CRACK PROPAGATION IN A STRIP
OF MATERIAL UNDER PLANE EXTENSION

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ABSTRACT

The problem of a uniformly propagating crack in a strip of elastic material is solved using the dynamic equations of elasticity in two-dimensions. Two specific conditions of loading on the strip with finite width are discussed. In the first case, the rigidly clamped edges are pulled apart in the opposite directions. The second case considers equal and opposite tractions applied to the crack surface. By varying the strip width to the crack length ratio, the amplitude of the dynamic stresses ahead of the running crack is determined as a function of the crack velocity. The local dynamic stresses are found to be lower than the corresponding static values for the displacement loading condition and higher for the stress loading condition. This effect becomes increasingly more important as the crack length to strip width ratio is enlarged. Numerical results for the dynamic crack opening displacements are also presented.

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INTRODUCTION

In a previous paper by Sih and Irwin [1], they analyzed a system of radial cracks emanating from a common point. The leading edges of these cracks terminate on a circular locus expanding at a constant speed. Fracture patterns of this type are commonly observed in the shattering of glass plates owing to projectile penetration. The amount of energy released by this system of expanding cracks was estimated by an approximate procedure in [1] which did not account for the coupling effects of dynamic unloading and crack-opening displacement. This is mainly because of the lack of an available method for solving the problem of a series of equally spaced parallel cracks ending at a moving line. From symmetry considerations, the foregoing problem is equivalent to that of a single crack moving in a strip whose boundaries are fixed. The formulation presented in this paper is specifically developed to treat this class of problems.

Previous work [2-5] on dynamic crack propagation has mostly been concerned with the geometry of an infinite medium. The influence of neighboring boundaries on crack speeds presents additional complexities to the problem which in many instances cannot be neglected. Sih and Chen [6] have examined this effect in the tearing of strip specimens. Results on the energy required to tear a specimen at a given rate were reported. Unfortunately the method...
of analysis in [6] based on the usage of the Schwarz-Christoffel transformation is not applicable to elastodynamic problems in plane elasticity.

Considered in this paper is the model of a constant length crack moving at a uniform velocity in a strip of finite width the boundaries of which are subjected to displacement and stress conditions. The assumption of a constant length crack is expected to be reasonable for long cracks in a narrow strip where the reflected stress waves from the boundaries are of primary concern as compared to the waves generated by the trailing end of the crack. Detailed studies are made on the variations of the intensity of the local dynamic stress field with parameters such as strip width, crack velocity, etc.

ELASTODYNAMIC EQUATIONS IN TWO DIMENSIONS

Consider a strip of width 2h with a Yoffé [2] crack of length 2a travelling along a straight line bisecting the strip. The Yoffé crack is defined as one which is self-sealing at the trailing end by an amount equal to the extended portion at the leading edge of the crack. This preserves the condition of a constant length crack at all time t. The position of the crack at a given time t is referred to the stationary coordinates axes X, Y by the distance vt with v being the crack velocity. Refer to Figure 1 for the moving crack geometry.
The two dimensional equations of dynamic elasticity which neglects wave reflection* in the direction normal to the XY-plane can be formulated in terms of two scalar functions \( \phi \) and \( \psi \) each of which is a function of \( X, Y \) and \( t \). They are related to the displacement components as

\[
[u_X, v_Y, w_Z] = \left[ \frac{\partial \phi}{\partial X} + \frac{\partial \psi}{\partial Y}, \frac{\partial \phi}{\partial Y} - \frac{\partial \psi}{\partial X}, 0 \right]
\]  

(1)

Satisfaction of the equations of motion requires that for plane strain \( \phi \) and \( \psi \) are to be governed by the following pair of wave equations

\[
\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} = \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2}, \quad \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2}
\]

(2)

in which \( c_1 \) is the compression (irrotational) wave speed and \( c_2 \) the shear (equivoluminal) wave speed in an infinitely extended medium. They are related to the elastic constants by the expressions

\[
c_1 = \left[ \frac{(\lambda + 2\mu)}{\rho} \right]^\frac{1}{2}, \quad c_2 = \left( \frac{\mu}{\rho} \right)^\frac{1}{2}
\]

(3)

Here, \( \lambda \) and \( \mu \) are the Lamé coefficients and \( \rho \) is the mass density of the elastic medium. Once \( \phi \) and \( \psi \) are determined from eq. (5), the two-dimensional stress components can be found from

* For this reason, dynamic stress solutions are not meaningful when applied to plate specimens in which wave reflection in the thickness direction cannot be ignored.
\[ \sigma_x = \lambda \nabla^2 \phi + 2\mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right) \]

\[ \sigma_y = \lambda \nabla^2 \phi + 2\mu \left( \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right) \]

\[ \sigma_{xy} = \mu \left( 2 \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \]

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplacian operator in two variables.

For a constant velocity crack, it is convenient to introduce a Galilean transformation

\[ x = X - vt, \quad y = Y, \quad t' = t \]

with \( x, y \) being the translating coordinate system attached to the moving crack. In the transformed system, the wave equations become independent of the time variable \( t' \), i.e.,

\[ s_1 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \quad s_2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]

The parameters \( s_1 \) and \( s_2 \) are defined as

\[ s_1 = \left[ 1 - (v/c_1)^2 \right]^{\frac{1}{2}}, \quad s_2 = \left[ 1 - (v/c_2)^2 \right]^{\frac{1}{2}} \]

In a similar fashion, the displacement and stress components can also be transformed into the new coordinate system. The results are
This completes the preliminary mathematical formulation which will be used subsequently to solve the dynamic crack problem.

DESCRIPTION OF LOADING

The method of solution can be best illustrated by solving two basic problems involving displacement and stress loading conditions.

Case (1) - Let the edges of the strip be clamped rigidly and displaced by an amount \( v_0 \) in the direction normal to the propagating crack and hence

\[
\begin{align*}
\sigma_x &= \lambda \nabla^2 \phi + 2\mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right) \\
\sigma_y &= \lambda \nabla^2 \phi + 2\mu \left( \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right) \\
\sigma_{xy} &= \mu \left( 2 \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)
\end{align*}
\]

Because of symmetry, the following conditions on the stress components may be written down

\[
\begin{align*}
\sigma_x(x, \pm h) &= 0, \quad -\infty < x < \infty \\
v_y(x, 0) &= 0, \quad |x| > a; \quad v_y(x, \pm h) = v_0, \quad -\infty < x < \infty
\end{align*}
\]
\[ \sigma_y(x,0) = 0, \quad |x| < a \]  
\[ \sigma_{xy}(x,0) = 0, \quad -\infty < x < \infty \]  

In order to use the technique of integral transforms, it is necessary to solve an alternate but equivalent problem involving loadings on the crack surface which can be obtained from the problem of a clamped strip (without a crack) subjected to a uniform strain. The equivalent stress conditions on the crack are

\[ \sigma_y(x,0) = \frac{E v_0}{1 - \nu^2}, \quad |x| < a \]  
\[ \sigma_{xy}(x,0) = 0, \quad -\infty < x < \infty \]  

and the displacements must satisfy

\[ u_x(x, \pm h) = v_y(x, \pm h) = 0, \quad -\infty < x < \infty \]  
\[ v_y(x,0) = 0, \quad |x| > a \]  

where \( \nu \) is the Poisson's ratio and \( E \) the Young's modulus. The solution to the original problem is then obtained by adding the solution of the equivalent problem and that of

\[ \sigma_x = \frac{\nu E v_0}{1 - \nu^2}, \quad \sigma_y = \frac{E v_0}{1 - \nu^2}, \quad \sigma_{xy} = 0 \]
Case (2) - If a uniform stretching load of intensity \( p_0 \) is applied to the upper and lower edges \( y = \pm h \) of the strip, then the equivalent problem for this case involves the application of tractions \( -p_0 \) to the crack surface. The boundary conditions along the strip edges are

\[
\sigma_y(x, \pm h) = \sigma_{xy}(x, \pm h) = 0 , \quad -\infty < x < \infty \quad (13)
\]

while the shear stress \( \sigma_{xy} \) vanishes along the entire \( x \)-axis i.e.,

\[
\sigma_{xy}(x, 0) = 0 , \quad -\infty < x < \infty \quad (14)
\]

The conditions leading to a system of dual integral equations are

\[
\sigma_y(x, 0) = -p_0 , \quad |x| < a \quad (15)
\]

\[
v_y(x, 0) = 0 , \quad |x| > a
\]

Other conditions similar to those described in Cases (1) and (2) may also be treated for the strip problem. It is now more pertinent to discuss the solution of eqs.(6).

**INTEGRAL REPRESENTATION**

The present method of solution for the strip problem follows that described in [7] for the corresponding strip problem in elastostatics. The basic structure of the integral solution is the same except that in the dynamic
problem the parameters $s_1$ and $s_2$ related to the wave speeds come into play through the exponential functions $\exp(\pm s_j \xi y)$, $(j = 1, 2)$. Following the work in [7], the integral representations of $\phi$ and $\psi$ are

$$
\phi(x,y) = \frac{2}{\pi} \int_0^\infty \left[ A(\xi) \exp(s_1 \xi y) + B(\xi) \exp(-s_1 \xi y) \right] \cos(\xi x) d\xi
$$

(16)

$$
\psi(x,y) = \frac{2}{\pi} \int_0^\infty \left[ C(\xi) \exp(s_2 \xi y) + D(\xi) \exp(-s_2 \xi y) \right] \sin(\xi x) d\xi
$$

which can be easily shown to satisfy eqs.(6). Here, $A(\xi)$, $B(\xi)$, etc. are the four unknowns in terms of the single variable $\xi$ whereas originally the problem contained two unknowns in terms of two variables, namely $x$ and $y$. Putting eqs.(16) into eqs.(7) and (8), it is found that the displacements are

$$
u_x = \frac{2}{\pi} \int_0^\infty \left[ -A(\xi) \exp(s_1 \xi y) - B(\xi) \exp(-s_1 \xi y) \\
+ s_2 \left[ C(\xi) \exp(s_2 \xi y) - D(\xi) \exp(-s_2 \xi y) \right] \right] \sin(\xi x) d\xi
$$

(17)

$$
u_y = \frac{2}{\pi} \int_0^\infty \left[ s_1 \left[ A(\xi) \exp(s_1 \xi y) - B(\xi) \exp(-s_1 \xi y) \right] \\
- C(\xi) \exp(s_2 \xi y) - D(\xi) \exp(-s_2 \xi y) \right] \cos(\xi x) d\xi
$$

and the stresses become

$$
\sigma_x = \frac{4u}{\pi} \int_0^\infty \xi^2 \left[ -\frac{1}{2}(1-s_2^2+2s_2^2)[A(\xi) \exp(s_1 \xi y) + B(\xi) \exp(-s_1 \xi y) \\
+ s_2 \left[ C(\xi) \exp(s_2 \xi y) - D(\xi) \exp(-s_2 \xi y) \right] \right] \cos(\xi x) d\xi
$$
\[
\sigma_y = \frac{4\mu}{\pi} \int_0^\infty \xi^2 \left\{ \frac{1}{2}(1+s_2^2)[A(\xi)\exp(s_1 \xi y) + B(\xi)\exp(-s_1 \xi y)] \\
- s_2 [C(\xi)\exp(s_2 \xi y) - D(\xi)\exp(s_2 \xi y)] \right\} \cos(\xi x) d\xi
\]

\[
\sigma_{xy} = \frac{4\mu}{\pi} \int_0^\infty \xi^2 \left\{ -s_1 [A(\xi)\exp(s_1 \xi y) - B(\xi)\exp(-s_1 \xi y)] \\
+ \frac{1}{2}(1+s_2^2)[C(\xi)\exp(s_2 \xi y) + D(\xi)\exp(-s_2 \xi y)] \right\} \sin(\xi x) d\xi
\]

Applying the boundary conditions stated in the previous section into the appropriate displacement and stress expressions in eqs. (17) and (18), the governing system of dual integral equations for Case (1) and (2) is the same as given by

\[
\frac{2}{\pi} \int_0^\infty B_j(\xi) \cos(\xi x) d\xi = 0, \quad x > a
\]

\[
\frac{2}{\pi} \int_0^\infty \xi F_j(\xi) B_j(\xi) \cos(\xi x) d\xi = -p_j, \quad 0 < x < a
\]

where symmetry in x has been implied and j = 1, 2 with

\[
p_1 = \frac{E \nu_0}{2\mu(1-\nu^2)}; \quad p_2 = \frac{p_0}{2\mu}
\]

With the exception of the definitions of \(s_1\) and \(s_2\), the subscripts 1 and 2 are used to distinguish those quantities corresponding to Case (1) from Case (2). The functions \(F_j(\xi)\) (j = 1, 2) in eqs. (19) are known in the problem and...
they are

\[
F_1(\xi) = \left\{ \begin{array}{l}
\frac{s_2[(1+s_2^2)\alpha_1(\xi) - (1+s_1 s_2)\beta_1(\xi) + (1-s_1 s_2)\gamma_1(\xi)]}{2s_1 s_2 \delta_1(\xi) - (1+s_1 s_2)\beta_1(\xi) + (1-s_1 s_2)\gamma_1(\xi)} \\
\end{array} \right. 
\]

\[
F_2(\xi) = \left\{ \begin{array}{l}
\frac{s_2 [2(1+s_2^2)^2 \alpha_2(\xi) + 8s_1 s_2 \sinh(s_2 \xi_2)\beta_2(\xi)}{-2(1+s_2^2)^2 \cosh(s_2 \xi_2)\gamma_2(\xi)]}{4s_1 s_2 (1+s_2^2) \delta_2(\xi)} \\
-8s_1 s_2 \cosh(s_2 \xi_2)\beta_2(\xi) + 2(1+s_2^2) \sinh(s_2 \xi_2)\gamma_2(\xi) \\
\end{array} \right. 
\]

(21)

In view of the complexity of \( F_1(\xi) \) and \( F_2(\xi) \), the following contractions on \( \alpha_j(\xi), \beta_j(\xi), \) etc. have been made:

\[
\alpha_j(\xi) = f_j(\xi) + 1, \quad \delta_j(\xi) = f_j(\xi) - 1 \quad (j = 1,2)
\]

\[
\beta_1(\xi) = f_1(\xi) \exp[(s_1 - s_2)\xi_2] - \exp[-(s_1 - s_2)\xi_2] 
\]

\[
\beta_2(\xi) = f_2(\xi) \exp(s_1 \xi_1) - \exp(-s_2 \xi_1) \quad (22)
\]

\[
\gamma_1(\xi) = f_1(\xi) \exp[(s_1 + s_2)\xi_2] - \exp[-(s_1 + s_2)\xi_2] 
\]

\[
\gamma_2(\xi) = f_2(\xi) \exp(s_1 \xi_1) + \exp(-s_1 \xi_1) 
\]

Furthermore, the functions \( f_1(\xi) \) and \( f_2(\xi) \) stand for

\[
f_1(\xi) = \left\{ \begin{array}{l}
\frac{4s_1 s_2 - (1+s_2^2) \exp[(1+s_1 s_2)\exp[-(s_1 - s_2)\xi_2]] + (1-s_1 s_2) \exp[-(s_1 + s_2)\xi_2]}{4s_1 s_2 - (1+s_2^2)} \\
\end{array} \right. 
\]

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\[ f_2(\xi) = (4s_1s_2 [1 - \cosh(s_2 \xi h) \exp(-s_1 \xi h)] - (1 + s_2^2) s_2 \sinh(s_2 \xi h) \)
\cdot \exp(-s_1 \xi h) / \left[ 4s_1s_2 [1 - \cosh(s_2 \xi h) \exp(s_1 \xi h)] \right]
+ (1 + s_2^2) s_2 \sinh(s_2 \xi h) \exp(s_1 \xi h). \]

Note that in eqs.(19), there is only one unknown \( B_j(\xi) \) to each of the boundary value problems, Case (1) for \( j = 1 \) or Case (2) for \( j = 2 \). The original unknowns \( A(\xi), B(\xi), \) etc. in eqs.(17) and (18) are related to \( B_j(\xi) \) \((j = 1, 2)\) as follows:

Case (1) - For the case of displacement loading it is found that

\[ A(\xi) = f_1(\xi) B(\xi) \]

\[ B(\xi) = \left[ 2s_2 B_1(\xi) \right] / \left[ (1 + s_1 s_2) f_1(\xi) \exp(s_1 \xi h) + (1 - s_1 s_2) \exp(-s_1 \xi h) \right] \]

\[ C(\xi) = \left[ (1 + s_1 s_2) f_1(\xi) \exp(s_1 \xi h) + (1 - s_1 s_2) \exp(-s_1 \xi h) \right] \cdot (2s_2)^{-1} \exp(-s_2 \xi h) B(\xi) \]

\[ D(\xi) = -\left[ (1 - s_1 s_2) f_1(\xi) \exp(s_1 \xi h) + (1 + s_1 s_2) \exp(-s_1 \xi h) \right] \cdot (2s_2)^{-1} \exp(s_2 \xi h) B(\xi) \]
Case (2) - A different set of relations are obtained for the stress loading condition.

\[ A(\xi) = f_2(\xi)B(\xi) \]

\[ B(\xi) = \left[ 4s_2(1+s^2_1)B_2(\xi) \right] / \left[ \xi[4s_1s_2(1+s^2_2)] + 8s_1s_2\cosh(s_2\xi)B_2(\xi) + 2(1+s^2_2)^2\sinh(s_2\xi)\gamma_2(\xi) \right] \]

\[ C(\xi) = [4s_1s_2B_2(\xi) + (1+s^2_2)^2\gamma_2(\xi)][4s_2(1+s^2_2)]^{-1} \exp(-s_2\xi)B(\xi) \]  

\[ D(\xi) = [4s_1s_2B_2(\xi) - (1+s^2_2)^2\gamma_2(\xi)][4s_2(1+s^2_2)]^{-1} \exp(s_2\xi)B(\xi) \]

The mathematical details that lead to the solution of the dual integral equations (19) have already been covered by Sih et al [7] and they will not be repeated here. The form of \( B_j(\xi) \) (\( j = 1, 2 \)) satisfying both conditions in eqs. (19) is

\[ B_1(\xi) = \frac{\pi a E v_0}{4u(1-v^2)} \xi^{-1} \left[ \phi(1)J_1(\xi a) \right] - \int_0^1 \tau J_1(\xi a \tau) \frac{d}{d\xi} \left[ \frac{\phi(\tau)}{\sqrt{\tau}} \right] d\tau \]

for \( j = 1 \) and

\[ B_2(\xi) = \frac{\pi a p_0}{4u} \xi^{-1} \left[ \psi(1)J_1(\xi a) \right] - \int_0^1 \tau J_1(\xi a \tau) \frac{d}{d\xi} \left[ \frac{\psi(\tau)}{\sqrt{\tau}} \right] d\tau \]

in which \( J_1 \) is the first order Bessel function of the first
kind and $\kappa$ is the factor

$$\kappa = \frac{4s_1s_2 - (1+s_2^2)^2}{2s_1(1-s_2^2)}$$

The quantities $\Phi(1)$ and $\Psi(1)$ represent the values of the functions $\Phi(\xi)$ and $\Psi(\xi)$ evaluated at the crack tip $\xi = 1$. They also depend on parameters such as crack speed, crack length, strip width, etc. and can in general be found from the Fredholm integral equations

$$\Phi(\xi) = \sqrt{\xi} + \int_0^1 K_1(\xi, \eta) \Phi(\eta) d\eta$$

$$\Psi(\xi) = \sqrt{\xi} + \int_0^1 K_2(\xi, \eta) \Psi(\eta) d\eta$$

for $0<\xi<1$ and $0<\eta<1$. The kernels $K_j(\xi, \eta)$ ($j = 1, 2$) in eqs.(28) are given by

$$K_j(\xi, \eta) = \kappa^{-1} \sqrt{\xi} \int_0^\infty \frac{s}{s^2 + (\xi^2 - 1)\eta^2} J_0(s\xi) J_0(sn) ds, \quad j = 1, 2$$

where $J_0$ is the zero-order Bessel function of the first kind and $G_j(s)$ are related to the known functions $F_j(s)$ in eqs.(21) as

$$G_j(s) = F_j(s) + \kappa, \quad j = 1, 2.$$
DYNAMIC STRESS FIELD AHEAD OF THE MOVING CRACK

It has been shown by Sih [9] that the amount of energy released per unit crack extension can be associated with the amplitude of the local dynamic stress field. This information is useful in studying the ability of the structure to arrest crack motion. Hence, it suffices to consider only the local character of the stresses around the leading edge of the moving crack.

Substituting eq.(26) into (24) and eq.(27) into (25), the stresses at every point in the elastic strip can be obtained from eqs.(18). Furthermore, it is not difficult to show that the singular behavior of the dynamic stresses is described by the leading terms in eqs.(26) and (27). The integrals from zero to unity in $B_j(\xi) (j = 1,2)$ remain bounded at the crack tips $x = \pm a$. Thus, by retaining only the highest-order terms in the Laurent series expansion of the integrands in eqs.(18) for the stresses and applying appropriate Bessel integral identities, the crack-tip stress field referred to the polar coordinates $(r, \theta)$ is

$$
\sigma_x = \frac{k_1(i)}{\sqrt{2r}} \frac{4s_1s_2 - (1+s_2^2)}{[4s_1s_2 - (1+s_2^2)]^{-1}} \begin{bmatrix}
(1+s_2^2)(2s_1^2 + 1 - s_2^2) f(s_1) \\
- 4s_1s_2 f(s_2)
\end{bmatrix}

- 4s_1s_2 f(s_2)
$$

$$
\sigma_y = \frac{k_1(i)}{\sqrt{2r}} \frac{4s_1s_2 - (1+s_2^2)}{[4s_1s_2 - (1+s_2^2)]^{-1}} \begin{bmatrix}
4s_1s_2 f(s_1) \\
- (1+s_2^2) f(s_1)
\end{bmatrix}
$$

(30)
\[ \sigma_{xy} = \frac{k_1^{(i)}}{\sqrt{2r}} \left[ 4s_1 s_2 - (1 + s_2^2) \right]^{-1} \left[ 2s_1 (1 + s_2^2) [g(s_1) - g(s_2)] \right] \]

where

\[ r = \left[ (x-a)^2 + y^2 \right]^{\frac{1}{2}}, \quad \theta = \tan^{-1} \left[ \frac{y}{x-a} \right] \]

The functions \( f(s_j) \) and \( g(s_j) \) describe the angular distribution of the dynamic stress field:

\[ f^2(s_j) + g^2(s_j) = \sec \theta (1 + s_j^2 \tan^2 \theta)^{-\frac{1}{2}} \]

\[ f^2(s_j) - g^2(s_j) = \sec \theta (1 + s_j^2 \tan^2 \theta)^{-1}, \quad j = 1, 2 \]

which agrees with the general result reported in [9].

Note that the inverse square root of \( r \) singularity is preserved in this problem while the stress variations in \( \theta \) is distorted by the speed of crack propagation. The amplitude of this distortion is governed by the dynamic stress-intensity factors

\[ k_1^{(1)} = \Phi(1) \frac{E\nu}{(1-\nu^2)} \sqrt{a} \quad \text{for Case (1)} \quad (31) \]

and

\[ k_1^{(2)} = \Psi(1) p_0 \sqrt{a} \quad \text{for Case (2)} \quad (32) \]

The numerical values of \( \Phi(1) \) and \( \Psi(1) \) have been obtained for \( \nu = 0.25 \) and various values of \( a/h, \nu/c_2 \) under the condition of plane strain. The results of Case (1) and (2) will now be discussed separately.
Case (1) - Figure 2 gives a plot of the normalized stress-intensity factor \((1-v^2)k_1^{(1)}/E\sqrt{a}\) versus the crack length to strip width ratio, \(a/h\). The dimensionless crack speed \(v/c_2\) is permitted to vary from 0.0 to 0.8 in increments of 0.2. It is seen that \(k_1^{(1)}\) decreases more rapidly with \(a/h\) as the crack speed is increased. Another plot of \((1-v^2)k_1^{(1)}/E\sqrt{a}\) against \(v/c_2\) is shown in Figure 3 for \(a/h = 0.0, 0.25, 0.50, 0.75, 1.00,\) and 2.00. The straight line marked \(a/h = 0.0\) corresponds to the solution of a crack in an infinite medium. Thus, the strip width has the tendency to lower the amplitude of the dynamic stress field. More specifically, Figures 2 and 3 indicate that the amount of load transferred to the crack tip region decreases with decreasing strip width.

Case (2) - Similar plots of \(k_1^{(2)}/p_0\sqrt{a}\) versus \(a/h\) and \(v/c_2\) as given in eq. (32) are displayed in Figures 4 and 5. In the case of stress loading condition, the opposite effects are observed. Here, the intensity of the dynamic stress field increases with \(a/h\). The curves in Figure 4 are seen to be steeper for higher crack velocities. Figure 5 shows that \(k_1^{(2)}\) is always higher than that corresponding to a crack in an infinite medium. The effect of narrowing the strip width is to increase the load transfer to the crack tip region. The curves in Figure 5 appear to possess vertical asymptotes which suggest that for each ratio of \(a/h\) there exists a limiting crack speed.
DYNAMIC CRACK OPENING DISPLACEMENT

In assisting laboratory measurements, it is often useful to have a knowledge of the shape of the displaced crack by varying such parameters as the crack speed, specimen size, etc. This information can be obtained from the second of eqs.(17). By following the procedure of substituting the unknowns $A(\xi)$, $B(\xi)$, etc. into the expression for the vertical displacement $v_y$ as it was done for the derivation of the stresses, it is found that

$$2\mu \nu v_y^{(1)}(x,0) = \frac{E \nu a}{1-\nu^2} \int_{x/a}^{1} \frac{\sqrt{\tau} \phi(\tau)}{[\tau^2-(x/a)^2]^{1/2}} \, d\tau, \quad 0<x<a \quad (33)$$

for Case (1) and

$$2\mu \nu v_y^{(2)}(x,0) = \rho_0 a \int_{x/a}^{1} \frac{\sqrt{\tau} \psi(\tau)}{[\tau^2-(x/a)^2]^{1/2}} \, d\tau, \quad 0<x<a \quad (34)$$

for Case (2).

When the strip is clamped rigidly and displaced symmetrically with respect to the crack plane, the crack opening decreases with increasing crack speed. This result is illustrated numerically in Figure 6 for $a/h = 1.0$, $\nu = 0.25$ and the condition of plane strain. Figure 7 also plots $2\mu(1-\nu^2)\nu v_y^{(1)}(x,0)/E\nu a$ versus $x/a$ but this time for a fixed value of $\nu/c_2 = 0.2$ while $a/h$ is varied. Note that as the strip width is reduced (i.e., $a/h$ increased) the crack opening displacement becomes smaller.
Finally, if the strip is pulled apart by uniform tractions, the opposite trends are observed. Figures 8 and 9 show that the crack opening increases with increasing crack velocity and decreasing strip width.
REFERENCES


FIGURE CAPTIONS

Figure 1 - Moving Crack in a Finite Strip

Figure 2 - Stress-Intensity Factor Versus Crack Length to Strip Width Ratio (Displacement Loading).

Figure 3 - Stress-Intensity Factor Versus Crack Velocity (Displacement Loading).

Figure 4 - Dynamic Stress Amplitude Against Crack Length to Strip Width Ratio (Applied Tractions).

Figure 5 - Dynamic Stress Amplitude Against Crack Speed (Applied Tractions).

Figure 6 - Deformed Crack at Various Velocities (Clamped Strip).

Figure 7 - Deformed Crack for Different Strip Width (Clamped Strip).

Figure 8 - Crack Opening Displacement at Various Velocities (Stress Loading).

Figure 9 - Crack Opening Displacement for Different Strip Width (Stress Loading).
Figure 1 - Moving Crack in a Finite Strip
Figure 2 - Stress-Intensity Factor Versus Crack Length to Strip Width Ratio (Displacement Loading).
Figure 3 - Stress-Intensity Factor Versus Crack Velocity (Displacement Loading)
Figure 4 - Dynamic Stress Amplitude Against Crack Length to Strip Width Ratio (Applied Trawctions).
Figure 5 - Dynamic Stress Amplitude Against Crack Speed (Applied Traction).
Figure 6 - Deformed Crack at Various Velocities (Clamped Strip).
Figure 7 - Deformed Crack for Different Strip Width (Clamped Strip).
Figure 8 - Crack Opening Displacement at Various Velocities (Stress Loading).

- $v/c_2 = 0$
- $v = \frac{1}{4}$
- Plane Strain

$a/h = 1.0$

$2\mu k_y^{(2)}(x, 0)/p_0 a$
Figure 9 - Crack Opening Displacement for Different Strip Width (Stress Loading).
The problem of a uniformly propagating crack in a strip of elastic material is solved using the dynamic equations of elasticity in two-dimensions. Two specific conditions of loading on the strip with finite width are discussed. In the first case, the rigidly clamped edges are pulled apart in the opposite directions. The second case considers equal and opposite tractions applied to the crack surface. By varying the strip width to the crack length ratio, the amplitude of the dynamic stresses ahead of the running crack is determined as a function of the crack velocity. The local dynamic stresses are found to be lower than the corresponding static values for the displacement loading condition and higher for the stress loading condition. This effect becomes increasingly more important as the crack length to strip width ratio is enlarged. Numerical results for the dynamic crack opening displacements are also presented.