POST-BUCKLING DEFLECTION MODES OF LINING IN CONTAINMENT VESSELS

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ABSTRACT

A theoretical study is made of the post-buckling deflection modes of a clamped infinitely long rectangular plate subjected to biaxial compressive edge strains and leaning against a flat rigid surface. This analysis is intended to be used as a phase in the determination of the ultimate strength of the side-wall liner plating in prestressed concrete containment vessels. The relative dimensions of the plates are such that elastic buckling will take place before the ultimate strength is reached. The most probable configurations of the buckled panel are a wave surface and a cylindrical surface. In this study the energy levels of these two potential deflection shapes are compared for a series of different strain combinations to determine which shape is most likely to form. The major parameters influencing the occurrence of one or the other shape were found to be the ratio of the edge strains in the two directions \( \varepsilon_1/\varepsilon_2 \) and the plate width to thickness ratio \( a/t \). The possibility of secondary buckling in the post-buckling range was also considered. It is concluded that the cylindrical surface is most likely to develop for the \( \varepsilon_1/\varepsilon_2 \) values typical in practical design.
NOTATION

\( a \) \quad \text{width of the panel plate}

\( D \) \quad \text{plate bending rigidity; } \frac{E t^3}{12(1-\nu^2)}

\( E \) \quad \text{Young's modulus}

\( p_1 \) \quad \text{average membrane stress in } x \text{ direction}

\( p_2 \) \quad \text{average membrane stress in } y \text{ direction}

\( U \) \quad \text{total strain energy}

\( U_B \) \quad \text{strain energy due to bending}

\( U_S \) \quad \text{strain energy due to membrane stresses}

\( U' \) \quad \text{strain energy per unit length of the panel plate in nondimensional form; } \frac{(1-\nu^2)}{E t a^2} U

\( t \) \quad \text{plate thickness}

\( u \) \quad \text{displacement in } x \text{ direction}

\( u_a \) \quad \text{edge displacement in } x \text{ direction}

\( v \) \quad \text{displacement in } y \text{ direction}

\( v_a \) \quad \text{edge displacement in } y \text{ direction}

\( V \) \quad \text{total potential}

\( w \) \quad \text{out-of-plane displacement}

\( w_0 \) \quad \text{unknown deflection parameter}

\( x, y \) \quad \text{cartesian coordinate axes}

\( \gamma_{xy} \) \quad \text{membrane shearing strain}

\( \varepsilon_1 \) \quad \text{edge strain in } x \text{ direction}

\( \varepsilon_2 \) \quad \text{edge strain in } y \text{ direction}

\( \varepsilon_x \) \quad \text{membrane strain in } x \text{ direction}
\( \epsilon_y \)  
membrane strain in y direction

\( \nu \)  
Poisson's ratio

\( \sigma_x \)  
membrane stress in x direction

\( \sigma_y \)  
membrane stress in y direction

\( \tau_{xy} \)  
membrane shearing stress

\( \Phi \)  
Airy's stress function

\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^3 \partial y} + \frac{\partial^4}{\partial y^4} \]
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1. INTRODUCTION

In prestressed concrete containment vessels of atomic reactors a lining is required to prevent chemical effects of hot carbon dioxide upon the inner concrete surface and to ensure gas-tightness. The most popular form of the liner is a steel membrane which absorbs dimensional changes by overall straining. The contribution of the lining to the structural strength of the vessel is considered to be negligible. Thus the lining is designed to accommodate the strains of the inside surface of the concrete and those due to the relative thermal expansion. Since these strains are mostly compressive, the lining will function in a compressive, biaxially-strained state.

The strain between the lining and the concrete is transmitted by rib anchors (shear connectors) which connect the lining to the inside concrete surface. Furthermore, rib anchors subdivide the lining into a number of panels. Each panel is subjected to loadings due to the dimensional changes of the pressure vessel, the temperature effects, and the internal pressure, and behaves substantially like an infinitely long rectangular plate under biaxial compression and normal pressure.

The relative and absolute magnitude of the loads on a panel may vary because of differences in material properties, changes in thickness from panel to panel, and different initial out-of-flatness. In order to have a safe design it is thus necessary to consider the worst case by assuming that the strongest and the weakest panels are adjacent and in a position in the vessel where the strains and strain gradients are at a maximum. The difference in loads carried by the adjacent panels is taken
by the rib anchors at the junction of the panels. Because of the flexibility of the rib anchors and the deformation of concrete under load, some slip occurs between the lining and the vessel. The slip results in an increased average strain in the weak panel and a decreased strain in the adjoining strong panels. This effect should be also included as a service condition.

A series of papers have been published on the design of lining for reactor vessels. Hardingham, Parker, and Spruce [1] describe the principal features of the lining used in the vessels for Oldbury-on-Severn power station, the factors governing its design, and the theoretical and experimental approaches which led to the particular design details adopted. A large portion of their work consisted of experimental and theoretical investigations into the behavior of panels under biaxial compression to strains beyond yielding. Young and Tate [2] paid particular attention to the design of the lining against the possibility of rupture and buckling. They also studied the anchorage design in some detail. Chapman and Carter [3] studied the load/slip properties of shear connectors. They performed a theoretical analysis in order to find the lining strains and shear connector slips in terms of the vessel strains and the stiffness of the lining and connectors. The lining discontinuity was assumed to be that of a buckled panel of minimum strength bounded by unbuckled panels of maximum strength. Bishop, Horseman, and White [4] described the loads induced by concrete stressing and temperature effects, gave a review of the possible general forms of construction, and discussed the design of penetrations and cooling systems. Chan [5] dealt with the case of a hollow cylinder of infinite length. Assuming that the liner
buckles into a cylindrical surface as the concrete is prestressed, he concluded that buckling can be suppressed by anchoring the lining to the concrete with the anchors spaced circumferentially at a distance $2.5t/\varepsilon$, where $t$ is the thickness of the liner and $\varepsilon$ is the circumferential strain imposed by prestressing.

In general, the relative dimensions of panel plates are such that elastic buckling will take place before the ultimate strength of the plate is reached and the post-buckling behavior of the panel plate must be analyzed. Under the existing straining conditions the two most likely configurations of the buckled panel are a wave surface and a cylindrical surface as shown in Figs. 1a and 1b. Difficulties of the post-buckling analysis necessitate that the type of the deflection surface be determined in advance. In this paper the more probable surface is found by comparing the strain energies of these two potential deflection modes.

In order to simplify the computation of the energy levels, the panel plate buckled into a wave surface is assumed to consist of a number of clamped square plates with the side equal to the panel width (Fig. 1a). With this assumption the strain energy of the panel plate buckled into a wave surface can be determined by considering one of these clamped square plates (Fig. 2). The energy computation of the panel plate buckled into a cylindrical surface is made by taking a portion of it with a unit width (Fig. 3). Large deformations of the post-buckling range are considered in both cases.

Schnadel [6] presented an approximate analysis of the post-buckling behavior of a simply supported plate loaded on two opposite
edges. Timoshenko [7], and Marguerre and Trefftz [8] formulated the strain energy of plates with large deflections. Marguerre [9] presented a more accurate analysis for the post-buckling behavior of simply supported rectangular plates where he used the principle of minimum total potential to determine the unknown parameter in the assumed expression for the deflection. Levy [10,11] gave solutions for simply supported and clamped rectangular plates under combined axial load and normal pressure. Although his solutions are theoretically of exact nature, numerical results can be obtained only very laboriously. Levy's solution has been extended by Hu, Lundquist, and Batdorf [12] to include plates with small initial deviations from flatness. Coan [13] has further extended Levy's solution to allow for non-uniform edge displacements.

In the study reported here, an approximate solution is given for a clamped square plate having large deflections under biaxial compression using some ideas of [9]. A suitable expression containing one unknown parameter is assumed for the deflected shape of the plate, and the unknown parameter is then determined by the principle of minimum total potential. Also, the post-buckling behavior of a plate buckled into a cylindrical surface under biaxial compression is analyzed.

In the comparison of the energies of the two potential deflection shapes, the lining material is assumed to have an infinite proportional limit and a Poisson's ratio equal to 0.3. The ratio of the strains in the two directions ($\varepsilon_1/\varepsilon_2$) and the ratio of the plate width to the plate thickness ($a/t$) are found to be the major parameters determining the relative probability of one or the other shape. In particular, the influence of the ratio ($\varepsilon_1/\varepsilon_2$) on the deflection shape is illustrated.
in Fig. 4 for a plate with a/t = 60. It can be observed that for some values of $\varepsilon_1 / \varepsilon_2$ there exists in the post-buckling range the possibility for secondary buckling. For example, the value of $\varepsilon_1 / \varepsilon_2 = 0.50$ leads to the buckling into a cylindrical surface, but after some additional straining the shape changes into a wave surface as indicated by the intersection of the curve with the $\varepsilon_1$-coordinate.

The energy levels of the two deflection modes are compared for the values of the strain ratio ($\varepsilon_1 / \varepsilon_2$) which represent the straining conditions of the weakest panel typical for practical design. It is found that the most likely configuration of the panel plate in the post-buckling range is a cylindrical surface.
2. BASIC EQUATIONS

The large deflection theory of thin plates is governed by the following two partial, non-linear differential equations [7]:

\[
\begin{align*}
\nabla^4 \hat{\Phi} &= E \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \right] \\
\nabla^4 w &= \frac{q}{D} + \frac{t}{D} \left[ \frac{\partial^2 \hat{\Phi}}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \hat{\Phi}}{\partial y^2} \cdot \frac{\partial^2 w}{\partial x^2} \\
&\quad - 2 \frac{\partial^2 \hat{\Phi}}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} \right]
\end{align*}
\]

(1a) (1b)

where \( D = E t^3 / 12(1-\nu^2) \), \( w \) is the out-of-plane deflection, and \( \hat{\Phi} \) is Airy's stress function. Equation (1a) expresses the in-plane equilibrium and compatibility, whereas Eq. (1b) expresses the out-of-plane equilibrium.

The pertinent strain displacement relationship including the second order terms are

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}
\end{align*}
\]

(2a) (2b) (2c)

where \( \varepsilon_x \) and \( \varepsilon_y \) are the membrane strains in \( x \) and \( y \) directions, respectively, \( \gamma_{xy} \) is the membrane shearing strain, \( u \) is the displacement in \( x \) direction, and \( v \) is the displacement in \( y \) direction.
The general solution of Eqs. (1a) and (1b) is not available. However, some approximate solutions can be obtained by assuming suitable functions for one or all of the three displacements \( u, v, \) and \( w \). The functions will contain a number of unknown parameters and the determination of these parameters constitutes the solution of the problem.

The principle of minimum total potential is here employed in the determination of the unknown parameters. In the application of the method, it is necessary to compute the strain energy of the plate. For large plate deflections strain energy consists of two contributions, \( U_s \) due to membrane stresses, and \( U_B \) due to bending. \( U_s \) is given by

\[
U_s = \frac{Et}{12(1-\nu^2)} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left( \epsilon_x a + \epsilon_y a + 2\nu \epsilon_x \epsilon_y + \frac{1-\nu^2}{2} \gamma_{xy} \right) dx \, dy
\]

which, in terms of stress function \( \phi \), can be written in the form

\[
U_s = \frac{t}{2E} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)^2 - 2(1+\nu) \left[ \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] \right\} dx \, dy
\]

\[
= \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)^2 \right\} dx \, dy
\]
and $U_B$ is given by

$$U_B = \frac{D}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-v) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \, dx \, dy$$

(4)

where $a$ and $b$ are the dimensions of a rectangular plate.

In the following sections, Eqs. (1a) and (1b) are solved for the two deflection modes (wave and cylindrical), and expressions for the energy levels are derived as functions of the edge displacements.
3. LARGE DEFLECTION ANALYSIS OF CLAMPED
SQUARE PLATES

A clamped square plate subjected to biaxial compression is shown in Fig. 2. It is assumed that the edges remain straight and the plate undergoes deformations as shown by the dotted outline. \( u_a \) and \( v_a \) designate the constant edge displacements. The average edge strains \( \varepsilon_1 \) and \( \varepsilon_2 \) are related to the edge displacements through

\[
\begin{align*}
    u_a &= \frac{\varepsilon_1 a}{2} \\
    v_a &= \frac{\varepsilon_2 a}{2}
\end{align*}
\]  (5)

where \( a \) is the original side length of the plate.

The boundary conditions are the following:

(1) \( w (\pm a/2, y) = w (x, \pm a/2) = 0 \)

(2) \( \frac{\partial w}{\partial x} (\pm a/2, y) = \frac{\partial w}{\partial y} (x, \pm a/2) = 0 \)

(3) The edges remain straight

\( \frac{\partial u}{\partial y} (\pm a/2, y) = \frac{\partial v}{\partial x} (x, \pm a/2) = 0 \)

(4) The shearing stress is zero along the edges. This condition together with condition (3) requires that

\( \frac{\partial v}{\partial x} (\pm a/2, y) = \frac{\partial u}{\partial y} (x, \pm a/2) = 0 \)

(5) Edge displacements are equal to \( u_a \) and \( v_a \), and are given by

\[
\begin{align*}
    u_a &= \frac{\varepsilon_1 a}{2} = \int_0^{a/2} \frac{\partial u}{\partial x} \, dx \\
    v_a &= \frac{\varepsilon_2 a}{2} = \int_0^{a/2} \frac{\partial v}{\partial y} \, dy
\end{align*}
\]
The deflection function $w$ satisfying boundary conditions (1) and (2) is assumed to be

$$w = w_0 \left(1 + \cos \frac{2\pi}{a} x\right) \left(1 + \cos \frac{2\pi}{a} y\right)$$

(6)

where $w_0$ is the unknown deflection parameter.

Substitution of Eq. (6) into Eq. (1a) results in

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -\frac{8E \pi^4 w_o^2}{a^4} \left[ \cos \frac{2\pi}{a} x \right.

+ \cos \frac{2\pi}{a} y + \cos \frac{4\pi}{a} x + \cos \frac{4\pi}{a} y + 2 \cos \frac{2\pi}{a} x

\cos \frac{2\pi}{a} y + \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y + \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y \right]$$

This differential equation together with the boundary conditions (3) and (4) is used next to determine the stress function $\phi$. The particular solution is found to be

$$\phi = -\frac{E w_o^2}{2} \left[ \cos \frac{2\pi}{a} x + \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y

+ \frac{1}{16} \left( \cos \frac{4\pi}{a} x + \cos \frac{4\pi}{a} y \right) + \frac{1}{25} \left( \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y

+ \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \right) \right]$$
and the complementary solution as proposed in [9] is taken in the form of

\[ \hat{\xi}_c = \frac{1}{2} \hat{p}_1 y^2 + \frac{1}{2} \hat{p}_2 x^2. \]

With these, the complete solution for \( \hat{\xi} \) becomes

\[
\hat{\xi} = \hat{\xi}_p + \hat{\xi}_c = \frac{1}{2} \hat{p}_1 y^2 + \frac{1}{2} \hat{p}_2 x^2 - \frac{E w_0^2}{2} \left[ \cos \frac{2\pi}{a} x \right.
\]

\[ + \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y + \frac{1}{16} \left( \cos \frac{4\pi}{a} x \right. \]

\[ + \cos \frac{4\pi}{a} y \left) + \frac{1}{25} \left( \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y + \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \right. \right] \]

(7)

where \( p_1 \) and \( p_2 \) are the average membrane stresses in \( x \) and \( y \) directions, respectively. Unknown at this point, they will be later determined in terms of the edge strains \( \epsilon_1 \) and \( \epsilon_2 \).

It remains to verify that this solution for \( \hat{\xi} \) satisfies boundary conditions (3) and (4). The in-plane stresses as obtained from \( \hat{\xi} \) are given by

\[
\sigma_x = \frac{\partial^2 \hat{\xi}}{\partial y^2} = p_1 + \frac{E w_0^2}{a^2} \left[ 2 \cos \frac{2\pi}{a} y + \cos \frac{2\pi}{a} x \right.
\]

\[ + \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} y + \frac{2}{25} \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y \]

\[ + \frac{8}{25} \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \left] \right) \]

(8a)
\[
\sigma_y = \frac{\partial^2 \varphi}{\partial x^2} = p_a + \frac{E\pi^2 w^2}{a^2} \left( 2 \cos \frac{2\pi}{a} x + \cos \frac{2\pi}{a} y \right)
\]

\[
\cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} x + \frac{8}{25} \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y
\]

\[
+ \frac{2}{25} \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \quad (8b)
\]

\[
\tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{E\pi^2 w^2}{a^2} \sin \frac{2\pi}{a} x \sin \frac{2\pi}{a} y +
\]

\[
+ \frac{4}{25} \sin \frac{2\pi}{a} y \sin \frac{4\pi}{a} x + \sin \frac{2\pi}{a} x \sin \frac{4\pi}{a} y \quad (8c)
\]

Membrane strains \(\varepsilon_x\) and \(\varepsilon_y\) are obtained by substituting Eqs. (8a) and (8b) into Hooke's stress-strain relationships

\[
\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) = \frac{1}{E} (p_1 - \nu p_a) + \frac{\pi^2 w^2}{a^2} \left( 2 \cos \frac{2\pi}{a} y - 2\nu \cos \frac{2\pi}{a} x \right)
\]

\[
+ (1-\nu) \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} y -
\]

\[
- \frac{\nu}{2} \cos \frac{4\pi}{a} x + \frac{2}{25} (1-4\nu) \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y +
\]

\[
+ \frac{2}{25} (4-\nu) \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \quad (9a)
\]
\[ \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) = \frac{1}{E} (\sigma_1 - \nu \sigma_2) + \frac{\pi^2 \omega_o^2}{a^2} \left[ 2 \cos \frac{2\pi}{a} x - 2\nu \cos \frac{2\pi}{a} y + \right. \\
+ (1-\nu) \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} x - \\
- \frac{\nu}{2} \cos \frac{4\pi}{a} y + \frac{2}{25} (4-\nu) \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y + \\
\left. + \frac{2}{25} (1-4\nu) \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \right] \] (9b)

Rearranging Eqs. (2a) and (2b) to

\[ \frac{\partial u}{\partial x} = \varepsilon_x - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \]
\[ \frac{\partial v}{\partial y} = \varepsilon_y - \frac{1}{2} \frac{\partial^2 v}{\partial y^2} \]

and introducing \( \varepsilon_x \) and \( \varepsilon_y \) from Eqs. (9a) and (9b) and \( w \) from Eq. (6) and integrating, displacements \( u \) and \( v \) are found to be given by

\[ u = \int \frac{\partial u}{\partial x} \, dx = \frac{1}{E} (\sigma_1 - \nu \sigma_2) x + \frac{\pi^2 \omega_o^2}{a^2} \left[ 2x \cos \frac{2\pi}{a} y - \\
- \frac{\nu a}{2} \sin \frac{2\pi}{a} x + \frac{a}{2} (1-\nu) \cos \frac{2\pi}{a} y \sin \frac{2\pi}{a} x + \\
+ \frac{1}{2} \cos \frac{4\pi}{a} y - \frac{\nu a}{8\pi} \sin \frac{4\pi}{a} x + \frac{a}{50\pi} (1-4\nu) \cos \frac{2\pi}{a} y \sin \frac{4\pi}{a} x + \\
\frac{a}{25\pi} (4-\nu) \cos \frac{4\pi}{a} y \sin \frac{2\pi}{a} x - \\
- \left( \frac{3}{2} + 2 \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} y \right) \left( x - \frac{a}{4\pi} \sin \frac{4\pi}{a} x \right) \right] + g(y) \]
\[
\begin{align*}
v &= \int \frac{\partial v}{\partial y} \ dy = \frac{1}{E} (p_a - \nu p_1) y + \frac{\pi^2 \omega^2}{a^3} \left[ 2y \cos \frac{2\pi}{a} x - \\
&\quad - \frac{va}{\pi} \sin \frac{2\pi}{a} y + \frac{a}{2\pi} (1-\nu) \cos \frac{2\pi}{a} x \sin \frac{2\pi}{a} y + \\
&\quad + \frac{1}{2} y \cos \frac{4\pi}{a} x - \frac{va}{8\pi} \sin \frac{4\pi}{a} y + \frac{a}{25\pi} (4-\nu) \cos \frac{4\pi}{a} x \sin \frac{2\pi}{a} y + \\
&\quad + \frac{a}{50\pi} (1-4\nu) \cos \frac{2\pi}{a} x \sin \frac{4\pi}{a} y - \\
&\quad - \left( \frac{3}{2} + 2 \cos \frac{2\pi}{a} x + \frac{1}{2} \cos \frac{4\pi}{a} x \right) \left( y - \frac{a}{4\pi} y \sin \frac{4\pi}{a} y \right) \right] + h(x)
\end{align*}
\]

where \( g(y) \) and \( h(x) \) are the unknown integration functions. Differentiation of \( u \) and \( v \) with respect to \( y \) and \( x \), respectively, gives

\[
\frac{\partial u}{\partial y} = -\frac{\pi^2 \omega^2}{a^3} \left[ \frac{4\pi}{a} x \sin \frac{2\pi}{a} y + (1-\nu) \sin \frac{2\pi}{a} x \sin \frac{2\pi}{a} y + \\
+ \frac{2\pi}{a} x \sin \frac{4\pi}{a} y + \frac{1}{25} (1-4\nu) \sin \frac{2\pi}{a} y \sin \frac{4\pi}{a} x + \\
+ \frac{4}{25} (4-\nu) \sin \frac{4\pi}{a} y \sin \frac{2\pi}{a} x - \\
- \left( \frac{4\pi}{a} \sin \frac{2\pi}{a} y + \frac{2\pi}{a} \sin \frac{4\pi}{a} y \right) \left( x - \frac{a}{4\pi} \sin \frac{4\pi}{a} x \right) \right] + g'(y)
\]
\[ \frac{\partial v}{\partial x} = -\frac{\pi^2 \omega^2}{a^2} \left[ \frac{4\pi}{a} y \sin \frac{2\pi}{a} x + (1-\nu) \sin \frac{4\pi}{a} x \sin \frac{2\pi}{a} y + \frac{2\pi}{a} y \sin \frac{4\pi}{a} x + \frac{4}{25} (4-\nu) \sin \frac{4\pi}{a} x \sin \frac{2\pi}{a} y + \frac{1}{25} (1-4\nu) \sin \frac{2\pi}{a} x \sin \frac{4\pi}{a} y - \left( \frac{4\pi}{a} \sin \frac{2\pi}{a} x + \frac{2\pi}{a} \sin \frac{4\pi}{a} x \right) \left( y - \frac{a}{4\pi} \sin \frac{4\pi}{a} y \right) \right] + h'(x) \]

which, at the plate boundaries, are equal to

\[ \frac{\partial u}{\partial y} (\pm a/2, y) = g'(y) \quad (11a) \]

\[ \frac{\partial u}{\partial y} (x, \pm a/2) = g'(\pm a/2) \quad (11b) \]

\[ \frac{\partial v}{\partial x} (\pm a/2, y) = h'(\pm a/2) \quad (11c) \]

\[ \frac{\partial v}{\partial x} (x, \pm a/2) = h'(x) \quad (11d) \]

On the other hand, it can be also verified from Eq. (8c) that the boundary condition (4) is satisfied by Eq. (7) for the stress function \( \hat{\tau} \), that is,

\[ \tau_{xy} (\pm a/2, y) = 0 \]

\[ \tau_{xy} (x, \pm a/2) = 0 \]

or in terms of displacements
These equations, after substitution of Eq. (6) for \( w \), reduce to

\[
\left[ \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial x} \right]_{x=\pm a/2} = 0
\]

Now, we introduce Eqs. (11a) and (11c) into Eq. (12a)

\[
g'(y) + h'(\pm a/2) = 0
\]

This implies that function \( g'(y) \) must be a constant, that is,

\[
g'(y) = c_1 \quad \text{and} \quad c_1 = -h'(\pm a/2)
\]

and by integration

\[
g(y) = c_1 y + c_2
\]

where \( c_2 \) is the integration constant and represents a rigid body translation. Since a rigid body motion is out of our consideration, \( c_2 \) may be set equal to zero and \( g(y) \) becomes

\[
g(y) = c_1 y
\]
However, due to the symmetry of the edge strains and the geometry of
the plate, displacement \( u \) must be symmetric about \( x \) axis and hence \( c_1 \)
must also be equal to zero. Therefore, Eqs. (11a) and (11b) become

\[
\frac{\partial u}{\partial y} (\pm a/2, y) = \frac{\partial u}{\partial y} (x, \pm a/2) = 0
\]

With these, Eqs. (12a) and (12b) lead to

\[
\frac{\partial v}{\partial x} (\pm a/2, y) = \frac{\partial v}{\partial x} (x, \pm a/2) = 0
\]

Thus the solution for the stress function \( \sigma \) obtained in Eq. (7) satisfies
boundary conditions (3) and (4) and may be used for further analysis.

The average membrane stresses \( p_1 \) and \( p_2 \) can now be determined
as functions of the edge strains \( \varepsilon_1 \) and \( \varepsilon_2 \). Introducing \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial y} \) from
Eqs. (10a) and (10b) into boundary condition (5) and carrying out the
integration,

\[
E\varepsilon_1 = p_1 - \nu p_2 - \frac{3}{2} \frac{E \varepsilon_0^2}{a^2}
\]

\[
E\varepsilon_2 = p_2 - \nu p_1 - \frac{3}{2} \frac{E \varepsilon_0^2}{a^2}
\]

or, after rearranging terms,

\[
p_1 = \frac{E}{(1-\nu^2)} \left[ \varepsilon_1 + \nu \varepsilon_2 + \frac{3}{2} (1+\nu) \frac{E \varepsilon_0^2}{a^2} \right] \quad (13a)
\]

\[
p_2 = \frac{E}{(1-\nu^2)} \left[ \varepsilon_2 + \nu \varepsilon_1 + \frac{3}{2} (1+\nu) \frac{E \varepsilon_0^2}{a^2} \right] \quad (13b)
\]
By introducing Eqs. (13a) and (13b) into Eqs. (8a) and (8b), \( \sigma_x \) and \( \sigma_y \) are found to be

\[
\sigma_x = \frac{E}{(1-\nu^2)} \left[ \epsilon_1 + \nu \epsilon_a + \frac{3}{2} \frac{1}{1+\nu} \frac{\pi^2 w_o^2}{a^2} \right] + 
\]

\[
+ \frac{E \pi^2 w_o^2}{a^2} \left[ 2 \cos \frac{2\pi}{a} y + \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} y + \frac{2}{25} \cos \frac{2\pi}{a} y \cos \frac{4\pi}{a} x + \frac{8}{25} \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \right] (14a)
\]

and

\[
\sigma_y = \frac{E}{(1-\nu^2)} \left[ \epsilon_1 + \nu \epsilon_a + \frac{3}{2} \frac{1}{1+\nu} \frac{\pi^2 w_o^2}{a^2} \right] + 
\]

\[
+ \frac{E \pi^2 w_o^2}{a^2} \left[ 2 \cos \frac{2\pi}{a} x + \cos \frac{2\pi}{a} x \cos \frac{2\pi}{a} y + \frac{1}{2} \cos \frac{4\pi}{a} x + \frac{8}{25} \cos \frac{4\pi}{a} x \cos \frac{2\pi}{a} y + \frac{2}{25} \cos \frac{2\pi}{a} x \cos \frac{4\pi}{a} y \right] (14b)
\]

The unknown parameter \( w_o \) is now determined by using the principle of minimum total potential. For this purpose the total potential \( V \) must be expressed as a function of \( w_o \), and the first variation of it set equal to zero.

\[
\delta V = 0 \quad \text{or} \quad \frac{\partial V}{\partial w_o} = 0 \quad (15)
\]

where the total potential \( V \) consists of two parts; the external potential \( V_e \) and the internal potential \( V_i \).
The external potential $V_e$ does not depend on $w_o$ since the edge displacement parameters $\varepsilon_1$ and $\varepsilon_2$ are prescribed

$$V_e = - (p_1 \varepsilon_1 + p_2 \varepsilon_2) a^2 t$$

and, therefore, $\partial V_e / \partial w_o = 0$. Equation (15) then reduces to

$$\frac{\partial V_1}{\partial w_o} = 0$$

The internal potential $V_1$ is equal to the strain energy, and Eq. (15) thus becomes

$$\frac{\partial U}{\partial w_o} = \frac{\partial (U_s + U_B)}{\partial w_o} = 0$$

(16)

Introducing Eqs. (6) and (7) into Eqs. (3) and (4) and substituting Eqs. (13a) and (13b) for the average membrane stresses $p_1$ and $p_2$, the total strain energy $U$ is obtained as a function of edge strains $\varepsilon_1$ and $\varepsilon_2$ and the unknown parameter $w_o$

$$U = U_s + U_B = \frac{Et a^2}{(1-\nu^2)} \left[ \frac{\varepsilon_1^2 + \varepsilon_2^2}{2} + v \varepsilon_1 \varepsilon_2 + \frac{3\pi^3 w_o^2}{2a^2} (1+\nu) (\varepsilon_1 + \varepsilon_2) + \frac{4\pi^4 t^2 w_o^2}{3a^4} + \frac{\pi^4 w_o^4}{a^4} \left( \frac{1966}{400} + \frac{9}{4} \nu - \frac{1066}{400} \nu^2 \right) \right]$$

(17)
With Eq. (17) substituted into Eq. (16) the following equation for \( w_o \) is obtained in terms of \( \varepsilon_1 \) and \( \varepsilon_2 \):

\[
\left( \frac{v_o}{a} \right)^2 = -\left( \frac{1966}{400} + \frac{9}{4} \nu - \frac{1066}{400} \nu^2 \right) \left[ \frac{2}{3} \left( \frac{t'}{a} \right)^2 + \frac{3}{4n^2} (1+\nu)(\varepsilon_1 + \varepsilon_2) \right]
\]  

(18)

With \( w_o \) known, the stress function \( \xi \) and the deflection surface \( w \) are defined, and thus the solution is completed.
4. LARGE DEFLECTION ANALYSIS OF PLATES BUCKLED INTO CYLINDRICAL SURFACE

The energy computation of the panel plate buckled into a cylindrical surface is made for a portion of the plate with a unit width. The plate geometry and coordinate system are shown in Fig. 3. The biaxial compression causing the post-buckling deformation of the plate is defined by edge strains $\varepsilon_1$ and $\varepsilon_2$ or edge displacements $u_a$ and $v_a$ related to each other through

$$u_a = \frac{\varepsilon_1}{2} \quad v_a = \frac{\varepsilon_2}{2}$$

The boundary conditions are for the unit plate strip under consideration are:

(1) The out-of-plane deflection is equal to zero at the clamped edges

$$w (+a/2, y) = 0$$

(2) The slope is equal to zero at the clamped edges and the out-of-plane deflection is independent of the coordinate $y$

$$\frac{\partial^2 w}{\partial y^2} (+a/2, y) = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial^3 w}{\partial y^3} = \ldots = 0$$
(3) The edges remain straight

\[ \frac{\partial u}{\partial y} (\pm a/2, y) = \frac{\partial v}{\partial x} (x, \pm 1/2) = 0 \]

(4) The shearing stress is zero at the edges. This condition together with condition (3) requires that

\[ \frac{\partial v}{\partial x} (\pm a/2, y) = \frac{\partial u}{\partial y} (x, \pm 1/2) = 0 \]

(5) The edge displacements are given by

\[ u_a = \frac{e_a}{2} = \int_0^{a/2} \frac{\partial u}{\partial x} \, dx \]

\[ v_a = \frac{e_a \cdot 1}{2} = \int_0^{1/2} \frac{\partial v}{\partial y} \, dy \]

The deflection surface of the plate is approximated by

\[ w = w_o \left( 1 + \cos \frac{2\pi}{a} x \right) \quad (19) \]

where \( w_o \) is the arbitrary parameter to be determined. Boundary conditions (1) and (2) are satisfied by this expression.

Introducing \( w \) into Eq. (1a)

\[ \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \]
A valid solution for this homogeneous differential equation is

\[ \phi = \frac{1}{2} p_1 y^2 + \frac{1}{2} p_2 x^2 \]

where constants \( p_1 \) and \( p_2 \) are the membrane stresses in \( x \) and \( y \) directions, respectively. The procedure of the proceeding section can be again used to show that this solution for \( \phi \) satisfies boundary conditions (3) and (4).

From Eq. (2) the in-plane stresses are now

\[ \sigma_x = \frac{\partial^2 \phi}{\partial y^2} = p_1 \quad (20a) \]

\[ \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = p_2 \quad (20b) \]

\[ \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad (20c) \]

Substituting these \( \sigma_x \) and \( \sigma_y \) into Hooke's stress-strain relations, the membrane strains are found to be

\[ \epsilon_x = \frac{1}{E} (p_1 - \nu p_2) \]

\[ \epsilon_y = \frac{1}{E} (p_2 - \nu p_1) \]

Introducing these expressions for \( \epsilon_x \) and \( \epsilon_y \), and Eq. (19) for \( w \) into Eqs. (2a) and (2b)
\[
\frac{\partial u}{\partial x} = \frac{1}{E} \left( p_1 - \nu p_2 \right) - \frac{\pi^2 w_0^2}{a^3} \left( 1 - \cos \frac{4\pi x}{a} \right)
\]

\[
\frac{\partial v}{\partial y} = \frac{1}{E} \left( p_2 - \nu p_1 \right)
\]

and substituting \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial y} \) into boundary condition (5) and performing the integration

\[
E \epsilon_1 = p_1 - \nu p_2 - \frac{E \pi^2 w_0^2}{a^3}
\]

\[
E \epsilon_2 = p_2 - \nu p_1
\]

or after rearranging terms

\[
p_1 = \frac{E}{(1-\nu^2)} \left[ \epsilon_1 + \nu \epsilon_2 + \frac{\pi^2 w_0^2}{a^3} \right]
\]

\[
p_2 = \frac{E}{(1-\nu^2)} \left[ \epsilon_2 + \nu \epsilon_1 + \nu \frac{\pi^2 w_0^2}{a^3} \right]
\]

With these \( p_1 \) and \( p_2 \) the stress function becomes

\[
\psi = \frac{1}{2} \frac{E}{(1-\nu^2)} \left( \epsilon_1 + \nu \epsilon_2 + \frac{\pi^2 w_0^2}{a^3} \right) y^2 + \frac{1}{2} \frac{E}{(1-\nu^2)} \left( \epsilon_2 + \nu \epsilon_1 + \nu \frac{\pi^2 w_0^2}{a^3} \right) x^2
\]

(21)

The unknown parameter \( w_0 \) can now be determined again by using the principle of minimum total potential, that is, from...
\[
\frac{\partial V}{\partial w_0} = 0
\]

Since the external potential \( V \) does not depend on \( w_0 \), this condition becomes

\[
\frac{\partial U}{\partial w_0} = 0 \tag{22}
\]

Introducing the stress function \( \phi \) from Eq. (21) and the deflection function \( w \) from Eq. (19) into Eqs. (5a) and (5b), the total strain energy \( U \) is obtained as a function of the edge strains \( \varepsilon_1 \) and \( \varepsilon_2 \), and the unknown parameter \( w_0 \).

\[
U = U_s + U_B = \frac{E\alpha}{(1-\nu^2)} \left[ \frac{\varepsilon_1^2 + \varepsilon_2^2}{2} + \nu \varepsilon_1 \varepsilon_2 + \right.
\]
\[
+ \frac{\pi^2 \omega_0^2}{a^2} (\varepsilon_1 + \nu \varepsilon_2) + \frac{\pi^4 \alpha^2 \omega_0^2}{3a^4} + \frac{\pi^4 \omega_0^4}{2a^4} \right] \tag{23}
\]

Differentiation of \( U \) according to Eq. (22) results in the following relation for \( w_0 \) in terms of the prescribed edge strains \( \varepsilon_1 \) and \( \varepsilon_2 \):

\[
\left( \frac{\omega_0}{a} \right)^2 = - \frac{1}{3} \left( \frac{\alpha}{a} \right)^2 - \frac{1}{\pi^2} (\varepsilon_1 + \nu \varepsilon_2) \tag{24}
\]

With \( \omega_0 \) known, the stress function \( \phi \) and the deflection function \( w \) are completely defined.
5. POST-BUCKLING DEFLECTION MODES

The large deflection solutions obtained in the preceding sections can be now utilized to compute the energy levels of the two probable deformation patterns for a given set of edge strains $\varepsilon_1$ and $\varepsilon_2$. The lower level will indicate the pattern, wave or cylindrical, which the plate will develop. The procedure, although valid in general, must be performed each time for specific values of $\varepsilon_1$, $\varepsilon_2$, and $(t/a)$. In the following this procedure is demonstrated for some values typical in practical design of reactor containment vessels.

Design values of the strains imposed on the lining at significant service stages in the reactor life are given in [1] and repeated in the Table. The vertical strain refers to the strains in the longitudinal direction of the panel plate $\varepsilon_2$ and the circumferential strain refers to the strains across the rib anchors $\varepsilon_1$, that is, across the width. These strains were calculated by ignoring the contribution of the lining to the deformations of the concrete wall. Where appropriate, they include a component due to the restrained thermal expansion of the lining corresponding to a design temperature of 65° C.

In addition to the strains listed in the Table, consideration should be given to the possibility that the panel under investigation is a weaker panel located between two stronger panels. The difference
in strength may be due to a variation of material properties, plate thickness or the magnitude of initial out-of-flatness. Motion of the panel edge due to a greater force in the stronger panel will be resisted by the weaker panel and by the rib anchors at the junction of the panels. Because of the flexibility of the shear connections (rib anchors) and local deformations of the concrete under this additional load, some slip will occur between the lining and the vessel. The slip will result in an increase in the average strain of the weak panel and a reduction in the average strain of the strong panels.

The problem of these additional studies was studied theoretically in [3]. Lining strains and connector slips were found for an assumed lining discontinuity in terms of the vessel service strains and stiffnesses of the plate and connectors. The lining discontinuity was assumed to coincide with the buckled panel of the minimum thickness and yield strength, bounded by unbuckled panels of the maximum thickness and yield strength. It was also assumed that the vessel is sufficiently rigid compared to the lining for the vessel strains not to be affected by the lining. This investigation has shown that the strength and stiffness of the rib anchors are sufficient to limit the average strain in the buckled panel to 3000 microstrains when the concrete strain is 1000 microstrains.

In accordance with this study, a magnification factor of three is assumed for the values of the circumferential strains in the Table (the second number in each block). A comparison of the magnified circumferential strains with longitudinal strains indicates that essentially all service conditions fall within the range of the
strain ratios \((\varepsilon_1/\varepsilon_2) = 2.0\) and \((\varepsilon_1/\varepsilon_2) = 3.0\). These ratios are therefore used in the study here.

In the comparison of the energy levels, the strain energy per unit length of the panel plate is used. For the panel plate buckled into a wave surface, the strain energy per unit length is obtained by dividing the strain energy of the clamped square plate considered in Section 3 by the side length \(a\). The strain energy per unit length of the panel plate buckled into a cylindrical surface is given by Eq. 23. The strain energy levels per unit length for the two assumed deflection modes were obtained here in nondimensional form for \(a/t\) ratios of 40, 60, 80 and for values of \(\varepsilon_1\) up to 3000 microstrains with \((\varepsilon_1/\varepsilon_2) = 2.0\) or 3.0.

Sample calculations are made in the following for \(\varepsilon_1/\varepsilon_2 = 2.0\) and \(\nu = 0.3\). From Eq. (17) the strain energy of the clamped square plate is given by

\[
U = \frac{E a^2}{(1-\nu^2)} \left[ 0.775 \varepsilon_1^2 + 28.80 \varepsilon_1 \frac{w_0}{a} \left( \frac{w_0}{a} \right)^2 + 129.50 \left( \frac{t}{a} \right)^2 \left( \frac{w_0}{a} \right)^2 + 522.0 \left( \frac{w_0}{a} \right)^4 \right]
\]

from which the strain energy per unit length of the panel plate buckled into a wave surface is obtained in nondimensional form

\[
U' = \frac{1-\nu^2}{Et a^2} \left( \frac{U}{a} \right) = \left[ 0.775 \varepsilon_1^2 + 28.80 \varepsilon_1 \frac{w_0}{a} \left( \frac{w_0}{a} \right)^2 + 129.50 \left( \frac{t}{a} \right)^2 \left( \frac{w_0}{a} \right)^2 + 522.0 \left( \frac{w_0}{a} \right)^4 \right]
\]

where parameter \(w_0\) is found from Eq. (18)

\[
\left( \frac{w_0}{a} \right)^2 = -0.1245 \left( \frac{t}{a} \right)^2 - 0.0247 \varepsilon_1
\]
Analogous equations can be derived also for the panel plate buckled into a cylindrical surface from Eqs. (23) and (24).

The strain energies per unit length for the two potential deflection modes are plotted against the circumferential strain $\varepsilon_1$ for strain ratios of $\varepsilon_1/\varepsilon_2 = 2.0, 3.0$ and for $a/t$ ratios of 40 and 80 in Figs. 5a, b, and 6a, b. As can be seen from these figures, in the post-buckling range the strain energy of the panel plate buckled into a cylindrical surface is always smaller than the strain energy of the panel plate with a wave mode. Therefore, it can now be concluded that the deflection shape of the panel plate is a cylindrical surface when the values of $\varepsilon_1/\varepsilon_2$ typical for practical design are used. Of course any other combination of $\varepsilon_1, \varepsilon_2,$ and $(a/t)$ can be readily investigated following the above described procedure.

It should be observed that the equations derived for the deflections in the post-buckling range can be used directly to compute the intensities of the edge strains (or stresses) which would cause buckling. All that is needed is to set $w_0 = 0$ in Eq. (18) for the wave surface (or in Eq. (24) for the cylindrical surface) and solve for the critical value of the edge strain. The critical stresses are then computed from the strains.
6. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The study presented here is concerned primarily with the determination of the deflection mode which the lining plate is most likely to develop. As the alternate modes a wave and a cylindrical surfaces were considered and the lower strain energy level for the same edge displacement was used as a criterion.

The wave surface was assumed to have the length of each buckle equal to the panel width. Thereby the large deflection analysis of the wave surface was simplified to analyzing a clamped square plate. A one-term deflection function was assumed and the principle of minimum total potential was used to develop an equation for the deflection in terms of the plate slenderness ratio (a/t) and the magnitude of the edge strains in the two directions. Strain energy in the post-buckling range could then be computed for a unit length of the plate.

An analogous, but simpler, procedure was needed to analyze the plate for cylindrical deflection mode and compute the strain energy.

Strain ranges due to service conditions encountered in typical practical designs were reviewed to establish a range for numerical computations. A modification of these strains was introduced to accommodate the critical condition when a weaker plate panel buckles between two stronger panels and the shear anchors allow a
slip of the edges, thereby increasing circumferential strains $(\varepsilon_1)$ in the weaker panel. It was concluded that strain ratios between $(\varepsilon_1/\varepsilon_2) = 2.0$ and $(\varepsilon_1/\varepsilon_2) = 3.0$ with $\varepsilon_1$ going up to 3000 microstrains represent a typical situation.

Plots of strain energy are shown in Figs. 5 and 6 for these conditions with $(a/t) = 40$ and 80. It can be seen there that the cylindrical mode of post-buckling deflection requires a lower energy level and thus is the one to be expected. This is very fortunate since a cylindrical deformation surface can be analyzed for the ultimate strength much more easily than a wave surface. This is also fortunate from another point of view and that is that in previous designs only the possibility of cylindrical deformation has been considered [18].

A study for lower strain ratios, that is, $(\varepsilon_1/\varepsilon_2)$ less than 2.0, indicates in Fig. 4 that a possibility may exist not only for a wave surface but also for a change-over from a cylindrical surface to a wave surface (secondary buckling).

Although the analysis was performed assuming only a one-term deflection function, it is believed that this was sufficiently adequate for the intended purpose of determining which deflection mode should be expected in the plate lining of prestressed concrete containment vessels.

An improvement in the analysis will be obtained by introducing more terms in the $w$-deflection function, both for the cylindrical and wave surfaces. The wave surface analysis will also be considerably improved by treating the lining not as a series of square
plates but as a series of rectangular plates with the length being another unknown parameter to be optimized. Also, unknown functions for $u$ and $v$ displacements would be helpful. Of course, other methods, such as, finite element, lumped parameter, etc., may be employed in solving this problem, especially in the plastic range.
7. ACKNOWLEDGMENTS

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8. REFERENCES


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(Compression is positive)

Table: Typical Strains (x10^6) in Lining of Reactor Vessels
Figure 1: Assumed Deflection Shapes of the Panel Plate
Figure 2: Clamped Square Plate under Biaxial Edge Strains
Figure 3: Unit Length of a Rectangular Plate Buckled into a Cylindrical Surface
Figure 4: Deflection Modes for Different Strain Ratios ($\epsilon_1/\epsilon_2$) and ($a/t$) = 60

$\left( \frac{U_{\text{wave}} - U_{\text{cyl.}}}{U_{\text{cyl.}}} \right) \times 100 \%$
Figure 5: Energy Levels of Wave and Cylindrical Surfaces for 
(a/t) = 40

(a) Energy Levels for \( \frac{\epsilon_1}{\epsilon_2} = 2.0 \)

(b) Energy Levels for \( \frac{\epsilon_1}{\epsilon_2} = 3.0 \)
Figure 6: Energy Levels of Wave and Cylindrical Surfaces for $(a/t) = 80$

(a) Energy Levels for $(\epsilon_1/\epsilon_2) = 2.0$

(b) Energy Levels for $(\epsilon_1/\epsilon_2) = 3.0$