[ c_{16} = 2 \pi^4 \left( \omega \Gamma_3^4 + 4\nu \Gamma_3^2 \right) \]
\[ c_{17} = \pi^4 \left( \omega \Gamma_3^4 + (4 - 3\nu) \Gamma_3^2 + 4 \right) \]
\[ c_{18} = -\pi^4 \left( c \Gamma_3^4 + 4(1 - \nu) \Gamma_3^2 \right) \]
\[ c_{19} = \pi^4 \left( c \Gamma_3^4 + 8(1 - 2\nu) \Gamma_3^2 + 4 \right) \]

B.4.2. **Analytical Solutions**

The analytical solution is given in the form:
\[
F_{11} = 0
\]
or
\[
F_{11} F_{22} - F_{12} F_{21} = 0
\]
where
\[
F_{11} = (g_4^2 (\nu_{110}) + (\nu_{111})
\]
\[
F_{12} = F_{21} = (g_4^2 (\nu_{120}) + (\nu_{121})
\]
\[
F_{22} = (g_4^2 (\nu_{220}) + (\nu_{221})
\]

B.4.2.a. Elastic Buckling

\[
(\nu_{110}) = u(t_{130})
\]
\[
(\nu_{120}) = u(t_{230})
\]
\[
(\nu_{220}) = u(t_{430})
\]
\[
(\nu_{111}) = -k_1(t_{140}) - k_2(t_{151}) + k_3(t_{152}) + k_4(t_{160})
\]
\[
(\nu_{121}) = -k_1(t_{240}) - k_2(t_{251}) + k_3(t_{252}) + k_4(t_{260})
\]
\[
(\nu_{221}) = -k_1(t_{440}) - k_2(t_{451}) + k_3(t_{452}) + k_4(t_{460})
\]
\[
(t_{130}) = \frac{E}{c_{11}}
\]
\[
(t_{140}) = \Gamma_0 + \frac{1}{2\pi} \sin \pi \Gamma_0 + \frac{1}{2\pi} \sin 2\pi \Gamma_0
\]
\[(t 151) = (\Gamma_2 - \Gamma_0) + \frac{1}{\pi} \left[ \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right] + \frac{1}{2 \pi} \left[ \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_0 \right] \]

\[(t 152) = \frac{1}{2} (\Gamma_2^2 - \Gamma_0^2) + \frac{1}{\pi} \left[ \Gamma_2 \sin \pi \Gamma_2 - \Gamma_0 \sin \pi \Gamma_0 \right] + \left(\frac{1}{\pi}\right)^2 \left[ \cos \pi \Gamma_2 - \cos \pi \Gamma_0 \right] + \left(\frac{1}{2 \pi}\right)^2 \left[ \cos 2 \pi \Gamma_2 - \cos 2 \pi \Gamma_0 \right] \]

\[(t 160) = -1 + \left(\frac{1}{\pi}\right) \sin \pi \Gamma_2 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_2 \]

\[(t 230) = \frac{\mathcal{E}}{h} \]

\[(t 240) = \Gamma_0 + \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_0 + \frac{1}{3 \pi} \sin 3 \pi \Gamma_0 \]

\[(t 251) = (\Gamma_2 - \Gamma_0) + \frac{1}{\pi} \left[ \sin \pi \Gamma_2 - \sin \pi \Gamma_0 \right] + \frac{1}{2 \pi} \left[ \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_0 \right] + \frac{1}{3 \pi} \left[ \sin 3 \pi \Gamma_2 - \sin 3 \pi \Gamma_0 \right] \]

\[(t 260) = \Gamma_2 - 1 + \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2 \pi} \sin \pi \Gamma_2 + \frac{1}{3 \pi} \sin 3 \pi \Gamma_2 \]

\[(t 430) = \frac{\mathcal{E}}{25} \]

\[(t 440) = \Gamma_0 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_0 + \frac{1}{4 \pi} \sin 4 \pi \Gamma_0 \]

\[(t 451) = (\Gamma_2 - \Gamma_0) + \frac{1}{\pi} \left[ \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_0 \right] + \frac{1}{4 \pi} \left[ \sin 4 \pi \Gamma_2 - \sin 4 \pi \Gamma_0 \right] \]

\[(t 452) = \frac{1}{2} (\Gamma_2^2 - \Gamma_0^2) + \frac{1}{2 \pi} \left[ \Gamma_2 \sin 2 \pi \Gamma_2 - \Gamma_0 \sin 2 \pi \Gamma_0 \right] + \left(\frac{1}{2 \pi}\right)^2 \left[ \cos 2 \pi \Gamma_2 - \cos 2 \pi \Gamma_0 \right] \]

\[(t 460) = \Gamma_2 - 1 + \frac{1}{2 \pi} \sin 2 \pi \Gamma_2 + \frac{1}{4 \pi} \sin 4 \pi \Gamma_2 \]
B.4.2.b. Elastic-plastic Buckling

(Based on the deformation theory)

\[(v_1) = u_1(t_1) - u_2(t_2) + u(t_3)\]
\[(v_2) = u_1(t_2) - u_2(t_3) + u(t_4)\]
\[(v_3) = u_1(t_3) - u_2(t_4) + u(t_5)\]
\[(v_4) = u_1(t_4) - u_2(t_5) + u(t_6)\]

\[(v_11) = - (t_1) + k_1(t_2) + k_2(t_3) + k_3(t_4)\]
\[(v_12) = - (t_2) + k_1(t_3) + k_2(t_4) + k_3(t_5)\]
\[(v_21) = - (t_3) + k_1(t_4) + k_2(t_5) + k_3(t_6)\]

\[(v_110) = c_{11}^{p} \gamma_0 + \frac{1}{4\pi} c_{12}^{p} \sin \pi \gamma_0 + \frac{1}{2\pi} c_{13}^{p} \sin 2\pi \gamma_0\]
\[(v_120) = c_{11}^{p} \log \frac{1}{1} - \frac{1}{6} + c_{12}^{p} S_1\]
\[(v_210) = c_{11}^{E} (1 - \gamma_1) - \frac{1}{4\pi} c_{12}^{E} \sin \pi \gamma_1 - \frac{1}{2\pi} c_{13}^{E} \sin 2\pi \gamma_1\]
\[(v_220) = 3\gamma_1 + \frac{4}{2\pi} \sin 2\pi \gamma_1 + \frac{1}{2\pi} \sin 2\pi \gamma_1\]
\[(v_111) = 3\gamma_1 + \frac{4}{2\pi} \sin 2\pi \gamma_1 + \frac{1}{2\pi} \sin 2\pi \gamma_1\]
\[(v_112) = \frac{1}{2} (\gamma_2 - \gamma_1) + \frac{1}{2\pi} \left[ \gamma_2 \sin 2\pi \gamma_1 - \gamma_1 \sin 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \cos 2\pi \gamma_1 - \cos 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \sin 2\pi \gamma_1 - \sin 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \cos 2\pi \gamma_1 - \cos 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \sin 2\pi \gamma_1 - \sin 2\pi \gamma_1 \right]
\[(v_121) = \frac{1}{2} (\gamma_2 - \gamma_1) + \frac{1}{2\pi} \left[ \gamma_2 \sin 2\pi \gamma_1 - \gamma_1 \sin 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \cos 2\pi \gamma_1 - \cos 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \sin 2\pi \gamma_1 - \sin 2\pi \gamma_1 \right]
\[(v_211) = \frac{1}{2} (\gamma_2 - \gamma_1) + \frac{1}{2\pi} \left[ \gamma_2 \sin 2\pi \gamma_1 - \gamma_1 \sin 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \cos 2\pi \gamma_1 - \cos 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \sin 2\pi \gamma_1 - \sin 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \cos 2\pi \gamma_1 - \cos 2\pi \gamma_1 \right] + \frac{1}{2\pi} \left[ \sin 2\pi \gamma_1 - \sin 2\pi \gamma_1 \right]"
\[\text{t220} = \frac{p}{14} \log \frac{1 - \frac{\Gamma_0}{\Gamma_6}}{1 - \frac{\Gamma_1}{\Gamma_6}} + \frac{p}{15} S_1 + \frac{p}{16} S_2 + \frac{p}{17} S_3\]

\[\text{t230} = \frac{c}{14} (1 - \frac{\Gamma_1}{\Gamma_6}) - \frac{c}{15} \sin \pi \Gamma_1 - \frac{c}{16} \sin 2 \pi \Gamma_1 - \frac{c}{17} \sin 3 \pi \Gamma_1\]

\[\text{t240} = -2 \frac{\Gamma_1 - \frac{\Gamma_1}{\Gamma_6}}{\pi} \sin \pi \Gamma_1 + \frac{\Gamma_1}{\pi} \sin 2 \pi \Gamma_1 + \frac{\Gamma_1}{3 \pi} \sin 3 \pi \Gamma_1\]

\[\text{t250} = -2 \left( \Gamma_2 - \Gamma_1 \right) - \frac{2}{\pi} \left( \sin \pi \Gamma_2 - \sin \pi \Gamma_1 \right) + \frac{2}{\pi} \left( \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_1 \right) + \frac{2}{3 \pi} \left( \sin 3 \pi \Gamma_2 - \sin 3 \pi \Gamma_1 \right)\]

\[\text{t252} = -2 \left( \Gamma_2 - \Gamma_1 \right)^2 - \frac{2}{\pi} \left( \Gamma_2 \sin \pi \Gamma_2 - \Gamma_1 \sin \pi \Gamma_1 \right) - \frac{2}{\pi} \left( \cos \pi \Gamma_2 - \cos \pi \Gamma_1 \right)\]

\[\text{t260} = \Gamma_2 - 1 + \frac{\Gamma_1}{\pi} \sin \pi \Gamma_2 + \frac{\Gamma_1}{2 \pi} \sin 2 \pi \Gamma_2 + \frac{\Gamma_1}{3 \pi} \sin 3 \pi \Gamma_2\]

\[\text{t410} = \frac{c}{14} \Gamma_0 + \frac{1}{4 \pi} \frac{c}{15} \sin 2 \pi \Gamma_0 + \frac{1}{4 \pi} \frac{c}{16} \sin 4 \pi \Gamma_0\]

\[\text{t420} = \frac{c}{14} \log \frac{1 - \frac{\Gamma_0}{\Gamma_6}}{1 - \frac{\Gamma_1}{\Gamma_6}} + \frac{c}{15} S_1 + \frac{c}{16} S_2 + \frac{c}{17} S_3\]

\[\text{t430} = \frac{c}{14} (1 - \frac{\Gamma_1}{\Gamma_6}) - \frac{c}{15} \sin \pi \Gamma_1 - \frac{c}{16} \sin 2 \pi \Gamma_1 - \frac{c}{17} \sin 3 \pi \Gamma_1\]

\[\text{t440} = -3 \Gamma_1 - \frac{2}{\pi} \sin 2 \pi \Gamma_1 + \frac{1}{4 \pi} \sin 4 \pi \Gamma_1\]

\[\text{t451} = 3 \left( \Gamma_1 - \Gamma_0 \right) - \frac{2}{\pi} \left( \sin 2 \pi \Gamma_2 - \sin 2 \pi \Gamma_1 \right) + \frac{1}{4 \pi} \left( \sin 4 \pi \Gamma_2 - \sin 4 \pi \Gamma_1 \right)\]

\[\text{t452} = \frac{3}{2} \left( \Gamma_1 - \Gamma_0 \right)^2 - \frac{2}{\pi} \left( \Gamma_2 \sin 2 \pi \Gamma_2 - \Gamma_1 \sin 2 \pi \Gamma_1 \right) - \frac{1}{2 \pi} \left( \cos 2 \pi \Gamma_2 - \cos 2 \pi \Gamma_1 \right)\]

\[\text{t452} = - \frac{4}{3 \pi} \left( \Gamma_2 \sin 4 \pi \Gamma_2 - \Gamma_1 \sin 4 \pi \Gamma_1 \right) - \frac{1}{4 \pi} \left( \cos 4 \pi \Gamma_2 - \cos 4 \pi \Gamma_1 \right)\]

\[\text{t460} = -3 \left( \Gamma_2 - 1 \right) - \frac{2}{\pi} \sin 2 \pi \Gamma_2 + \frac{1}{4 \pi} \sin 4 \pi \Gamma_2\]
B.4.2.6. Elastic-plastic Buckling

(Based on the flow theory)

\[(v110) = u_0(t110) + u(t130)\]

\[(v120) = u_0(t210) + u(t230)\]

\[(v220) = u_0(410) + u(t430)\]

\[(v111) = -(t140) - k_2(t151) + k_3(t152) + k_4(t160)\]

\[(v121) = -(t240) - k_2(t251) + k_3(t252) + k_4(t260)\]

\[(v221) = -(t440) - k_2(t451) + k_3(t452) + k_4(t460)\]

\[(t110) = c_{i11}^{P} \sin \pi \Gamma_1 + \frac{(\frac{1}{2})}{c_{i2}^{P}} \sin 2\pi \Gamma_1 + \frac{(\frac{1}{2})}{c_{13}^{P}} \sin 2\pi \Gamma_1\]

\[(t130) = c_{i10}^{E} (1-\Gamma_1) - \frac{1}{c_{i12}^{E}} \sin \pi \Gamma_1 - \frac{1}{2\pi} c_{13}^{E} \sin 2\pi \Gamma_1\]

\[(t210) = c_{i14}^{P} + \frac{1}{c_{i15}^{P}} \sin \pi \Gamma_1 + \frac{1}{2\pi} c_{i16}^{P} \sin 2\pi \Gamma_1 + \frac{1}{3\pi} c_{i17}^{P} \sin 3\pi \Gamma_1\]

\[(t230) = c_{i4}^{E} (1-\Gamma_1) - \frac{1}{\pi} c_{i15}^{E} \sin \pi \Gamma_1 - \frac{1}{2\pi} c_{16}^{E} \sin 2\pi \Gamma_1 - \frac{1}{3\pi} c_{i17}^{E} \sin 3\pi \Gamma_1\]

\[(t410) = \frac{1}{2\pi} c_{25}^{P} \sin 2\pi \Gamma_1 + \frac{1}{4\pi} c_{26}^{P} \sin 4\pi \Gamma_1\]

\[(t430) = \frac{1}{2\pi} c_{25}^{E} (1-\Gamma_1) - \frac{1}{2\pi} c_{26}^{E} \sin 2\pi \Gamma_1 - \frac{1}{4\pi} c_{27}^{E} \sin 4\pi \Gamma_1\]

\[(t140),(t151),(t152),(t160),(t240),(t251),(t252),(t260),(t440),
(t451),(t452)\) and \((t460)\) are the same as in Section B.4.2.
B.4.2.d. Plastic Buckling

(Based on the deformation theory)

\[(v_{110}) = u_1(t_{110}) - u_2(t_{120}) + u_3(t_{130})\]

\[(v_{120}) = u_1(t_{210}) - u_2(t_{220}) + u_3(t_{230})\]

\[(v_{220}) = u_1(t_{410}) - u_2(t_{420}) + u_3(t_{430})\]

\[(v_{111}) = -3\]

\[(v_{121}) = 2\]

\[(v_{221}) = -3\]

\[(t_{110}) = c_{11}^P \Gamma_0 + \frac{1}{\pi} c_{12}^P \sin \pi \Gamma_0 + \frac{1}{2 \pi} c_{13}^P \sin 2\pi \Gamma_0\]

\[(t_{120}) = c_{12}^P \log \frac{\Gamma_2 - \Gamma_0}{\Gamma_0} + c_{12}^P \Gamma_0 + c_{13}^P \Gamma_0^2 + c_{14}^P \Gamma_0^3\]

\[(t_{130}) = 1 - \Gamma_2 \sin \pi \Gamma_2 - \frac{1}{2 \pi} \sin 2\pi \Gamma_2^0\]

\[(t_{210}) = c_{14}^P \Gamma_0 + \frac{1}{\pi} c_{15}^P \sin \pi \Gamma_0^0 + \frac{1}{2 \pi} c_{16}^P \sin 2\pi \Gamma_0 + \frac{1}{3 \pi} c_{17}^P \sin 3\pi \Gamma_0\]

\[(t_{220}) = c_{14}^P \log \frac{\Gamma_2 - \Gamma_0}{\Gamma_0} + c_{15}^P \Gamma_0^2 + c_{16}^P \Gamma_0^2 + c_{17}^P \Gamma_0^3\]

\[(t_{230}) = c_{14}^P (1 - \Gamma_2^0) - \frac{1}{2 \pi} c_{15}^P \sin \pi \Gamma_2^0 + \frac{1}{4 \pi} c_{16}^P \sin 2\pi \Gamma_2^0 - \frac{1}{3 \pi} c_{17}^P \sin 3\pi \Gamma_2^0\]

\[(t_{410}) = c_{25}^P \Gamma_0 + \frac{1}{2 \pi} c_{26}^P \sin 2\pi \Gamma_0 + \frac{1}{4 \pi} c_{27}^P \sin 4\pi \Gamma_0\]

\[(t_{420}) = c_{25}^P \log \frac{\Gamma_2 - \Gamma_0}{\Gamma_0} + c_{26}^P \Gamma_0^2 + c_{27}^P \Gamma_0^2\]

\[(t_{430}) = c_{25}^P (1 - \Gamma_2^0) - \frac{1}{2 \pi} \sin 2\pi \Gamma_2^0 - \frac{1}{4 \pi} \sin 4\pi \Gamma_2^0\]
10. TABLES AND FIGURES
TABLE 4.1

Distribution of Stress and Strain, and Secant and Tangent Moduli in Plates

Original State

<table>
<thead>
<tr>
<th>Domain</th>
<th>Strain</th>
<th>Stress</th>
<th>(E_s)</th>
<th>(E_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 - b_0)</td>
<td>(-\sigma_{r_1}/E)</td>
<td>(-\sigma_{r_1})</td>
<td>(E)</td>
<td>(E)</td>
</tr>
<tr>
<td>(b_0 - b_2)</td>
<td>(-\phi(y)/E)</td>
<td>(-\phi(y))</td>
<td>(E)</td>
<td>(E)</td>
</tr>
<tr>
<td>(b_2 - b)</td>
<td>(\sigma_{r_2}/E \leq \varepsilon_Y)</td>
<td>(\sigma_{r_2} \leq \sigma_Y)</td>
<td>(E)</td>
<td>(E)</td>
</tr>
</tbody>
</table>

Elastic Buckling

<table>
<thead>
<tr>
<th>Domain</th>
<th>Strain</th>
<th>Stress</th>
<th>(E_s)</th>
<th>(E_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 - b_0)</td>
<td>[-\left(\sigma_{r_1}/E + \varepsilon_c\right)]</td>
<td>[-\left(\sigma_{r_1} + \sigma_c\right)]</td>
<td>(E)</td>
<td>(E)</td>
</tr>
<tr>
<td>(b_0 - b_2)</td>
<td>[-\left(\phi(y)/E + \varepsilon_c\right)]</td>
<td>[-\left(\phi(y) + \sigma_c\right)]</td>
<td>(E)</td>
<td>(E)</td>
</tr>
<tr>
<td>(b_2 - b)</td>
<td>(\sigma_{r_2}/E - \varepsilon_c)</td>
<td>(\sigma_{r_2} - \sigma_c)</td>
<td>(E)</td>
<td>(E)</td>
</tr>
</tbody>
</table>

Elastic-Plastic Buckling

<table>
<thead>
<tr>
<th>Domain</th>
<th>Strain</th>
<th>Stress</th>
<th>(E_s)</th>
<th>(E_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 - b_0)</td>
<td>[-\left(\sigma_{r_1}/E + \varepsilon_c\right)]</td>
<td>(-\sigma_Y)</td>
<td>(E) (\sigma_Y/\sigma_{r_1} + \sigma_c)</td>
<td>(0)</td>
</tr>
<tr>
<td>(b_0 - b_2)</td>
<td>[-\left(\phi(y)/E + \varepsilon_c\right)]</td>
<td>(-\sigma_Y)</td>
<td>(E) (\sigma_Y/\phi(y) + \sigma_c)</td>
<td>(0)</td>
</tr>
<tr>
<td>(b_2 - b)</td>
<td>(\sigma_{r_2}/E - \varepsilon_c)</td>
<td>(\sigma_{r_2} - \sigma_c)</td>
<td>(E)</td>
<td>(E)</td>
</tr>
</tbody>
</table>

Plastic Buckling

<table>
<thead>
<tr>
<th>Domain</th>
<th>Strain</th>
<th>Stress</th>
<th>(E_s)</th>
<th>(E_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 - b_0)</td>
<td>[-\left(\sigma_{r_1}/E + \varepsilon_c\right)]</td>
<td>(-\sigma_Y)</td>
<td>(E) (\sigma_Y/\sigma_{r_1} + \sigma_c)</td>
<td>(0)</td>
</tr>
<tr>
<td>(b_0 - b_2)</td>
<td>[-\left(\phi(y)/E + \varepsilon_c\right)]</td>
<td>(-\sigma_Y)</td>
<td>(E) (\sigma_Y/\phi(y) + \sigma_c)</td>
<td>(0)</td>
</tr>
<tr>
<td>(b_2 - b)</td>
<td>(\varepsilon_c - \sigma_{r_2}/E)</td>
<td>(-\sigma_Y)</td>
<td>(E) (\sigma_Y/\sigma_{r_2} + \sigma_{r_2})</td>
<td>(0)</td>
</tr>
</tbody>
</table>

where \(\phi(y) = \sigma_{r_1} - (y - b_0) \frac{\sigma_{r_1} + \sigma_{r_2}}{b_2 - b_0} = \frac{\sigma_{r_1} b_2 + \sigma_{r_2} b_0}{b_2 - b_0} - \frac{\sigma_{r_1} + \sigma_{r_2}}{b_2 - b_0} y\)

\(\sigma_c = E \varepsilon_c\)
### TABLE 6.1

**Dimensions of Columns**

<table>
<thead>
<tr>
<th></th>
<th>Column S1</th>
<th>Column S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness of Plate (2t) in.</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Width of Plate (2b) in.</td>
<td>11.25</td>
<td>16.25</td>
</tr>
<tr>
<td>Length of Column (L) in.</td>
<td>50.0</td>
<td>80.0</td>
</tr>
<tr>
<td>b/t</td>
<td>45.0</td>
<td>65.0</td>
</tr>
</tbody>
</table>

**Result of Tension Coupon Tests**

<table>
<thead>
<tr>
<th></th>
<th>Column S1</th>
<th>Column S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Y$ (Parent Material) ksi</td>
<td>39.0</td>
<td>38.5</td>
</tr>
<tr>
<td>$\sigma_Y$ (Welded Portions) ksi</td>
<td>45.0</td>
<td>43.0</td>
</tr>
</tbody>
</table>

**Result of Residual Stress Measurements.**

<table>
<thead>
<tr>
<th></th>
<th>Column S1</th>
<th>Column S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compressive $\sigma_r$, ($\sigma_{r1}$) ksi</td>
<td>12.5</td>
<td>10.5</td>
</tr>
<tr>
<td>Tensile $\sigma_r$, ($\sigma_{r2}$) ksi</td>
<td>39.0</td>
<td>38.5</td>
</tr>
<tr>
<td>$\sigma_{r1}/\sigma_Y$</td>
<td>0.320</td>
<td>0.273</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.500</td>
<td>0.500</td>
</tr>
</tbody>
</table>

**Result of Local Buckling Tests**

<table>
<thead>
<tr>
<th></th>
<th>Column S1</th>
<th>Column S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical Load, $P_{cr}$, kips</td>
<td>340</td>
<td>261</td>
</tr>
<tr>
<td>Ultimate Load, $P_u$, kips</td>
<td>357</td>
<td>337</td>
</tr>
<tr>
<td>Critical Stress, $\sigma_{cr}$, kips</td>
<td>30.2</td>
<td>16.0</td>
</tr>
<tr>
<td>Ultimate Strength, $\sigma_u$, kips</td>
<td>31.7</td>
<td>20.7</td>
</tr>
<tr>
<td>$\sigma_{cr}/\sigma_Y$</td>
<td>0.775</td>
<td>0.415</td>
</tr>
<tr>
<td>$\sigma_u/\sigma_Y$</td>
<td>0.813</td>
<td>0.538</td>
</tr>
</tbody>
</table>

**Theoretical Prediction**

| $\sigma_{cr}/\sigma_Y$ Corresponding to $\sigma_{r1}/\sigma_Y$ | 0.795 | 0.390 |
Fig. 3.1 STRESS-STRAIN RELATIONSHIP

(a) GENERAL CASE

(b) IDEALIZED RELATIONSHIP FOR STEEL (ELASTIC PERFECTLY PLASTIC MATERIAL)
Fig. 3.2 PLATE SUBJECTED TO FORCES

(a) PLATE SUBJECTED TO $\sigma_x, \sigma_y$ AND $\tau$

(b) PLATE SUBJECTED TO $\sigma_{x,y}^{(1)}$ AND $\sigma_{x,y}^{(2)}$
Fig. 4.1 USUAL RESIDUAL STRESS DISTRIBUTIONS IN EDGE WELDED PLATES

Fig. 4.2 SIMPLIFIED RESIDUAL STRESS DISTRIBUTION

Fig. 4.3 SIMPLIFIED RESIDUAL STRESS DISTRIBUTION USED IN THE ANALYSIS
Fig. 4.4 RELATIONSHIP BETWEEN STRESS AND STRAIN IN A LOADED PLATE CONTAINING RESIDUAL STRESSES
Fig. 4.5 COORDINATE AXES FOR PLATE ELEMENTS
Fig. 4.6 BOUNDARY CONDITIONS OF THE PLATES

Fig. 4.7 COORDINATE AXES FOR A COLUMN OF RECTANGULAR CROSS SECTION
Fig. 5.1 RELATIONSHIP BETWEEN RESIDUAL STRESS DISTRIBUTION AND REDUCTION IN BUCKLING STRENGTH

Fig. 5.2 RELATIONSHIP BETWEEN RESIDUAL STRESS MAGNITUDE AND REDUCTION IN BUCKLING STRENGTH
Fig. 5.3 BUCKLING STRENGTH OF PLATES WITH RESIDUAL STRESSES (DEFORMATION THEORY)
Fig. 5.4 BUCKLING STRENGTH OF PLATES WITH RESIDUAL STRESSES (FLOW THEORY)
Fig. 5.5 CRITICAL BUCKLING STRAIN OF PLATE WITH RESIDUAL STRESSES (DEFORMATION THEORY)
Fig. 5.6 CRITICAL BUCKLING STRAIN OF PLATE WITH RESIDUAL STRESSES (FLOW THEORY)

Fig. 5.7 RELATIONSHIP BETWEEN \( \frac{b}{t_{cr}} \) AND \( \frac{L}{b} \) FOR \( \frac{\sigma_{cr}}{\sigma_{Y}} = 0.125 \)
Fig. 5.8 RELATIONSHIP BETWEEN \( \left( \frac{b}{t} \right) \) AND \( \left( \frac{L}{B} \right) \) FOR \( \sigma_{cr} \). 

Fig. 5.9 RELATIONSHIP BETWEEN \( \left( \frac{b}{t} \right) \) AND \( \left( \frac{L}{B} \right) \) FOR \( \sigma_{cr} \).
Fig. 5.10 RELATIONSHIP BETWEEN \( \frac{b}{t} \) AND \( \frac{L}{B} \) FOR \( \frac{\sigma_r}{\sigma_Y} = 0.75 \)

Fig. 5.11 RELATIONSHIP BETWEEN \( \frac{b}{t} \) AND \( \frac{L}{B} \) FOR \( \frac{\sigma_r}{\sigma_Y} = 1.0 \)
Fig. 6.1 EXPERIMENTAL AND SIMPLIFIED RESIDUAL STRESS DISTRIBUTIONS
Fig. 6.2 LOAD-STRAIN CURVES
Fig. 6.3 LOAD-DEFLECTION CURVES
Fig. 6.4 LOAD - $\delta^2$ CURVES
Fig. 6.5 CRITICAL LOCAL BUCKLING STRENGTH OF COLUMNS
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12. V I T A

The author was born in Osaka, Japan on April 12, 1932, the first child of Torao and Komae Ueda.

He attended Osaka University from April 1951 to March 1955, and received the degree of Bachelor of Engineering in Naval Architecture in March 1955. He continued his study in the graduate school at Osaka University and in March 1957 was awarded the degree of Master of Engineering. At the same time he joined the staff of Osaka University as a research instructor.

He was appointed to a research fellowship at Fritz Engineering Laboratory, Lehigh University, in September 1959. In February 1960, he was appointed research assistant and in July 1962 research associate. He has been associated with research projects on residual stresses and column instability for both high strength and mild steels.