Space Frames with Biaxial Loading in Columns

A NEW APPROACH TO THE SOLUTION OF ECCENTRICALLY LOADED COLUMNS

by

Wai F. Chen
Sakda Santathadaporn

Fritz Engineering Laboratory Report No. 331.2
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The eccentrically loaded column of perfectly plastic material has been studied analytically in considerable detail, but the approach to date always begins by solving the basic differential equations for the deflection curve. As a result, the solution of the deflection curve is too complicated in form even for the simple cases or impossible to obtain at all, because of the nonlinear nature of the differential equations. Simple illustrations demonstrate that the curvature curve of the column plays exactly the same role as deflection did; while the necessary steps in obtaining the solutions are reduced considerably. Using this concept, expressions for the curvature curve of an eccentrically loaded column of rectangular cross-section at all stages of plasticity are derived. It is shown how solutions for the critical loads of the column may be obtained directly from this approach. The analytically obtained results of column curves are compared with the existing solutions and good agreement is observed.
A B S T R A C T

The eccentrically loaded column of perfectly plastic material has been studied analytically in considerable detail, but the approach to date always begins by solving the basic differential equations for the deflection curve. As a result, the solution of the deflection curve is too complicated in form even for the simple cases or impossible to obtain at all, because of the nonlinear nature of the differential equations. Simple illustrations demonstrate that the curvature curve of the column plays exactly the same role as deflection did; while the necessary steps in obtaining the solutions are reduced considerably. Using this concept, expressions for the curvature curve of an eccentrically loaded column of rectangular cross-section at all stages of plasticity are derived. It is shown how solutions for the critical loads of the column may be obtained directly from this approach. The analytically obtained results of column curves are compared with the existing solutions and good agreement is observed.
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1. INTRODUCTION

All columns to be considered are assumed to be made of an idealized material which is termed perfectly plastic. Perfectly plastic material is elastic up to the yield point, and then flows under constant stress. The eccentrically loaded columns are assumed to fail by excessive bending in the plane of the applied moments. The analysis of this type of column in the elastic range is well known. The basic differential equations for bending of the column are usually expressed in terms of the lateral deflection of the column. A solution is obtained by solving the differential equations for the deflection curve. By proper differentiation of this curve, moments, shears, slope, and etc. can then be calculated. Analytical solutions can be obtained in most cases.¹ Similar types of calculations have proved equally successful for other structural elements. It is natural to expect that solutions could be obtained in a similar manner when the columns are stressed beyond the elastic limit. Unfortunately, such an attempt has had little success. The direct solutions for the deflection are not possible to obtain because of the nonlinear nature of the basic differential equations which are too complicated in form even for the most elementary cases.²,³,⁴

The most important feature of such problems is to develop the relation between the slenderness ratio and the critical load. To obtain the deflection curve is merely an intermediate step.
It will be seen in the following discussion that the curvature of the column can serve the same purpose for the inelastic analytical solution as deflection did for an elastic analysis. The steps to obtain solutions are reduced considerably.

Before returning to this point, a brief recapitulation of the general behavior of the eccentrically loaded column may be helpful. An initially straight column of rectangular cross-section of width $b$ and depth $h$, which is loaded by a compressive load $P$ acting on the lever arm $e$ in the plane of the cross-sectional axis $1-1$, is shown in Fig. 1a. A typical load vs. mid-section curvature curve $\xi_m$ is shown in Fig. 2. The characteristics of such a $(P, \xi_m)$-curve is that it consists of an ascending and descending branch with a definite apex which defines the ultimate strength of the eccentrically loaded column (Fig. 2, point A).

When the load $P$ is gradually increased to its maximum and then drops off steadily beyond the maximum point A, the sections at the middle portion of the column go through the successive stages of elastic action, partial yield on one-side of the column (Fig. 1b) and eventually yielding on both sides of the column (Fig. 1c). In Fig. 2, point 1 corresponds to the value of load $P_1$ at which the yield point is just reached in the most compressed fiber of the mid-section, and point 2 corresponds to the value of $P$ for which the fiber of maximum tensile stress of the mid-section on the convex side of the column has also reached its yield point, while the compressive stress on the concave side spreads the plastic regions toward the axis and along the column on each side of the mid-section. The $(P, \xi_m)$-curve consists of three portions marked 0-1, 1-2, 2-3, which corresponds to the different regions in Figs. 1a, 1b, and 1c.
which are described as "elastic", and "one-side plastic", and "two-side plastic" respectively. The plastic zones in the column corresponding to one-side yield is referred to as "Primary Plastic", that of two-side yield as "Secondary Plastic". It is worthwhile to note that the maximum value of $P$, given by the ordinate $P_c$ of the highest point of the $(P, \frac{r}{r_m})$-curve, may be reached in any one of those six possible stages of loading as indicated in Fig. 1, depending upon the relative magnitude of eccentricity and slenderness ratio of the column. The basic differential equations differ in the "Elastic", "Primary Plastic", and "Secondary Plastic" zones of the same column. Hence, the solutions corresponding to each stage of loading have to be considered separately.

Consider first the analysis of an eccentrically loaded column subjected to a uniformly distributed lateral load. The technique discussed at the beginning of the paper will be employed so that the essence and the nature of the approach will be demonstrated clearly by means of this basic elastic stability problem. A complete solution will then be sought for the problem described in Fig. 1, as well as expressions for the curvature curve of the elastic-plastic column at all stages of plasticity.
2. THEORETICAL ANALYSIS

As an example of the basic concepts of the approach, let us consider an eccentrically loaded column of length \( l \) on two simple supports (Fig. 3), and carrying a uniformly distributed load \( q \), which acts in the direction of the \( y \) axis. When the \( q \) is zero, the problem will then be reduced to the Case 1 as shown in Fig. 1a.

The equation of equilibrium for bending of the beam-column is, in the usual notation

\[
\frac{d^2M}{dx^2} - p \frac{d^2y}{dx^2} = -q
\]  

(1)

The expression for the curvature of the axis of the column is

\[
\frac{d^2y}{dx^2} = \frac{M}{EI}
\]  

(2)

The quantity \( EI \) represents the flexural rigidity of the column in the plane of bending. For simplification, the following notation is introduced:

\[
k^2 = \frac{p}{EI}
\]  

(3)

Combining Eq. 1 with 2 and 3, the basic differential equation for bending of beam-column is obtained

\[
\frac{d^2\phi}{dx^2} + k^2 \phi = -\frac{q}{EI}
\]  

(4)
A general solution of this equation is

\[ \ddot{\phi} = A \sin kx + B \cos kx - \frac{q}{p} \]  

(5)

The constants of integration \( A \) and \( B \) are now determined from the conditions at the ends of the column. Since the curvature at the ends of the column is \( \frac{M_0}{EI} \), one concludes that

\[ A = \left( \frac{M_0}{EI} + \frac{q}{p} \right) \tan \frac{1}{2} k\ell \] \[ B = \left( \frac{M_0}{EI} + \frac{q}{p} \right) \]  

(6)

Substituting into Eq. 5 the values of these constants, one obtains the curvature curve:

\[ \ddot{\phi} = \frac{M_0}{EI} \cos k\left( \frac{\ell}{2} - x \right) + \frac{q}{p} \left[ \frac{\cos k\left( \frac{\ell}{2} - x \right)}{\cos \frac{\ell}{2}} - 1 \right] \]  

(7)

The maximum bending moment, which in this case is at the center of the span, is

\[ M_{\text{max}} = EI \ddot{\phi}_{x = \frac{\ell}{2}} = M_0 \sec \frac{\ell}{2} + \frac{q}{k^2} \left( \sec \frac{\ell}{2} - 1 \right) \]  

(8)

The first term on the right hand side of Eq. 7 or Eq. 8 is the solution if the end moments \( M_0 \) act alone. The second term is the solution if the uniform load \( q \) acts alone. This shows that the principle of superposition, which is widely used when lateral loads act alone on a beam, can also be applied in the case of the combined action of lateral and eccentrically loaded column, provided the same axial force acts on the column for each individual applied load. This conclusion was, of course, obtained by Timoshenko.\(^1\)
It is also seen from Eq. 8 that \( \sec \frac{k_f}{2} \) becomes infinite when \( \frac{k_f}{2} \) approaches \( \frac{\pi}{2} \). When \( \frac{k_f}{2} = \frac{\pi}{2} \), one finds from Eq. 3

\[
p = \frac{\pi^2 EI}{\ell^2}
\]

thus it can be concluded that when the axial compressive force approaches the limiting value given by Eq. 9, the curvature at the center of the column will become very large no matter how small the lateral load may be. This load is defined as the critical load of the column according to Timoshenko's definition.
3. ELASTIC - PLASTIC BEHAVIOR

In general, there are three different distributions of stress possible as shown in Fig. 4a, 4b, and 4c. It is convenient to define the following initial yield quantities for moment, axial force, and curvature of the section

\[ M_y = \frac{\sigma_o b h^2}{6}, \quad P_y = \sigma_o b h, \quad \phi_y = \frac{2 \epsilon_o}{h} = \frac{2 \sigma_o}{h E} \quad (10) \]

where \( \sigma_o \) is the yield point in tension or compression and \( \epsilon_o \) is the corresponding strain. One further defines the dimensionless variables by

\[ m = \frac{M}{M_y}, \quad p = \frac{P}{P_y}, \quad \phi = \frac{\phi}{\phi_y} \quad (11) \]

In the following, it is desired to derive the nondimensional expressions for moment in terms of axial force and curvature. Thus, the combination of these relations with the equation of equilibrium will then lead to the basic differential equation of the problem for different stress zones.

3.1 Elastic Zone

In nondimensional form, the moment-curvature relation of Eq. 2 has the form

\[ m = \phi \quad \text{for} \quad 0 \leq \phi \leq 1 - p \quad (12) \]
and the equation of equilibrium Eq. 1 is reduced to

\[ \frac{d^2}{dx^2} m + k^2 \phi = 0 \]  \hspace{2cm} (13)

when the lateral load \( q \) is absent. Substituting into Eq. 13 the value of the moment from Eq. 12, one obtains the basic differential equation of the column for the elastic zone.

\[ \frac{d^2}{dx^2} \phi + k^2 \phi = 0 \]  \hspace{2cm} (14)

### 3.2 Primary Plastic Zone

The region of yielding is assumed to extend a distance \((1 - \beta)h\) into the cross section (see Fig. 4b). The resultant axial force acting on the section is

\[ p = 1 - \beta^2 \phi \]  \hspace{2cm} (15)

or

\[ \beta^2 = (1 - p)/\phi \]  \hspace{2cm} (16)

and the moment about the center line of the section is

\[ m = 3(1 - p) - 2(1 - p)^{3/2}/\phi^{1/2} \]  \hspace{2cm} (17)

which is valid for the range

\[ 1 - p \leq \phi \leq \frac{1}{1 - p} \]  \hspace{2cm} (18)

Combining Eq. 17 with the equilibrium equation (13), the basic differential equation for this zone has the form
\[ \phi \frac{d^2 \phi}{dx^2} - \frac{3}{2} \left( \frac{d\phi}{dx} \right)^2 + n \phi^{7/2} = 0 \]  

(19)

in which \( n \) is defined as

\[ n = \frac{k^2}{(1 - p)^{3/2}} \]  

(20)

3.3 **Secondary Plastic Zone**

Referring to Fig. 4c, the region of yielding is assumed to extend distances \( \gamma h \) and \( (1 - \frac{1}{\phi}) - \gamma h \) respectively, into the cross section. The resultant of the axial force acting on the section is

\[ p = 1 - 2 \gamma - \frac{1}{\phi} \]  

(21)

or

\[ \gamma = \frac{1}{2} \left[ 1 - p - \frac{1}{\phi} \right] \]  

(22)

and the moment is

\[ m = \frac{3}{2} \left( 1 - p^2 \right) - \frac{1}{2} \frac{1}{\phi^2} \]  

(23)

which is valid for

\[ \frac{1}{1 - p} \leq \phi \]  

(24)

Combining Eq. 23 with Eq. 13 gives the basic differential equation for this zone

\[ \phi \frac{d^2 \phi}{dx^2} - 3 \left( \frac{d\phi}{dx} \right)^2 + k^2 \phi^5 = 0 \]  

(25)
4. **SOLUTION OF ELASTIC - PLASTIC COLUMN**

**ONE - SIDE PLASTIC**

There are two different distributions of the primary plastic zone possible, indicated in Fig. 1b as Case 2 and Case 3. Since the procedure for solutions are the same for both cases, the Case 2 only will be discussed in details here and the solutions for the Case 3 will be given in Appendix B.

Referring to Fig. 1b (Case 2), the primary plastic zone extends a distance \( \rho_1 \) from the ends. The curvature at the center of the column is denoted by \( \Theta_m \). Since the column and the loading are symmetrical about the mid-section of the column, one needs only to consider half of the column.

The elastic end portions of the column will be investigated first. The general solution of Eq. 14 is

\[
\Theta = A \cos kx + B \sin kx
\]

(26)

The constants of integration A and B are now determined from the conditions at the end and at the elastic-primary plastic boundary. Since the curvature at the end of the bar is \( m_o \), one concludes

\[
A = m_o
\]

(27)
At \( x = \rho_1 \), the curvature is \((1 - p)\). This condition gives

\[
B = \frac{1 - p - m_o \cos k \rho_1}{\sin k \rho_1}
\]  

(28)

Substituting into Eq. 26 the values of the constants from Eq. 27 and Eq. 28, one obtains the following equation for the curvature in the elastic zone:

\[
\phi = m_o \cos x + \frac{1 - p - m_o \cos k \rho_1}{\sin k \rho_1} \sin x
\]  

(29)

valid for

\[
0 \leq x \leq \rho_1
\]

The unknown distance \( \rho_1 \) will be determined after the solution for the primary plastic zone is obtained.

For the primary plastic portion of the column, Eq. 19 is non-linear. However, if one introduces the new variable

\[
u = \frac{d\phi}{dx}
\]

(30)

Equation 19 can be reduced to the exact differential equation

\[
\phi u \frac{du}{d\phi} - \frac{3}{2} u^2 + n \phi^{7/2} = 0
\]  

(31)

where \( \phi^{-4} \) is the integral factor. The general solution of Eq. 31 is

\[
u^2 = (c - 4n \phi^{1/2}) \phi^3
\]  

(32)
in which $c$ is an arbitrary constant. Since $\phi$ is always a positive quantity in Eq. 32, it follows that

$$c - 4n\phi^{1/2} > 0$$  \hspace{1cm} (33)

The constant $c$ can be determined from the condition of symmetry of the curvature curve at the mid-section of the column. Since the tangent to the curvature curve is zero at the mid-section,

$$c = 4n\phi^{1/2} = \frac{4k^2\phi_m^{1/2}}{(1 - p)^{3/2}} > 0$$  \hspace{1cm} (34)

Hence, Eq. 32 becomes

$$u = \frac{d\phi}{dx} = \frac{2k}{(1 - p)^{3/4}}(\phi_m^{1/2} - \phi^{1/2})^{1/2}\phi^{3/2}$$  \hspace{1cm} (35)

The unknown quantity, $\phi_m$, which denotes the curvature at the mid-section of the column can now be determined from conditions at the section $x = \rho_1$. At this section, the two portions of the curvature curve as given by Eq. 29 and Eq. 35, have a common tangent. This condition, together with $\phi = \rho_1 = 1 - p$, gives

$$\phi_m = (1 - p)\eta^2$$  \hspace{1cm} (36)

in which $\eta$ is defined as

$$\eta = 1 + \frac{1}{4}(\cot k\rho_1 - \frac{m_0}{(1 - p)}\csc k\rho_1)^2$$  \hspace{1cm} (37)

In order to solve the curvature curve, one has to integrate Eq. 35 for the solution. It is difficult to express $\phi$ explicitly in terms
of \( x \). Alternatively, it is more convenient to express the curvature curve implicitly in the form \( x = x(\theta) \). Hence,

\[
x = \frac{(1 - p)^{3/4}}{2k} \int \frac{d\theta}{(\theta_1^{1/2} - \theta^{1/2})^{1/2} \theta^{3/2}} + D
\]

(38)

From which, one obtains

\[
x = D - \frac{1}{k} \left( \frac{1 - p}{\theta_m} \right)^{3/4} \left[ \left( \frac{\theta_m}{\theta} - \frac{\theta_m^{1/2}}{\theta^{1/2}} \right)^{1/2} + \tanh^{-1} \left( 1 - \frac{\theta_m^{1/2}}{\theta_m^{1/2}} \right) \right]
\]

(39)

The constant \( D \) and the value \( \rho_1 \) are found from the conditions that \( \theta \) in Eq. 39 must be equal to \( \theta_m \) and \( 1 - p \) for \( x = \frac{\ell}{2} \) and \( x = \rho_1 \), respectively. Hence,

\[
D = \frac{\ell}{2}
\]

(40)

and

\[
\frac{k \ell}{2} = k \rho_1 + \left( \frac{\eta - 1}{\eta} \right)^{1/2} + \frac{1}{\eta^{3/2}} \tanh^{-1} \left( \frac{\eta - 1}{\eta} \right)^{1/2}
\]

(41)

Equation (39) becomes

\[
\frac{x}{\ell} = \frac{1}{2} - \frac{1}{k \ell} \frac{1}{\eta^{3/2}} \left[ \left( \frac{(1 - p) \eta}{\theta} - \frac{(1 - p)^{1/2} \eta}{\theta^{1/2}} \right)^{1/2}
\]

\[
+ \tanh^{-1} \left[ 1 - \frac{\theta^{1/2}}{(1 - p)^{1/2} \eta} \right]^{1/2} \right]
\]

(42)

valid for

\[
\rho_1 \leq x \leq \frac{\ell}{2}
\]

in which \( \rho_1 \) is given by Eq. 41.
It is difficult to solve for $\rho_1$, from Eq. 41, in terms of the given values of $p$, $m_0$, $k$, and $\ell$. Alternatively, one can assume a value of $kp_1$ and obtain the values of $k\ell$ and $\phi_m$ from Eq. 41 and Eq. 36 respectively. With these values, the curvature curve of the column can be computed in a straightforward manner from Eq. 29 for the elastic portions of the column and from Eq. 42 for the primary plastic portion of the column.

It is important to note that an implicit assumption has been made, however, that the curvature curve at $x = \rho_1$ and $x = \frac{\ell}{2}$ is a continuous differential function of $x$. This assumes that there exists a unique tangent to the curvature curve at $x = \rho_1$ and $x = \frac{\ell}{2}$. The possibility that the tangent to the curvature are discontinuous must be reckoned with. This is always so, for instance, when a concentrated load is applied laterally to the column, a discontinuous "jump" of the tangent to the curvature curve at the load point must be properly taken into account. A detailed discussion of the "jump" condition is given in Appendix A.
5. **SOLUTION OF ELASTIC-PLASTIC COLUMN YIELDED ON TWO-SIDES**

In addition to the primary plastic zone just discussed, there also exists a secondary plastic zone in the column. There are three different distributions of the primary plastic zone combining with the secondary plastic zone possible, indicated in Fig. 1c as Case 4, Case 5, and Case 6. Again, the Case 4 only will be discussed in details here and the solutions for the Case 5 and Case 6 will be recorded in Appendix B.

Referring to Fig. 1c (Case 4), the secondary plastic zone extends a distance $\rho_2$ from the ends. The procedure for solving the problem is similar as before.

The provisions (Eq. 29) for the elastic end portions of the column are still applicable if the appropriate value of $\rho_1$ is used.

As for the primary plastic zone of the bar, for which, $\rho_1 \leq x \leq \rho_2$, the general solution of Eq. 31 is modified as

$$u^2 = (c^* - 4 n \phi^{1/2}) \phi^3$$

in which $c^*$ is a new integration constant. Since

$$c^* - 4 n \phi^{1/2} \geq 0$$

it will be more convenient, in the following computation, to solve the
secondary plastic zone first, for which, \( \rho_2 \leq x \leq \frac{L}{2} \), and then determine the constant, \( c^* \).

For the secondary plastic zone of the column the differential equation (25) is again nonlinear. However, with the aid of Eq. 30, one can transform it to the exact differential equation

\[
\phi \ u \ \frac{du}{d\phi} - 3 \ u^2 + k^2 \ \phi^5 = 0
\]

(45)

where \( \phi^{-7} \) will be the integral factor. The general solution of Eq. 45 is

\[
u^2 = G \ \phi^6 + 2 \ k^2 \ \phi^5
\]

(46)

The constants \( G \) and \( c^* \) in Eq. 46 and Eq. 43 are now determined from the condition of symmetry of the curvature curve at the section \( x = \frac{L}{2} \), and the condition of common tangent to the curvature curve at the section \( x = \rho_2 \). That is,

\[
\frac{d\phi}{dx} = u = 0
\]

For \( x = \frac{L}{2} \), which implies

\[
G = - \frac{2 \ k^2}{\phi_m}
\]

(47)

and the condition

\[
u_x = \rho_2 = \left. \frac{d\phi}{dx} \right|_{x = \rho_2} = \left. \frac{d\phi}{dx} \right|_{x = \rho_2}
\]

primary secondary

when \( x = \rho_2 \).
Using the fact that \( \phi = \rho_2 = \frac{1}{1 - p} \) and Eq. 47, one concludes

\[
c^* = \frac{2k^2}{(1 - p)^2} \left[ 3 - \frac{1}{(1 - p) \phi_m} \right] > 0
\]  

(48)

With the value of \( G \) from Eq. 47, Eq. 46 may be rewritten as

\[
u = \frac{d\phi}{dx} = \sqrt{2} k \left( 1 - \frac{\phi}{\phi_m} \right)^{1/2} \phi_m^{5/2}
\]  

(49)

Integrating this equation with respect to \( x \), taking account of the fact that \( \phi = \phi_m \) for \( x = \frac{\rho_1}{2} \), and \( \phi = \frac{1}{1 - p} \) for \( x = \rho_2 \) one finds the curvature for the secondary plastic zone expressed implicitly as a function of \( x \)

\[
\frac{x}{\ell} = \frac{1}{2} - \frac{\sqrt{2}}{3} \frac{1}{k \ell} \left( \frac{1}{\phi} - \frac{1}{\phi_m} \right)^{1/2} \left( \frac{1}{\phi} + \frac{2}{\phi_m} \right)
\]  

(50)

valid for

\[ \rho_2 \leq x \leq \frac{\rho_1}{2} \]

and also the relation

\[
k \rho_2 = \frac{k \ell}{2} - \frac{\sqrt{2}}{3} \left( 1 - p - \frac{1}{\phi_m} \right)^{1/2} \left( 1 - p + \frac{2}{\phi_m} \right)
\]  

(51)

From Eq. 48 and the condition that the two portions of \( u \), as given by Eqs. 29 and 43, have a common value at the section \( x = \rho_1 \), the mid-section curvature can be expressed as

\[
\phi_m = \frac{1}{(1 - p)[3 - 2(1 - p) \eta]}
\]  

(52)
Hence, Eq. 43 then reduces to

\[ u = \frac{2 \, k}{(1 - p)^{1/2}} \left[ \eta - \frac{\phi^{1/2}}{(1 - p)^{1/2}} \right]^{1/2} \phi^{3/2} \]  

(53)

Integrating Eq. 43 with the condition that \( \phi = 1 - p \), for \( x = \rho_1 \)
the curvature curve for the primary plastic zones of the column becomes

\[ \frac{x}{k} = \frac{k \, \rho_1}{k \, \ell} + \frac{1}{k \, \ell} \frac{1}{\eta^{3/2}} \left\{ (\eta^2 - \eta)^{1/2} - \left[ (1 - p) \frac{\eta^2}{\phi} - (1 - p)^{1/2} \frac{\eta}{\phi^{1/2}} \right]^{1/2} \right. 

+ \left. \tanh^{-1} \left( 1 - \frac{1}{\eta} \right)^{1/2} \right. 

\left. - \tanh^{-1} \left[ 1 - \frac{\phi^{1/2}}{\eta(1 - p)^{1/2}} \right]^{1/2} \right\} \]  

(54)

valid for

\[ \rho_1 \leq x \leq \rho_2 \]

and the desired formula for the elastic-primary plastic boundary \( \rho_1 \),
using the relations given in Eq. 51 and Eq. 52

\[ \frac{k \, \ell}{2} = \frac{2}{3} (1 - p)^{3/2} \left[ (1 - p) \eta - 1 \right]^{1/2} \left[ 7 - 4(1 - p)\eta \right] + k \, \rho_1 \]

\[ + \frac{1}{\eta^{3/2}} \left\{ (\eta^2 - \eta)^{1/2} - \left[ (1 - p)^2 \eta^2 - (1 - p)\eta \right]^{1/2} \right. 

- \left. \tanh^{-1} \left( 1 - \frac{1}{\eta} \right)^{1/2} \right. 

\left. - \tanh^{-1} \left[ 1 - \frac{1}{\eta(1 - p)} \right]^{1/2} \right\} \]  

(55)

Again, it is convenient to obtain numerical results by first assuming a
value of \( k \rho_1 \) and obtain the corresponding value of \( k \ell \) from Eq. 55.
Once this is done, the desired curves for curvature are readily obtained.
6. COLUMN CURVES

It has been shown in the previous sections that the length and mid-section curvature of the column present themselves as functions of the parameter, \( k \rho_1 \), for the constant values of \( p \) and \( m_0 \) (or \( \frac{e}{r} \))

\[
    k \ell = f(k \rho_1), \quad \varphi_m = g(k \rho_1)
\]  

(56)

as \( k \rho_1 \) varies, the points \( (k \ell, \varphi_m) \) describes a curve. A set of such curves with eccentricity ratio \( \frac{e}{r} = 1 \) are shown in Fig. 5 from which the applied load versus mid-section curvature relationships given in Fig. 6 were obtained. The small circles in Fig. 6 are plotted as corresponding small circles in Fig. 5 for the case \( \frac{e}{r} = 160 \) as an illustration. The regions representing different stages of plasticity of the columns are also indicated in the figure. The peak, A, of the \( (p, \varphi_m) \)-curve defines the critical load of the eccentrically loaded column. Accordingly, the stability criterion is defined as

\[
    \frac{dp}{d\varphi_m} = 0
\]  

(57)

Alternatively, it may be more convenient to obtain the critical length of the column for which the given load \( p \) will become the critical load. An equivalent expression can be expressed as (see Fig. 5)
from which, one obtains

\[
\frac{dk_\ell}{d\phi_m} = \frac{dk_\ell}{dkp_1} = 0
\]

(58)

\[
\text{for } \frac{d\phi_m}{dkp_1} \neq 0
\]

(59)

this condition may be used to derive the critical length of eccentrically loaded columns.

For the case of one-side plastic (Fig. 1b, Case 2) for example, Eq. 59 requires that the value of \( kp_1 \) must satisfy the relation

\[
\tan kp_1 = \frac{2 \left[ 3 \eta - \eta^2 - 3(\eta^2 - \eta)1/2 \tan^{-1}(1 - \frac{1}{\eta})1/2 \right]}{\left[ \eta(2\eta + 3)(\eta - 1)1/2 + 3\eta^{1/2}\tanh^{-1}(1 - \frac{1}{\eta})1/2 \right]}
\]

(60)

in which \( \eta \) is a function of \( kp_1 \) as defined in the expression in Eq. 37. Once the value of \( kp_1 \) is obtained, the corresponding values of the physical characteristics of the column at the point of collapse can be computed in a rather straight forward manner. Similar relation can be derived for the case where both sides are plastic. Simple expressions for other cases will be given in Appendix B.

The maximum loads obtained in this way for various values of the slenderness ratio \( \frac{L}{r} \) and eccentricity \( e \) are plotted in Fig. 7. They were computed for columns with rectangular cross-section and a yield point
\[ \sigma_0 = 34 \text{kips/in.}^2 \]. In particular, the curves with eccentricity ratios 
\[ \frac{e}{r} = 0.05 \text{ and } \frac{e}{r} = 1.0, \] 
are compared with both the exact curves\(^5,6,7\) determined from the real stress-strain curve of the structural steel having 
a proportional limit \( \sigma_p = 27 \text{kips/in.}^2 \) and a yield point \( \sigma_0 = 34 \text{kips/in.}^2 \) and the Jezek's approximate curves\(^8,9\) based upon the perfectly plastic 
idealization for the steel as well as by assuming the sine curve shape 
of the deflected column axis. The significance of this comparison is 
that they brought insight into the degree of approximation involved as 
influenced by the use of the idealized stress-strain diagram (the 
difference between the present solution and the exact solution) and as 
influenced by the use of the half wave of a sine curve for the deflected 
column (the difference between the present solution and the Jezek's 
solution).
7. CURVATURE CURVES

The curvature curves obtained in the previous sections may be plotted in a single diagram for various values of $p$. Figures 8 and 9 show two families of such curves, one for $\frac{e}{r} = 1$ and $\frac{L}{r} = 160$, the other for $\frac{e}{r} = 0$, and $\frac{L}{r} = 110$. In each case, the yield stress $\sigma_o = 34$ ksi and $E = 30,000$ ksi were used. The extent of the elastic, primary plastic and secondary plastic zones of the column corresponding to different stages of loading, $p$, are denoted by light solid lines, dashed lines, and heavy solid lines respectively in the figures.

It may be observed from Figs. 8 and 9 that the curvature curves resemble a sine curve up to the critical load (or maximum load). However, it is clear from the figures that this is not true when the columns are loaded beyond the critical load. This explains why the critical loads computed from the approximate theory of Jezek's, for all practical purposes, give satisfactory results for all kinds of material having a sharply defined yield strength.

When the deflection of the eccentrically loaded column is sought, it is convenient to compute directly from the equilibrium relationship, that is

$$M_x = Py + Mo \quad (61)$$
or, in nondimensional form

\[
\frac{y}{\left(\frac{h}{\theta}\right)} = \frac{m_x - m_0}{p} \tag{62}
\]

in which \(m_x\) denotes the bending moment at the section \(x\).

Since the value of curvature is known along the elastic-plastic column, the moment, \(m_x\), can be computed easily from \(m - \theta - p\) relations for any given values of \(p\). Hence, Eq. 62 furnishes the relationship for the determination of deflection at any point on the column and at any stage of loading. In fact, direct integration of curvature relationships are possible, but the resulting algebraic expression cannot be simplified sufficiently to warrant recording.
8. CONCLUSIONS

Curvature is seen to be a more appropriate variable than deflection for the determination of the load-deformation relationship and for the derivation of the collapse loads of an eccentrically loaded column in the elastic as well as in the elastic-plastic range for many cases. It is expected that such an approach can be applied to the solution of a variety of problems involving load-deformation analysis. In the present paper, the approach was demonstrated clearly by the analysis of eccentrically loaded columns with perfectly plastic idealization. An immediate extension of the approach to include the cases of wide flange sections with lateral load and initial imperfection will prove to be very worthwhile. As the purpose of this paper is to elucidate the approach and to demonstrate the points and not to solve problems, the general considerations have not been attempted here.
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10. FIGURES
Case I

(a) Elastic

Case 2

(b) One-Side Plastic

Fig. 1 Eccentrically Loaded Column
Fig. 1 Eccentrically Loaded Column
Fig. 2 Typical Load vs. Mid-Section Curvature

Fig. 3 Eccentrically Loaded Beam-Column
\( \phi \leq 1-p \)

\( 1-p \leq \phi \leq \frac{1}{1-p} \)

\( \frac{1}{1-p} \leq \phi \)

(a) Elastic Stress Distribution

(b) Primary-Plastic Stress Distribution

(c) Secondary-Plastic Stress Distribution

Fig. 4 Three Possible Stress Distributions
Fig. 5 A Sample of $\ell/r$ vs. $\phi_m$

- $e/r = 1$
- $E = 30,000$ ksi
- $\sigma_0 = 34$ ksi
Fig. 6 Load vs. Mid-Section Curvature Curve
Fig. 7 Comparison of Various Solutions of Slenderness Ratio vs. Critical Average Stress Curves
(Rectangular Section)

\[ \frac{P_c}{A} \text{ vs. SLENDERNESS RATIO } \frac{f}{f} \]

- Exact Sol.
- Jezek's Sol.
- Present Sol.

\( E = 30,000 \text{ ksi} \)
\( \sigma_0 = 34 \text{ ksi} \)
Fig. 8 Curvature Along the Length of a Column

\[ \phi \left( \frac{x}{\ell} \right) \]

- Elastic
- Primary Plastic
- Secondary Plastic

\[ \frac{e}{r} = 1 \]
\[ \frac{k}{r} = 160 \]
\[ \sigma_0 = 34 \text{ ksi} \]
\[ E = 30,000 \text{ ksi} \]

\[ p = 0.16 \]
\[ p = 0.18 \]
\[ p = 0.20 \]
\[ \text{max } p = 0.203 \]
\[ p = 0.186 \]
Fig. 9 Curvature Along the Length of a Column
Fig. 10 Beam-Column with a Concentrated Lateral Load
11. APPENDIX A

DISCONTINUITY IN THE DERIVATIVE OF THE CURVATURE CURVE

It has been pointed out in the discussion of the solution for Case 2, that one has considered only the curvature curves in which the derivative of the curvature are continuous functions of \( x \). On the other hand, when concentrated load is present there is discontinuity in slope to the curvature curve at the section where the concentrated load is applied. The "jump condition" instead of the continuity condition for the derivation of the curvature curve should then be used for the determination of integration constants.

A simple example is furnished by an elastic beam-column which is loaded laterally by a single concentrated load \( Q \) at distance \( c \) from the right end (Fig. 10). A moment diagram is also sketched in the figure. The two sides of the curvature curve (or moment diagram) as separated by the concentrated load \( Q \) will be labelled 1 and 2 as in the figure, and, wherever this is necessary, the subscripts 1 and 2 will be used to distinguish the values which a quantity assumes on these two sides. Clearly, the equilibrium of the external, and internal moments about the concentrated load section requires

\[
\begin{align*}
\frac{d\phi_1}{dx} &= \frac{dM_1}{dx} = \frac{Qc}{\ell} + p \frac{dy}{dx} \\
\frac{d\phi_2}{dx} &= \frac{dM_2}{dx} = -\frac{Q(\ell - c)}{\ell} + p \frac{dy}{dx}
\end{align*}
\]

(63)
Hence, the "jump condition" of the slope to the curvature curve at the load section is

\[ \frac{d\phi_1}{dx} - \frac{d\phi_2}{dx} = \frac{Q}{EI} \]  

(64)

The general solutions of the basic differential equation (14) is

\[ \phi_1 = A_1 \cos kx + B_1 \sin kx \quad \text{for} \quad 0 \leq x \leq \ell - c \]  

(65)

\[ \phi_2 = A_2 \cos kx + B_2 \sin kx \quad \text{for} \quad \ell - c < x < \ell \]

The boundary conditions at \( x = 0 \) and \( x = \ell \) require that the curvature be zero; that is

\[ A_1 = 0 \]

\[ A_2 = -B_2 \tan k\ell \]  

(66)

At the point of application of the load \( Q \) the two portions of the curvature curve, as given by Eq. 65, have the same curvature (continuity condition) but a jump in slope as given by Eq. 64. These conditions give

\[ B_1 \sin (\ell - c) = -B_2 \frac{\sin k\ell}{\cos k\ell} \]  

(67)

\[ B_1 \cos (\ell - c) = B_2 \frac{\cos k\ell}{\cos k\ell} + \frac{Q}{kEI} \]

From which

\[ B_1 = \frac{k Q \sin k\ell}{P \sin k\ell} \]  

(68)

\[ B_2 = \frac{-k Q \sin (\ell - c)}{P \tan k\ell} \]
Substituting these results back to Eq. 65, the curvature curve has the form

\[ \dot{\gamma}_1 = \frac{k Q \sin k c}{P \sin k \ell} \sin kx \quad \text{for} \quad 0 \leq x \leq \ell - c \]  

(69)

\[ \dot{\gamma}_2 = \frac{k Q \sin (\ell - c)}{P \sin k \ell} \sin k(\ell - x) \quad \text{for} \quad \ell - c \leq x \leq \ell \]

agreeing with the results obtained by Timoshenko.¹

When the sections under consideration are stressed beyond the elastic limit, the following "jump conditions" in nondimensional form are given below for reference.

**Primary plastic zone**

\[ \frac{d\phi_1}{dx} - \frac{d\phi_2}{dx} = \frac{Q}{M_y} \frac{\phi^{3/2}}{(1 - p)^{3/2}} \quad x = \text{load point} \]  

(70)

**Secondary plastic zone**

\[ \frac{d\phi_1}{dx} - \frac{d\phi_2}{dx} = \frac{Q}{M_y} \phi^3 \quad x = \text{load point} \]  

(71)
Case 3 (Fig. 1b)

The curvature curve expressed implicitly in Eq. 39 is valid for the entire column. The constant $D$ and the value $\phi_m$ are found from the conditions that $\phi$ in Eq. 39 must be equal to $\phi_m$ and $\phi_o$ for $x = \ell/2$ and $x = \phi$ respectively. Hence

$$k\ell = 2(1 - p)^{3/4}\left[\left(\frac{\phi_m}{\phi_o} - \frac{1}{\phi_m^{1/2}}\right)^{1/2} + \tanh^{-1}\left(1 - \frac{\phi_o^{1/2}}{\phi_m^{1/2}}\right)^{1/2}\right]^{1/2}$$

and

$$D = \ell/2$$

where

$$\phi_o = \frac{4(1 - p)^3}{[3(1 - p) - m_o]^2} \quad \text{(73)}$$

valid for

$$1 - p \leq \phi_o$$

and

$$1 - p \leq \phi_m \leq \frac{1}{1 - p}.$$
Case 5 (Fig. 1c)

Equation 50 is still applicable for the secondary plastic zone but the curvature curve for the primary plastic zone is modified as

\[
x = \frac{2}{c^*} \left[ \frac{(c^* - 4 n \phi_o^{1/2})^{1/2} (c^* - 4 n \phi)^{1/2}}{\phi_o^{1/2}} - \frac{(c^* - 4 n \phi)^{1/2}}{\phi^{1/2}} \right]
\]

\[
+ \frac{8 n}{(c^*)^{3/2}} \left[ \tanh^{-1} \left( 1 - \frac{4 n \phi^{1/2}}{c^*} \right)^{1/2} - \tanh^{-1} \left( 1 - \frac{4 n \phi_o^{1/2}}{c^*} \right)^{1/2} \right]
\]

(74)

where the values \( n, c^*, \) and \( \phi_o \) are given in Eqs. 20, 48, and 73 respectively. The values \( k\rho_2 \) and \( \phi_m \) can be determined by the relations

\[
k\rho_2 = \frac{\sqrt{2} (1 - p)}{\left[ 3 - \frac{1}{(1 - p) \phi_m} \right]^{1/2}} \left\{ \left( 1 - \frac{4 n \phi_o^{1/2}}{c^*} \right)^{1/2} \right\}
\]

\[
- \left[ 1 - \frac{4 n \phi^{1/2}}{c^* (1 - p)^{1/2}} \right]^{1/2} \left( 1 - p \right)^{1/2}
\]

\[
+ \frac{2 \sqrt{2} (1 - p)^{3/2}}{\left[ 3 - \frac{1}{(1 - p) \phi_m} \right]^{3/2}} \left\{ \tanh^{-1} \left( 1 - \frac{4 n \phi_o^{1/2}}{c^*} \right)^{1/2} \right\}
\]

\[
- \tanh^{-1} \left[ 1 - \frac{4 n \phi^{1/2}}{c^* (1 - p)^{1/2}} \right]^{1/2}
\]

(75)

\[
k\ell = 2k\rho_2 + \frac{2\sqrt{2}}{3} \left[ (1 - p) - \frac{1}{\phi_m} \right]^{1/2} \left[ (1 - p) + \frac{2}{\phi_m} \right]
\]

(76)
valid for

\[ 1 - p \leq \phi_o \leq \frac{1}{1 - p} \]

and

\[ \frac{1}{1 - p} \leq \phi_m \]

Case 6 (Fig. 1c)

The curvature curve expressed implicitly in Eq. 50 is valid for the entire column and the value \( \phi_m \) can be determined from the equation

\[ kL = \frac{2}{3} \left[ \frac{1}{\phi_o} - \frac{1}{\phi_m} \right]^{1/2} \left[ \frac{1}{\phi_o} + \frac{2}{\phi_m} \right] \]  

(77)

The critical length of the column is obtained by applying the condition

\[ \frac{dkL}{d\phi_m} = 0 \]

to Eq. 77, one obtains

\[ \phi_m = \frac{4 \phi_o}{4 - \phi_o} \]  

(78)

valid for

\[ \phi_o \leq 4 \]

Substitute Eq. 78 into Eq. 77, one obtains

\[ (kL)_{cr} = \frac{\sqrt{2}}{6} \left\{ 6 \left[ 3(1 - p^2) - 2m_o \right]^{1/2} - 1 \right\} \]  

(79)
valid for
\[ \frac{1}{1 - p} \leq \varnothing \leq 4 \]

and
\[ \frac{1}{1 - p} < \varnothing_m \]
13. **NOTATION**

A, B, C, etc. = constants of integration

\( b \) = width of rectangular section of column

\( E \) = modulus of elasticity

\( e \) = eccentricity

\( h \) = depth of rectangular section of column

\( I \) = moment of inertia of section about the axis of bending

\( k \) = \( \sqrt{P/ET} \)

\( L \) = length of column

\( M \) = bending moment

\( M_0 \) = applied moment at the end of column

\( M_y \) = moment which causes first yielding in the section

\( m \) = \( M/M_y \)

\( m_0 \) = \( M_0/M_y \)

\( p \) = axial force

\( p_y \) = axial yield load

\( P \) = \( P/p_y \)

\( Q \) = concentrated lateral load

\( q \) = lateral distributed load

\( r \) = radius of gyration about the axis of bending

\( x, y \) = coordinate axes
$\varepsilon_0$ = strain at yield point

$\rho_1$ = distance from the end to the primary plastic zone of the column

$\rho_2$ = distance from the end to the secondary plastic zone of the column

$\sigma_0$ = yield stress

$\phi$ = curvature

$\phi_m$ = midspan curvature

$\phi_y$ = curvature at initial yielding for pure bending moment

$\phi_m / \phi_y$

$\phi_m / \phi$
14. REFERENCES

1. Timoshenko, S. P. and Gere, J. M.
THEORY OF ELASTIC STABILITY, Chapter 1, 2nd Edition,

2. Horne, M. R.
THE ELASTIC-PLASTIC THEORY OF COMPRESSION MEMBERS,
Journal of the Mechanics of Physics of Solids,
Vol. 4, 1956, p. 104

3. Ellis, J. S.
PLASTIC BEHAVIOR OF COMPRESSION MEMBERS,
Journal of the Mechanics of Physics of Solids,
Vol. 6, 1958, p. 282

4. Hauck, G. F. and Lee, S. L.
STABILITY OF ELASTIC-PLASTIC WIDE-FLANGE COLUMNS,
Journal of the Structural Division, ASCE, Vol. 89,
No. ST6, Proceeding Paper 3738, December 1963,
pp. 297-324

5. Von Karman, T.
UNTERSUCHUNGEN UBER KNICKFESTIGKEIT, MITT. FORSCH.
GESELLSCHAFT VON INGENIEURWESENS, No. 81, 1910; also in
"Collected Works of Theodor Von Karman, Butterworths

6. Von Karman, T.
DIE KNICKFESTIGKEIT GERADER STAB, Phys. Zeitschr,
Vol. 9, 1908, p. 136

7. Bleich, F.
BUCKLING STRENGTH OF METAL STRUCTURES, Chapter 1,

8. Jezek, K.
DIE TRAGFÄHIGKEIT DES GLEICHMÄSSIG QUERBELASTETEN
DRUCKSTABES AUS EINEM IDEAL-PLASTISCHEN STAHL,
STAHLBAU, Vol. 8, 1935, p. 33

9. Jezek, K.
NAHERRUngsBerechnung Der Tragkraft Exzentrisch
Gedruckter Stahlstabe, STAHLBAU, Vol. 8, 1935, p. 89