Plastic Design in High Strength Steel

THE INELASTIC BEHAVIOR OF BEAMS UNDER MOMENT GRADIENT

by
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I. **INTRODUCTION**

It is the purpose of this report to discuss the behavior of beams under moment gradient when loaded into the inelastic range. The moment gradient on a beam will be defined as $V$, where

$$ V = \frac{dM}{dZ} \quad (1) $$

$M$ is the moment and $Z$ the distance measured along the length of the beam. From elementary strength of materials, $V$ is also the shear force. The term moment ratio, $\phi$, will apply to the ratio of the end-moments on a beam. The moments are of the same sign if the curvatures, $\psi$, that they produce are of the same sign, and are taken such that $\phi$ does not exceed one.

The in-plane behavior of beams under moment gradient is well known, both theoretically\(^1\) and experimentally\(^2\). This behavior is, however, terminated by the effects of lateral and local buckling and much less is known of this aspect of the problem. As these effects will determine the deformation capacity of a beam, it is important that they be understood.

This report will present a study of the influence of lateral and local buckling on the rotation capacity of inelastic steel beams under moment gradient.
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II. BASIC BEHAVIOR

It has been shown elsewhere\textsuperscript{3,4} that the actual stress-strain diagram for any location in a steel member is the discontinuous curve given in Fig. 1. This presents some difficulties when beams under uniform moment are being considered\textsuperscript{4}, as the location of the discontinuous jumps in strain within the uniform moment region cannot be defined.

These strain jumps correspond to the formation of the slip planes\textsuperscript{3} within the material. However in discussing the beam under moment gradient there is no difficulty in defining the location of the slip planes, that is, the yielded region of the beam.

The region in which yielding occurs is not only well defined but it is also of limited extent. Consider the simply-supported beam under a central concentrated load, as shown in Fig. 2. Let $M_{ps}$ ($< M_p$, the plastic moment) be the moment at which slip planes form across the flanges. When the central moment reaches $M_{ps}$ there will be (in the ideal case) a jump in flange strain from yield to strain hardening. This will occur over the width of a yield plane which will be negligible relative to the length of the beam, and therefore unobservable in external measurements. Any further deformation of the beam can only occur if the central moment increases above $M_{ps}$ (Fig. 2b) to a moment $M_o$. The length of the yielded region $2 \zeta L$, is then given by

$$\zeta = 1 - \frac{M_{ps}}{M_o}$$

(2)
The maximum possible value of $M_o$ is $\sigma_u Z$, where $\sigma_u$ is the ultimate tensile stress of the material and $Z$ is the plastic modulus. If $M_{ps} = M_p = Z\sigma_y$ and $\sigma_u = \frac{5}{3} \sigma_y$, where $\sigma_y$ is the yield stress, it is seen that the maximum value of $\psi$ in Fig. 2 is 0.4. In reality, the moment $\sigma_u Z$ will never be attained as the large strains will cause prior failure due to local buckling or tensile fracture.

The form of the load-deformation curve may be obtained by assuming that $M_{ps} = M_p$. The curvature diagram is then as shown in Fig. 2c, where $s$ and $h$ are strain-hardening properties defined in Fig. 1. The angle of rotation, $\theta$, between the ends of the beam is the area of this diagram and is

$$\frac{\theta}{\gamma_p L} = \frac{M_p}{M_0} + \left(1 - \frac{M_p}{M_0}\right)\left(2s + h\left[\frac{M_o}{M_p} - 1\right]\right)$$

(3)

where

$$\gamma_p = \frac{M_p}{EI}$$

(4)

This equation is plotted in Fig. 3 for $s = 11.5$ and $h = 33$ (corresponding to A36). This curve is typical of those obtained from tests on beams under moment gradient. Note (1) that the curve continues to increase after yielding has occurred, in contrast to the beam under uniform moment, (2) that the dominant term in Eq. 3 is the curvature jump term, $2s$, (3) that the curve is closely linear, and its slope is almost $1/2s$ times the elastic slope, and (4) that at $M_o = 1.5M_p$ the mid-span strains are 2.4 times the strain hardening strain.
III. REVIEW OF PREVIOUS STUDIES

The elastic lateral buckling of beams under moment gradient has been conclusively studied\textsuperscript{6}. Studies of inelastic lateral buckling prior to 1956 have been summarized and extended by White\textsuperscript{7}.

White assumed that the beam was composed of two materials. Those portions of the beam subjected to moments less than $0.915M_p$ were assumed to possess elastic properties, and for moments above $0.915M_p$ the properties were taken as the strain-hardened values. The value of $0.915M_p$ was chosen from a visual examination of some moment-curvature curves. The buckling solution was found by the finite difference method, with each of the two portions being divided into three segments. Solutions for the cases of pure warping and pure St. Venant torsion can also be obtained, as the characteristic values of a $4 \times 4$ matrix whose elements are Bessel functions and their derivatives.

White's solution has been modified by Kusuda et al., and these modifications form the basis of the current AISC provisions\textsuperscript{(1)(13)} for the bracing of plastically designed steel beams.

Sawyer\textsuperscript{9} has recently experimentally investigated the behavior of beams in which the moment gradient was relatively high. These beams were also tested at an undefined but appreciable strain rate, and it is difficult to estimate the true significance of the results presented\textsuperscript{10}. The analytical work consists of evaluating certain rather arbitrary
parameters from these results. Such an approach is not amenable to wider applications.
IV. LATERAL BUCKLING SOLUTIONS

White's finite difference approach has certain disadvantages. The necessary numerical solutions make it impossible to extend the calculations to other steels and conditions, without undertaking complete and lengthy recalculations. It is, similarly, difficult to assess the influence of various parameters, and some inconsistencies can be noted in the presented results. In addition, the assumptions introduced by Kusuda et al., in order to present the solutions in a form suitable for design purposes, tend to obscure the relevancy or otherwise of the final design rules. The same comments would apply to a solution based on the matrix of Bessel functions. Finally, it is difficult to introduce the effect of local buckling into these analyses. This is a serious difficulty as local, and not lateral buckling will be the cause of failure in most cases. This makes any elaborate lateral buckling analysis rather unrealistic.

For these reasons, this report will present an analysis based on certain simplified assumptions. These will lead to a result which is capable of easy algebraic manipulation, and which indicates the significance of the various parameters involved. Because of the assumptions involved, the analysis is not exact, but it will provide a sufficiently close estimate of actual conditions.

Firstly, it is necessary to define the moment $M_{ps}$ at which slip planes occur. The following analysis will use the beam-model postulated
by the authors in previous reports. This considers the compression half of the beam to act as an isolated T column (Fig. 4). To be consistent with this model, $M_{ps}$ will be taken as the moment at which the yield strain is reached at the interior face of the flanges (that is, the flanges are fully yielded). It can easily be shown that, under this assumption,

$$\frac{M_{ps}}{M} = 1 - \frac{1}{3} \frac{1}{1 + \frac{hbt(d-t)}{w(d-2t)^2}}$$

(5)

where $b$ and $d$ are the beam breadth and depth, $t$ is the flange thickness and $w$ the web thickness. The value of $\frac{M_{ps}}{M}$ from Eq. 5 averages 0.94 for WF beam sections and this value will be taken for $M_{ps}/M$ in the following work. It is higher than the corresponding value used by White (0.915).

If the beam-model is used, it is necessary to consider that yielding occurs when the moment exceeds $0.94M_p$, and that the axial force varies from zero at zero moment to $0.94\sigma_y/2$ at $M_p$, where $A$ is the cross-sectional area. However, White has shown that the influence of moment variation on lateral buckling is small relative to the influence of the extent of yielding. Therefore the following analysis will assume that the force on the compression T of the model remains constant at $0.94\sigma_y/2$. This will be conservative in the regions where $M < M_p$ and unconservative when $M > M_p$. The latter region is small relative to the former, and so the assumption that the force is $0.94\sigma_y/2$ may be regarded as conservative.
The problem to be solved is shown in Fig. 5. No end restraint is considered at the yielded end of the beam; the end restraint at the elastic end is expressed in terms of the length and stiffness of the span under consideration. The buckling solution to this problem is given in Reference 11 as

\[
\frac{\tan \frac{\lambda \pi \gamma}{\sqrt{\frac{E}{c}}}}{\tan \frac{\lambda \pi (1-\gamma)}{\sqrt{\frac{E}{c}}}} + \sqrt{\frac{\lambda \pi + S (\frac{1}{\lambda \pi} - \cot \lambda \pi (1-\gamma))}{\lambda \pi + S (\frac{1}{\lambda \pi} + \tan \lambda \pi (1-\gamma))}} = 0
\]  

(6)

where \( c \) is the ratio of the lateral bending stiffness in the yielded region to its value in the elastic region. \( SEI_{yy}/L \) is the stiffness of the span adjacent to the elastic end of the beam. \( \lambda \) is the slenderness factor of the beam, defined by

\[
\lambda = \frac{\sqrt{\varepsilon_y}}{r}\frac{L}{r_y}
\]  

(7)

where \( \varepsilon_y \) is the yield strain and \( r_y \) the weak axis radius of gyration.

The value of \( c \) to be used is a consequence of the formation of flexural yield planes and is given by \( c_b \) where

\[
c_b = \frac{2}{h+\sqrt{h}}
\]  

(8)

It is assumed that \( h = 33 \) for A36 steel and \( 45 \) for A441.

The graphical solution of Eq. (6) is shown in Fig. 6 for three typical end-restraint values of \( S = 0, 3 \) and \( 6 \). A beam of a given slenderness factor, \( \lambda \), will therefore buckle laterally when the yielded portion of its length, \( \frac{\varepsilon}{h} L \), reaches the value given in Fig. 6. This yielded portion can then be found as those portions of the beam in which the bending moment exceeds \( M_{ps} \). It is noted that the predictions of Fig. 6 are conservative.
V. LOCAL BUCKLING

As the yielded zone is concentrated into a restricted region, local buckling is very likely to occur\textsuperscript{12}. The problem offers some interesting comparisons with the same problem for a beam under uniform moment\textsuperscript{11}. When the beam is under moment gradient the local buckling criterion is the progression of yielding along the length of the beam, whereas for uniform moment it is the attainment of strain-hardening strains across the half-flange\textsuperscript{12}.

The half wave length of a local buckle was given in Reference \textsuperscript{12} as 1, where

\[
\frac{L}{b} = 0.71 \frac{t}{w} \sqrt{\frac{A_w}{A_f}}
\]

and where \(A_f = bt\) and \(A_w = (d-2t)w\). The derivation of Eq. (9) assumed that the section meets the definition of a compact section\textsuperscript{12,13} with respect to local buckling. This means that the section will not local buckle until it is fully yielded. For a beam under moment gradient, one end of the yielded region will be adjacent to the relatively stiff elastic portion and the other end will be adjacent to the load point or connection. These will provide relatively stiff end restraints and the local buckle will then require the full wave length in which to form. The distinction between this situation and the uniform moment case is shown in Fig. 7. Therefore the criterion to be applied here will be that local buckling occurs when the yielded length equals the full
length of the local buckle, that is, from Eq. 9, when

$$\gamma_{lb} \ell = 2 \ell = 1.42 \frac{t}{\sqrt{v}} \frac{4A_{y}}{A_{y}}$$

where $\gamma_{lb} \ell$ is the yielded length associated with local buckling and

from Eq. (7)

$$\gamma_{lb} \lambda = 1.42 \frac{\sqrt{\varepsilon_{y}}}{\varepsilon_{y}} \frac{b}{r_{y}} \frac{t}{\sqrt{v}} \frac{4A_{y}}{A_{y}}$$

Using the typical value of $l = 1.2b$, and $b = 2\sqrt{3} r_{y}$, gives

$$\gamma_{lb} \lambda = 2.65 \sqrt{\varepsilon_{y}}$$

This equation is plotted in Fig. 6 for A36 steel and A441 steel ($\varepsilon_{y} = .00122$ and .00169 respectively).

Fig. 6 illustrates that for practical cases, failure will be initiated by local buckling. In all the cases shown lateral buckling requires larger yielded regions than local buckling. Thus local buckling will be critical.

Now the lateral buckling curves in Fig. 6 are based on a beam-model which completely neglects the St. Venant torsional resistance and assumes a uniform compressive force of $A \varepsilon_{y}/2$. These assumptions mean that Fig. 6 will conservatively estimate the lateral buckling strength of inelastic beams. However, even in this case, Fig. 6 still shows that local buckling, and not lateral buckling, will be the cause of failure. Thus the conservative assumptions used in obtaining the lateral buckling curves in Fig. 6 are not of importance, and further
studies of the behavior of beams under moment gradient must concentrate on the effect of local buckling.

Experimental confirmation of these statements has recently been given by Augusti who tested beams under moment gradient with $L/r_y$ between 100 and 200 ($\lambda$ between 1 and 2). The beams were constructed in the laboratory with dimensions such that local buckling would not occur under normal conditions*. Under these conditions the beams were able to reach $M_p$ and deliver a relatively large amount of deformation at this moment, although their slenderness ratios were in excess of the range plotted in Fig. 5.

Tests to be described later will further support the thesis that local buckling is the critical phenomenon in beams under moment gradient.  

* This choice of sections follows previous practice of the Cambridge research group².
VI. NECESSARY SUPPORT SPACING

The present AISC provision¹,¹³ for the spacing of lateral supports in A36 beams under moment gradient is

\[ \frac{L}{r_y} = 60 - 40 f \geq 35 \]  

(13)

This equation is now seen to have little relevance, and has never possessed any experimental justification⁸.

The necessary support spacing must be sufficient to allow the plastic moment, \( M_p \), to be maintained until local buckling occurs. There is no point in bracing for deformations larger than this. Augusti's experiments¹⁴ indicate that, with normal end restraints, this support spacing will be quite large.

Now it is known that lateral buckling is critical for a beam under uniform moment. Therefore it is necessary to guard against the possibility of a braced span being subjected to this loading. The AISC Specifications¹³ have used the following rule

\[ L \leq 13b \]  

(14)

to ensure that \( M_p \) is attained in a span. This rule is based on tests¹⁵ on A36 beams under uniform moment.

The authors have shown elsewhere¹⁴ that the relation between the unbraced length of a beam and the rotation capacity, \( R \), that it will deliver is given by
\[ k \frac{\lambda}{\sqrt{1 + 0.7 \frac{LR}{5}}} = 1 \]  

(15)

where \( R \) is defined as the ratio of the rotation which occurs at the moment \( M_p \) to the elastic rotation corresponding to \( M_p \). The factor \( k \) is the lateral effective length factor of the unbraced span. If it is assumed that the beam should be able to carry \( M_p \) for a rotation equal to the elastic rotation (\( R = 1 \)), then Eq. (15) becomes

\[ k \frac{\lambda}{1} = 0.55 \quad (A36) \]
\[ = 0.49 \quad (A441) \]

(16)

The value of \( k \) can also be evaluated \(^{11}\). The resulting unsupported \( L/r_y \) values are given in the following Table.

**UNSupported LENGTHS**

<table>
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<tr>
<th></th>
<th>Simply Supported Beam</th>
<th>Continuous Beam</th>
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<tr>
<td>( k )</td>
<td>0.70</td>
<td>0.54</td>
</tr>
<tr>
<td>A36</td>
<td>( 70r_y )</td>
<td>( 90r_y )</td>
</tr>
<tr>
<td>A441</td>
<td>( 55r_y )</td>
<td>( 70r_y )</td>
</tr>
</tbody>
</table>

It should be pointed out that the unsupported lengths in the above Table are not based on the lateral buckling of a beam under moment gradient. They are based, rather, on guarding against the possibility of the beam being subjected to uniform moment. If the above provisions were adopted there would be a rotation capacity of one before unloading occurred.
In an earlier report the authors discussed the dimensional requirements for braces. Although these provisions were derived directly for a beam under uniform moment, it was shown that they could be conservatively applied to a beam under moment gradient. These provisions will not be further discussed in this report.
VII. FAILURE CRITERIA

The yielded portion of a beam span, $\gamma L$, can be found from the beam bending moment diagram. The value of $\gamma L$ to cause local buckling is known and hence the applied load at local buckling may be found. This gives the point on the load-deformation curve of the beam at which local buckling will occur. Now there is no proven reason why local buckling in a cross-section will cause unloading of the member. In light-gage sections post-local buckling strength is utilized in design, and there is experimental evidence that it also exists in at least some wide-flange sections. However there is also experimental evidence that a combination of local and lateral buckling together will result in large out-of-plane deformations and consequent unloading of a member. This unloading due to out-of-plane deformations will be assumed to represent failure of the member.

Consider a beam under moment gradient which has a local buckle in the in-plane condition. Both halves of the compression flange participate in the local buckle. When lateral deflections occur the half-flange which undergoes bending compression will be inactive due to the local buckle, whereas the other half-flange will behave in a manner which is very similar to that of a non-locally buckled flange. The bending stiffness of such a flange is $c_b(EI_y)$ and so the bending stiffness of a half-flange is approximated by $c_b(EI_y)/2^2$. When this value is substituted in Eq. (6) a curve is obtained for lateral buckling after a
local buckle has occurred. This curve is plotted in Fig. 7, together with the local buckling curve from Fig. 5 and Eq. (12). It can be seen that the beam is now considerably weakened and lateral buckling will occur. The difference between the two curves is sufficiently great to avoid any necessity for refining the value of $c_b/\theta$ used in the calculations.

Thus it may be concluded that local buckling in a beam under moment gradient will lead to lateral buckling, and that these two effects in combination will probably cause unloading of the applied moment.

It is possible that the bracing might be spaced so as to prevent this post-local buckle lateral buckling, and thus gain further rotation capacity. However there are two objections to this approach. Firstly, it can be seen from Fig. 7 that the bracing would need to be very close and, secondly, the post-local-buckling strength of wide flange members is not yet well understood.
VIII. ROTATIONS AND ROTATION CAPACITY

In plastic design the beam hinge is assumed to occur at a point location on the ZZ axis of a beam, and all rotations are assumed to occur at this point. The preceding portions of this report have indicated that this assumption is not very far removed from reality when beams under moment gradient are being considered. Yielding will occur over a length, $\zeta_{lb}$, which is independent of the span of the beam and the moment gradient. It was shown in Section II that most of the rotation that occurs in a yielded beam will be a result of the curvature jump effect $(2(s-1)\varepsilon_{y}/d)$ occurring in the yielded portion.

In the following discussion, the rotation $\theta_h$, over the yielded length and due to the curvature jump, will be called the hinge rotation. It is actually the rotation across the yielded portion, and neglects the additional rotation due to the curvature increasing with moment at the strain-hardening modulus. The hinge rotation will be less than the relative rotation of the ends of the beam by an amount equal to the elastic rotation before yielding, and further elastic deformations caused by the increase in moment due to strain-hardening. The definition of $\theta_h$ is illustrated in Fig. 9.

Now, the fact that $\zeta_{lb}$ is independent of span length and moment gradient, means that $\theta_h$ will be similarly independent. It therefore becomes misleading to use the rotation capacity symbolism, as the denominator of the rotation capacity ratio will vary with span and
loading. The following work will consider only the absolute hinge angle, $\theta_H$.

Under the above assumptions, the hinge angle is given by

$$\theta_H = \frac{2(\xi - 1)E}{d} \left\{ \gamma_L L_L + \gamma_R L_R \right\}$$

where $(\gamma L)$ is defined in Fig. 9, and the subscripts L and R refer to the portions to the left and right of the maximum moment, respectively. Unloading will occur when one of the $(\gamma L)$'s reaches the local buckling value given by Eq. 10. This will be the value under the smaller moment gradient and the sum may then be given by using Eq. 10 and Fig. 9 as

$$\gamma L L_L + \gamma R L_R = 1.42 \frac{E}{\omega} \sqrt{\frac{A_H}{A_f}} \cdot b \cdot \left( 1 + \frac{V_L}{V_R} \right)$$

where $V_1 \leq V_2$ (i.e., $V_1 = V_L$, $V_2 = V_R$, if $V_L < V_R$; and vice versa) and substituting in Eq. 17 gives

$$\frac{\theta_H}{(\xi - 1)E} = 2.84 \frac{E}{\omega} \sqrt{\frac{A_H}{A_f}} \cdot \frac{b}{d} \cdot \left( 1 + \frac{V_L}{V_R} \right)$$

For an A36 10WF25 section, this predicts that the inelastic hinge angle is 0.070 radian if $V_1 = V_2$. It can be seen that Eq. (19) shows that the hinge angle is a simple function of the material properties, section properties, and bending moment diagram. The fact that $\theta_H$ can be readily determined will be useful in checking rotation capacities of plastically designed structures.
IX. LIMITS OF THE ROTATION ANALYSIS

The beam under moment gradient behaves in a considerably different manner than the beam under uniform moment. For example, it was shown in Section VIII that an A36 10WF25 beam can be expected to deliver a hinge rotation of 0.070 radian, and Section VI showed that the required support spacing with elastic adjacent spans was 90r_y. Now it can be shown that the same beam would need a spacing of 57r_y to achieve the same hinge angle under uniform moment.

The difference between the two cases arises from the fact that the yielded regions of a beam under moment gradient are in a relatively non-critical location (at the end of the span) and therefore lateral buckling is less likely to occur than in a beam under uniform moment with a uniform distribution of yielding. It is therefore necessary to ensure that the moment gradient equations are not used in the more critical uniform moment case. To achieve this, it is proposed that the smaller of the two end moments does not exceed 0.7M_y, the moment at which yielding would occur with the standard residual stress pattern. Thus the maximum value of $f$ over a span would be 0.7/$f$ or about 0.65. For $f$ greater than this the beam should be considered to be under uniform moment.

There is a similar limit to the moment $M_0$, at the center of the beam. In order to limit strains, this will be arbitrarily restricted to a moment of $\frac{1}{2} \left( \frac{0u}{0y} + 1 \right) M_p$ (for A36 steel this would be about $1.4M_p$).
Consequently, if \( V \) exceeds a certain limit of \( V_m \), say, then it will not be possible for the wavelength of the local buckle to yield as the moment gradient will be too high. The value of \( V_m \) can be found, then, from

\[
M_o - M_{ps} = \frac{V_m}{2\ell_{LB}L}
\]  

(20)

and so from Eqs. (5) and (12),

\[
V_m = \frac{M_p}{4.6b} \left\{ \frac{\sigma_u}{\sigma_y} - 0.88 \right\}
\]  

(21)

An upper limit will also be placed on \( V \) by the shear stresses. This limit is well known to be

\[
V \leq \sigma_y A_w / \sqrt{3}
\]  

(22)

However, when the ratio of the two limiting \( V \) values in Eqs. (21) & (22) is taken, it is found that \( V_m \) will always be the smaller and therefore the more critical limit. (For most rolled sections \( \sigma_u / \sigma_y \) would need to approach two for the shear stress limit to be critical).

If \( V \) exceeds \( V_m \), then it is obvious from the derivation in Section VIII that

\[
\sqrt{1}, \sqrt{2} > V_m : \frac{\theta_H}{(s-1)\epsilon_y} = 2.84 \sqrt{A_w} \sqrt{\frac{b}{A_i}} \left( \frac{V_m}{V_1} + \frac{V_m}{V_2} \right)
\]  

(23)
X. COMPARISON WITH TEST RESULTS

Of all the tests that have been conducted on beams under moment gradient, relatively few can be used to confirm either the rotation predictions or the bracing predictions of this chapter. It is interesting to consider the reason for this. The test results obtained were all similar in form to Fig. 3, but the large majority of tests were stopped before unloading occurred. This stoppage was because the deflections of the deformed beams exceeded either the available machine clearances or the experimenter's opinion of the useful deformation range of the beam.

It is worth noting here that this may be an indication that beams under moment gradient will deliver rotations greater than would frequently seem necessary.

The comparisons between test and theory are given in Table I. The basic comparison made is between the hinge angle predictions and measurements. When rotations were not measured, they have been estimated as

\[ \Theta_H = \delta_H \left( \frac{1}{L_L} + \frac{1}{L_R} \right) \]  

(24)

where \( \delta_H \) is the hinge deflection, and \( L_L \) and \( L_R \) are the lengths of beam between hinge and support (Fig. 9). The support spacing is also given in the Table. Only those tests which meet the limits of Eq. 21, and in which a definite decrease in load capacity was noted, are recorded in Table I.
Seventeen tests are tabulated and of these fourteen had hinge rotations greater than the lower bound predictions of Eq. 19. In the first eight tests in the Table there is reasonable agreement between test and theory; however, four of the tests from Ref. 9 showed hinge rotations close to twice the predicted amount. It is very likely that this is due to the rapid loading rate used in those tests. It has been pointed out elsewhere\(^{10}\) that this can lead to an apparent strengthening of a member. The effect is dependent on the strain rate (which was not recorded in Ref. 9 except to note that the tests took between 0.5 to 1.5 hr. It is noted that the tests G1, 2 and 5 in Table I took approximately 8 hr.).

Three tests showed hinge angles less than the predicted value. In test G5 local buckling occurred in only one span in such a manner that little yielding took place in the other span. Hence \(V_1/V_2 < 1\) should have been used in Eq. 19. The effect was probably caused by some initial structural imperfection. The same effect probably accounts for the 27\% underprediction for test LB5 (Ref. 8). It is difficult to explain the behavior of test 18 from Ref. 9 except to note that the \(d/w\) value was high, and the shear local buckling may have occurred. Unfortunately Sawyer gives no indication of the mode of failure and describes the test only "as being trivial" because \(M_o\) was less than \(M_p\).

The \(V_m\) reduction was applied in two tests (HT28, HT43) and in each case reduced the initial prediction of 0.08\(M_o\) to results which were conservative estimates of the actual behavior.
With regard to bracing, it can be seen that the tests in Table I were all too closely braced to give any definitive information on the usefulness of Eq. (16). The very interesting tests of Augusti were discussed earlier in Section V, however there is obviously a need for tests on more slender beams of realistic proportions in order to confirm the conclusions of Section V.

Therefore, similar conclusions can be reached with respect to both the hinge angle and the bracing spacing. In each case the theory is adequate as a lower bound for the tests conducted, but the tests cannot be considered to delineate the range of validity of the theories, particularly the bracing theory.
XI. SUMMARY

This chapter has examined the behavior of beams under moment gradient. It has been shown that they behave very much as assumed in simple plastic theory. However, the existing analysis is based on lateral buckling considerations, whereas this chapter has shown that local buckling will be the predominant effect. Hence it has been necessary to suggest changes in the existing provisions. For a large range of support spacing, a theory based on local buckling has been used to show that the hinge angle which can be delivered is constant, and a simple lower bound expression has been presented (Eq. (19)). Test results confirm the lower bound nature of the hinge angle theory, but there is an obvious need for further tests to define the limits within which the theory is applicable.

It has been shown that high shear forces in inelastic beams may have effects other than those previously considered. A prime consequence of high shear forces is to reduce the rotation capacity of the beam.
XII. ACKNOWLEDGEMENTS

This study is part of a general investigation "Plastic Design in High Strength Steel" currently being carried out at Fritz Engineering Laboratory, Department of Civil Engineering. Professor V. J. Eney is Head and Professor Beedle the Director of the Laboratory. The investigation is sponsored jointly by the Welding Research Council, and the Department of the Navy, with funds furnished by the American Institute of Steel Construction, the American Iron and Steel Institute, Lehigh University Institute of Research, the Bureau of Ships and the Bureau of Yards and Docks. The Column Research Council acts in an advisory capacity.

Miss Barbara Williams very carefully typed the report and Mr. H. Izquierdo and R. Weiss did the drawings.
XIII. NOMENCLATURE

b     breadth of beam

\( c_b \)     ratio of fully yielded lateral stiffness of a beam to the corresponding elastic value

d     depth of beam

f     shape factor

h     ratio of Young's modulus to strain hardening modulus

k     effective length factor

l     half wavelength of local buckle

\( r_y \)     weak axis radius of gyration

s     ratio of strain at strain hardening to yield strain

t     flange thickness

w     web thickness

z     length along beam

A     cross-sectional area

\( A_f \)     bt

\( A_w \)     (d-2t)w

E     Young's modulus

I     moment of inertia

L     unsupported length

M     moment

\( M_{ps} \)     moment at which flange yields

\( M_p \)     plastic moment
$M_o$  maximum moment in a beam
$R$  rotation capacity
$V$  moment gradient (shear)
$V_m$  maximum permissable shear force
$Z$  plastic modulus
$\delta_H$  deflection at a hinge
$\varepsilon_y$  yield strain
$f$  moment ratio
$\lambda$  slenderness factor (Eq. (7))
$\zeta$  ratio of yielded length of beam to total length
$\zeta_{Lb}$  value of $\zeta$ at local buckling
$\theta$  beam rotation
$\phi_H$  hinge rotation
$\sigma_u$  ultimate stress
$\sigma_y$  yield stress
$\gamma$  curvature
$\gamma_p$  $M_p/EI$
XIV. TABLES AND FIGURES
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(1) Effective \( V_1/V_2 < 1 \) due to unsymmetrical yielding about load point  
(2) \( d/w = 58 \)
Fig. 1  Discontinuous stress-strain curve
Fig. 2  Beam under moment gradient
Fig. 3  Load-rotation curve for a beam under moment gradient
Fig. 4  Lateral buckling model
Fig. 5 Statement of problem
Fig. 6  Buckling of a partially yielded beam

Solid Curves: Lateral Buckling
Dashed Curves: Local Buckling

$c_b = 0.0517$

PROPORTION YIELDED 0.6

SLENDERNESS FACTOR

Eq. 6

A441
A36
Eq. 12

$S = 0$

$S = 5$

$S = 2.5$

$S = 0.5$

$S = 0$
a) LOCAL BUCKLE UNDER UNIFORM MOMENT

b) LOCAL BUCKLE UNDER MOMENT GRADIENT

Fig. 7  Local buckling conditions
Local Buckling, Eq. 12

Lateral Buckling after local buckle

Fig. 8 Lateral buckling after local buckling
Fig. 9  Behavior of a beam under moment gradient
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