Soil Mechanics and Theories of Plasticity

ON THE COULOMB YIELD SURFACE AND RATE OF DISSIPATION OF ENERGY

by

WAI-FAH CHEN

January 1968

Fritz Engineering Laboratory Report No. 355.1
SOIL MECHANICS AND THEORIES OF PLASTICITY

On The Coulomb Yield Surface and
Rate of Dissipation of Energy

by

WAH-FAH CHEN

This work has been carried out as part
of an investigation sponsored by The Institute
of Research, Lehigh University

Fritz Engineering Laboratory
Department of Civil Engineering
Lehigh University
Bethlehem, Pennsylvania

January 1968

Fritz Engineering Laboratory Report No. 355.1
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. THE FLOW RULE AND DISSIPATION FUNCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. APPENDIX - THE COULOMB YIELD SURFACE IN PRINCIPAL STRESS SPACE</td>
<td>6</td>
</tr>
<tr>
<td>3. REFERENCES</td>
<td>9</td>
</tr>
<tr>
<td>4. SYMBOLS</td>
<td>10</td>
</tr>
<tr>
<td>5. FIGURES</td>
<td>11</td>
</tr>
<tr>
<td>6. ACKNOWLEDGMENTS</td>
<td>15</td>
</tr>
</tbody>
</table>
1. THE FLOW RULE AND DISSIPATION FUNCTION

The purpose of this report is to show that the rate of dissipation of energy which is uniquely determined by the plastic strain rate is given by

\[ D = 2c \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \Sigma \hat{\varepsilon}_t \]  

(1)

for the idealized perfectly plastic soils obeying Coulomb's yield criterion and its associated flow rule \([1]^*\) where \(c\) is the cohesion, \(\theta\) is the angle of internal friction of the soil and \(\hat{\varepsilon}_t\) denotes a tensile principal component of the plastic strain rate tensor.

According to Coulomb's criterion, plastic flow can occur under a constant state of stress which is represented by a point on the right hexagonal pyramid equally inclined to the \(\sigma_1, \sigma_2, \sigma_3\) axes \([2]\) (Fig. 1), or for example, by a point on the hexagon in Fig. 2 which is the intersection of the pyramid with a plane perpendicular to the \(\sigma_2\)-axis and at

\[ \text{Numbers in brackets designate references at end of report.} \]
a distance $\sigma_2$ from the origin. Since the soil is isotropic, the principal axes of the plastic strain rate must coincide with the principal axes of stress and the principal components of the strain rate in the $\sigma_1$, $\sigma_2$, $\sigma_3$ directions will be denoted by $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$, $\dot{\varepsilon}_3$ in the $(\sigma_1, \sigma_2, \sigma_3)$ diagram.

The rate of dissipation of energy is given by

$$ D = \sigma_1 \dot{\varepsilon}_1 + \sigma_2 \dot{\varepsilon}_2 + \sigma_3 \dot{\varepsilon}_3 $$

or

$$ D = c \cot \theta (\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) \quad (2) $$

which is seen most easily if one takes the end of stress vector at the vertex of the pyramid.

Consider a point on the pyramid in Fig. 1 which does not coincide with an edge. The concept of perfect plasticity [3] requires that the associated plastic strain rate vector must be normal to the face of the pyramid. For example, on the face V-C-D, the associated plastic strain rate vector is normal to $\sigma_2$-axis so that $\dot{\varepsilon}_2 = 0$. Hence

$$ \dot{\varepsilon}_1 = -\dot{\varepsilon}_3 \tan^2 (\pi/2 - \theta/2) $$

from the geometrical relation of line C' - D' in Fig. 2. Substituting the strain rates into (2) the results of (1) is then followed. Here $\Sigma \dot{\varepsilon}_t = \dot{\varepsilon}_3 > |\dot{\varepsilon}_1|.$

For a stress point which coincides with an edge, however, the plastic strain rate vector must lie between the directions of the normals to the two faces of the pyramid which meet at the edge. For example, the plastic strain rate vectors drawn in the face bounded by the normals to the sides which meet at the corner of the hexagon, e.g., point C' in Fig. 2, are the projections on the plane of Fig. 2 of possible plastic
strain rates for stress points lying on the edge V - C of the yield pyramid in Fig. 1.

For the stress point \((\sigma_1, \sigma_2, \sigma_3)\) on the edge V - C, one can choose a neighboring point \([\sigma_1 + p, \sigma_2 + p, \sigma_3 + p \tan^2(\pi/4 - \theta/2)]\) on the same edge and since the two points should result in a same dissipation of energy, it follows that

\[
\dot{\varepsilon}_1 + \dot{\varepsilon}_2 = -\dot{\varepsilon}_3 \tan^2 \left(\frac{\pi}{4} - \frac{\theta}{2}\right)
\]  

(3)

substituting \(\dot{\varepsilon}_1 + \dot{\varepsilon}_2\) of (3) into (2), and again the result of (1) is followed because the components of the plastic strain rate on edge V - C are \(\dot{\varepsilon}_1 \leq 0, \dot{\varepsilon}_2 \leq 0, \dot{\varepsilon}_3 > 0\) and also \(\dot{\varepsilon}_3\) has the numerically largest value among those three components.

As for the special edges V - B, V - D, V - F, where two non-zero tensile components of the plastic strain rate exist, one can proceed in a similar way as for edge V - C and obtain the corresponding expression of (3). When, for example, edge V - B is selected, the expression is

\[
\dot{\varepsilon}_2 = - (\dot{\varepsilon}_1 + \dot{\varepsilon}_3) \tan^2 \left(\frac{\pi}{4} - \frac{\theta}{2}\right)
\]  

(4)

Again, the result of (1) is followed on account of (4).

In the same way, it can be shown that Equation (1) holds at every stress point of the pyramid since the components of the plastic strain rates satisfy the condition

\[
\tan^2 \left(\frac{\pi}{4} - \frac{\theta}{2}\right) \sum \dot{\varepsilon}_t + \sum \dot{\varepsilon}_c = 0
\]  

(5)
where $\dot{\varepsilon}_c$ denotes the principal compressive component of plastic strain rate tensor.

Tresca yield criterion may be considered as a soil for which $\phi = 0$ and $c = k$ where $k$ is the shear yield stress, it follows from the expression (1) that the rate of energy dissipation reduces to

$$D = 2k \max |\dot{\varepsilon}|$$  \hspace{1cm} (6)

where $\max |\dot{\varepsilon}|$ denotes the absolute value of the numerically largest principal component of the plastic strain rate. Expression (5) then becomes the familiar incompressibility condition

$$\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3 = 0$$  \hspace{1cm} (7)

Expression (6) for the rate of dissipation of energy was first obtained by Hodge and Prager [4] for the special case of plane stress, $\sigma_2 = 0$ and later extended by Shield and Drucker [5] to the general case.

For the particular case of plane strain, one of the principal components of the plastic strain rate is always zero, and the other two components have the relation

$$\dot{\varepsilon}_t = -\dot{\varepsilon}_c \tan \left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$  \hspace{1cm} (8)

so that the maximum rate of engineering shear strain is given by

$$\dot{\gamma}_{\text{max}} = \dot{\varepsilon}_t - \dot{\varepsilon}_c = \frac{2\dot{\varepsilon}_t}{1 + \sin \phi}$$  \hspace{1cm} (9)
thus for plane strain

\[ D = c \cos \phi \cdot \dot{\gamma}_{\text{max}} \]  

agreeing with the expression obtained by Drucker and Prager [1].
2. **APPENDIX**

**THE COULOMB YIELD SURFACE IN PRINCIPAL STRESS SPACE**

Shield [2], following upon related work by Drucker [6], extended Coulomb's Law of Failure in two-dimensional problems to a unique yield surface appropriate for the general treatment of three-dimensional problems. The purpose here is to outline a geometrical method of constructing such a surface in principal stress space showing that this yield surface is a right hexagonal pyramid equally inclined to the \( \sigma_1 \), \( \sigma_2 \), \( \sigma_3 \) axes (see Fig. 1). Its intersection with the plane of Fig. 2, perpendicular to the \( \sigma_2 \)-axis and at a distance \( \sigma_2 \) from the origin, is a hexagon. In the following the method of constructing the yield curve in the plane of Fig. 2 will be chosen. The value of \( \sigma_2 \) is fixed for the curve in the \((\sigma_1, \sigma_3)\) coordinate plane.

According to the Mohr's graphical representation of stresses in \((\sigma, \tau)\) coordinates, it is well-known that all points representing possible pairs of \((\sigma, \tau)\) values at a point of the soil are within the shaded curvilinear triangle bounded by the three principal stress circles as shown in Fig. 3(a). Failure of the soil can occur only when the largest of the circles touches the two straight lines while the intermediate principal stress can have any value between the largest and smallest principal stresses. The determination of the critical circle requires the consideration of the relative magnitudes of the three principal stresses. There are six possible orderings of the relative magnitudes of the stresses \( \sigma_1 \), \( \sigma_2 \), \( \sigma_3 \) which determine the six yield lines \( A'B' \), \( B'C' \), \( C'D' \), \( D'E' \), \( E'F' \) and \( F'A' \) as shown in Fig. 3. For
example, for the line C'D' in the figure, the stress points on the line having the ordering \( \sigma_3 \geq \sigma_2 \geq \sigma_1 \) and the critical circle is the one which passes through the points \( \sigma_3, \sigma_1 \), of Fig. 3(a). The geometrical relations shown in the Mohr diagram in this case give

\[
\sigma_3 - \sigma_1 = 2 \cos \phi - (\sigma_3 + \sigma_1) \sin \phi
\]

Equation (11) can be written

\[
\sigma_1 = \sigma_3 \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) - 2 \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right)
\]

This equation is represented by the straight line \( V' - 2 \) in the figure. Geometrically, Coulomb's law of failure states that the circles which represent critical states of stress can move to the right or to the left in the wedge-shaped region and have the lines \( M_0 M, M_0 M_1 \) as common envelope as shown in Fig. 3(a). Since the principal stress \( \sigma_2 \) has a fixed value, the critical circles can not move arbitrarily in the wedge-shaped region but are restricted to the right as well as to the left. The two right and left extreme circles are shown in the figure by dotted lines which correspond to the stress points C' and D' respectively on the straight line \( V' - 2 \). By proceeding similarly for the lines B'C', E'D', E'F', F'A', A'B' it can be found that the yield surface intersects the \( (\sigma_1, \sigma_3) \) plane along the hexagon A'B'-C'D'-E'F'-A'; and from the interchangeability of the \( \sigma_1, \sigma_2, \sigma_3 \) for example \( \sigma_3, \sigma_1 \) in (a), \( \sigma_2, \sigma_3 \) in (b) and \( \sigma_1, \sigma_2 \) in (c), it can be concluded that the yield hexagon in the figure is symmetrical in shape about the line \( \sigma_1 = \sigma_3 \). It can be seen now that the yield surface is
a right hexagonal pyramid equally inclined to the $\sigma_1$, $\sigma_2$, $\sigma_3$ axes and with its vertex $V$ at the point $\sigma_1 = \sigma_2 = \sigma_3 = c \cot \theta$ (see Fig. 1).

Clearly, such a pyramid is fully defined by the hexagon $A'B'C'D'E'F'A'$ in the $(\sigma_1, \sigma_3)$ coordinate plane. Figure 1 shows the hexagonal pyramid with the line $VO$ as its center line and every two faces of the pyramid opposite to each other is parallel to a corresponding axis. The stress point $V$ in the figure corresponds to a state of stresses $\sigma_1 = \sigma_2 = \sigma_3 c \cot \theta$ that is, the point $M_0$ of figure 3(a).
3. REFERENCES


4. SYMBOLS

\( c \)  
Cohesion

\( \phi \)  
Angle of internal friction

\( D \)  
Rate of dissipation of energy per unit volume

\( D_A \)  
Rate of dissipation of energy per unit area

\( \dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3 \)  
Principal component of the plastic strain rate tensor

\( \dot{\varepsilon}_t \)  
Tensile principal component of the plastic strain rate tensor

\( \dot{\varepsilon}_c \)  
Compressive principal component of the plastic strain rate tensor

\( \dot{\gamma}_{\text{max}} \)  
Maximum rate of engineering shear strain

\( \tau \)  
Shearing stress

\( \sigma \)  
Normal stress

\( \sigma_1, \sigma_2, \sigma_3 \)  
Principal component of the stress tensor
5. FIGURES
\[ \text{OA} = \text{OC} = \text{OE} = 2c \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \]

Fig. 1 The Coulomb Yield Surface
Fig. 2 Section of the Coulomb Yield Surface in Fig. 1.
The Yield Curve in a Plane Perpendicular to the 
\( \sigma_2 \)-axis and at a Distance \( \sigma_2 \) from the Origin

0-1 = 0-4 = 2c \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right)

0-2 = 0-3 = 2c \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right)
Fig. 3 Mohr's Representation of a Stress and the Coulomb Yield Criterion
6. ACKNOWLEDGMENTS

This report is part of a research project on Soil Mechanics and Theories of Plasticity carried out at Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania. Dr. L. S. Beedle is Director of the Laboratory. The project sponsor is the Institute of Research, Lehigh University. Professor G. R. Jenkins is Director of the Institute. The author is thankful for his support.

The author is appreciative of the encouragement of Dr. H. Y. Fang and also indebted to Mrs. Fielding for typing the manuscript.