Plastic Design of Multi-Story Frames

COMBINED LOAD ANALYSIS OF UNBRACED FRAMES

by

J. Hartley Daniels

Fritz Engineering Laboratory Report No. 338.2
Plastic Design of Multi-Story Frames

COMBINED LOAD ANALYSIS OF UNBRACED FRAMES

by

J. Hartley Daniels

This work has been carried out as a part of an investigation entitled "Plastic Design of Multi-Story Frames" with funds furnished by an American Iron and Steel Institute Doctoral Fellowship.

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

July 1967

Fritz Engineering Laboratory Report No. 338.2
ACKNOWLEDGMENTS

The author wishes to acknowledge the special advice and assistance that he received from Professor John W. Fisher who supervised this dissertation. Sincere thanks are also extended to each member of the special committee which directed the author's doctoral work. The committee comprises: Professors Lynn S. Beedle, Chairman, John W. Fisher, Supervisor, Le-Wu Lu, V. V. Latshaw and D. A. VanHorn.

The author is particularly indebted to his wife and family for their patience, kindness and encouragement during the last three years and to Professor Le-Wu Lu for the inspiration and advice received from him during the early development of this work.

The work described in this dissertation was conducted as part of a general investigation into the plastic design of multi-story frames at Fritz Engineering Laboratory, Department of Civil Engineering, Lehigh University. Dr. Lynn S. Beedle is Acting Head of the Civil Engineering Department and Director of the Laboratory. This investigation was sponsored jointly by the Welding Research Council and the Department of the Navy with funds furnished by the American Institute of Steel Construction, American Iron and Steel Institute, Naval Ships Systems Command and Naval Facilities Engineering Command. Technical guidance
was provided by the Lehigh Project Subcommittee of the Structural Steel Committee of the Welding Research Council, under the chairmanship of Dr. T. R. Higgins. Their support is acknowledged.

The author gratefully acknowledges the assistance of the American Iron and Steel Institute who provided a doctoral fellowship so that a year of full time study could be devoted to this work.

Sincere appreciation is also extended to the author's many colleagues for their many criticisms and suggestions. Special thanks are due to Professor G. C. Driscoll, Jr. who encouraged the early development of this work and permitted it to be introduced at the Summer Conference on "Plastic Design of Multi-Story Frames" held in September 1965 at Lehigh University. Appreciation is also extended to Professors A. Ostapenko and T. V. Galambos, and to Drs. B. M. McNamee, E. Yarimci, P. F. Adams, W. C. Hansell and B. P. Parikh.

The manuscript was typed by Mrs. D. Eversley and Mrs. D. Kroohns and the illustrations were prepared by Miss. S. D. Gubich and Mr. J. M. Gera, Jr.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>1</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>1.1 The Basic Design Process</td>
<td>4</td>
</tr>
<tr>
<td>1.2 PΔ Effects and Instability</td>
<td>12</td>
</tr>
<tr>
<td>1.3 Preliminary Design Methods</td>
<td>14</td>
</tr>
<tr>
<td>1.4 Sway Analysis Methods</td>
<td>16</td>
</tr>
<tr>
<td>1.5 Purpose and Scope of the Dissertation</td>
<td>21</td>
</tr>
<tr>
<td>1.6 Definitions and Assumptions</td>
<td>22</td>
</tr>
<tr>
<td>1.6.1 Frame Layout</td>
<td>22</td>
</tr>
<tr>
<td>1.6.2 Loading Conditions</td>
<td>24</td>
</tr>
<tr>
<td>1.6.3 Secondary Failures</td>
<td>26</td>
</tr>
<tr>
<td>1.6.4 Materials</td>
<td>26</td>
</tr>
<tr>
<td>1.6.5 Application of the Sway Subassemblage Method</td>
<td>27</td>
</tr>
<tr>
<td>1.7 Summary of the Dissertation</td>
<td>28</td>
</tr>
<tr>
<td>2. THE SWAY SUBASSEMBLAGES IN AN UNBRACED MULTI-STORY FRAME</td>
<td>30</td>
</tr>
<tr>
<td>2.1 The Role of Subassemblages</td>
<td>30</td>
</tr>
<tr>
<td>2.2 Possible Plastic Hinge Locations and Failure Mechanisms</td>
<td>31</td>
</tr>
<tr>
<td>2.3 Sign Convention</td>
<td>35</td>
</tr>
<tr>
<td>2.4 The One-Story Assemblage at Level n</td>
<td>35</td>
</tr>
<tr>
<td>2.5 The Half-Story Assemblage at Level n</td>
<td>41</td>
</tr>
<tr>
<td>2.6 The Sway Subassemblages at Level n</td>
<td>44</td>
</tr>
<tr>
<td>3. THE RESTRAINED COLUMN IN A SWAY SUBASSEMBLAGE</td>
<td>47</td>
</tr>
<tr>
<td>3.1 Nature of the Restraint</td>
<td>47</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.2 Equations of Equilibrium and Compatibility</td>
<td>48</td>
</tr>
<tr>
<td>3.2 Moment-Rotation Relationship</td>
<td>50</td>
</tr>
<tr>
<td>3.4 Load-Deflection Equation for Constant Restraint Stiffness</td>
<td>52</td>
</tr>
<tr>
<td>3.5 Load-Deflection Behavior for Constant Restraint Stiffness</td>
<td>54</td>
</tr>
<tr>
<td>3.6 Load-Deflection Behavior for Variable Restraint Stiffness</td>
<td>58</td>
</tr>
<tr>
<td>3.6.1 Constant - Zero Restraint Stiffness</td>
<td>60</td>
</tr>
<tr>
<td>3.6.2 Constant - Constant Restraint Stiffness</td>
<td>61</td>
</tr>
<tr>
<td>3.7 Design Charts</td>
<td>67</td>
</tr>
<tr>
<td>4. RESTRANING COEFFICIENTS OF STEEL BEAMS AND COLUMNS</td>
<td>70</td>
</tr>
<tr>
<td>4.1 Initial Restraint</td>
<td>70</td>
</tr>
<tr>
<td>4.2 Initial Restraint Coefficients</td>
<td>72</td>
</tr>
<tr>
<td>4.3 Reduced Restraint Coefficients</td>
<td>81</td>
</tr>
<tr>
<td>4.3.1 Plastic Hinges Outside the Sway Subassemblage</td>
<td>82</td>
</tr>
<tr>
<td>4.3.2 Plastic Hinges Within the Sway Subassemblage</td>
<td>83</td>
</tr>
<tr>
<td>5. RESTRAINING CHARACTERISTICS OF COMPOSITE BEAMS AND STEEL COLUMNS</td>
<td>85</td>
</tr>
<tr>
<td>5.1 Flexural Behavior Under Combined Loads</td>
<td>85</td>
</tr>
<tr>
<td>5.2 Initial Restraint</td>
<td>87</td>
</tr>
<tr>
<td>5.3 Initial Restraint Coefficients</td>
<td>87</td>
</tr>
<tr>
<td>5.3.1 Slope-Deflection Coefficients</td>
<td>91</td>
</tr>
<tr>
<td>5.4 Reduced Restraint Coefficients</td>
<td>95</td>
</tr>
<tr>
<td>5.5 Ultimate Strength Behavior of Composite Beams Under Combined Loads</td>
<td>97</td>
</tr>
</tbody>
</table>
6. LOAD-DEFLECTION BEHAVIOR OF A STORY  101

6.1 Load-Deflection Curve of a Sway Subassemblage  101

6.1.1 Evaluation of \( M_r' \) in a Sway Subassemblage With Steel Beams  101

6.1.2 Evaluation of \( M_r' \) in a Sway Subassemblage With Composite Beams  106

6.2 Construction of a Typical Load-Deflection Curve  108

6.3 Load-Deflection Curve of a Story  111

7. FUTURE RESEARCH  113

7.1 Analytical Studies  114

7.2 Experimental Studies  116

8. SUMMARY AND CONCLUSIONS  119

9. NOMENCLATURE  122

10. FIGURES  124

11. REFERENCES  144

12. VITA  148
LIST OF FIGURES

1. Instability Under Combined Loads
2. Design Conditions for Unbraced Frames
3. Possible Plastic Hinge Locations and Failure Mechanisms
4. One-Story Assemblage
5. Half-Story Assemblage
6. The Sway Subassemblages
7. The Restrained Column in a Sway Subassemblage
8. Load-Deflection Curve of a Restrained Column With Constant Restraint Stiffness
9. Load-Deflection Curve of a Restrained Column With Constant - Zero Restraint Stiffness
10. Load-Deflection Curve of a Restrained Column With Constant - Constant Restraint Stiffness
11. Superposition of Load-Deflection Curves
12. Derivation of Initial Restraint Coefficients for Steel Beams
13. Possible Plastic Hinge Locations
14. Derivation of Initial Restraint Coefficients for Composite Beams
15. Derivation of Slope-Deflection Coefficients
16. Distribution of Bending Moments in the Sway Subassemblage
17. Construction of Load-Deflection Curve - Steel Beams
18. Construction of Load-Deflection Curve - Composite Beams
19. Construction of Load-Deflection Curve for a Story
ABSTRACT

An approximate method of analysis which is suitable for the combined load analysis of an unbraced frame on a maximum strength basis is presented in this dissertation. It is called the sway-subassemblage method of analysis and is particularly useful for developing the lateral-load versus sway-deflection curve of a story in the middle and lower stories of an unbraced frame.

It has been assumed that a preliminary design of the frame has been made, preferably by the moment balancing method. The preliminary design should provide not only the tentative beam and column sizes but also the distribution of axial forces in the columns corresponding to either the maximum lateral load capacity of the frame or the plastic mechanism load.

The method is based on the concept of sway subassemblages and uses directly the results of previous research on the strength and behavior of restrained columns permitted to sway. In the analysis, a story with known member sizes is subdivided into a number of sway subassemblages, each consisting of a restrained column and either one or two adjacent restraining beams. The restraining beams together with assumed realistic boundary conditions constitute the restraining system. The restraining
system represents the rotational restraint provided by the steel columns in the story as well as by the steel or composite steel-concrete beams. An analysis of each sway subassemblage is made to determine its load-deflection behavior using specially prepared design charts. The resulting load-deflection curves of all the sway subassemblages in a story are then combined to determine the complete load-deflection curve of the story. This curve may be obtained up to and beyond the deflection corresponding to the maximum load and mechanism load capacities.

The adequacy of the preliminary design may be determined on the basis of strength (maximum or mechanism load for example) or deflection (working load, maximum load or mechanism load for example).

The sway subassemblage method of analysis accounts for the reduction in strength of a frame due to PΔ effects. It also considers plastification of the columns including residual stresses, as well as plastic hinges in the beams. A recent pilot study on the ultimate strength behavior of composite beams under combined loads has provided experimental evidence that a combination of plastic analysis and ultimate strength theory may be used for the design of such beams. The approximate sequence of formation of plastic hinges in a story may also be determined from the analysis.
The sway subassemblage method as developed in this dissertation does not consider unbraced frames with significantly large initial sway deflections under factored gravity loads alone. The effect of differential column shortening on the strength and deflection of the frame is also neglected.
1. INTRODUCTION

This dissertation considers the theoretical development of an approximate method of analysis for unbraced multi-story frames which are subjected to combined loads. Throughout the dissertation the method of analysis will be referred to as the sway subassemblage method. The discussion in this Chapter will center initially on the required steps in the basic design process for unbraced frames and on how these steps are carried out. The available methods for executing these steps will then be discussed. It will be shown that a need exists for the further development of methods of analysis. The purpose and scope of this dissertation will then be presented. The chapter will conclude with a discussion of the restrictions regarding the frame considered in this dissertation, the applicable loading conditions, the frame material and the restrictions regarding the application of the sway subassemblage method.

1.1 The Basic Design Process

The direct design of an unbraced multi-story frame for the combined load condition is a problem of great complexity and is virtually impossible to perform with "exactness" for tall frames. For this reason a large number of approximate methods
have been developed which supposedly give reasonably good designs and at the same time simplify the work involved.

Ideally, a direct design method would be used to determine the final "exact" member sizes starting only with a knowledge of frame geometry, loading conditions and material properties and working towards certain strength, stiffness and economic criteria. Plastic design methods approach this ideal. However, the complex interrelation between strength and stiffness in problems concerning unbraced multi-story frames demands a method which would yield a direct solution to a non-linear problem. Such a method has not yet been found.

From a practical standpoint the complete design of an unbraced frame requires methods which will achieve a solution, step by step, from a gradually converging trial-and-error procedure. Such methods would require three well defined steps to complete each trial design cycle:

**Step 1.** The preliminary design; the selection of tentative beam and column sizes.

**Step 2.** The analysis; the determination of the adequacy of the members selected in Step 1 based on strength and stiffness.

**Step 3.** The revision; the revision of one or more members based on the results of the
1.1 analysis or on other factors such as economy. Any revision constitutes another preliminary design.

Because of the tedious, uneconomical and time consuming work involved in such a trial-and-error procedure, it is necessary to develop methods which will produce final designs in only one or two cycles. This imposes considerable demands on preliminary design methods which must determine the member sizes with a relatively high degree of accuracy. The demands placed on the methods of analysis depend to a considerable degree on the design philosophy; that is whether allowable stress or maximum strength criteria are employed. Analyses for strength and stiffness at working loads can be considerably less involved than those for maximum strength. Very little attention is usually given to rational means of making the necessary revisions. The designer often relies only on his design experience and intuitive ability. Quite often, if the indicated revisions are relatively minor, the resulting frame constitutes the final design, further analysis deemed unnecessary because of the conservative nature of other factors not considered in the analysis such as the stiffening effects of cladding and interior partitions.

Present design procedures for unbraced multi-story frames are based on the allowable stress concept.¹ That is, the
members selected are considered adequate for strength if, under working loads, the allowable stresses are not exceeded anywhere in the frame. Although many design procedures are used which vary from one office to another, a complete design based on the above three steps is rarely made. This is due to the complexity of performing an "exact" elastic analysis. Most methods of analysis are based on approximations and are derived from an assumed frame behavior under load. Usually, Steps 1 and 2 are combined into one operation. The well-known portal and cantilever methods (and variations of them) for wind loads are examples. Approximate methods are also used to determine the distribution of moments under gravity loads and to calculate the wind induced sway deflections at working loads.

From the point of view of the three steps required in the basic design process, these procedures constitute nothing more than preliminary design methods coupled with an approximate working load sway analysis. However, many years of design experience, built upon observed frame behavior have established procedures which result in satisfactory structures from the point of view of serviceability. The recent introduction of electronic computation has reduced many of the approximations involved but has not changed the basis for the design (allowable stress).
With the successful application of plastic design methods to the design of low building frames,\(^3\) and to the design of braced multi-story frames\(^4,5,6\) much interest has been aroused in the possibility of extending these methods to the design of unbraced multi-story frames. Plastic design methods employ the concept of the maximum or plastic strength of a structure as the basis for design. They are founded on the unique ductility exhibited by the structural steels and on the ability of structures to redistribute internal forces (moments) as plastification occurs. They usually result in more efficient use of material, a more uniform factor of safety, and what is equally important - relatively simple design procedures.\(^3\)

Investigations are also presently directed towards extending plastic design methods to composite steel-concrete structures.\(^7\) The resulting methods would require simple rules for determining the ultimate moment of resistance as well as the rotation capacity of a cross-section when the slab is in tension or compression.

The current specifications\(^8\) for the design of steel structures limit application of plastic design to one- and two-story rigid frames and to beams in multi-story frames where the columns have been designed by the allowable stress method. This limitation is a consequence of the well-known assumptions of the
simple plastic theory.\textsuperscript{9} Two assumptions stated in Ref. 9, which prevent direct application of simple plastic design to unbraced multi-story frames are:

"The deformations are so small that the equilibrium equations can be formulated for the undeformed structure (as in ordinary elastic analysis)"; and

"No instability will occur prior to the attainment of the ultimate load".

Both of these assumptions imply that the effects of gravity loads on the behavior of a frame can be neglected. Unfortunately, axial forces play a dominant role in the behavior of tall unbraced multi-story frames and cannot be ignored; this is especially so when the frames are subjected to combined gravity and wind loads. Designers are becoming increasingly aware of this fact with the current trend towards lighter and more economical structures.

The chief concern with the gravity loads is the magnitude of the additional overturning moment which can be produced in each story of a frame. As the frame sways under the action of the combined gravity and wind loads, the total gravity loads, $P$, above a story act through the story sway displacement, $\Delta$, to produce an additional overturning moment that the story must carry. This is commonly referred to as the $P\Delta$ moment. The effect on the load-deflection behavior of the frame is called
the PΔ effect. Because PΔ moments result from the gravity loads, the total wind shear in a story will be unchanged. Thus, it is clear that for a given value of combined ultimate loads (working load times the load factor) the required shear capacity will not only be a function of the plastic strength of its members but also of the sidesway stiffness of each story of the frame.

The significance of PΔ moments has not generally been recognized by designers using the allowable stress methods for 3 reasons:

1. The PΔ moments are relatively small at working loads, and at first yield,\(^1,10\)

2. Indirectly, the PΔ effect has been accounted for in design specifications by the use of an effective length concept\(^1,8\) and,

3. The stiffening effect of the exterior cladding and the interior partitions has, in the past, reduced the sidesway deflections of the frames in a building to lower values than the calculations have indicated; the PΔ moments therefore are even smaller.

With the trend in modern building designs towards light curtain-wall construction, larger areas of glass and re-
movable interior partitions the bracing effect from these sources is becoming small or unreliable. Consequently, the bare structural frame must then resist the total combined loads. Three conditions are therefore imposed on unbraced frames which are designed by plastic methods:

1. They must be able to resist the combined working loads within acceptable sidesway deflection limitations, \(^1,11\)

2. They must be able to resist the combined ultimate loads, and,

3. For efficient use of material, the shear capacity should not greatly exceed the required shear capacity under the combined ultimate loads.

From the previous discussion it is apparent that condition 1 will depend on the availability of a suitable method of calculating working load deflections. Conditions 2 and 3 will require a suitable method for calculating the maximum shear capacity of the frame which in turn is a function of the sidesway deflection corresponding to the maximum shear capacity. In effect, an analysis procedure is required which will predict the complete load-deflection behavior of the frame at least until the maximum shear capacity has been reached and preferably beyond. In keeping with one of the advantages of the plastic
methods stated earlier, the method should be relatively simple to apply. It may also be approximate, so long as it gives reasonably dependable results.

1.2 Effects and Instability

References 10, 12 and 13 contain extensive discussions of PA effects and instability. They also include many additional references. A further elaboration will not be attempted in this dissertation. However, in order to bring into sharper perspective the importance of sidesway deflections and to give background for later development, a brief review will be necessary.

Figure 1 illustrates the PA effects and instability under combined loads. It is assumed that an unbraced multi-story frame is subjected to proportional gravity and wind loads. Consider a story in the middle or lower regions of the frame. The story height will be taken as h and the relative sidesway deflection between the top and the bottom of the story is assumed to be \( \Delta \). It is also assumed that the story will eventually develop a plastic mechanism. It is further supposed that instability does not occur in another story prior to the formation of this collapse mechanism, and that the material exhibits elastic, elasto-plastic, perfectly plastic moment-curvature behavior.
If PΔ effects were absent, the wind load H (Fig. 1) would attain the value predicted by the simple plastic theory. The load-deflection curve for the story would then be curve O-a-b. This curve would be linear from point O until first yielding at point a.

The actual load deflection curve for the story is shown as curve O-c-d. This curve will be non-linear from point O and will reach a peak value at point c. The plastic mechanism of the story will form at point d where the load-deflection curve intersects the second-order rigid-plastic mechanism curve for the story. It is characteristic of those stories in unbraced multi-story frames where the member sizes are controlled by the combined loading condition that the maximum shear capacity (point c in Fig. 1) will be attained prior to the formation of the collapse mechanism.¹⁰,¹²,¹⁴

The important concept to gain from Fig. 1 is that the PΔ moments induced by the gravity loads acting through the side-sway displacements cause a significant reduction in the shear capacity of the frame. Also failure can occur by instability rather than by the formation of a failure mechanism.

It is quite apparent from this introductory discussion that any design method for unbraced multi-story frames subjected to combined loading which is based on the maximum shear capacity
of each story of the frame must satisfactorily account for PA effects. To be a complete design method it must enable a designer to execute each of the 3 steps in the basic design process (Art. 1.1) in order to arrive at a final design. Such a method would be of great value if it also allowed a final design to be made within only one or two cycles.

1.3 Preliminary Design Methods

Any method which results in a distribution of trial beam and column sizes throughout an unbraced multi-story frame constitutes a preliminary design. A wide variety of methods satisfy this condition. They vary in complexity from nothing more than educated guesses based on extensive experience and intuitive ability to more involved techniques relying upon only a knowledge of the frame geometry, material properties and loading conditions as well as an assumed distribution of forces within the frame. From this point of view any of the methods employing the allowable stress concept would be adequate for the preliminary design step in a 3-step design process based on maximum shear capacity.

However, relatively simple preliminary design procedures have recently been developed which consider the inelastic behavior of frames. The most successful are plastic moment
Plastic moment distribution is, in some respects, similar to elastic moment distribution. Plastic moment balancing is described in Ref. 10 as a "refined formulation of equilibrium for unbraced multi-story frames", where "the refinements are formulated to consider the influence on frame statics of sway deflection, finite joint size, and plastic girder mechanisms or restricted girder hinge patterns". Both of these methods employ equilibrium conditions, but in a different manner.

Plastic moment balancing (or plastic moment distribution) is ideally suited for the preliminary design of unbraced frames because it can include an approximate \( P_A \) effect. An initial sway deflection estimate is made and then the resulting \( P_A \) moments are included when equilibrium is established. The initial sway deflection estimate can be made (guessed) either of two ways; (1) on the basis of the expected sway deflection corresponding to the formation of a mechanism in each story, or (2) on the basis of the expected sway deflection at the maximum shear capacity of each story. Either way, a sway analysis should be performed to verify the initial sway estimate. In addition, the sway deflection should be calculated at working loads to complete the analysis. The need for a relatively simple method of predicting all three sway deflections is therefore indicated.
The moment balancing method described in Ref. 10 is also well suited to perform part of the last step in the basic design process, Step 3 (Art. 1.1). If a new sway deflection estimate is indicated, the resulting revised PA moments, when added to the moments produced by the combined loads alone, dictate directly the revised beam sections. The revised column sizes are then dictated by the formulation of equilibrium, and by consideration of stability as before.

The remaining part of the last step in the basic design process is mainly concerned with economy. Although a final design may be made on the basis of strength and stiffness, it may not be the most economical. The moment balancing method cannot in itself, lead to the most economical frame although some steps in this direction have been indicated in Ref. 10.

A preliminary design by moment balancing may also be extended to unbraced multi-story frames utilizing composite beams. Reference 10 suggests this possibility. Such an extension requires only the knowledge of the plastic moment capacity of composite beams which are subjected to positive and negative end moments as well as transverse loads.

1.4 Sway Analysis Methods

The importance of performing a sway analysis as a major step in the complete design, on a maximum strength basis,
of an unbraced multi-story frame subjected to combined loads, has now been established. The obvious question to be considered is: Are there relatively simple methods available which will predict the sway deflection of each story of such a frame at working loads, at design ultimate loads, at maximum shear capacity and at the formation of a mechanism? Particularly, methods are required which are suitable for use in those stories where the member sizes are controlled by the combined loading case.

An introduction to the literature on the analysis and design of unbraced multi-story frames may be found in Refs. 10 and 13. There is unanimous agreement that sway induced effects must be considered. There is also considerable agreement that an "exact" analysis which is suitable for design purposes is not in sight. Surprisingly, little has been done to provide an approximate analysis which would predict the load-deflection behavior of a story in an unbraced multi-story frame.

Much research has been devoted to frames of the order of three stories or less. Either a compatibility analysis or a second-order elastic-plastic analysis is used to obtain the load-deflection curves. The compatibility analysis is sufficiently involved that it has not been applied to other than very simple frames. 13

Parikh, 17 applied a second-order elastic-plastic anal-
ysis to unbraced multi-story frames up to 25 stories in height. Although plastification of beams and columns was partially included, the effects of strain hardening were not considered. However, the analysis did consider the formation of plastic hinges, the effect of axial loads on the plastic moment capacity and stiffness of the columns, axial shortening and instability of the columns including residual stresses and the PÅ effect. The loads were assumed proportional for the combined load case.

Parikh's analysis began with members selected from a previous preliminary design and predicted the load-deflection curve for each story of the frame. The curves all terminated at the load corresponding to the story with the smallest shear capacity. This was a consequence of the divergence occurring in the iterative calculations when a stable equilibrium could not be found somewhere in the frame. As a result, the shear capacity of the remaining stories was unknown. If the highest load obtained corresponded to failure of a story by instability, then the mechanism load and deflection for that and all other stories are unknown. Although the maximum shear capacity of the frame can be determined, the method is not suitable for checking the sway estimates used in the preliminary design except, possibly, for the story which failed first.

The analysis in Ref. 17 relies on electronic computa-
tion, and is fairly complex. Even though an efficient computational technique was used it was not of a general formulation and considerable demands would be placed on the capacity of a digital computer when analyzing other moderately large multi-story frames. No attempt was made in Ref. 17 to include frames with composite beams.

Many investigators have been interested in designs which approximately satisfy sway conditions at one or two points on the load-deflection curve. Of the four mentioned earlier in this article, those at working load and at mechanism are used extensively.

Heyman\textsuperscript{18,19} considered the design of unbraced multi-story frames, although little attention was paid to the problem of instability. It was suggested that stability calculations would not be realistic because of the "enormous" amount of stiffening due to the cladding. The analysis was therefore based on simple plastic theory, with checks to assure compliance with the relevant assumptions. In particular, methods were developed

"to give reasonable estimates of working load deflections and bending moments for the resulting design, and the design may be modified if necessary in the light of these estimates".

Although the girders were selected in this method on the basis of the calculated full plastic moments, the columns were propor-
tioned to remain elastic when the calculated full plastic moments and axial thrust were applied.

Holmes and Gandhi\textsuperscript{19,20} introduced a two stage procedure using an analysis which was assisted by electronic computation. Columns and girders were proportioned according to the simple plastic theory in the first stage and then increased if necessary in the second stage to allow for sway deflection and instability effects. The analysis initially assumes that inflection points occur at mid-height of the columns. This assumption is then later modified to account for unequal column end moments. The mechanism condition establishes the maximum frame capacity.

Stevens\textsuperscript{21,22} recognized that the emphasis in design must be placed not only on strength and safety but also on satisfactory deflection behavior at working loads. He suggested that design methods must meet two conditions; (1) deformations at working loads must be less than some acceptable value, and (2) the critical loading condition must also be less than some acceptable value. He suggested that condition (2) must also include some limit on deformations at the maximum load.

Gent\textsuperscript{23} also suggested that adequate design procedures must meet strength, stability and deflection criteria. He noted that complete design methods which would consider all three criteria are at an elementary stage of development.
1.5 Purpose and Scope of the Dissertation

It is the purpose of this dissertation to develop an approximate analytical method which will predict the load-deflection curve of each story in the middle and lower stories of an unbraced multi-story frame (Art. 1.6.5). The method will be applicable to the combined load condition only (Art. 1.6.2). This analytical method will be particularly useful for performing Step 2 (Art. 1.1) of the basic design process. The assumption is made throughout the development that a preliminary design of the frame (Step 1) has already been made, preferably by the moment balancing method.10,16

The basic concepts of the approximate analytical method were previously discussed by the writer in Refs. 24 to 28. Because the method is based on the idea of sway subassemblages29 and uses directly the results of recent studies of restrained columns permitted to sway30 the method is referred to as the sway subassemblage method. The load-deflection curve of a story is obtained through the application of a semigraphical analytical procedure using specially-prepared design charts.28 The sway subassemblage method accounts for the \( P \Delta \) effect on both the columns and the story. It also considers plastification of the columns including residual stresses,26 as well as plastic hinges in the beams. The approximate sequence of formation of plastic hinges in a story may be obtained from the analysis.
The sway subassemblage method may also be extended to develop procedures for performing Step (3) (Revision) in the basic design process. Such an extension will not be included in the scope of this dissertation.

1.6 Definitions and Assumptions

1.6.1 Frame Layout

Unbraced multi-story frames will be defined in this dissertation as that class of plane rectangular frames, more than one story in height and of one or more bays in width, which derive their resistance to in-plane forces from the bending resistance of the frame members themselves. These frames may be constructed of steel columns and either steel beams or composite steel-concrete beams. The steel members may be rolled or welded shapes such as wide-flange and I sections or other sections with a similar distribution of material over the cross-section. Welded hybrid shapes may also be used. The slabs of composite beams will be assumed to meet the requirements of Ref. 8 and are further assumed to be continuous and continuously reinforced at all interior columns. The shear connection will be assumed to extend throughout the length of each beam. The span lengths of the beams will be taken as the distance between the centroidal axes of the columns. The length of each column will be equal to the story height and will be taken as the distance between the centroidal axes of the beams, which will be assumed co-linear.
The following additional assumptions will also be made:

1. The frames are regular in geometry. No "missing" columns or beams will be permitted and column footings are assumed to be at the same elevation.

2. The connections between steel beams and steel columns are rigid, and may be made by welding or bolting. The joint is assumed to transmit the full plastic moment capacity of the beams and the reduced plastic moment capacity of the columns without local buckling or excessive distortion.

3. No bracing or cladding is used in the plane of the frame to resist sway deflection.

4. The column bases are assumed to be fixed in the plane of the frame.

5. The minor axis of each member is assumed to lie in the plane of the frame which is a plane of symmetry.

6. Biaxial bending of columns does not occur.

7. Axial and curvature shortening of beams and columns will be neglected.
8. Concrete cannot resist tensile forces.

9. Complete interaction is assumed for composite steel-concrete beams.

1.6.2 Loading Conditions

Unbraced multi-story frames may be subjected to two types of static loads; (1) gravity, and (2) wind. Gravity loads are assumed to be vertical uniformly distributed or concentrated loads applied to the beams by the floor system. These loads consist of both dead (DL) and live (LL) loads. Wind loads (WL) are distributed horizontal loads applied by the exterior wall system and assumed to be concentrated at the exterior joints of the frame. The static loading conditions can be represented by the following cases:

1. DL + LL (all beams)

2. DL + LL (some beams - - checkerboard)

3. DL + LL (all beams) + WL

4. DL + LL (some beams) + WL

Cases 1 and 2 constitute the gravity load conditions while 3 and 4 are the combined load conditions. The design ultimate values of load are obtained by multiplying the working loads
by a load factor (LF). Where indicated, load factors will be chosen in accordance with those established in Ref. 31 namely:

Gravity load conditions: LF = 1.70
Combined load conditions: LF = 1.30

All loads are assumed to be either horizontal or vertical and lying in the plane containing the minor axes of the members. Thus the loads and the frame form a co-planar system.

This dissertation will consider only the combined load conditions, Cases 3 and 4. It is unlikely that a practical frame, subjected to both gravity and wind loads, would be loaded proportionally. It is more likely that the applied gravity loads will remain virtually unchanged as the wind loads are applied. Therefore, the following non-proportional loading sequence will be assumed:

(1) The factored uniformly distributed gravity loads, $1.30w$, are applied first, where $w$ is the working load value.

(2) The factored wind loads are then applied, increasing monotonically from zero to the design ultimate load $H$.

The probability that full gravity live load plus wind load will not act together is accounted for by live load reduc-
1.6.4

Section factors\(^{31}\) which are applicable only to the live gravity loads. In general, \(w\) will be different for each beam. In addition, the column loads will not likely be in equilibrium with the total beam loads. This condition will be acceptable when performing an analysis by the sway subassemblage method.

1.6.3 Secondary Failures

It will be assumed that all secondary failures of the frame are prevented. These include lateral-torsional and local buckling of the steel sections as well as splitting and diagonal tension failures of the concrete slab and failure of the shear connection. These failures may be prevented by adequate bracing, minimum width-thickness and depth-thickness ratios of projecting steel elements, adequate slab reinforcement and a sufficient number of shear connectors to develop the flexural capacity of the member.

1.6.4 Materials

Only ASTM A36 and A441 steels are considered. The design charts in Ref. 28 have been prepared for A36 steel \((\sigma_y = 36\ \text{ksi})\) but can be used with little modification for A441 steel. This restriction may not ultimately be necessary but is imposed here in view of the fact that research into the behavior of the higher strength steels is still in progress.
1.6.5 Application of the Sway Subassemblage Method

The design of an unbraced multi-story frame must consider both the gravity load and the combined load conditions. The design must also consider wind loads from both directions. It will be found that the gravity load conditions will control the selection of beam and column sizes for a limited number of stories at the top of the frame. The number of stories comprising this region is not definite and will depend on many factors such as frame geometry, material properties, load factors, and live load reduction factors. The number of stories may also vary from one bay to another across the frame. The combined load conditions control the selection of beam and column sizes in the middle and lower stories of the frame. Between the regions controlled by the gravity load and combined load conditions there will be a transition zone where both may govern in any one story. Figure 2 shows a typical distribution of the three regions for a three-bay unbraced multi-story frame.

It will be assumed in this dissertation that the upper
region and to some extent the transition zone has been analyzed by other methods. The sway subassemblage method will be particularly useful for analyzing the middle and lower stories where the member sizes are dictated by the combined load conditions. The method may also be used in the transition zone but with a possible lower degree of accuracy.

Finally, it will be assumed that the frame and its gravity loads are sufficiently symmetrical that the effects of initial sidesway deflection may be ignored.

1.7 Summary of the Dissertation

Chapter 2 begins with a discussion of the role of subassemblages. It is shown that with certain simplifying assumptions, a single story of the frame may be successively reduced to an assemblage of beams and columns, then to sway subassemblages each containing a restrained column. Chapter 3 discusses the behavior of restrained columns and illustrates the method of obtaining the load-deflection curve when the restraint is known. Chapters 4 and 5 discuss the restraint provided by the steel columns and composite or non-composite beams in a story. Chapter 6 illustrates the method of obtaining the load-deflection behavior of a story from the behavior of the restrained column and the restraining system, and discusses some of the
assumptions used in the analysis. Chapter 7 discusses the future research required and Chapter 8 summarizes the results of this study.
2. THE SWAY SUBASSEMBLAGES IN AN UNBRACED MULTI-STORY FRAME

2.1 The Role of Subassemblages

A subassemblage of a multi-story frame is a limited assemblage of beams and columns analyzed remote from the frame, the behavior of which, under realistic loads and boundary conditions, can be assumed to approximate the true behavior of that portion of the frame. The subassemblage is usually determined from the point of view of ease of analysis. The case for the study of subassemblages has been clearly expressed in Ref. 30 as follows:

1. "The analysis and design of an entire multi-story, multi-bay frame is almost prohibitive if stability and deflection effects are predominant considerations;" and

2. "Subassemblages can be used in the analysis and design of individual members and of member groups, when conservative assumptions are made for end conditions".

The use of subassemblages to assist in the analysis and design of frames is already widespread. For example:

1. The portal and cantilever methods for wind analysis are subassemblage methods; a subassemblage being bounded by assumed points
of inflection above and below, and to
left and right of a joint.

2. The determination of the effective length
of columns in braced and unbraced multi-
story frames for use in allowable stress
design procedures is based on a sub-
assemblage analysis.\textsuperscript{1,8}

3. A proposed method of designing braced
multi-story frames by plastic methods
is also based on the use of sub-
assemblages.\textsuperscript{4,5,6,34,35}

The configuration and extent of a subassemblage de-
pends on the design philosophy, the available methods of anal-
ysis and whether the frame is braced or unbraced. The term
\textit{sway subassemblage} will define a particular configuration of
subassemblage which will be found useful in predicting the load-
deflection curve of a story in an unbraced multi-story frame.
This chapter will be concerned with developing the sway sub-
assemblages that will be used in the sway subassemblage method
of analysis.

2.2 \textbf{Possible Plastic Hinge Locations and Failure Mechanisms}

Consider the unbraced multi-story frame shown in
Fig. 3(a). For simplicity, only the centroidal axes of the members are shown. The factored distributed gravity loads, \( 1.30 \, w \), are assumed to be applied first where \( w \) is the working load value. In Fig. 3(a) the subscripts, AB, --- etc., refer to the particular beam. The factored wind loads are then assumed to increase monotonically from zero to their design ultimate value \( H \). The subscripts 1, 2, --- etc., refer to the level number. As previously discussed (Art. 1.6.2) this loading sequence will likely be more realistic than proportional loading for practical frames. In the middle and lower stories of the frame, the beam and columns will likely be stiff enough that under the factored gravity loads alone the joint rotations will be negligibly small. Consequently, the beams will behave as fixed-ended for gravity loads. The resulting distribution of bending moments in the beams and columns in the vicinity of level \( n \) has been shown in Fig. 3(b). The initial application of wind loads introduces additional bending moments such as those shown in Fig. 3(c). When these two bending moment diagrams are combined, the bending moments at the leeward ends of the beams will increase while those at the windward ends will decrease. The leeward ends of the beams are therefore the potential locations for the first plastic hinges. This will be true for both composite and non-composite beams. Similarly, the bending moments at the ends of certain columns will initially increase. Since the columns in
2.2

this region of a frame will likely be bent in a double curvature configuration\textsuperscript{17} the first plastic hinges can also form at the ends of these columns. As the wind loads are increased additional plastic hinges will form at the ends of other columns and elsewhere in the beams. Finally, a story will fail either by instability or by the formation of a mechanism.

Figures 3(d) and 3(e) show two possible failure mechanisms which can result. In a weak-girder, strong column design\textsuperscript{32} the column strengths may be sufficient to force all plastic hinges to develop in the beams. Similarly, in a weak-column, strong beam design\textsuperscript{32} the beam strengths may be sufficient to force all plastic hinges to develop in the columns. The resulting mechanisms may be called either a combined mechanism (Fig. 3(d) or a sway mechanism (Fig. 3(e). A combined mechanism may also be formed with plastic hinges in the beams and the columns. Such a mechanism would result if the design was between the extreme cases cited.

In a well-proportioned regular frame, the member sizes in a region containing the middle and lower stories will likely increase at a relatively uniform rate with increasing distance from the top of the frame. Since the gravity loads within each bay will likely be constant in these regions of the frame, the beams will increase in size due to the increasing wind and PA moments which must be carried by the lower stories. The columns
will increase in size due to the increasing wind and PΔ moments as well as the accumulation of the gravity loads on the beams. Although the wind loads may not be uniformly distributed over the height of the frame, and the sway index will not likely be uniform for each story, the variation over a limited number of stories could be considered small for most frames. As a result, if level n is within this region (Fig. 3) the load deflection behavior at levels n+1, n, and n-1 could be expected to be nearly the same. Consequently, it can be assumed that the behavior of the frame in the region of these several stories can be represented by the behavior of a small assemblage which contains level n.

At this point therefore, an assemblage of beams and columns at level n can be isolated from the frame. The equilibrium and compatibility conditions at the boundary can be approximated conservatively. There are two principle approaches which can be followed in choosing the assemblage:

1. Isolate the beams along two adjacent levels and the columns between those levels, or

2. Isolate the beams along one level and isolate a certain length of each column above and below that level.
2.4

It will be found easier to approximate the boundary conditions if the second approach is used because of the available studies of restrained columns permitted to sway.\textsuperscript{30} The resulting assemblage will then consist of the beams at level \text{n} which forms the boundary between stories \text{m} and \text{m+1} (Fig. 2), plus a portion of each column in these two stories.

2.3 \textbf{Sign Convention}

The sign convention which will be adopted in this dissertation has been stated in Ref. 30 and may be summarized as follows:

1. External moments acting at a joint are positive when clockwise.

2. Internal moments acting at a joint are positive when counterclockwise.

3. Moments and rotations at the ends of members are positive when clockwise, and

4. Horizontal shear is positive if it causes a clockwise moment about the opposite joint.

2.4 \textbf{The One-Story Assemblage at Level \text{n}}

Parikh\textsuperscript{17} studied the locations of the inflection
points in the columns in stories 5 and 15 of a regular, 24 story, 3 bay, unbraced, unsymmetrical steel frame (Frame C of Ref. 5). These stories were immediately below levels 20 and 10 respectively. If \( h \) equals the story height of Frame C, then the following results were obtained in the study:

<table>
<thead>
<tr>
<th>Location of Inflection Points</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Table 1</strong></td>
</tr>
<tr>
<td><strong>Distance Below Level Above</strong></td>
</tr>
<tr>
<td><strong>Story</strong></td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>15</td>
</tr>
</tbody>
</table>

Three observations are significant:

1. The average position of the inflection points did not vary appreciably in 10 stories. The variation was only about 2% of the story height.

2. The maximum variation in the position of the inflection point across a story was relatively small; about 10% of the story height.

3. All of the inflection points were at or above mid-height of the stories.
These results were not unexpected. Consider the middle and lower stories of a tall unbraced multi-story frame such as the one discussed above. Suppose that the frame is well proportioned and behaves as suggested in Art. 2.2. The following general statements may then be made:

1. The gravity load moments will be relatively small compared with the plastic moment capacities of the members.

2. The relative distribution of the stiffnesses of the beams and columns at a joint will be nearly the same for several consecutive joints along one column.

3. The distribution of column stiffness above and below a joint will be nearly the same.

Under these conditions and assuming that the joint rotations are nearly zero under gravity loads alone, the inflection points in the columns will lie approximately at mid-height of each story above and below level \( n \). As wind loads are applied the inflection points must shift upward to account for the greater wind shear in the story below. If the elastic and inelastic behavior of several consecutive stories containing level \( n \) are nearly the same then the inflection points must
remain at a relatively constant position for all values of monotonically increasing wind loads.

The 10% maximum variation in the position of the inflection points across the two stories of the frame studied in Ref. 17 was likely caused by small differences in the behavior of the frame in the region of these stories. The distribution of moments in stories 5 and 15 corresponds to general instability near the top of the frame under the proportional gravity and wind loads. This instability load was less than the maximum load capacities of stories 5 and 15. Although a considerable number of plastic hinges had formed in story 15, only one had developed in story 5. The pattern of plastic hinges was also slightly different in the stories above and below stories 5 and 15. If the design of this frame were to be improved somewhat, the variation in the position of the inflection points could be expected to decrease. It would also be expected from the previous discussion that the inflection points would remain above mid-height in each story.

On the basis of this brief analysis, it will be assumed in the following development that the point of inflection in each column above and below level n (Fig. 3(a)) is at mid-height of the story. It will be shown in Art. 2.5 that this assumption is conservative providing the behavior of the frame reasonably
approximates the ideal behavior assumed in Art. 2.2. Consider again the frame shown in Fig. 3(a). A one-story assemblage can now be isolated from the frame by passing horizontal cuts through the assumed inflection points above and below level \( n \). The resulting one-story assemblage is shown in Fig. 4. Also shown in this figure are the forces acting on the members and the resulting deformations. Center-to-center beam spans are shown as \( L_{AB} \), etc., where the subscripts refer to the bay in which the beam occurs. The lengths of the half-story columns above and below level \( n \) are designated \( h_{n-1}/2 \) and \( h_{n}/2 \). The total shear between levels \( n \) and \( n-1 \) is \( \Sigma H_{n-1} \). Similarly, the total shear between levels \( n \) and \( n+1 \) is \( \Sigma H_{n} \) where

\[
\Sigma H_{n} = \Sigma H_{n-1} + H_{n} \quad (1)
\]

and \( H_{n} \) is the concentrated wind load at level \( n \). The constants \( \lambda_{A} \), \( \lambda_{B} \), etc., define the distribution of the total wind shear force to each column in a story. It will be assumed that these constants have the same value in each story above and below level \( n \). Referring again to Fig. 4, the sideways displacement of the top of each column above level \( n \) relative to level \( n \) is assumed to be equal to \( \Delta_{n-1}/2 \). Similarly, the sideways displacement of each column below level \( n \) is equal to \( \Delta_{n}/2 \). These assumptions are a consequence of the previous discussion of the
behavior of the frame in the region of level \( n \). The axial forces in the columns above level \( n \) are designated as \( P_{n-1} \) and are assumed to remain constant for all values of \( H \). For a particular column, \( P_{n-1} \) is calculated as the algebraic sum of the following loads:

1. The total gravity loads coming from the tributary length of each beam connected to the column above level \( n \), taking into account the live load reductions \( 31 \) for the columns.

2. The shear forces at the two column faces which come from each beam connected to the column above level \( n \). These forces are to be associated with the design ultimate wind moments and the \( P \Delta \) moments in each beam, which were assumed in the preliminary design.\(^{10}\)

The axial loads \( P_n \) in the columns below level \( n \) are also to be calculated in a similar manner but they must include the gravity loads and shear forces from the beams at level \( n \).

The analysis of the frame at level \( n \) has now been reduced to the analysis of the one-story assemblage shown in Fig. 4.
It will be assumed that the load-deflection behavior of the one-story region of the frame which is symmetrical about level \( n \) can be represented by the load-deflection behavior of this one-story assemblage. At this point, it will no longer be necessary to consider the analysis of frame as a whole. Instead, the analytical method will consider only the load-deflection behavior of the one-story assemblage.

2.5 The Half-Story Assemblage at Level \( n \)

An "exact" analysis of the load-deflection behavior of the one-story assemblage would still be a formidable problem especially if the assemblage contained many bays. Even if such an analysis could be carried out it would not likely be favored by designers. Furthermore, application of the results to the frame itself would not be exact but would reflect the approximations already used in developing the one-story assemblage. For this reason, an approximate analysis of the one-story assemblage would be satisfactory providing it gave dependable results and was conservative. It is proposed that the sway subassemblage method can be used for such an approximate analysis.

Consider again the one-story assemblage of beams and columns at level \( n \) which is shown in Fig. 4. Assume now that the shear forces \( \Sigma H_{n-1} \) and \( \Sigma H_n \) are replaced by shear forces \( \Sigma Q_{n-1} \) and \( \Sigma Q_n \). Previously, the shear forces \( \Sigma H_{n-1} \) and \( \Sigma H_n \) were
used in a restricted sense. That is, the maximum value of $\Sigma H_n$, for instance, was equal to the smallest value of

1. The maximum horizontal shear capacity (instability or mechanism load) of the first story in the frame to fail, or

2. The sum of the factored wind loads for the story immediately below level $n$.

A similar restriction was placed on the maximum value of $\Sigma H_{n-1}$. The shear forces $\Sigma Q_{n-1}$ and $\Sigma Q_n$ will not be subjected to these restrictions. The maximum values of these forces will be limited only by the maximum horizontal shear capacity of the one-story assemblage.

It will now be necessary to find the relationship between the horizontal shear forces, $\Sigma Q$, and the sway deflections, $\Delta$. Before proceeding further, it is apparent that the one-story assemblage in Fig. 4 may be simplified. Each column above level $n$ applies a vertical force, a horizontal shear force and a bending moment to the joint at the base of the column. The horizontal shear forces $\lambda \Sigma Q_{n-1}$ at each joint may be combined with the applied force $Q_n$ to give a total force at level $n$ equal to $\Sigma Q_n$ where

$$\Sigma Q_n = \Sigma Q_{n-1} + Q_n$$ (2)
2.5

The bending moment at each joint will have a magnitude of (Art. 2.3)

\[ M_{n-1} = -{(\lambda \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{n-1} \frac{\Delta_{n-1}}{2}} \]  

(3)

Figure 5 shows the simplified half-story assemblage together with the forces and bending moments which are applied by the columns above level \( n \).

It is apparent from the previous discussions that both \( \Sigma Q_n \) and \( M_{n-1} \) will increase monotonically while all other loads and forces will remain constant. From statics, the internal moments, \( M_n \), in the columns immediately below level \( n \) will be given by (Art. 2.3)

\[ M_n = -{(\lambda \Sigma Q_n) \frac{h_n}{2} - P_n \frac{\Delta_n}{2}} \]  

(4)

Now, referring to Art. 2.4, the subscripts \( \lambda \) were assumed to have the same value in each story above and below level \( n \). In addition, \( h_n \) was assumed to be approximately equal to \( h_{n-1} \). Since \( \Sigma Q_n > \Sigma Q_{n-1} \) and \( P_n > P_{n-1} \) then if \( \Delta_n \) is approximately equal to \( \Delta_{n-1} \), \( M_n \) will likely be greater than \( M_{n-1} \). In fact, if \( h_n > h_{n-1} \) and \( \Delta_n > \Delta_{n-1} \) then \( M_n \) will always be greater than \( M_{n-1} \). With the ideal behavior of the middle and lower stories
of a well proportioned frame which was assumed in Art. 2.2, it can be conservatively assumed that $M_{n-1}$ is equal to $M_n$ and acts in the same sense. The previous assumption in Art. 2.5 that the inflection point will be at mid-height of the story now becomes conservative providing the actual inflection points are at or above mid-height. Therefore, the load-deflection behavior of the half-story assemblage in Fig. 5 will be assumed to predict conservatively the load-deflection behavior of the one-story assemblage in Fig. 4.

2.6 The Sway Subassemblages at Level $n$

To facilitate the load-deflection analysis of the half-story assemblage shown in Fig. 5, it will be necessary to further subdivide this assemblage into a number of subassemblages. Assume now that the total shear force $\Sigma Q_n$ is distributed to each column such that for any magnitude of sway deflection $\Delta_n/2$ no axial forces exist in the beams at level $n$. (Disregard the effect of gravity loads for the present). For columns A, B, --- etc., these shear forces will be designated $Q_{nA}$, $Q_{nB}$; --- etc. It should then be apparent that, in general $Q_{nA} \neq \lambda_A \Sigma Q_n$, $Q_{nB} \neq \lambda_B \Sigma Q_n$, etc. As a result of this assumption, each column top in a half-story assemblage will be restrained only by the rotational restraint provided by the adjacent rigidly attached beams. The opposite ends of these beams will be re-
strained by the adjacent columns. It is now possible to define the subassemblages in the half-story assemblage.

The simplest subassemblage would consist of one column plus the adjacent beams at the column top. Such an arrangement is shown in Fig. 6. Because the subassemblages can sway laterally under the applied loads $Q_n$, they will be called sway subassemblages. In each sway subassemblage the beams are considered as the restraining members which provide rotational restraint to the top of the column. There are 3 types of sway subassemblages which are possible in a half-story assemblage and they may be defined as follows:

1. **Windward Sway Subassemblage (Fig. 6(a))**: That portion of the half-story assemblage (Fig. 5) containing the windward column plus the adjacent beam at level $n$.

2. **Interior Sway Subassemblage (Figs. 6(b) and 6(c))**: That portion of the half-story assemblage (Fig. 5) containing an interior column plus the two adjacent beams at level $n$.

3. **Leeward Sway Subassemblage (Fig. 6(d))**: That portion of the half-story assemblage (Fig. 5) containing the leeward column plus the adjacent beam at level $n$. 
It is apparent from the above definitions and from an examination of Fig. 6 that each beam in a half-story assemblage will always be common to two sway subassemblages. For this reason each sway subassemblage must be analyzed as a separate structure distinct from the other members comprising the half-story assemblage. However, rotational restraints will be imposed at the free ends of the beams in each sway subassemblage in order to account for the restraining effects of the members outside the sway subassemblage. These rotational restraints are shown in Fig. 6 as springs at the ends of the beams. Since axial forces in the beams will be zero for all values of sway, the ends of the beams in each sway subassemblage can be assumed supported by rollers. The bottom of the column is assumed pinned (inflection point).

It should be apparent that if the load-deflection (Q vs. Δ) curves for all of the sway subassemblages can be accurately obtained, then they may be combined in some way to give the load-deflection curve for the half-story assemblage. The problem has thus been reduced to the load-deflection analysis of each sway subassemblage.
3. **THE RESTRAINED COLUMN IN A SWAY SUBASSEMBLAGE**

3.1 **Nature of the Restraint**

In general, a sway subassemblage will initially sway to the right or left when the gravity loads (DL+LL) are applied to the beams. It will be found convenient in this Chapter to neglect the effect of gravity loads so that the initial sway will be zero. The effect of gravity loads will be considered in Chapter 6.

From the discussion in Art. 2.6 it is apparent that the column in a sway subassemblage is able to resist shear forces $Q_n$ (Fig. 6) because of the rotational restraint which is applied to the top of the column by the adjacent beams. This rotational restraint is a function not only of the stiffnesses of the adjacent beams but also of the stiffnesses of all the other beams and columns in the half-story assemblage, whose actions are represented in Fig. 2.6 by restraints at the free ends of the beams in the sway subassemblage. The load-deflection behavior of a sway subassemblage will then be defined by the load-deflection behavior of the restrained column in the sway subassemblage providing that the rotational restraint at the top of the column is known for all values of sway deflection.
The study of restrained columns in this Chapter will consider the effect of the rotational restraint in two steps. First, the load-deflection behavior of a restrained column will be formulated when the rotational restraint is linearly elastic for all values of sway deflection. Then, the load-deflection behavior will be studied when the rotational restraint is allowed to decrease at discrete intervals of sway deflection. The results of this study can then be applied to determine the load-deflection behavior of restrained columns in a sway subassemblage where the beams initially provide elastic rotational restraint which then decreases at discrete intervals of sway deflection with the formation of plastic hinges.

3.2 Equation of Equilibrium and Compatibility

The restrained column in a sway subassemblage is shown in Fig. 7(a). The column is subjected to a constant vertical force, \( P_n \) (Art. 2.4) as well as monotonically increasing lateral force, \( Q_n \), and moment \( M_n \). The rotational restraint at the column top is represented by a spring which applies a restraining moment, \( M_r \). The spring is assumed to have a constant restraint stiffness so that the rotational restraint will be linearly elastic for any value of sway deflection.

The forces acting on the restrained column are shown again in Fig. 7(b) together with the resulting deformations.
From statics the moment at the upper end of the column will be given by (Art. 2.3)

\[ M_n = - \left[ Q_n \frac{h_n}{2} + P_n \frac{\Delta_n}{2} \right] \]  

(5)

The external moment applied to the column top (from the column above level \( n \)) was previously assumed to be equal to \( M_n \) and acting in the same sense (Art. 2.5). Equilibrium of moments at the joint then requires that

\[ 2M_n + M_r = 0 \]  

(6)

For small angles the rotations \( \theta, \gamma \) and \( \Delta_n / h_n \) in Fig. 7(b) are all related by the compatibility condition (Art. 2.3)

\[ \frac{\Delta_n}{h_n} = \theta - \gamma \]  

(7)

where \( \theta \) defines the column top (or joint) rotation and \( \gamma \) the column chord rotation. The angle \( \Delta_n / h_n \) is commonly referred to as the deflection index or story drift.

The load-deflection relationship, \( Q_n \) versus \( \Delta_n / 2 \) of the restrained column with constant restraint stiffness can be determined by solving Eqs. 5 to 7 together with the known moment-
rotation relationship, $M_n$ versus $\gamma$, for the restrained column.

3.3 Moment-Rotation Relationship

For small values of the deflection index it can be assumed that the restrained column in Fig. 7 is axially loaded by the vertical forces $P_n$. The column will then be subjected to single curvature bending under the system of forces shown in Fig. 7(b). However, this column is only the upper half of the column in the story below level $n$ which (under the assumption regarding the position of the inflection point) will be subjected to symmetrical double curvature bending. The moment-rotation relationship for such a column can be calculated following the procedures described in Ref. 36 or it can be obtained from curves such as those shown in Charts III-1 to III-7 of Ref. 25 for specified values of the axial load ratio $P_n / P_y$ and slenderness ratio $h_n / r_x$. $P_y$ is the axial load corresponding to full yielding of the column cross-section and $r_x$ is the radius of gyration of the column for strong axis bending. The charts shown in Ref. 25 give the moment-rotation curves for columns which have moment applied at one end, are pinned at the other end (end moment ratio $q = 0$) and are of length $h_n$. These curves can be used for columns which are bent in symmetrical double curvature ($q = 1.0$) where the half-length is $h_n / 2$, by using an equivalent slenderness ratio equal to one-half the actual slenderness ratio of the column.
The basic information which was used to prepare the moment-rotation curves was column deflection curves (CDC's) and moment-curvature-thrust relationships (M-Ø-P) which are both discussed in Ref. 36. A numerical integration procedure was used to obtain the moment-rotation curves. The curves in Ref. 25 include the effect of the residual stresses present in the column (assumed pattern) and plastification of the column. They are valid for ASTM A36 steel columns which are subjected to strong axis bending but can also be used with slight modifications for columns having yield stresses up to 50 ksi.25

In the middle and lower stories of unbraced multi-story frames the slenderness ratios of the columns will likely fall below 30. If A441 steel columns are used the equivalent slenderness ratios (slenderness ratio modified to account for yield stress different from 36 ksi), may lie between 30 and 40 for columns near the transition zone. An examination of the curves in Charts III-1 to III-7 of Ref. 25 indicates that for q = 1.0 and P_n/P_y equal to 0.80 or 0.90, the maximum value of M_n/M_pc will be slightly less than 1.0 for h_n/r_x = 40. For h_n/r_x = 30, it is apparent that the maximum value of M_n/M_pc will lie close to 1.0. Therefore, the analytical development in this dissertation will consider columns where h_n/r_x ≤ 30 and P_n/P_y ≤ 0.90. Hence, it will be assumed that a plastic hinge will develop at the top of the column and a minimum ro-
3.4 Load-Deflection Equation for Constant Restraint Stiffness

Since Eqs. 5 to 7 are valid for any restrained column in the story below any level \( n \), the subscript \( n \) can be deleted from these equations. Equation 5 can then be re-arranged and written

\[
\frac{Q_h}{2} = -\left[ M + \frac{PA}{2} \right] \tag{8}
\]

or equivalently

\[
\frac{Q_h}{2M_{pc}} = -\left[ \frac{M}{M_{pc}} + \frac{PA}{2M_{pc}} \right] \tag{9}
\]

where \( M_{pc} \) is the reduced plastic moment capacity of the restrained column corresponding to the applied axial load ratio \( P/P_y \) (Part II, Ref. 25).

For \( 0.15 < P/P_y < 1.0 \), the reduced plastic moment capacity of the column can be approximated by the equation

\[
M_{pc} = 1.18(1-P/P_y)M_p \tag{10}
\]
where $M_p$ is the plastic moment capacity of the unloaded column. If $M_p$ is taken as

$$M_p = \sigma_y fS = 2P_y f \frac{r_x}{d}$$  \hspace{1cm} (11)

where $\sigma_y$ is the yield stress level of the steel column, $f$ is the shape factor, $S$ is the section modulus and $d$ the column depth in the plane of the weak axis, then

$$\frac{P h d A}{2M_p} = \frac{\frac{P}{P_y} \frac{h}{r_x} \frac{d}{2r_x} \frac{A}{h}}{2.36 f(1 - \frac{P}{P_y})}$$  \hspace{1cm} (12)

Substituting Eq. 12 into Eq. 9 the non-dimensional load-deflection relationship of the restrained column leads to

$$\frac{Qh}{2M_p} = - \left[ \frac{M}{M_p} + \frac{\frac{P}{P_y} \frac{h}{r_x} \frac{d}{2r_x} \frac{A}{h}}{2.36 f(1 - \frac{P}{P_y})} \right]$$  \hspace{1cm} (13)

Equation (13) may be simplified by noting that $f$ and $d/2r_x$ can be approximated by their average values of 1.11 and 1.15 respec-
3.5

For wide-flange shapes normally used for columns (Ref. 8). With these substitutions, Eq. (13) becomes

$$\frac{Q_h}{2M_{pc}} = - \left[ \frac{M}{M_{pc}} + \frac{P_p h \Delta}{2.28 (1 - \frac{P}{P_y})} \right]$$

(14)

For a particular restrained column of length $h/2$, the load-deflection relationship given by Eq. (14) will be a function of the slenderness ratio $h/r_x$ of the column, the axial load ratio $P/P_y$ and the restraining moment $M_r$ (Eq. (6)). Equation (14) may be further simplified to

$$\frac{Q_h}{2M_{pc}} = - \left[ \frac{M}{M_{pc}} + C \frac{\Delta}{h} \right]$$

(15)

where $C$ is a constant given by

$$C = \frac{P_p h \Delta}{2.28 (1 - \frac{P}{P_y})}$$

(16)

3.5 Load-Deflection Behavior for Constant Restraint Stiffness

It was assumed in the derivation of Eq. (15) that the
restraint stiffness was constant for any sway deflection. Consequently for all values of \( \theta \)

\[
M_R = \bar{k} \theta \tag{17}
\]

or equivalently

\[
M_R = k \theta M_{pc} \tag{18}
\]

where \( \bar{k} \) is the restraint stiffness and \( k = \bar{k}/M_{pc} \).

Since the moment, \( M \), at the end of the restrained column cannot be expressed in terms of the end rotation, \( \gamma \), (Art. 3.3) except in the elastic range, Eq. (15) cannot be solved explicitly. However, a tabular form of the solution is possible and was presented by the writer in Appendix 1 of Ref. 26.

The non-dimensional load-deflection relationship, \( Qh/2M_{pc} \) versus \( \Delta/h \), for a particular restrained column with slenderness ratio, \( h/r_x \), constant axial load ratio \( P/P_y \), and constant restraint stiffness, \( k \), is shown by curve 0-a-b-c-e in Fig. 8. The load-deflection behavior of this restrained column may be summarized as follows:

1. When \( Q = 0, \Delta/2 = 0 \) (point 0)
2. Point \( a \) represents first yielding of the column.

3. Plastification of the column after point \( a \) results in significant non-linear load-deflection behavior.

4. The maximum shear resistance of the column is attained at point \( b \).

5. Equilibrium can be formulated beyond point \( b \) only by a reduction of the applied shear force \( Q \).

6. Points on the curve segment \( 0-a-b \) represent stable equilibrium configurations of the deflected column.

7. Points on the curve segment \( b-c-e \) represent unstable equilibrium configurations of the deflected column.

8. The restraining moment, \( M_r \), at all points on curve \( 0-a-b-c-e \) is given by Eq. (18).

The maximum value of restraining moment will be reached when a plastic hinge forms at the top of the column, or when

\[
M = - \frac{M_p}{2} = - M_{pc}
\]  
(19)
The load-deflection equation for the restrained column after the formation of this plastic hinge (and thus a mechanism) can then be found from Eqs. 15 and 19 as

\[
\frac{Qh}{2M_{pc}} = 1 - C \frac{\Delta}{h}
\]  

Equation (20) is shown in Fig. 8 as the straight line segment d-e which passes through the point \( Qh/2M_{pc} = 1 \) when \( \Delta/h = 0 \). Curves O-a-b-c-e and d-c-e intersect at point \( c \) when a plastic hinge forms at the top of the column. The restraining moment corresponding to point \( c \) can be found from Eq. (19) as

\[
M_r = 2M_{pc}
\]

The angle, \( \theta_p \), corresponding to the formation of the plastic hinge may be found by equating Eqs. (18) and (21) so that

\[
\theta = \theta_p = \frac{2}{k}
\]

It should be apparent from the derivation of Eq. (15) that curves O-d and d-c-e in Fig. 8 define the second-order, rigid-plastic load-deflection curves for the column shown in the figure. It is evident then that the restraining moment, \( M_r \),
will have a constant value, \( M_r' \) everywhere on \( d-c-e \). From Eq. (21).

\[
M_r = M_r' = 2M_{pc}
\]  

along the line \( d-c-e \).

Additional load-deflection curves may also be obtained for the column shown in Fig. 8 which has a slenderness ratio \( h/r_\times \) and a constant axial load ratio \( P/P_y \). Each curve would correspond to a different value of restraint stiffness, \( k \), where \( 0 \leq k \leq \infty \). All curves would be similar in shape to \( O-a-b-c-e \) and all would pass through point \( O \). In addition, all of the load-deflection curves would intersect curve \( d-c-e \) (or its extension for greater values of \( \Delta/h \)), since the maximum restraining moment, \( M_r' \) for all curves is independent of the restraint stiffness \( k \), (Eq. (23)).

3.6 Load-Deflection Behavior for Variable Restraint Stiffness

In general, the restraint stiffness, \( k \), will not remain constant for all values of joint rotation, \( \theta \), but will decrease as \( \theta \) increases due to the successive formation of plastic hinges in the sway subassemblage and in the beams and columns outside the sway subassemblage. It will be assumed that \( k \) de-
creases in value at discrete values of \( \theta \) but remains constant between those values of \( \theta \). This implies elastic-plastic behavior of all members except the restrained column which is in the sway subassemblage under analysis. Of the infinite number of \( k - \theta \) relationships possible, only two of them are fundamental to the sway subassemblage method of analysis. These two may be described as follows:

1. **Constant - Zero Restraint Stiffness:**

\[
k = k_1 \quad (0 \leq \theta \leq \theta_1) \tag{24}
\]

\[
k = k_2 = 0 \quad (\theta_1 \leq \theta \leq \infty) \tag{25}
\]

where \( \theta_1 < \theta_p \) (Eq. (22))

2. **Constant - Constant Restraint Stiffness:**

\[
k = k_1 \quad (0 \leq \theta \leq \theta_1) \tag{26}
\]

\[
k = k_2 \quad (\theta_1 \leq \theta \leq \infty) \tag{27}
\]

where \( k_1 > k_2 \)
3.6.1 Constant - Zero Restraint Stiffness

The restraining moment at the column top will be defined by the equations

\[ M_r = k_1 \theta M_{pc} \quad (0 \leq \theta \leq \theta_1) \] (28)

\[ M_r = M'_r = k_1 \theta M_{pc} = p_1 M_{pc} \quad (\theta_1 \leq \theta \leq \infty) \] (29)

where \( p_1 \) is a constant and by the definition of \( \theta_1 \), \( 0 \leq p_1 \leq 2 \), (Eq. (22)). The solution of Eq. (15) for the restraining moment defined by Eq. (28) will give the load-deflection curve \( O - g \) in Fig. 9. For equal values of \( P/P_y \), \( h/r_x \) and \( k = k_1 \) curve \( O - g \) would be a segment of the complete load-deflection curve shown as \( O-a-b-c-e \) in both Figs. 8 and 9. At point \( g \) however, the restraint stiffness becomes zero. Therefore additional restraining moment cannot be generated and a mechanism condition will result. The restraining moment after the formation of the mechanism will be the constant, \( M'_r \), given by Eq. (29). Using Eqs. (6) and (29), the maximum column moment will be given by

\[ M = -\frac{p_1}{2} M_{pc} \] (30)

The load-deflection curve following the formation of the mechanism
3.6.2 can be obtained by substituting Eq. (30) into Eq. (15)

\[
\frac{Qh}{2M_{pc}} = \frac{P_1}{2} - C\frac{A}{h}
\]  

Equation (31) is shown in Fig. 9 as the straight line f-g-h. Since the derivatives with respect to \(\Delta/h\) of Eqs. (20) and (31) are equal then f-g-h will be parallel to d-o-e. It should be apparent by recalling the discussion in Art. 3.6 that 0-f and f-g-h in Fig. 9 will define the second-order rigid-plastic load-deflection curves for the rigid-plastic column with maximum restraining moment given by Eq. (29). It will be further evident from Eq. (6) that the moment at the top of the column, \(M\), and thus the column chord rotation, \(\gamma\), will be constant everywhere on f-g-h.

3.6.2 Constant - Constant Restraint Stiffness

The load-deflection behavior of a restrained column with constant - constant restraint stiffness is also fundamental to the sway subassemblage method of analysis. The restraining moment at the column top will now be defined by the equations

\[
M_r = k_1 \theta M_{pc} \quad (0 \leq \theta \leq \theta_1)
\]  

\[
M_r = k_2 \theta M_{pc} \quad (\theta_1 \leq \theta \leq \infty)
\]
The solution of Eq. (15) when the restraining moment is defined by Eq. (32) will give the load-deflection curve O-g in Fig. 10. For equal values of $P/P_y$, $h/r_x$ and $k = k_1$, curve O-g would be the same segment of the load-deflection curve O-a-b-c-e which is shown in Figs. 9 and 10. However, at point $g$ in Fig. 10, the restraint stiffness reduces to $k_2$ which is greater than zero. Additional restraining moment can be developed after point $g$ with increased sway deflection but at a smaller rate than before. The load-deflection curve after point $g$ is shown as curve $g$-j-m in Fig. 10. Curves g-j-m and d-c-e will intersect at point $m$ with the formation of a plastic hinge at the top of the column.

The load-deflection curve O-g-j-m in Fig. 10 has considerable practical significance. Such a curve would be obtained if one plastic hinge formed somewhere in the beams and columns providing restraint to the top of the restrained column and the next plastic hinge formed at the column top. In reality, more than one plastic hinge could develop in the restraining system thus leading to more than one reduction in stiffness. Perhaps a sufficient number of plastic hinges could form in the restraining system to reduce the restraint stiffness to zero before a plastic hinge forms at the column top. The load-deflection curve for the restrained column would then be a combination of Fig. 10 with more than one kink in the curve, and Fig. 9.
where a mechanism curve results with $p < 2$.

A simple procedure is needed to determine the segments of a load-deflection curve, such as the segment $g-j-m$ in Fig. 10. Consider the two load-deflection curves which are shown in Fig. 11(c). Curve $O-a-b-c$ represents the load-deflection curves for a column where the restraint stiffness decreases from $k_1$ to $k_2$. Curve $O-a'-b'-c'$ however, represents the load-deflection curve for the same column ($h/r_x$ and $P/P_y$ constant) but with constant restraint stiffness $k_2$. Segment $O-a$ of curve $O-a-b-c$ and the complete curve $O-a'-b'-c'$ may be obtained by solving Eq. (15) where the restraining moments are defined by $k_1$ and $k_2$ respectively. Recall that Eq. (15) was derived by considering the equilibrium of a column such as the one shown in Fig. 11(b).

The load-deflection equation for segment $a-b-c$ of curve $O-a-b-c$ can be derived in a similar manner. The forces acting on the restrained column are shown in Fig. 11(a) together with the resulting deformations. The initial conditions (point $a$ in Fig. 11(c)) are given by $Q_1$, $\Delta_1$, $\theta_1$, $\gamma_1$, $M_1$ and $M_{rl}$. From statics the moment at the top of the column will be given by

$$M_1 + M = - \left[ (Q_1 + Q) \frac{h}{2} + P \left( \frac{\Delta_1 + \Delta}{2} \right) \right]$$

Equilibrium of moments at the joint then requires that

$$2(M_1 + M) + M_{rl} + M_r = 0$$
and the compatibility condition becomes

\[
\frac{\Delta_1 + \Delta}{h} = \theta_1 + \theta - (\lambda_1 + \lambda)
\]  

(36)

The load-deflection relationship can be determined (for constant restraint stiffness) by solving Eqs. (34) to (36) with the known moment-rotation relationship \((M - \gamma)\) for the restrained column.

Eq. (34) can be written in non-dimensional form as

\[
\frac{(Q_1 + Q)h}{2M_{pc}} = -\left[\frac{(M_1 + M)}{M_{pc}} + \frac{P(\Delta_1 + \Delta)}{2M_{pc}}\right]
\]  

(37)

and further reduced to the form of Eq. (15) so that

\[
\frac{(Q_1 + Q)h}{2M_{pc}} = -\left[\frac{(M_1 + M)}{M_{pc}} + C \frac{(\Delta_1 + \Delta)}{h}\right]
\]  

(38)

It should be apparent that a linear transformation of axes in Fig. 11(c) with the new origin at point \(a\) will result in Eq. (38) reducing to Eq. (15). Equation (15) then gives the load-deflection relationship for curve a-b-c in Fig. 11(c) with the new origin at point \(a\) and with the restraining moment defined
by the restraint stiffness $k_2$. Obviously, Eq. (15) also applies
to the segment $a'-b'-c'$ with a new origin at point $a'$.

The question is now raised: If points $a$ and $a'$ in
Fig. 11(c) both lie on the straight line defined by $M_r' = p_2 M_{pc}$
where $0 < p_2 < 2$, will the curves $a-b-c$ and $a'-b'-c'$ be iden-
tical. To show that they are, it will be necessary to prove
that each curve has the same slope at points which intersect
the straight line defined by $M_r' = p_1 M_{pc}$ where $p_2 \leq p_1 \leq 2$
(Fig. 11(c)).

Using Eqs. (6) and (18) where $k = k_2$, Eq. (15) may
be written

$$\frac{Qh}{2M_{pc}} = \frac{k_2 \theta}{2} - C \frac{\Delta}{h}$$

Substituting for $\theta$ from Eq. (7) gives

$$\frac{Qh}{2M_{pc}} = \left(\frac{k_2}{2} - C\right) \frac{\Delta}{h} + \frac{k_2}{2} \gamma$$

(40)

The slope at any point then may be found by differentiating
both sides of Eq. (40) with respect to $\Delta/h$ giving

$$\frac{\partial}{\partial \left(\frac{\Delta}{h}\right)} \left( \frac{Qh}{2M_{pc}} \right) = \frac{k_2}{2} - C + \frac{k_2}{2} \frac{\partial \gamma}{\partial \left(\frac{\Delta}{h}\right)}$$

(41)
3.6.2

It should be apparent from Eq. (41) that each curve will have the same slope at the two points on the curve \( M'_r = p_1 M_{pc} \) only if \( \frac{\partial y}{\partial (\frac{A}{h})} \) is the same at each point.

Let \( B \) and \( B' \) be two identical restrained columns \((h/r, P/P_y \text{ constant})\) with shear forces, \( Q \), and sway deflections, \( \Delta \), defined by the points \( b \) and \( b' \) respectively in Fig. 11(a). From the discussions in Art. 3.6.1, it will be evident that the end moment, \( M \), and the chord rotation, \( \gamma \), will be the same for both columns. Assume now that the deflection indexes of columns \( B \) and \( B' \) are incremented by the amounts \( \delta (\frac{\Delta}{h})_B \) and \( \delta (\frac{\Delta}{h})_B' \) respectively such that the increment in restraining moment, \( \delta M_r \), is the same for each column. Then from Eq. (33)

\[
\delta M_r = k_2 \delta \theta_B M_{pc} = k_2 \delta \theta_{B'} M_{pc} \tag{42}
\]

where \( \delta \theta_B \) and \( \delta \theta_{B'} \) are increments of joint rotation. It is apparent from Eq. (42) that \( \delta \theta_B = \delta \theta_{B'} = \delta \theta \). Thus the increments in column top moment, \( \delta M \), and the increments in chord rotation \( \delta \gamma \) will be the same for each column.

Writing Eq. (7) in the form

\[
\delta (\frac{\Delta}{h}) = \delta \theta - \delta \gamma \tag{43}
\]
then since

\[
\delta(\frac{A}{h})_B = \delta(\frac{A}{h})_{B'} = \delta(\frac{A}{h})
\]

Equation (43) may be re-arranged to give

\[
\frac{\delta y}{\delta (\frac{A}{h})} = \frac{\delta \theta}{\delta (\frac{A}{h})} - 1
\]

Since \(\frac{\delta \theta}{\delta (\frac{A}{h})}\) will be the same for each column, then

\(\frac{\delta y}{\delta (\frac{A}{h})}\) will also be the same. Consequently as \(\delta(\frac{A}{h})\) approaches zero, the term \(\frac{\delta y}{\delta (\frac{A}{h})}\) in Eq. (41) will be the same at points b and b' (Fig. 11(c)).

It is apparent from this study that curves a-b-c and a'-b'-c' in Fig. 11(c) are identical. The significant conclusion is that it will not be necessary to derive the load-deflection equation corresponding to each reduced value of restraint stiffness, k. Instead the load-deflection curve may be built up from segments of complete load-deflection curves which are given by Eq. (15) for the appropriate values of k.

3.7 Design Charts

The solution of Eq. (15) has been presented in Ref. 28
in the form of 78 design charts. These charts have been prepared for use with ASTM A36 steel wide-flange column shapes but can also be adapted for use with A441 columns. Each chart contains up to 27 closely spaced load-deflection curves for arbitrarily chosen values of \( k (0 \leq k \leq \infty) \) which define the relationship \( \frac{Q_h}{2M_p} \) versus \( \frac{A}{h} \) for a restrained column with slenderness ratio \( h/r_x \) and constant axial load ratio \( P/P_y \) where:

1. \( 0.30 \leq \frac{P}{P_y} \leq 0.90 \) intervals of 0.50 \( \frac{P}{P_y} \)

2. \( 20 \leq \frac{h}{r_x} \leq 30 \) intervals of 2 \( \frac{h}{r_x} \)

Each load-deflection curve was constructed for a constant value of restraining moment, \( M_r \), which is defined by Eq. (18). Also shown on each chart in Ref. 28, are the straight lines representing constant values of maximum restraining moment, \( M'_r \). These lines have been constructed for arbitrarily chosen values of \( p \) where \( 0 \leq p \leq 2 \).

The two sets of curves contained in each chart may be used as described in Arts. 3.6.1 and 3.6.2 for constructing the complete load-deflection curve of a restrained column when the restraint has been completely defined. The restraining characteristics of steel beams and columns and of composite beams must now be determined. A knowledge of these characteristics will
enable the set of values of $M_r$ and $M'_r$ to be obtained for any restrained column in the half-story assemblage.
4. RESTRAINING CHARACTERISTICS OF STEEL BEAMS AND COLUMNS

4.1 Initial Restraint

The term "initial restraint" will be used in this chapter to denote the rotational restraint provided to the top of a restrained column by the restraining system prior to the formation of the first plastic hinge in the restraining system. The restraining system will consist of the beams and columns on both sides of the restrained column.

In Art. 2.2 it was assumed that under the factored gravity loads, 1.3 w, the joint rotations could be taken as zero. Thus fixed-end moments were assumed to occur at the ends of each beam. Since the beams must also carry factored loads, 1.7 w, these fixed-end moments will always be less than the plastic moment capacities of the beams. For the beams in the lower stories of a frame the differences between the fixed-end moments and the plastic moment capacities will be relatively large. Therefore, initial sway will always be accompanied by elastic restraint from the beams.

The initial restraint provided by the columns is more involved. Due to residual stresses, the magnitude of the axial forces calculated in Art. 2.4 would indicate plastification of the leeward exterior column and perhaps one or two adjacent col-
4.1

umns. The latter situation could arise in a frame with many bays, each having approximately the same beam size at a given level, and where consecutive narrow and wide bays occurred on the leeward side of the frame. In reality, under zero sway conditions, each column would carry only its share of the factored gravity load, 1.3 \( w \). Since the leeward columns must also be designed to carry the additional forces produced either by the factored wind loads, \( H \), or by the factored gravity loads, 1.7 \( w \), little or no plastification of these columns would be expected at 1.3 \( w \). This would depend upon the axial load ratio, \( P/P_y \), at 1.3 \( w \) and upon the magnitude of the residual stresses. Therefore, it will be assumed that the initial restraint provided by all the columns in a half-story assemblage can be computed on the basis of the effective moment of inertia under the factored gravity loads, 1.3 \( w \). In addition, it will be assumed that the calculated effective moments of inertia of the columns remain constant as the sway deflection of the restrained column increases. This assumption will generally lead to sufficiently accurate calculations of the initial restraint, providing that combined mechanisms govern the maximum shear capacity of the story and providing that the maximum shear capacity does not greatly exceed the factored wind shear in the story. This implies that plastic hinges will form at the leeward ends of the beams in each sway subassemblage at relatively low values of
the applied shear force thus isolating the effect of the columns outside the sway subassemblage.

If a sway mechanism occurs, the axial load ratio of the leeward exterior columns (and one or two adjacent columns) could reach a value which would result in a considerable reduction of the effective moment of inertia of the column. Under this condition, the initial restraint should be re-calculated at discrete increments of the applied shear force. If the increments are sufficiently small, the initial restraint can be assumed constant over the increment.

4.2 Initial Restraint Coefficients

The interior region of a half-story assemblage is shown in Fig. 12(a) together with the vertical forces, P, and joint moments, M, which were determined in Chapter 2. The deflected configuration is consistent with a relatively small applied shear force, \( \Sigma Q \), acting towards the right. The behavior of all the beams and columns is assumed to be elastic. The deflection index, \( \Delta/h \), will also be relatively small.

Now consider the restrained column at joint \( i \). It is desired to calculate the initial elastic value of restraint stiffness, \( k_i \) (Art. 3.5) which is provided by the beams and columns of the half-story assemblage. The restraining moment, \( M_r \),
at joint \( i \) will be the sum of the restraining moments on either side of the joint and can be written

\[
M_r = M_{i(i-1)} + M_{ij} = \left[ K_{i(i-1)} \frac{E I_{i(i-1)}}{L_{i(i-1)}} + K_{ij} \frac{E I_{ij}}{L_{ij}} \right] \theta_i \quad (46)
\]

where

- \( M_{i(i-1)} \) = moment at \( i \) in beam \( i(i-1) \)
- \( M_{ij} \) = moment at \( i \) in beam \( ij \)
- \( K_{i(i-1)} \) = initial restraint coefficient at \( i \) in beam \( i(i-1) \)
- \( K_{ij} \) = initial restraint coefficient at \( i \) in beam \( ij \)

Also, \( I_{i(i-1)} \) and \( I_{ij} \) are the moments of inertia of beams \( i(i-1) \) and \( ij \);

- \( L_{i(i-1)} \) and \( L_{ij} \) are the center to center lengths of beams \( i(i-1) \) and \( ij \);

\( \theta_i \) is the rotation of joint \( i \) and \( E \) is the modulus of elasticity.

Equation (46) may also be written in non-dimensional form

\[
M_r = \left[ K_{i(i-1)} \frac{E I_{i(i-1)}}{L_{i(i-1)} M_{pci}} + K_{ij} \frac{E I_{ij}}{L_{ij} M_{pci}} \right] \theta_i M_{pci} \quad (47)
\]
where \( M_{pci} \) is the reduced plastic moment capacity of the restrained column at joint \( i \) corresponding to the axial load ratio \( P/P_y \) of column \( i \). Equating Eqs. 18 and 47 yields

\[
k_i = K_{i(i-1)} \frac{E I_i(i-1)}{L_i(i-1) M_{pci}} + K_{ij} \frac{E I_{ij}}{L_{ij} M_{pci}}
\]

The solution of Eq. (48) requires only the determination of the initial restraint coefficients \( K_{i(i-1)} \) and \( K_{ij} \) since all other terms are known. These coefficients will be a function of the flexural stiffnesses of all the beams and columns in the half-story assemblage on either side of joint \( i \). A reasonably accurate solution may be obtained by considering only a limited number of members in the vicinity of joint \( i \).

The initial restraint coefficients can be evaluated approximately by considering only the members shown in Fig. 12(b). The following assumptions will be made:

1. The restraining moment \( M_{i(i-1)} \) on the windward side of joint \( i \) is known.

2. The restraining effect of the members to the right of joint \( (j+1) \) will be approximated by taking \( \theta_{(j+1)} = \theta_j \).
3. The effects of the axial loads in the columns (except as discussed in Art. 4.1) and the effects of the gravity loads on the beams may be neglected.

The initial restraint coefficient $K_{ij}$ may be determined if the relationship between $M_{ij}$ and $\theta_i$ can be found when joints $i$, $j$ and $(j+1)$ each undergo a small sway displacement equal to $\Delta/2$. At joint $i$, the moment, $M_i$, at the top of the restrained column, the joint rotation, $\theta_i$, and the deflection index, $\Delta/h$, are related as

$$M_i = \frac{6EI_i}{h} \left[ \theta_i - \frac{\Delta}{h} \right]$$

where $I_i$ is the moment of inertia of the restrained column.

The moment $M_i$, above joint $i$ will also be given by Eq. (49). The restraining moments in the two beams at joint $i$ can be obtained from Eq. (46). Equilibrium of moments at joint $i$ yields

$$2M_i + M_r = 0$$

so that

$$12 \frac{EI_i}{h} \left[ \theta_i - \frac{\Delta}{h} \right] + M_{ij} + K_{i(i-1)} \frac{EI_{i(i-1)}}{I_{i(i-1)}} \theta_i = 0$$
At joint $j$ the stress-resultants can be expressed as

$$M_j = \frac{6EI_{ij}}{h} \left[ \theta_j - \frac{\Delta}{h} \right]$$

(52)

$$M_{ji} = \frac{EI_{ij}}{L_{ij}} \left[ 4\theta_j + 2\theta_i \right]$$

(53)

$$M_{j(j+1)} = \frac{6EI_{j(j+1)}}{L_{j(j+1)}} \theta_j$$

(54)

Equilibrium of moments at joint $j$ then requires that

$$12 \frac{EI_{ij}}{h} \left[ \theta_j - \frac{\Delta}{h} \right] + \frac{EI_{ij}}{L_{ij}} \left[ 4\theta_j + 2\theta_i \right] + \frac{6EI_{j(j+1)}}{L_{j(j+1)}} \theta_j = 0$$

(55)

The deflection index, $\Delta/h$, can be evaluated from Eqs. 51 and 55 as

$$\frac{\Delta}{h} = \left[ 1 + \frac{\alpha'}{12} K_i(i-1) \right] \theta_i + \frac{M_{i,h}}{12EI}$$

(56)

and

$$\frac{\Delta}{h} = \frac{6\theta_i}{6} + \left[ 1 + \frac{\theta}{3} + \frac{n}{2} \right] \theta_j$$

(57)
where

\[ \alpha' = \frac{h I_i(i-1)}{L_i(i-1) I_i} \]  \hspace{1cm} (58)

\[ \beta = \frac{h I_{ij}}{L_{ij} I_{ij}} \]  \hspace{1cm} (59)

\[ \eta = \frac{h I_{j(j+1)}}{L_{j(j+1) I_{j}}} \]  \hspace{1cm} (60)

Now \( \theta_j \) can be expressed as a function of \( \theta_i \)

\[ \theta_j = \frac{M_{ij} L_{ij}}{2E L_{ij}} - 2\theta_i \]  \hspace{1cm} (61)

Equating Eqs. 56 and 57 yeilds the moment \( M_{ij} \) as

\[ M_{ij} = 6 \left[ \frac{3 + 0.5\beta + \eta + \frac{\alpha'}{12} K_i(i-1)}{3 - 0.5\alpha + \beta + 1.5\eta} \right] \frac{E L_{ij} I_{ij}}{L_{ij}} \theta_i \]  \hspace{1cm} (62)

where

\[ \alpha = \frac{h I_{ij}}{L_{ij} I_{ij}} \]
4.2

Hence,

\[
K_{ij} = 6 \left[ \frac{3 + 0.5\beta + \eta + \frac{\alpha'}{12} K_{i(i-1)}}{3 - 0.5\alpha + \beta + 1.5\eta} \right]
\]  \hspace{1cm} (63)

The initial restraint coefficient to the left of joint \( j \), \( K_{ji} \), is related to \( K_{ij} \). The stress resultants for beam \( ij \) can be written as

\[
M_{ij} = \frac{E I_{ij}}{L_{ij}} \left[ 4\theta_i + 2\theta_j \right] = K_{ij} \frac{E I_{ij}}{L_{ij}} \theta_i
\]  \hspace{1cm} (64)

\[
M_{ji} = \frac{E I_{ij}}{L_{ij}} \left[ 4\theta_j + 2\theta_i \right] = K_{ji} \frac{E I_{ij}}{L_{ij}} \theta_j
\]  \hspace{1cm} (65)

Thus

\[
K_{ij} \theta_i = 4\theta_i + 2\theta_j
\]  \hspace{1cm} (66)

\[
K_{ji} \theta_j = 4\theta_j + 2\theta_i
\]  \hspace{1cm} (67)

Equations (66) and (67) yield

\[
K_{ji} = 4 \left[ \frac{K_{ij} - 3}{K_{ij} - 4} \right]
\]  \hspace{1cm} (68)
4.2

If joint \( i \) is an interior joint, then \( K_{i(i-1)} \) is also given by

\[
K_{i(i-1)} = 4 \left[ \frac{K_{(i-1)i} - 3}{K_{(i-1)i} - 4} \right]
\] (69)

If joint \( i \) is the windward exterior joint, then \( K_{i(i-1)} = 0 \).

It should be apparent that the initial restraint coefficients must be calculated starting at an exterior joint and progressing left or right across the assemblage. Since Eq. (63) provides an approximate value of \( K_{ij} \), the use of this equation will lead to cumulative errors in \( K_{i(i-1)} \) and \( K_{ij} \). A method of evaluating the relative error (but not the absolute error) would be to calculate \( K_{i(i-1)} \) and \( K_{ij} \) first by starting at the windward exterior joint and then by starting at the leeward exterior joint, each time proceeding across the assemblage and then to compare the two sets of initial restraint coefficients obtained.

An alternate approach would be to derive a more accurate expression for \( K_{ij} \). This could be achieved by including one more bay to the right of joint \((j+1)\) assuming that
\[ \theta_{j+2} = \theta_{j+1} \] The resulting expression for \( K_{ij} \) can be written

\[
K_{ij} = 6 \left[ \frac{3 + \frac{8}{12}(6 + 2\tau + 3\xi) + \frac{\eta}{12}(6 + 2\tau + 4\xi) + 1.5\xi + \tau}{3 - \frac{\alpha}{12}(6 - \eta + 2\tau + 3\xi) + \frac{\beta}{12}(12 + 4\tau + 6\xi)} \right]
\]

(70)

\[
+ \frac{\alpha'}{72} K_{i(i-1)}(6 - \eta + 2\tau + 3\xi) \]

\[
+ \frac{\eta}{12}(12 + 3\tau + 6\xi) + \tau + 1.5\xi
\]

where

\[
\tau = \frac{h I_{j+1}(j+1)}{L j(j+1)^2 (j+1)} \]

(71)

\[
\xi = \frac{h I_{j+1}(j+2)}{L (j+1) (j+2) j^2 (j+1)} \]

(72)

For well proportioned frames with no unusually short stiff beams, the parameters \( \alpha', \alpha, \beta, \ldots \) etc., will generally range from near 0 to less than 1.0. For example, in Frame C (Ref. 5) these parameters vary from about 0.10 to about 0.70. If it is assumed that \( K_{ij} \) is almost exact by Eq. (70) then the error involved in using Eq. (63) will generally be less than 6% for a range of parameters from 0 to 1.0.
4.3 The procedure discussed in this article may also be followed to obtain exact expressions for $K_{ij}$ in frames with up to 3 bays. These expressions are readily obtained and will not be developed in this dissertation.

4.3 Reduced Restraint Coefficients

As the shear force $EQ$ on a half-story assemblage increases in magnitude from zero, the successive formation of plastic hinges in the beams and columns will reduce the restraint stiffness at the top of each restrained column. Considering a particular restrained column in a sway subassemblage it should be evident that the plastic hinges within the sway subassemblage will have the greatest effect on the reduction of the restraint stiffness. The plastic hinges nearest the sway subassemblage should have a somewhat reduced effect. For the purposes of the following discussions it will be assumed that all of the beams exhibit elastic-plastic behavior when the plastic moment capacity, $M_p$, has been reached. Plastic hinges may form within a beam span or at the intersection of the beam and column axes. Columns outside the sway subassemblage will also be assumed to exhibit elastic-plastic behavior where plastic hinges develop at the top of the column (intersection of beam and column axes) when the reduced plastic moment capacity, $M_{pc}$, has been reached. It is of interest to examine the effect of
4.3.1 Plastic Hinges Outside the Sway Subassemblage

Assume that the nearest plastic hinges to the interior sway subassemblage containing the restrained column at joint i (Fig. 12(a)) are those just to the left of joints (i-1) and (j+1), excluding those in the columns below joints (i-1) and j. Assume also that the first plastic hinge occurs just to the left of joint (j+1) and prior to any plastic hinges within the interior sway subassemblage. The expression for $K_{ij}$ after the first plastic hinge forms may be found as before. Referring to Fig. 12(b) the end of beam j(j+1) can be assumed pinned at (j+1) when the plastic hinge forms. The exact expression for $K_{ij}$ will then be given by

$$K_{ij} = 6 \left[ \frac{3 + 0.5\beta + 0.5\eta + \frac{\alpha'}{12} K_{i(i-1)}}{3 - 0.5\alpha + \beta + 0.75\eta} \right]$$

(73)

If Eq. (73) is compared with Eqs. (63) and (70) for a range of parameters from 0 to 1.0, the difference in $K_{ij}$ will generally be less than 5% to 10%. The error in neglecting the plastic hinge just to the left of joint (i-1) is also small. Therefore
plastic hinges forming outside the sway subassemblage can be neglected.

4.3.2 Plastic Hinges Within the Sway Subassemblage

Figure 13 shows the locations of the possible plastic hinges within an interior sway subassemblage. Also shown are the plastic hinges which can form at the columns at (i-1) and j. The sequence in which these plastic hinges form will be a function of the relative member stiffnesses, plastic moment capacities and the intensity of the factored gravity loads, \( w \). Plastic hinges 1, 3, 4, 6 and 7 will usually be the first to form and will occur at the ends of the members. In certain cases, plastic hinges 2 and 5 may also form at the windward ends of the beams. A method of determining the positions of plastic hinges 2 and 5 is discussed in Ref. 16. Although all the plastic hinges shown are possible for interior sway subassemblages only 4, 5, 6, and 7 can occur in windward sway subassemblages while 1, 2, 3 and 4 are possible plastic hinge locations for leeward sway subassemblages.

The initial restraint coefficients \( K_{i(i-1)} \) and \( K_{ij} \) are associated with the restraint stiffness, \( k_i \), given by Eq. (48). Since the load-deflection behavior of the restrained column in Fig. 13 will be described completely by the values of \( k_i \)
(Chapter 3) then it will also be described by the values $K_{i(i-1)}$ and $K_{ij}$ and the reduced restraint coefficients. Referring to the numbered locations of plastic hinges shown in Fig. 13, and assuming that 3 and 6 will form before 2 and 5, respectively, (Art. 2.2) the reduced restraint coefficients can be determined as follows:

1. **1 occurs before 3:** Since additional moment cannot be developed at joint (i-1), beam i(i-1) may be considered pinned at (i-1). Thus $K_{i(i-1)}$ reduces to 3.0.

2. **3 occurs after 1:** $K_{i(i-1)}$ reduces from 3.0 to 0.

3. **3 occurs before 1:** $K_{i(i-1)}$ reduces to zero.

4. **6 or 7 occurs:** $K_{ij}$ reduces to 3.0.

5. **5 occurs after 6 or 7:** $K_{ij}$ reduces from 3.0 to 0.

6. **4 occurs:** $K_{i(i-1)}$ and $K_{ij}$ remain unchanged from their values at the time 4 develops.
5. RESTRAINING CHARACTERISTICS OF COMPOSITE BEAMS AND STEEL COLUMNS

5.1 Flexural Behavior Under Combined Loads

The basic concepts developed in Chapter 4 are also generally applicable to a restraining system comprising composite beams and steel columns. In some respects, the interpretation of those concepts is not the same. As in Chapter 4, the restraint stiffness, $k_i$ (Art. 3.5) at the top of column $i$ may be computed considering only those members within one or two bays either side of the sway subassemblage containing column $i$. Also, the influence of plastic hinges outside the sway subassemblage on the restraint stiffness is small and may be neglected. However, although the restraint provided by the columns of the restraining system will be as discussed in Chapter 4, the restraint provided by the composite beams will be significantly different.

Under the factored gravity loads, $1.3w$, positive and negative bending moments will develop over the span length of each composite beam in the half-story assemblage. If the usual assumption is made that concrete cannot resist tensile forces (Art. 1.6.1), then the moment of inertia of each beam will vary over the span length. The determination of the extent of the positive and negative moment regions is complicated by the fact
that the boundaries of these regions and the elastic properties of the composite beams are inter-dependent. For the first very small increment of lateral shear force, $\delta \Sigma Q$, applied to the half story assemblage, the restraint at the top of column $i$, due to the composite beams will be closely determined by their rotational stiffnesses when the variation in the moment of inertia of each beam is consistent with the bending moments under the factored gravity loads. However, as the lateral shear force $\Sigma Q$ increases, the relative lengths and positions of the positive and negative moment regions in each beam will change because of the changing bending moments which are induced. Consequently, the restraint provided by the beams will change. During the increments of loading between the formation of consecutive plastic hinges, the bending moments will vary continuously resulting in a continuous variation of restraint. Furthermore, during any specified increment of lateral shear force $\delta \Sigma Q$ between zero and maximum load, the restraint provided by the beams may increase or decrease independently of the formation of plastic hinges in the restraining system. Since the restraint provided by steel beams remains constant with increasing values of $\Sigma Q$ and reduces only with the formations of plastic hinges (Chapter 4), this represents a significant difference in the behavior of composite and non-composite beams which are subjected to combined loads.
The restraint provided by composite beams will also change with the formation of plastic hinges as was the case for steel beams. However, the determination of the plastic hinges in composite beams subjected to combined loads is somewhat more involved. A brief discussion of this problem is contained in Art. 5.5.

5.2 Initial Restraint

The term initial restraint is also used in this Chapter to denote the rotational restraint at the top of a restrained column prior to the formation of the first plastic hinge in the restraining system. On the basis of the assumptions in Art. 5.1, the initial restraint will apply up to the formation of the first plastic hinge in the sway subassemblage containing the restrained column. Since the initial restraint will vary continuously with increasing values of $\Sigma Q$ it must be re-computed for each increment $\delta \Sigma Q$. If electronic computation is used to calculate the initial restraint coefficients, $K$, (Art. 5.3) the increments can be taken sufficiently small that $K$ can be assumed constant over the increment.

5.3 Initial Restraint Coefficients

The initial restraint coefficients can be evaluated approximately by considering only the members shown in Fig. 14
(which is similar to Fig. 12(b)). The only difference between the two figures occurs in the moments of inertia of beams $ij$ and $j(j+1)$. The moment of inertia, $I$, in the negative moment regions is computed from a cross-section consisting of the steel beam and the longitudinal slab reinforcement. The moment of inertia, $\tilde{I}$, in the positive moment regions is computed from a cross-section consisting of the steel beam, longitudinal slab reinforcement and the concrete slab. The assumptions used in Art. 4.2 during the derivation of Eq. 63 will also be applicable to the members and forces shown in Fig. 14. In addition, it will be assumed that the concrete slab cannot take tensile forces and that complete interaction exists between the steel beam and the concrete slab. The initial restraint coefficient $K_{ij}$ may be determined if the relationship between $M_{ij}$ and $\theta_j$ (Fig. 14) can be found when joints $i$, $j$ and $(j+1)$ all undergo small equal sway displacements, $\Delta/2$. The slope-deflection equations for beams $ij$ and $j(j+1)$ can be written in the general form

\begin{align}
M_{ij} &= \frac{E}{L_{ij}} \tilde{I}_{ij} \left[ A_1 \theta_i + A_2 \theta_j \right] \quad (74) \\
M_{ji} &= \frac{E}{L_{ij}} \tilde{I}_{ij} \left[ A_3 \theta_j + A_2 \theta_i \right] \quad (75)
\end{align}
where the coefficient $A_1$, $A_2$, $A_3$, $B_1$ and $B_2$ are functions of the moments of inertia in the positive and negative moment regions as well as functions of the relative lengths of those regions. These slope-deflection coefficients will be evaluated in Art. 5.3.1. The slope-deflection equations for the columns at $i$ and $j$ were given by Eqs. 49 and 52. Derivation of the initial restraint coefficient $K_{ij}$ following the procedure used to determine Eq. 63 will give

$$K_{ij} = 6 \left[ \frac{(A_1 + A_2)}{6} + \frac{B}{24} (A_1 A_3 - A_2^2) + \frac{\eta}{24} A_1 (B_1 + B_2) + \frac{\alpha'}{24} A_2 K_{i(i-1)} \right] \frac{3 - \frac{\alpha}{4} A_2 + \frac{B}{4} A_3 + \frac{\eta}{4} (B_1 + B_2)}{1 - \frac{\alpha}{4} A_2 + \frac{B}{4} A_3 + \frac{\eta}{4} (B_1 + B_2)}$$

(77)

where $K_{i(i-1)}$ can be determined from Eq. 69 and

$$\alpha' = \frac{h \bar{I}_{i(i-1)}}{L_{i(i-1)}^2}$$

(78)

$$\alpha = \frac{h \bar{I}_{ij}}{L_{ij}^2}$$

(79)
\[ \theta = \frac{h}{L_{ij}} \bar{I}_{ij} \] 
\[ \eta = \frac{h}{L_{j(j+1)}} \bar{I}_{j(j+1)} \] 

A more accurate although much more complicated expression for \( K_{ij} \) may be determined by including one more bay and assuming that \( \theta_{(j+2)} = \theta_{(j+1)} \), as follows:

\[
K_{ij} = (A_1 + A_2) \left[ 3 + \frac{\alpha}{12} \left( \frac{3(A_1 A_2 - A_2^2)}{(A_1 + A_2)} + \frac{\tau}{4} \frac{(A_1 A_2 - A_2 A_1)}{(A_1 + A_2)} \right) + \frac{\xi}{4} \frac{A_2^2}{A_1} \right] \\
+ \frac{\xi}{4} \frac{(A_1 A_3 - A_2^2)}{(A_1 + A_2)} + \frac{\eta}{12} \left[ 3(A_1 - A_2) + \frac{\tau}{4} \frac{(A_1 A_2 - A_2 A_1)}{(A_1 + A_2)} + \frac{\xi}{4} \frac{A_2^2}{A_1} \right] \\
+ \frac{\xi}{4} \frac{(A_1 A_2 - A_2^2)}{(A_1 + A_2)} + \frac{\eta}{12} \left[ 3A_3 + \frac{\tau}{4} A_3^2 + \frac{\xi}{4} A_3 \left(A_1 + A_2\right) \right] \\
+ \frac{\tau}{4} \frac{(A_1 A_3 + A_2 A_3)}{(A_1 + A_2)} + \frac{\xi}{4} \frac{(A_1 A_2)}{(A_1 + A_2)} + \frac{\alpha}{72} K_{i(i-1)} \left[ \frac{\eta}{12} \left( 3A_1 + \frac{\tau}{4} (A_1 A_3 - A_2^2) + \frac{\xi}{4} A_1 (A_1 + A_2) \right) \right] + \]
5.3.1 Slope-Deflection Coefficients

Under the factored gravity loads alone, the bending moments in each composite beam will be symmetrical with fixed-end moments developing at each joint. The moments of inertia in the positive moment regions of beam \( ij \) are shown in Fig. 15(a). The length of each negative moment region is designated \( bL_{ij} \), where \( L_{ij} \) is the total length of beam \( ij \). The coefficients \( A_1, A_2 \) and \( A_3 \) in Eq. 77 may be found by deriving the slope-deflection equations for beam \( ij \), Fig. 15(a), which yield

\[
A_1 = A_3 = \frac{12C_1}{4C_1^2 - C_2^2} \quad \text{(82)}
\]

\[
A_2 = \frac{6C_2}{4C_1^2 - C_2^2} \quad \text{(83)}
\]
where
\[
C_1 = \frac{I_{ij}}{I_{ij}} - (1-b)^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] + b^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] \tag{84}
\]

\[
C_2 = \frac{I_{ij}}{I_{ij}} - (1-b)^2 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] (2b+1) + b^2 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] (3-2b) \tag{85}
\]

If the negative moment at joint \( j \) is numerically larger than the negative moment at joint \( i \), the moments of inertia in these two regions will be distributed as shown in Fig. 15(b). The length of the shortest negative moment region is designated \( a \) while that of the longest negative moment region is shown as \( b \). The slope-deflection equations yield

\[
A_1 = \frac{12C_1}{4C_1C_3-C_2^2} \tag{86}
\]

\[
A_2 = \frac{6C_2}{4C_1C_3-C_2^2} \tag{87}
\]

\[
A_3 = \frac{12C_3}{4C_1C_3-C_2^2} \tag{88}
\]
5.3.1

where

\[ C_1 = \frac{I_{ij}}{I_{ij}} - (1-b)^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] + a^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] \]  

(89)

\[ C_2 = \frac{I_{ij}}{I_{ij}} - (1-a)^2 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] (2a+1) + b^2 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] (3-2b) \]  

(90)

\[ C_3 = \frac{I_{ij}}{I_{ij}} - (1-a)^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] + b^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] \]  

(91)

If positive moment exists at joint i and negative moment at joint j, the moments of inertia in these two regions will be distributed as shown in Fig. 15(c). The length of the negative moment region is designated as \( bL_{ij} \). For this case the slope-deflection coefficients \( A_1, A_2 \) and \( A_3 \) will also be given by Eqs. 86 to 88 where

\[ C_1 = \frac{I_{ij}}{I_{ij}} - (1-b)^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] \]  

(92)

\[ C_2 = 1+b^2 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] (3-2b) \]  

(93)
5.3.1

\[ C_3 = 1 + b^3 \left[ \frac{I_{ij}}{I_{ij}} - 1 \right] \]  

(94)

Similar expressions may be obtained for the coefficients \( B_\perp \) and \( B_2 \).

The initial restraint coefficient to the left of joint \( j \), \( K_{ji} \), is related to \( K_{ij} \). The stress resultants for beam \( ij \) can be written as

\[ M_{ij} = \frac{E I_{ij}}{L_{ij}} \left[ A_1 \theta_i + A_2 \theta_j \right] = K_{ij} \frac{E I_{ij}}{L_{ij}} \theta_i \]  

(95)

\[ M_{ji} = \frac{E I_{ij}}{L_{ij}} \left[ A_3 \theta_j + A_2 \theta_i \right] = K_{ji} \frac{E I_{ij}}{L_{ij}} \theta_j \]  

(96)

Thus \( K_{ij} \theta_i = A_1 \theta_i + A_2 \theta_j \)  

(97)

\[ K_{ji} \theta_j = A_3 \theta_j + A_2 \theta_i \]  

(98)

Equations 97 and 98 yield

\[ K_{ji} = A_3 \left[ \frac{K_{ij} + \frac{A_2^2}{A_3} - A_1}{K_{ij} - A_1} \right] \]  

(99)
5.4

A similar expression can be derived for $K_{i(i-1)}$

5.4 Reduced Restraint Coefficients

The reduced restraint coefficients for composite beams may be evaluated using the concepts discussed in Art. 4.3. As before, it will be considered sufficiently accurate to recompute $K_{ij}$ based only on the plastic hinges which form within the sway subassemblage and at the top of the columns below joints $(i-1)$ and $j$ as shown in Fig. 13. Referring to the numbered locations of plastic hinges shown in Fig. 13, and assuming that 3 and 6 will form before 2 and 5, respectively, (Art. 2.2) then the reduced restraint coefficients can be determined as follows:

1. 1 occurs before 3: Since additional moment cannot be developed in beam $i(i-1)$ at joint $(i-1)$ that end may be considered pinned. The relationship between $M_{i(i-1)}$ and $\theta_i$ may be determined from the general slope deflection equations.

\[
M_{i(i-1)} = \frac{E I_{i(i-1)}}{L_{i(i-1)}} \left[ \frac{-\theta_i}{3} + \frac{\theta_i}{2} \right] (100)
\]
\[ M_{i(i-1)} = \frac{EI_{i(i-1)}}{L_{i(i-1)}} \left[ \bar{A}_1 \theta_{i(i-1)} + \bar{A}_2 \theta_i \right] \]  

(101)

where the coefficients \( \bar{A}_1, \bar{A}_2 \) and \( \bar{A}_3 \) may be determined as discussed in Art. 5.3.1 with the necessary change in subscripts where referring to beam \( i(i-1) \) instead of beam \( ij \). Solving Eqs. 95 and 96 yield

\[ K_{i(i-1)} = \bar{A}_3 - \frac{\bar{A}_2^2}{\bar{A}_1} \]  

(102)

2. 3 occurs after 1: \( K_{i(i-1)} \) reduces from the value given by Eq. 97 to zero.

3. 3 occurs before 1: \( K_{i(i-1)} \) reduces to zero.

4. 6 or 7 occurs: \( K_{ij} \) may be determined from the general slope-deflection equations for beam \( ij \), Eqs. 74 and 75. Solving these equations and considering beam \( ij \) to be pinned at \( j \)

\[ K_{ij} = \bar{A}_1 - \frac{\bar{A}_2^2}{\bar{A}_3} \]  

(103)
where $A_1$, $A_2$ and $A_3$ are given in Art. 5.3.1.

5. $5$ occurs after $6$ or $7$: $K_{ij}$ reduces from the value given by Eq. 98 to zero.

6. $4$ occurs: $K_{i(i-1)}$ and $K_{ij}$ remain unchanged from their values at the time $4$ develops.

5.5 Ultimate Strength Behavior of Composite Beams Under Combined Loads

Investigations\(^7\) have shown that continuous composite beams which are subjected to gravity loads may be designed on the basis of the plastic method for steel frames combined with simple rules for determining the ultimate moment of resistance of a cross-section when the slab is in tension or compression. The ultimate moment of such beams is determined in the positive moment regions by plastification of the steel beam and by crushing of the concrete slab over the full slab width. In the negative moment regions the ultimate moment is determined by plastification of both the steel beam and the longitudinal reinforcement over the slab width. The results of these investigations may be used for the design of composite beams in unbraced frames which are subjected to factored gravity loads. However, they may not be generally applicable for the design of composite beams in unbraced frames which are subjected to combined loads.
Generally, under factored gravity loads, positive bending moments will occur in the interior regions of the beams while negative bending moments will occur in the vicinity of the joints. Under combined loads, wind induced shear forces in the columns will result in concentrated bending moments applied to each joint of the frame. In the middle and lower stories this moment will be sufficiently large that positive bending moments will develop in the beam at the leeward side of a joint while negative bending moments will remain on the windward side. The interior regions of the beams will generally remain under positive bending moments. Such a distribution of bending moments will determine four main regions which must be considered in an investigation of the ultimate strength behavior of composite beams under combined loads. These regions may be defined as:

1. The interior regions which are subjected to positive bending moments. The leeward boundary of this region will be determined by the inflection point; the windward boundary by the first cross-section on which compressive forces act over the full slab width at the ultimate moment capacity.

2. The positive moment regions adjacent to the leeward side of a joint. Unless provision
is made to distribute compressive forces over a greater slab width, compressive forces will be developed only between the column face and the concrete slab, which is assumed to be in contact with the column.

3. The negative moment regions adjacent to an interior joint.

4. The negative moment region adjacent to the windward side of an exterior joint.

Regions 1 and 3 do not differ appreciably from similar regions of composite beams under gravity loads. It might be expected that the results of the previous investigations (Ref. 7) could be used to predict the ultimate moment capacity of cross-sections within these regions. However, these investigations can provide no information on the expected capacities within regions 2 and 4.

Since no studies are available on the ultimate strength behavior of composite beams under combined loads, a small pilot investigation was recently initiated\textsuperscript{37} to provide preliminary experimental data on the behavior of composite beams in regions 2, 3 and 4. This investigation was also intended to determine the feasibility of applying the concepts of plastic design and
ultimate strength theory to the design of composite beams under combined loads. Although the studies have not been completed the following tentative observations can be made:

1. The ultimate moment capacity at the column face in region 2 can be based on a cross-section consisting of the steel beam plus a width of slab equal to the column face width.

2. The ultimate moment capacity in region 3 can be based on the steel beam plus the longitudinal slab reinforcement in the slab.

3. In region 4 the plastic moment capacity of the steel beam will determine the ultimate moment capacity. The longitudinal slab reinforcement cannot be used. (This may not be the case if the slab overhangs the exterior column).

4. Considerable rotation capacity exists at the plastic hinge locations suggesting that plastic analysis may be used.
6. LOAD-DEFLECTION BEHAVIOR OF A STORY

6.1 Load-Deflection Curve of a Sway Subassemblage

Procedures were developed in Chapter 3 which, with the aid of prepared design charts (Ref. 28) could be used to determine the complete load-deflection curve for the restrained column in a half-story assemblage. It was shown that it was only necessary to evaluate the restraining moment, \( M_r \), corresponding to each change in restraint stiffness, \( k \), and then to determine the value of constant restraining moment, \( M'_r \), corresponding to each change in \( k \). Solutions for \( k \) and \( M_r \) were presented in Chapters 4 and 5. A procedure for determining \( M'_r \) will now be presented which will enable the load-deflection curve of a restrained column to be determined. The load-deflection curve of the restrained column will represent the load-deflection curve of the sway subassemblage containing the restrained column.

6.1.1 Evaluation of \( M'_r \) in a Sway Subassemblage With Steel Beams

The bending moments in an interior sway subassemblage which result from gravity loads alone on the beams are shown by
the dashed curves in Fig. 16. The maximum negative moments at the end of each beam will equal the fixed-end moments (Art. 2.2).

Assume for the present that the half story assemblage containing this sway subassemblage, as well as the distribution of the gravity loads on all the beams of the assemblage, is symmetrical. Then the initial sway deflection of the half-story assemblage or of any sway subassemblage within the assemblage will be zero whether the sway subassemblages are symmetrical or not. This implies the existence of initial horizontal shear forces acting on each sway subassemblage to maintain a zero initial sway condition. These forces will result from the interaction of the sway subassemblages and need not be considered when determining the load-deflection behavior of a particular sway subassemblage. The influence of initial column top rotation and the influence of initial sway deflection will be discussed in Chapter 7.

Consider now the effect of a small increment of lateral shear force \( \delta Q \) acting from the left on the sway subassemblage shown in Fig. 16. The fixed-end moments at the leeward ends of the beams will be increased by \( \delta M_{i(i-1)} \) and \( \delta M_{ji} \) while at the windward ends the fixed-end moments will decrease by \( \delta M_{(i-1)i} \) and \( \delta M_{ij} \). These small increments in moment will be related to the small increments of joint rotation as follows:
\[
\delta M_{ji} = K_{ji} \frac{E I_{ij}}{L_{ij}} \delta \theta_j \\
\delta M_{ij} = K_{ij} \frac{E I_{ij}}{L_{ij}} \delta \theta_i \\
\delta M_{i(i-1)} = K_{i(i-1)} \frac{E I_{i(i-1)}}{L_{i(i-1)}} \delta \theta_i \\
\delta M_{(i-1)i} = K_{(i-1)i} \frac{E I_{i(i-1)}}{L_{i(i-1)}} \delta \theta_{(i-1)}
\]

where \(K_{ji}, K_{ij}, \text{etc.},\) are the initial restraint coefficients which were determined in Chapter 4. The joint rotations \(\delta \theta_j\) and \(\delta \theta_i\) are related by the restraint coefficient \(K_{ij}\). This relationship can be determined from Eqs. 66, 67, and 68. Hence,

\[
\delta \theta_j = \left[ \frac{K_{ij} - 4}{2} \right] \delta \theta_i
\]

Similarly

\[
\delta \theta_{(i-1)} = \left[ \frac{2}{K_{(i-1)i} - 4} \right] \delta \theta_i
\]
The small increments in moment which are given by Eq. 104 to 107 can now all be written in terms of the small increment of joint rotation $\delta \theta_i$ at joint $i$ as follows:

\[
\delta M_{ji} = K_{ji} \frac{E I_{ij}}{L_{ij}} \left[ \frac{K_{ij} - 4}{2} \right] \delta \theta_i \quad (110)
\]

\[
\delta M_{ij} = K_{ij} \frac{E I_{ij}}{L_{ij}} \delta \theta_i \quad (111)
\]

\[
\delta M_{i(i-1)} = K_{i(i-1)} \frac{E I_{i(i-1)}}{K_{i(i-1)}} \delta \theta_i \quad (112)
\]

\[
\delta M_{(i-1)i} = K_{(i-1)i} \frac{E I_{(i-1)i}}{L_{(i-1)i}} \left[ \frac{2}{K_{(i-1)i} - 4} \right] \delta \theta_i \quad (113)
\]

Since the initial restraint coefficients, $K$, will be constant in Eqs. 110 to 113 prior to the formation of the first plastic hinge in the sway subassemblage, the increments of moment, $\delta M$, need not be small. Referring to Fig. 13 and assuming, as before, that plastic hinges 3 and 6 will form before plastic hinges 2 and 5, respectively, then $\delta M_{ji}$ and $\delta M_{i(i-1)}$ can be taken equal to the increments in moment required to form the plastic hinges at 6 and 3, respectively. The solutions of Eqs. 110 and 112 will yield two values for the increment of joint rotation $\delta \theta_i$. 

the minimum value will correspond to the formation of the first plastic hinge. The corresponding value of \( M_r' \) will then be given by

\[
M_r' = p M_{pc}
\]  \hspace{1cm} (114)

where

\[
p = k_i \theta_i \hspace{1cm} (0 \leq p \leq 2.0) \]  \hspace{1cm} (115)

Substitution of the minimum value of \( \delta \theta_i \) found above into Eqs. 110 to 113 will determine a set of moment increments which when added to the fixed-end moments will yield the total moment at the end of each beam corresponding to the formation of the first plastic hinge.

One or more values of initial restraint coefficient, \( K_i \), can now be reduced as was discussed in Art. 4.3. The reduced values are now used in Eqs. 110 to 113 when determining the second plastic hinge in the sway subassemblage. A procedure similar to the one described above is followed to determine each plastic hinge in the sway subassemblage and is further described in Ref. 26.

The end result after the sway subassemblage has been reduced to a mechanism will be values of \( \theta_i \) and \( k_i \) which will determine a set of values of \( M_r \) and \( M_{r}' \). The load-deflection
curve of the restrained column and thus of the sway subassemblage can then be determined as discussed in Chapter 3 from the appropriate design chart (Ref. 28) and the previously calculated values of $M_r$ and $M_r'$.

An alternate procedure to that described above could be used which would be more suitable for electronic computation. The joint rotation could be incremented by arbitrarily small values of $\delta \theta_i$ and the corresponding increments in moment computed from Eqs. 110 to 113. Using the original fixed end moments, the total bending moments in the sway subassemblage could then be computed after each increment, $\delta \theta_i$, and checked for the formation of plastic hinges. Following the formation of a plastic hinge, Eqs. 110 to 113 would be modified by substituting the reduced values of restraint coefficients, $K$. The joint rotation could again be incremented and this procedure continued until zero restraint stiffness had been obtained. The result would be a set of $\theta_i$ and $k_i$ values from which a set of $M_r$ and $M_r'$ values could be determined as before, for use with the appropriate design chart.

6.1.2 Evaluation of $M_r'$ in a Sway Subassemblage With Composite Beams

Equations 104 to 107 will also be valid for this case except that the restraint coefficients, $K$, will continuously
vary between the formation of consecutive plastic hinges as discussed in Chapter 5. The relationship between \( \delta \theta_j \) and \( \delta \theta_i \) can be determined from Eqs. 97, 98, and 99 so that

\[
\delta \theta_j = \left[ \frac{K_{ij} - A_1}{A_2} \right] \delta \theta_i
\]  

(115)

Similarly, if for beam \((i-1)i\)

\[
K(i-1)i \theta(i-1) = \bar{A}_1 \theta(i-1) + \bar{A}_2 \theta_i
\]

(116)

and

\[
K_i(i-1) \theta_i = \bar{A}_3 \theta_i + \bar{A}_2 \theta(i-1)
\]

(117)

then

\[
K_i(i-1) = \bar{A}_3 \left[ \frac{K(i-1)i + \frac{\bar{A}_2^2}{\bar{A}_3} - \bar{A}_1}{K(i-1)i - \bar{A}_1} \right]
\]

(118)

and

\[
\delta \theta(i-1) = \left[ \frac{\bar{A}_2}{K(i-1)i - \bar{A}_1} \right] \delta \theta_i
\]

(119)

The increments in moment given by Eqs. 104 to 107 can now be written in terms of \( \delta \theta_i \) as follows

\[
\delta M_{ji} = K_{ji} \frac{E I_{ij}}{L_{ij}} \left[ \frac{K_{ij} - A_1}{A_2} \right] \delta \theta_i
\]

(120)
The slope-deflection coefficients $A_1$ and $A_2$ may be found from the equations given in Art. 5.2.1. Similar equations may be used to determine the coefficients $\bar{A}_1$ and $\bar{A}_2$ with appropriate changes in subscripts to refer to beam $(i-1)i$ instead of beam $ij$.

The procedure for determining $M_r'$ will be the same as that discussed in Art. 6.1.1 although it will be found desirable to increment $\delta \theta_i$ rather than $\delta M$ due to the variable nature of the restraint coefficients $K$ between the formation of consecutive plastic hinges.

6.2 Construction of a Typical Load-Deflection Curve

Figure 17 illustrates the method of constructing a typical load-deflection curve for an interior sway subassemblage with steel restraining beams. Assume that an analysis of the sway subassemblage which is shown in Fig. 17(b) has been made. Three plastic hinges were required to form a mechanism and they
occurred at points $a$, $b$, and $c$ in that order. The analysis showed that the initial restraint stiffness was $k_1$ and that the first plastic hinge formed at a joint rotation $\theta_1$ so that $p_1 = k_1 \theta_1$ (Eq. 115). Similarly, prior to the second and third plastic hinges the restraint stiffness was found to be $k_2$ and $k_3$ respectively, and it was found that the second and third plastic hinges formed at joint rotations of $\theta_2$ and $\theta_3$. Therefore $p_2 = k_2 \theta_2$ and $p_3 = k_3 \theta_3$. The set of values of $k$ and $\theta$ and the set of values of $M_r$ and $M'_r$, which can be generated from them completely describe the load-deflection behavior of the restrained column or of the sway subassemblage.

A design chart can now be selected from the charts given in Ref. 28 which will correspond to the axial load ratio $P/P_y$, and slenderness ratio, $h/r$, of the restrained column. The set of $M_r$ values previously determined will define the three load-deflection curves $0-e$, $0-f$, and $0-g$ which are contained in the design chart and also shown in Fig. 17(a). Similarly, the set of $M'_r$ values will define the three sloping straight lines in Fig. 17(a) (and in the design chart) which intersect the vertical axis at $p_1/2$, $p_2/2$, and $p_3/2$. The initial segment $0-a$ of the load-deflection curve is shown in Fig. 17(a). This segment terminates at point $a$ which corresponds to the formation of the first plastic hinge at point $a$ in Fig. 17(b). The second segment of the load-deflection curve is shown as $a-b$ in Fig. 17(a)
where point $b$ corresponds to the formation of the second plastic hinge. This segment is obtained by translating segment $a'-b'$ of curve $0-f$ as demonstrated in Chapter 3. Similarly, segment $b-c$ is obtained by translating segment $b''-c''$ of curve $0-g$ and point $c$ corresponds to the formation of the third plastic hinge and a mechanism. The final segment $c-d$ of the load-deflection curve is the second-order plastic mechanism curve and follows the straight line $M_{b3}' = p_3 M_{pc}$.

The same procedure may be followed when constructing the load-deflection curve for a sway subassemblage with composite restraining beams. In this case, however, the values of $M_b$ and $M_c$ will be determined as a function of the increasing values of lateral shear force, $Q$, as well as with the formation of plastic hinges in the sway subassemblage. Consequently, the load-deflection curve will consist of more segments; the number will depend, of course, upon the degree of accuracy required.

A typical load-deflection curve is shown as curve $a-b-c-d$ in Fig. 18. As before, plastic hinges are assumed to occur at points $a$, $b$, and $c$ in that order. Only 2 or 3 segments were chosen between each of these points for clarity. The segments of the load-deflection curve and the corresponding segments of restrained column curves are shown by heavier lines.
Non-dimensional load-deflection curves similar to those shown in Figs. 17 and 18 must be constructed for each sway subassembly in the half-story assemblage. Before combining these curves to obtain the load-deflection curve for the half-story assemblage, it will be necessary to transform them to $Q$ versus $\Delta/h$ curves by multiplying the ordinates of each curve by the appropriate values of $2M_{pc}/h$.

6.3 Load-Deflection Curve of a Story

Figure 19 illustrates the method of combining the load-deflection curves of each of the sway subassemblies in a half-story assemblage. The method of determining the sway subassemblies which was discussed in Art. 2.6 requires that the ordinates $Q$, corresponding to a constant value of deflection index $\Delta/h$ be added algebraically to determine the total shear resistance $\Sigma Q$ of the half-story assemblage. It was shown in Art. 2.5 that the load-deflection curve $\Sigma Q$ versus $\Delta/h$ illustrated in Fig. 19 will be a conservative estimate of the actual load-deflection curve of the story containing the sway subassemblies providing all the assumptions made in the analysis are valid.

The sequence of formation of plastic hinges in the story, the maximum shear resistance, the shear resistance and sway deflection at working load, the mechanism load and deflection
etc., may all be obtained from this load-deflection curve. One or more of these may be used to determine the adequacy of the preliminary design of the story which was discussed in Chapter 1.
7. Future Research

The philosophy which has guided the development of the sway subassemblage method of analysis presented in this dissertation has been consistent with the purpose and scope stated in Art. 1.7. That is, an approximate method of analysis has been evolved which will be suitable for the combined load analysis of the middle and lower stories of unbraced frames. However, it is inevitable that in such a work as this, a few simplifying assumptions must be made. Therein lies the basis for much of the future research which should be undertaken.

Both analytical and experimental research will be required. Of immediate concern are the experimental studies of restrained columns permitted to sway and of one-story assemblages. Further, analytical studies should then be carried out in order to compare the load-deflection curve of a story of an unbraced frame with the predictions of the sway subassemblage method. Such studies would be similar to those of Ref. 17. Experimental studies of full scale unbraced frames would be virtually impossible to perform efficiently and economically. Studies should also be directed towards the removal of certain assumptions if they can be shown to be unnecessary or towards the introduction of further simplifying assumptions if the method of analysis would benefit from them. Efforts should be made to increase the
efficiency in applying the method and to the full use of electronic computation in obtaining the desired load-deflection curves.

7.1 Analytical Studies

A summary of a few of the areas which could be studied analytically would include the following:

1. The influence of initial sway and initial joint rotation on the load-deflection curve is required. A method of including their effect is needed. One approach may be to perform a moment distribution (perhaps of limited extent) using only the factored gravity loads, while providing horizontal holding forces to maintain zero sway. The resulting moments would be used in Chapter 6 instead of the moments assumed in Art. 2.2 when determining the load-deflection curve of a story. The distribution of holding forces would also be added algebraically to the distribution of forces \( Q \) found in the analysis. This procedure would account for the increase or decrease of the shear resistance of the story due to the initial sway deflection of the frame.

2. The influence of a plastic hinge in the restrained column should be studied further. When a plastic
hinge is allowed to form at the top of a restrained column before the restraint stiffness reaches zero, the additional restraining capacity of the adjacent beams is wasted. The maximum strength and deformation capacity of a story results when plastic hinges 3 and 5 in Fig. 13 form before 4. This implies that all the columns should remain elastic up to the maximum load or the mechanism load.

3. Reference 17 and other unpublished work elsewhere suggest that differential column shortening may have a detrimental effect on the lateral load resistance of tall unbraced frames. Its effects should be studied further and a procedure developed which would allow its inclusion in the sway subassemblage method of analysis.

4. Second-order elastic-plastic studies using the procedures described in Ref. 17 should be made to further substantiate the assumption regarding the position of the inflection points in the columns (Art. 2.4).

5. The distribution of axial forces in the columns which was suggested in Art. 2.4 should be studied further. Other distributions should be considered and a comparison of results made. It may also be
possible to vary the distribution of axial forces with increasing lateral load and to use more than one design chart in determining the load-deflection curve of a particular restrained column.

6. Additional theoretical studies should be made concerning the ultimate strength behavior of composite beams which are subjected to combined loads.

7. Preliminary studies have shown that the sway subassemblage concept can be extended to provide information on which the necessary revisions of the preliminary design can be made. Such an extension could allow revisions based on strength, deflection, and economy.

7.2 Experimental Studies

The first objective would be to obtain experimental data on the behavior of restrained columns where both constant and variable restraint stiffness was provided. Of particular interest, would be the lateral-load versus sway-deflection behavior of these restrained columns. In effect, such column tests would also be tests of sway subassemblages, the only difference being the boundary conditions at the far ends of the restraining beams. In the restrained column tests, these ends would likely be pinned. Therefore, the results of such tests
could be extrapolated to predict the experimental behavior of sway subassemblies with the realistic boundary conditions imposed.

The second objective would be to obtain experimental data on the behavior of a one-story assemblage. This behavior would be compared with the predictions of the sway subassemblage theory and also with predictions based on the previous experimental study or restrained columns. Of particular interest would again be the load-deflection versus sway-deflection behavior of the one-story assemblage.

A listing of individual problems to be studied experimentally would include:

1. The load-deflection behavior and the failure characteristics of restrained columns permitted to sway considering both constant and variable restraint stiffness.

2. The load-deflection behavior and the failure characteristics of the restraining beams.

3. The influence of initial gravity load moments on the load-deflection behavior of a restrained column.

4. The load-deflection behavior and failure characteristics of a one-story assemblage.
5. The influence of the distribution of axial loads in the columns on the load-deflection behavior of the one-story assemblage.

6. The influence of initial gravity load moments on the load-deflection behavior of the one-story assemblage.

7. Evaluation of the interaction of the restraint provided to the columns in the one-story assemblage and the formation of plastic hinges in the beams and columns.

8. Experimental studies of the ultimate strength of composite beams which are subjected to combined loads.

Preparations are now being made to conduct an experimental investigation on restrained columns permitted to sway. The problems to be investigated will therefore include the first three listed above. The restraining beams would be non-composite. It is hoped that on the conclusion of this investigation that further studies would be made of the remaining problems.
8. **SUMMARY**

The development of the moment balancing method \(10,16\) which can be used for the preliminary design phase of unbraced frames was a significant first step. It is ideally suited as a preliminary design method since it can include an approximate P\(\Delta\) effect. This is accomplished by estimating the sway deflection of each story of the frame at either the maximum load capacity or the mechanism load. After the determination of the preliminary beam and column sizes, a sway analysis should be performed to verify the estimated sway deflection. In addition to this, the sway deflection should be calculated at working loads.

The sway subassemblage method of analysis developed in this dissertation will enable the determination of the approximate lateral-load versus sway-deflection curve of a story in the middle and lower stories of an unbraced frame which is subjected to combined loads. Such a curve will allow the verification of the sway estimates used in the preliminary design phase. In addition, the working load sway deflection, the maximum lateral load capacity, the mechanism load, etc. are all determined. Therefore, the sway subassemblage method can be used for the second step in the complete design process.

The sway subassemblage method of analysis is based on the concept of sway subassemblies and uses directly the results
of previous research of the strength and behavior of restrained columns permitted to sway. In the analysis a story with known member sizes is subdivided into a number of sway subassemblages, each consisting of a restrained column and either one or two restraining beams. The base of the restrained column is pinned at the assumed inflection point at mid-height of the column. The near ends of the beams are rigidly attached to the top of the column. The far ends are subjected to restraints which are to simulate the action of the remaining members in the story. The restraining beams may be of steel or of composite steel-concrete construction.

The procedures developed in this dissertation allow the combined load analysis of each sway subassemblage to be made. The resulting load-deflection curves are determined using specially prepared design charts. The load-deflection curves of all the sway subassemblages in a story are then combined to give the load-deflection curve of the story.

The sway subassemblage method of analysis accounts for the reduction in strength of a frame due to PA effects. It also considers plastification of the columns including residual stresses as well as plastic hinges in the beams. A recent pilot study on the ultimate strength of composite beams under combined loads has provided experimental evidence that a combination of
plastic analysis and ultimate strength theory may be used for the design of frames containing composite beams. The approximate sequence of formation of plastic hinges in a story may also be determined from the analysis.

The sway subassemblage method as developed in this dissertation does not consider unbraced frames with significantly large initial sway deflections under factored gravity loads alone. The effect of differential column shortening on the strength and deflection of the frame is also not considered. These effects and others are suggested as possible areas for future analytical and experimental research.

The third phase of the complete design process - the means of making the necessary revisions to the preliminary design, based on strength, deflection, and economy - is yet to be accomplished. It is suggested that the sway subassemblage method can be extended in order to provide information on which the necessary revisions can be made.
9. NOMENCLATURE

A, Ā  Slope-deflection coefficient
B  Slope-deflection coefficient
d  Depth of section
E  Modulus of Elasticity
f  Shape factor
H  Horizontal wind load
h  Story height
I, Ī  Moment of Inertia
K  Restraint coefficient
k, ā  Restraint stiffness
L  Span length
Mp  Plastic moment
Mpc  Reduced plastic moment (modified plastic moment to account for axial forces).
Mr  Restraining moment
Mr'  Maximum restraining moment
m  Story
n  Level
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Axial force</td>
</tr>
<tr>
<td>$P_y$</td>
<td>Yield stress level of axial force</td>
</tr>
<tr>
<td>Q</td>
<td>Horizontal force</td>
</tr>
<tr>
<td>q</td>
<td>Moment ratio</td>
</tr>
<tr>
<td>$r_x$</td>
<td>Radius of gyration about x axis</td>
</tr>
<tr>
<td>S</td>
<td>Section Modulus</td>
</tr>
<tr>
<td>w</td>
<td>Distributed vertical load per unit length</td>
</tr>
<tr>
<td>$h/r_x$</td>
<td>Slenderness ratio about x axis</td>
</tr>
<tr>
<td>$P/P_y$</td>
<td>Axial load ratio</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Angle between tangent and chord of a column</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Relative lateral deflection of two consecutive stories</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Joint rotation</td>
</tr>
<tr>
<td>$\theta_p$</td>
<td>Joint rotation at $M = M_p$ in a sway subassemblage</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>Yield stress level</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Curvature</td>
</tr>
<tr>
<td>$\Delta/h$</td>
<td>Deflection index</td>
</tr>
</tbody>
</table>
10. FIGURES
**FIG. 1 INSTABILITY UNDER COMBINED LOADS**

- **LOAD H**
  - **SIMPLE PLASTIC - 1st ORDER**
  - **RIGID PLASTIC - 2nd ORDER**
  - **NO INSTABILITY**
  - **INSTABILITY**

- **Gravity Loads**
- **Wind Loads**

- **Maximum Load Capacity**
- **Reduction Due To \( P \Delta \) Moment**

- **DEFLECTION \( \Delta \)**
FIG. 2 DESIGN CONDITIONS FOR UNBRACED FRAMES
FIG. 3 POSSIBLE PLASTIC HINGE LOCATIONS AND FAILURE MECHANISMS

a) FRAME AND LOADING

b) MOMENT DUE TO GRAVITY LOAD

c) MOMENT DUE TO LATERAL LOAD

d) COMBINED MECHANISM

e) SWAY MECHANISM
FIG. 4 ONE-STORY ASSEMBLAGE
\[
\begin{align*}
M_{(n-1)A} &= -(\lambda_A \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)A} \frac{\Delta n-1}{2} \\
M_{(n-1)B} &= -(\lambda_B \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)B} \frac{\Delta n-1}{2} \\
M_{(n-1)C} &= -(\lambda_C \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)C} \frac{\Delta n-1}{2} \\
M_{(n-1)D} &= -(\lambda_D \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)D} \frac{\Delta n-1}{2}
\end{align*}
\]

FIG. 5  HALF-STORY ASSEMBLAGE
FIG. 6  THE SWAY SUBASSEMBLAGES
FIG. 7  THE RESTRAINED COLUMN IN A SWAY SUBASSEMBLAGE
FIG. 8 LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN WITH CONSTANT RESTRAINT STIFFNESS
FIG. 9 LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN WITH CONSTANT - ZERO RESTRAINT STIFFNESS
FIG. 10 LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN WITH CONSTANT - CONSTANT RESTRAINT STIFFNESS
FIG. 11 SUPERPOSITION OF LOAD-DEFLECTION CURVES
FIG. 12 DERIVATION OF INITIAL RESTRAINT COEFFICIENTS FOR STEEL BEAMS
FIG. 13  POSSIBLE PLASTIC HINGE LOCATIONS
FIG. 14 DERIVATION OF INITIAL RESTRAINT COEFFICIENTS FOR COMPOSITE BEAMS
FIG. 15 DERIVATION OF SLOPE-DEFLECTION COEFFICIENTS
FIG. 16 DISTRIBUTION OF BENDING MOMENTS IN THE SWAY SUBASSEMBLAGE
FIG. 17 CONSTRUCTION OF LOAD-DEFLECTION CURVE - STEEL BEAMS
FIG. 18 CONSTRUCTION OF LOAD-DEFLECTION CURVE - COMPOSITE BEAMS
Load-Deflection Curve of the Windward Sway Subassemblage

Load-Deflection Curve of an Interior Sway Subassemblage

Load-Deflection Curve of the Leeward Sway Subassemblage

Load-Deflection Curve of the Story

FIG. 19 CONSTRUCTION OF LOAD-DEFLECTION CURVE FOR A STORY
11. REFERENCES

1. Tall, et.al.  

2. Norris, C. H., and Wilbur, J. B.  

3. Beedle, L. S.  

4. Levi, V.  

5. Fritz Laboratory Staff  
   PLASTIC DESIGN OF MULTI-STORY FRAMES - LECTURE NOTES  
   Fritz Engineering Laboratory Report No. 273.20, Lehigh University, August 1965.

6. Yura, J. A.  

7. Daniels, J. H., and Fisher, J. W.  

8. American Institute of Steel Construction  

9. WRC-ASCE Joint Committee  

10. Hansell, W.  
    PRELIMINARY DESIGN OF UNBRACED MULTI-STORY FRAMES  
11. ASCE Subcommittee No. 31
WIND BRACING IN STEEL BUILDINGS (FINAL REPORT)

12. Horne, M. R.
INSTABILITY AND THE PLASTIC THEORY OF STRUCTURES,

13. Ostapenko, A.
Chapter 13, PLASTIC DESIGN OF MULTI-STORY FRAMES -
LECTURE NOTES, Fritz Engineering Laboratory Report
No. 273.20, Lehigh University, August 1965.

STRENGTH OF SINGLE STORY STEEL FRAMES, Proc. ASCE

15. Horne, M. R.
A MOMENT DISTRIBUTION METHOD FOR THE ANALYSIS AND
DESIGN OF STRUCTURES BY THE PLASTIC THEORY, Proc.

Chapters 14 and 16, PLASTIC DESIGN OF MULTI-STORY FRAMES-
LECTURE NOTES, Fritz Engineering Laboratory Report
No. 273.20, Lehigh University, August 1965.

17. Parikh, B. P.
THE ELASTIC-PLASTIC ANALYSIS AND DESIGN OF UNBRACED
MULTI-STORY STEEL FRAMES, Ph.D. Dissertation, Fritz
Engineering Laboratory Report No. 273.44, Lehigh
University, May 1966.

18. Heyman, J.
AN APPROACH TO THE DESIGN OF TALL STEEL BUILDINGS
Proc. Inst. Civil Engineers, Vol. 17, pp. 431-454,
1960.

The Design of Sway Frames in Britain
PLASTIC DESIGN OF MULTI-STORY FRAMES - GUEST LECTURES
Fritz Engineering Laboratory Report No. 273.46,
Lehigh University, August 1965.

20. Holmes, M., and Gandhi, S. N.
ULTIMATE LOAD DESIGN OF TALL STEEL BUILDING FRAMES
ALLOWING FOR INSTABILITY, Proc. Instn. Civil Engineers,
21. Stevens, L. K.

22. Stevens, L. K.
CONTROL OF STABILITY BY LIMITATION OF DEFORMATION
Proc. Instn. Civil Engineers, Vol. 19, May 1961,
DISCUSSION, Vol. 23, October 1962.

23. Gent, A. R.
THE DESIGN OF FRAME STRUCTURES CONSIDERING STRENGTH,
STABILITY AND DEFLECTIONS, Proc. Instn. Civil Engineers,

24. Lu, Le-Wu, Daniels, J. H.
Chapters 18 and 19, PLASTIC DESIGN OF MULTI-STORY FRAMES - LECTURE NOTES, Fritz Engineering Laboratory Report No. 273.20, Lehigh University, August 1965, Published by AISI.

25. Parikh, B. P., Daniels, J. H., and Lu, Le-Wu

26. Daniels, J. H., and Lu, Le-Wu

27. Daniels, J. H.


29. Levi, V., and others

Chapter 6, PLASTIC DESIGN OF MULTI-STORY FRAMES -
LECTURE NOTES, Fritz Engineering Laboratory Report
No. 273.20, Lehigh University, August 1965.

32. Horne, M. R.
SYMPOSIUM ON THE PLASTIC THEORY OF STRUCTURES -
Multi-Story Frames, British Welding Journal, Vol. 3,
No. 8, August 1956.

33. McNamee, B. M.
THE GENERAL BEHAVIOR AND STRENGTH OF MULTI-STORY
FRAMES UNDER GRAVITY LOADING, Ph.D. Dissertation,
Lehigh University, 1967, University Microfilms, Inc.
Ann Arbor, Michigan.

34. Levi, V., Driscoll, G. C., Jr., and Lu, Le-Wu
STRUCTURAL SUBASSEMBLAGES PREVENTED FROM SWAY, Proc.

35. Lu, Le-Wu
DESIGN OF BRACED MULTI-STORY FRAMES BY THE PLASTIC
METHOD, AISC Engineering Journal, Vol. 4, No. 1,
January 1967.

36. Lu, Le-Wu, Galambos, T. V., and others
Chapters 4 and 9, PLASTIC DESIGN OF MULTI-STORY FRAMES -
LECTURE NOTES, Fritz Engineering Laboratory Report
273.20, Lehigh University, August, 1965.

37. Daniels, J. H., and Fisher, J. W.
ULTIMATE STRENGTH BEHAVIOR OF COMPOSITE BEAMS UNDER
COMBINED LOADS, Fritz Engineering Laboratory Report
No. 338.3, Lehigh University, (In Preparation).

38. English, J. M.
DESIGN OF FRAMES BY RELAXATION OF YIELD HINGES, Trans.