THE INELASTIC ANALYSIS OF PRESTRESSED
AND REINFORCED CONCRETE BRIDGE BEAMS

BY THE FINITE ELEMENT METHOD

by

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ABSTRACT

This dissertation presents a nonlinear analysis technique for reinforced and prestressed concrete beams which was developed for use in the overload analysis of beam-slab highway bridges. A layered finite element is used to model the beam. The assumption of plane sections is assumed to adequately describe the cross-sectional strain field and the layers are assumed to be in a uniform state of stress. The Ramberg-Osgood law is used as the basis of stress-strain curves. The cracking, crushing, yielding and strain hardening of materials are included. A tangent stiffness, iterative, incremental numerical process is used to treat the nonlinear problem in a piecewise linear fashion. Comparisons between computed and experimental load-deflection curves are presented which demonstrate the accuracy of the method.

Selected results are presented from a parametric study on the effects of layering, elemental discretization, stress-strain curve parameters and numerical tolerances as seen in the load-deflection curves.
The nonlinear beam has been applied to the inelastic analysis of a four beam bridge subjected to a variety of loadings. Load-deflection curves have been plotted which approach the load predicted by applying ultimate strength theory to the cross-section of the four beam bridge and treating it as a beam. The lateral load distribution to the beams has also been studied as nonlinear action occurs.

Appendices to this dissertation consist of:

1. A discussion of the effects of neglecting the torsional stiffness of the nonlinear beams when acting as part of a bridge.

2. An extension of the layered beam element techniques to the study of inelastic beam-column problems.

3. A flow chart of the mainline of the computer program developed for nonlinear beam analysis.

This report is based on the doctoral research of the first author.
1. INTRODUCTION

1.1 Purpose and Scope

1.1.1 Problem Description

This research has been undertaken to develop an analysis technique which adequately describes the entire load-deflection behavior of reinforced and prestressed concrete beams. The technique includes material nonlinearities such as the cracking and crushing of concrete and the yielding of reinforcement. There were two basic requirements placed on the development of the solution technique as explained below.

1. The technique had to be applicable to the overload analysis of beam-slab highway bridge superstructures composed of a reinforced concrete deck supported on several prestressed concrete I-beams. This overload analysis is the intended use of the processes developed here for beams.

It is known that many permit-overload vehicles cross beam-slab bridges every year. There is currently no adequate analysis technique to predict the response of beam-slab superstructure to these overload vehicles. Therefore, the analysis technique developed here had to be efficient enough to provide application oriented solutions to more complicated problems than the analysis of single beams. It will be seen in the rest of this
chapter and in Chapter 2 that this application requirement was the basis for selecting the methods used in this research.

2. The analysis technique should also be applicable to the flexural (not stability) analysis of bridge superstructures using steel I-shaped beams.

Thus the resulting techniques are as independent of material and cross-section as possible within the limits discussed herein. However, only beams which act with the bridge deck in a fully composite manner can be treated.

Longitudinal bending of the bridge is assumed to be the dominant action causing a nonlinear response in the beams. Past research has shown that the torsional stiffness of prestressed concrete I-beams does not drastically effect the lateral distribution of vehicular loads in the linear range (Refs. 66,67,68). In fact, a conservative distribution of load to interior beams results from neglecting the torsional stiffness of the beams. Accordingly, the torsional stiffness of the I-shaped beams when acting as part of a bridge superstructure will be neglected. This subject is discussed further in Appendix A.

Thus the basic beam model under consideration is a simply supported, essentially prismatic beam subjected to loading in a plane of symmetry. The formulation is general enough to allow for a wide range of materials. Local or lateral-torsional buckling of the beam is not considered. The small deflection theory is used.
1.1.2 Selection of the Basic Method to be Used

The finite element method was chosen as the basis of this research for several reasons.

1. This method provides for the solution of a global equilibrium problem using a stiffness matrix which can be easily identified and operated upon. Thus many of the features connected with a nonlinear analysis could be handled by selective alterations of this stiffness matrix.

2. This method produces the displacement components at selected points along the beam directly. This is exactly what is needed to establish load-deflection behavior.

3. The finite element method provides a convenient way to analyze systems composed of several structural elements such as beam-slab bridges.

4. Since the finite element method utilizes a load vector with force components for each displacement component, the stress redistribution effects caused by cracking and crushing of concrete can be accommodated.

5. One of the well known advantages of the finite element method is the ease with which boundary conditions can be handled.
Previous research had already produced a finite element analysis technique and a computer program for the nonlinear analysis of eccentrically stiffened plates of a von Mises material (Refs. 66,69). This program provided the main-frame for showing that the method developed here for beams of a more general material would also apply to beam-slab bridges.

1.2 Development of the Finite Element Method with Material Nonlinearities

The finite element method is a recent extension of matrix analysis techniques to problems of stress analysis. It employs the following steps:

1. The region to be considered (in this context, a beam) is divided into subregions called finite elements.

2. A suitable description of the displacement field is made. A polynomial description is usually assumed.

3. Generalized stresses are related to generalized strains by a suitable stiffness matrix. This stiffness matrix reflects material properties.

Since the material properties used in stiffness matrices are stress dependent, solutions to problems with material nonlinearities usually require the employment of an iterative scheme and an incremental loading path.
The application of the finite element method to problems involving material nonlinearity has progressed along two different paths; the initial stiffness method and the tangent stiffness method (Refs. 19, 33, 46, 59, 62, 66, 75). These two paths are described below. Concepts from both approaches have been used in the research reported herein. The following discussion of the widely used techniques is presented to provide a better appreciation of the problem.

1.2.1 The Initial Stiffness Method

The initial stiffness method utilizes the original stiffness matrix of the system throughout the analysis. This matrix need be inverted only once in the entire process. Solution of a problem involves a series of linear analyses which requires the representation of previous load history as a state of accumulated stress and strain. This can be written in equation form as:

\[
\sigma = [\Gamma] \{F\} + [G] \{\varepsilon_1\}
\]

where

- \([\sigma]\) = Stress vector
- \([\Gamma]\) = Stress matrix
- \([F]\) = Force vector
- \([G]\) = A transformation matrix
- \([\varepsilon_1]\) = A vector of initial strains.
The initial strains are the plastic strains at the current load level. The obvious difficulty with Eq. 1.1 is finding \( \{e_1\} \) for the current step. This drawback can be overcome by assuming that the inelastic strains of the previous load cycle can be used to approximate the current inelastic strains. Equation 1.1 may then be rewritten as:

\[
\{\sigma^{(K)}\} = [\Gamma] \{\sigma^{(K)}\} + [G] \{e_1^{(K-1)}\}
\]

where \( K \) denotes the current and \( K-1 \) the previous load cycle.

There are several ways of incorporating the strain from the previous cycle. Two common methods are the constant stress method and the constant strain method.

1.2.1.1 Constant Stress Method

The \( K \) cycle of loading is started with the current applied loads, \( \{\sigma^{(K)}\} \), and the initial strains from the previous cycle \( \{e_1^{(K-1)}\} \). \( \{\sigma^{(K)}\} \) is found by using Eq. 1.2. \( \{e_1^{(K)}\} \) for use with \((K+1)\)th cycle is obtained by using a stress-strain curve to find \( \{e_1^{(K)}\} \) corresponding to \( \{\sigma_1^{(K)}\} \). This process is shown in Fig. 1-A. Similar sketches and more detailed descriptions are found in Ref. 19. Experience has shown that the constant stress method has a tendency to diverge at a problem dependent step size and is therefore an undesirable approach (Ref. 46).
1.2.1.2 Constant Strain Method

In this approach \( \{\sigma^{(K)}\} \) is again found from Eq. 1.2. \( \{\varepsilon_1^{(K)}\} \) is found using a stress-strain curve by locating a point whose coordinates are \( \sigma_i^{(K)} \) and \( \sigma_i^{(K)}/E_{\sigma,i} + \varepsilon_{1,i}^{(K-1)} \). The strain coordinate defines a total strain. A new estimate of \( \sigma_i^{(K)} \) is found using the total strain. This process is shown in Fig. 1-B. Experience has shown the constant strain method to be numerically stable but less accurate than the constant stress method (Ref. 46).

This discussion of the initial stiffness method serves only as an introduction. The concept used in this research is that nonlinearities may be mathematically imposed by some set of fictitious forces or displacements (stresses or strains) as will be shown in Section 2.4.

1.2.2 The Tangent Stiffness Method

In the tangent stiffness approach the global stiffness matrix is regenerated each time the global equilibrium equations are solved. The stiffness properties of the elements are continually updated to account for the ongoing stress history. Lansing and Gallagher (Ref. 46) state that the tangent stiffness method "appears to be favored by theorists in finite element plasticity analysis. This is presumably the consequence of the consistency of this approach with classical methods of plasticity analysis and because of computational efficiency as well." There
are no conceptional difficulties associated with perfect plasticity when using the tangent stiffness approach.

The tangent stiffness method has enjoyed wide application through the use of the elastic-plastic stiffness matrix. The von Mises yield condition and Prandtl-Reuss flow rule are usually assumed to hold. The incremental formulation proceeds as follows:

1. The global equilibrium equations for linear elastic behavior are written as:

\[ \{F\} = [K] \{\delta\} \]  \hspace{1cm} (1.3)

where \([K] = \int_V [B]^T [D] [B] \, dv\)

\([B]\) = Relates element strain to nodal displacements

\([D]\) = Is the elasticity matrix relating stresses to strains

\(\{\delta\}\) = Is the vector of nodal displacements

\(\{F\}\) = Is the nodal force vector

\(v\) = Volume

2. When an element (or part of one) becomes plastic Eq. 1.3 must be modified. By using small increments of load Eq. 1.3 may be replaced by an equation which relates the increment of stress to the increments of strain. The von Mises yield condition and Prandtl-Reuss flow rule will be used in this introductory discussion. Eqs. 1.4
may be written for plane stress problems. Their derivation can be found in any elementary plasticity text such as Ref. 51.

\begin{align*}
\text{de}^p_x &= \frac{d\sigma}{\sigma_0} (\sigma_x - \frac{1}{2} \sigma_y) = \frac{d\sigma}{\sigma_0} \eta \\
\text{de}^p_y &= \frac{d\sigma}{\sigma_0} (\sigma_y - \frac{1}{2} \sigma_x) = \frac{d\sigma}{\sigma_0} \zeta \\
\text{dy}_{xy}^p &= \frac{d\sigma}{\sigma_0} (3 \tau_{xy}) = \frac{d\sigma}{\sigma_0} \xi \\
\end{align*}

(1.4)

where

\begin{align*}
\tilde{\sigma} &= (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2)^{\frac{1}{2}} \\
\end{align*}

(1.5)

\begin{align*}
d\tilde{\sigma} &= \frac{1}{\tilde{\sigma}} \left[ \eta \, d\sigma_x + \xi \, d\sigma_y + \xi \, d\tau_{xy} \right] \\
\end{align*}

(1.6)

\( \tilde{\sigma} \) is defined as the effective stress and is given by Eq. 1.5. \( d\tilde{\sigma} \) is given by Eq. 1.6. \( H \) is defined as the instantaneous slope of the effective stress-strain curve. Substituting Eq. 1.6 into Eq. 1.4 results in a relation between plastic incremental strains and incremental stresses:

\[
\begin{bmatrix}
\text{de}^p_x \\
\text{de}^p_y \\
\text{dy}_{xy}^p
\end{bmatrix}
= \frac{1}{H\tilde{\sigma}^2}
\begin{bmatrix}
\eta^2 & \eta\xi & \eta\xi \\
\eta\xi & \xi^2 & \xi\xi \\
\xi\eta & \xi\xi & \xi^2
\end{bmatrix}
\begin{bmatrix}
d\sigma_x \\
d\sigma_y \\
d\tau_{xy}
\end{bmatrix}
\]

(1.7)
Defining total strain increments as the sum of elastic and plastic portions, $\Delta \varepsilon^e + \Delta \varepsilon^p$, and using the elasticity matrix, the elastic-plastic stress-strain relation can be defined as follows:

\[
\begin{pmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \gamma_{xy}
\end{pmatrix}
= \left( [D^{-1}] + [D^p]^{-1} \right)
\begin{pmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \tau_{xy}
\end{pmatrix}
\]

Equation 1.8 can be inverted and substituted into Eq. 1.3 to find the increments of nodal displacements corresponding to increments of applied loads. An explicit elastic-plastic stiffness matrix has been derived by Yamada et al. (Ref. 72) which eliminated the need for inversion.

An iterative process is required because the change in stress field during the current load step alters the material properties. Thus the stiffness matrix is a function of the unknown stress level. If this alteration in material properties is not included, a systematic error will be introduced. This process is repeated until a convergence criteria is met for each load increment.
1.3 Previous Work on Concrete Beams
Using the Finite Element Method

The process described above is usually employed in continuum analyses. This type of analysis has been employed by several investigators of reinforced concrete beam behavior. Some of the previous work is reviewed below.

Ngo and Scordelis (Ref. 53), 1967, for example used a continuum approach with triangular elements to model reinforced concrete beams. A pre-existing crack pattern was assumed. A load system was applied and the finite element method was used to find the resulting stress and displacement fields. There was no consideration of successive cracking of concrete, yielding of reinforcement or incrementally increasing the loading. Bond was included by finite spring elements with an assumed linear bond stress-bond slip relation. Approximately 170 simultaneous equations were used in this solution in each half-beam of symmetric problems.

Ngo, Franklin and Scordelis (Ref. 52), 1970, also studied reinforced concrete beams with pre-existing crack patterns. A linear finite element continuum analysis was used. Aggregate interlock and bond slip were accounted for by linear linkage elements. Cracks were pre-formed by disconnecting nodes. Dowel action was included by using two dimensional elements for reinforcing rods with the effective dowel length assumed as two inches. Two constrained, linear strain triangles were used to form a
quadrilateral element which was considered to be a refined element. This was used to allow a coarser finite element mesh. Approximately 640 degrees of freedom were used with each symmetric half-beam.

Nilson (Ref. 54), 1968, used four constant strain triangular elements to form a quadrilateral element for use with a continuum analysis. Saenz's concrete stress-strain curve (to be discussed in detail in Section 2.3.3) was used to find Young's moduli in two principal directions in an effort to account for the orthotropic nature of biaxially loaded concrete. Springs were used again to model bond action. The bond stress-bond slip relation was assumed to be a cubic parabola. It was noted, however, that the correct relations were not known. Cracking was accounted for by disconnecting the nodes where a cracking stress was reached and reloading the modified member from an unloaded state.

Valliappan and Doolan (Ref. 63), 1971, also studied the finite element analysis of reinforced concrete beams. As in the previously mentioned studies, prestressed concrete beams were not included. This study also employed a continuum approach but used the von Mises yield condition and Prandtl-Reuss flow rule together with simplified stress-strain curves. An initial stiffness approach was used to reduce computational effort. Perfect bond was assumed because of the incomplete state of knowledge about the bond stress-bond slip relationship. Approximately 270 simultaneous equations were used in a half-beam.
All of the continuum approaches discussed so far would lead to a three-dimensional problem if applied to the overload analysis of a bridge. It also seems plausible that a thousand or more continuum type elements with a comparable number of simultaneous equations might be necessary for a nonlinear analysis of a bridge superstructure. The following logic supports this conclusion:

1. It would be desirable to use a fine enough discretization through the depth of the beam to monitor crack growth.

2. Consideration of the ratio of sides of triangular elements, especially the constant strain triangle, would imply that many elements would be needed along the length of the beam.

3. The problem of connecting the beam and slab elements means that common nodes would be required so that a fine discretization of the slab would also be implied unless an approximate displacement field was assumed between plate element nodes.

This argument also contains the prior assumption that the displacement components of the beam and slab are compatible. Thus it can be seen that while continuum methods provide thorough, but laborious, tools for analyzing a single beam there are
significant disadvantages associated with their application to the overload analysis of beam-slab highway bridges.

Franklin (Ref. 23), 1970, studied the nonlinear analysis of reinforced concrete frames and panels by building up a structure from four basic elements; a plane stress quadrilateral for panels, a tie-link element to model slip relations such as bond in concrete, rod elements for use as reinforcement or truss members, and a frame or bending element. An iterative, variable stiffness approach was used with nonlinear stress-strain properties modeled as a series of straight line segments.

The frame element was a layered quadrilateral element with four nodes and two degrees of freedom per node (Cartesian displacements) and was used to model flexural action. The quadrilateral was made by joining two constrained, linear strain triangles and eliminating the internal node by static condensation. Transformation matrices were used to convert the Cartesian displacements into an axial elongation, a rotation and a transverse displacement. Transformations were also used to convert stiffness properties about a centroidal axis into stiffness properties about an arbitrary reference axis. Plane sections were assumed to remain plane. Relatively few elements were used in this study.

Application of this approach to the problem being studied would still require the solution of a three dimensional problem but it would be a simpler problem than the previous methods because each beam would be defined by only two lines of nodal
points. It will be shown in Chapter 2 that a still simpler model can be developed.

Related work has also been done on the nonlinear analysis of shear walls by the finite element method (Refs. 8,73).

1.4 Other Methods of Nonlinear Beam Analysis

There are certainly other methods which would be applicable to the nonlinear solution of single reinforced or prestressed concrete beams. The column-curvature curve method of Chen and Santathadaporn (Ref. 12) involving a solution of the governing differential equation of a beam in terms of curvature would seem possible. This method requires an analytic expression for the moment-thrust-curvature curve. Chen and Chen (Ref. 11) have extended the column-curvature technique to reinforced concrete beam-columns by using an approximate moment-thrust-curvature curve based on concrete having no tensile strength. Breen (Ref. 5) has presented analytic moment-curvature relations for columns bent about one principal axis. An iterative computer program was written to solve for the curvature corresponding to some combination of axial load and moment.

Numerical techniques for developing biaxial moment-thrust-curvature diagrams for reinforced concrete sections based on sectioning the cross-section and assuming plane sections and perfect bond have been presented by Warner (Ref. 65) and Gesund (Ref. 25). Slightly different compressive stress-strain curves
were used and Warner assumed that concrete had no strength in tension whereas Gesund allowed a tensile stress up to some limiting value. Above this value the concrete is assumed to be cracked.

Curvature integration techniques could be applied to find the load-deflection response of single nonlinear beams. Known boundary conditions would have to be employed. Indeterminate beams could be handled by techniques developed by Breen (Ref. 5), Gesund (Ref. 25), Cranston (Ref. 17) or Cranston and Chatterji (Ref. 18). These techniques involve numerical integration of a curvature distribution found from an assumed distribution of moment. The resulting slopes and/or deflections are checked against compatibility conditions. An iterative procedure is employed to alter geometry or moment distributions until both equilibrium and compatibility are satisfied.

Finite difference equations could also be used to solve nonlinear beam problems but the finite element method allows for more convenient handling of variable element length, arbitrary loading, boundary conditions and handling the effects of cracking and crushing.

The constraint that the developed method be applicable to beam-slab highway bridges makes these other methods less attractive than the finite element method.

This report is based on the doctoral research of the first author (Ref. 78).
2. THEORETICAL DEVELOPMENT

2.1 A Simplified Model

The reported research uses a method especially suited to the analysis of beams of those proportions usually found in bridges (Refs. 40, 44). The Bernoulli beam theory, which assumes that a plane section before bending remains a plane section after bending, is used instead of a continuum elasticity approach. A relatively small number of elements along the longitudinal axis of the beam are divided into layers (Figs. 2 and 3). The plane sections assumption is used to relate the strains in the layer to the displacements at the nodes. This implies that no relative slip between layers can occur. Therefore, perfect bond in reinforced and prestressed concrete beams has been assumed. If a sufficient number of layers is used each layer may be assumed to be in a state of uniaxial tension or compression for the purpose of including material properties. The centroid of the layer is assumed to be representative of the whole layer. This has the effect of reducing the plasticity equations to the simple substitution of the instantaneous slope of a stress-strain curve into the conventional elasticity matrix. These assumptions would become tenuous if high shearing stresses were present as in the case of an interior support of a continuous beam. This consideration might have to be included if this simplified model were to be extended to the detailed study of continuous beams.
The economy of solution via the tangent stiffness approach using the simplified layered model can be demonstrated with the following example. If 10 elements each having 15 layers are used with a plane section type analysis as explained in Section 2.2, there are 11 nodes each having 3 degrees of freedom. This results in 33 simultaneous equations. If on the other hand, a continuum approach as presented in Ref. 75 utilizing 300 elements with 2 degrees of freedom per node was used, there would be 352 simultaneous equations. Recognizing that the solution time increases by a factor approximately equal to the number of equations raised to the 2.5 power and that an incremental-iterative approach typically requires 200 to 300 solutions, it is apparent that the savings in computational effort is enormous and would allow for a fine tolerance on solution accuracy. The need for an efficient solution was one of the constraints on this research as discussed in Section 1.1. The number of elements used in this example was chosen to provide the same area subdivision as 10 elements of 15 layers or a total of 150 layers. In this case there would be two triangles corresponding to each rectangular layer.

Nonlinear behavior associated with tensile cracking and compressive crushing is included by applying fictitious forces to the surrounding structure to maintain equilibrium and redistribute the accumulated stresses. It is this portion of the research being reported which utilizes the basic concepts of the initial stiffness approach.
This simplified model does not, directly, provide a means for computing true shear stresses. This is a direct result of assuming that plane sections remain plane before and after bending. It will be shown in Section 2.6 that approximate shear stresses can be computed from equilibrium considerations such as those normally used in beam theory. Principal stresses can then be calculated from the normal stresses and the approximate shear stresses.

As mentioned in Section 1.1, it has been assumed that the dominant inelastic action for the type of beams being studied will be flexural in nature. Behavioral criteria will be based on normal stresses. This means that shear related failure modes will not be directly treated although the principal stress fields do provide a means of visually determining if such problems exist.

It will be shown in Section 2.2 that the beam element stiffness properties can be directly related to displacements occurring in an arbitrary reference plane. If this arbitrary reference plane is chosen as the mid-plane of plate elements which are composite with the beam elements, then a beam-slab highway bridge superstructure could be analyzed, approximately, using a two dimensional finite element approach. This will result in a considerable savings in computational effort.
2.2 Finite Element Formulation

Consistent with the finite element method (Ref. 19, 33, 62, 75) the structure to be studied is first subdivided into elements, and in this context also into layers. Figure 2 shows the type of elemental idealization and Fig. 3 the type of layering employed for most of the examples included here. Reasonable care should be taken to place the elements and layers in the points of most interest and/or highest strain gradient. This is definitely more important in nonlinear than in linear analysis.

A displacement function or functions is then chosen to represent the displacements within the element. In the current context two displacement functions were used.

\[
U = \alpha_1 + \alpha_2 X \\
W = \alpha_3 + \alpha_4 X + \alpha_5 X^2 + \alpha_6 X^3
\]

(2.1)  
(2.2)

\(U\) is the axial displacement and \(W\) is the transverse displacement. The \(\alpha's\) are constants to be determined. By using the deflection and slope at both ends of the beam element the four constants in \(W\) can be found. Furthermore, since the bending displacement function is unique and contains the possibility of constant strains this shape function guarantees convergence for bending.

The constants in Eqs. 2.1 and 2.2 are evaluated by using the nodal displacements at both ends of the element.
\[ \{ \delta^e \} = [C] \{ \alpha \} \]  
(2.3)

\[
\begin{pmatrix}
U_i \\
W_i \\
-\partial W/\partial X_i \\
U_k \\
W_k \\
-\partial W/\partial X_k
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -2 & -3 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{pmatrix}
(2.4)

Figure 4 shows a beam element, coordinates and positive sign conventions. The vector \( \{ \alpha \} \) is evaluated by matrix inversion. Explicit inversion of matrix \([C]\), resulted in Eq. 2.5 (Refs. 66, 68)

\[
C^{-1} =
\begin{pmatrix}
\alpha_1 & 0 & 1 & 0 & 0 & 0 & 0 & -\partial W/\partial X_i \\
\alpha_2 & -1 & 0 & 0 & 1 & 0 & 0 & -\partial W/\partial X_i \\
\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial W/\partial X_i \\
\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial W/\partial X_i \\
\alpha_5 & 0 & -3 & 0 & 0 & 0 & 0 & -\partial W/\partial X_i \\
\alpha_6 & 0 & 2 & 0 & 0 & 0 & 0 & -\partial W/\partial X_i \\
\end{pmatrix}
(2.5)

The generalized stresses, \( \{ \sigma \} \), are the normal force and bending moment at the plane of reference defined by \( Z = 0.0 \) in Fig. 4. The generalized strains, \( \{ \varepsilon \} \) are the axial strain and curvatures at the plane of reference. The generalized stresses
and strains are related by an elasticity matrix, \([D]\). These relations are expressed by Eqs. 2.6, 2.7 and 2.8.

\[
\{\sigma\} = \begin{bmatrix} N \\ M \end{bmatrix} \quad (2.6)
\]

\[
\{\varepsilon\} = \begin{bmatrix} \partial U/\partial X \\ \partial W/\partial X \end{bmatrix} \quad (2.7)
\]

\[
\{\sigma\} = [D] \{\varepsilon\} \quad (2.8)
\]

The strains can also be related to the coefficients \((\alpha_1, \ldots, \alpha_6)\) by substituting Eqs. 2.1 and 2.2 into Eq. 2.7. This relation is given by matrix \([Q]\), the elements of which will be defined later by Eq. 2.22. Further, the strains could be related to the nodal displacements by substituting Eq. 2.5 into Eq. 2.3 and solving for \((\alpha_1, \ldots, \alpha_6)\). These operations would lead to Eq. 2.9.

\[
\{\varepsilon\} = [Q] \{\alpha\} = [Q] [C^{-1}] \{\delta^e\} \quad (2.9)
\]

The global stiffness matrix could then be derived by equating internal and external virtual work. The standard forms are given by Eq. 2.10.

\[
[K] = \int_V [B^T] [D] [B] \, d\mathbf{V} = [C^{-1}]^T \int_0^L [Q]^T [D] [Q] \, d\mathbf{X} [C^{-1}] \quad (2.10)
\]

The layering technique is employed by supposing that each element is composed of layers such that the element
stiffness properties are summations of layer stiffness properties. Each layer has its own area, position coordinates X and Z, material properties such as stress-strain law, tensile and compressive strengths, modulus of elasticity and stress and strain fields. It is also possible that each layer could have different material properties. As mentioned in the introduction, continuity between layers is maintained by the assumption of plane sections (Section 2.1). This assumption provides two additional benefits:

1. The strain state in each layer can be found from the displacements of the node points at each end of the element. This materially reduces the number of unknowns as discussed in detail in Section 2.1.

2. The layers composing each element provide a bookkeeping technique to account for the spread of cracking, yielding or crushing.

The assumption of plane sections enables the problem to be handled by the usual equations of mechanics instead of the theory of elasticity. This is a sacrifice of some accuracy and geometric generality for far greater computational efficiency. Using the plane sections assumption and referring to Fig. 5, the state of strain in a layer can be defined as:

\[ U_z = U - Z \frac{\partial W}{\partial X} \]

\[ \varepsilon_x = \frac{\partial U_z}{\partial X} = \frac{\partial U}{\partial X} - Z \frac{\partial^2 W}{\partial X^2} \]  

(2.11)
Employing the assumption of uniform stress, a layer stress can easily be related to strain.

\[ \sigma_x = E \varepsilon_x \]  

(2.12)

\( E \) in Eq. 2.12 is an instantaneous modulus of elasticity. The generalized forces can now be computed as a summation of layer contributions.

\[ N_j = \sum_{i=1}^{n} \sigma_i A_i \]  

(2.13)

\[ M_j = \sum_{i=1}^{n} \sigma_i A_i Z_i + \sum_{i=1}^{n} M_i \]  

(2.14)

Substituting Eqs. 2.11 and 2.12 into Eqs. 2.13 and 2.14 yields:

\[ N_j = \sum_{i=1}^{n} E_i A_i \left( \frac{\partial U}{\partial X} - Z_i \frac{\partial^2 W}{\partial X^2} \right) \]

\[ = \sum_{i=1}^{n} E_i A_i \frac{\partial U}{\partial X} - \sum_{i=1}^{n} E_i A_i Z_i \frac{\partial^2 W}{\partial X^2} \]  

(2.15)

\[ M_j = \sum_{i=1}^{n} E_i A_i Z_i \frac{\partial U}{\partial X} - \sum_{i=1}^{n} E_i A_i Z_i \frac{\partial^2 W}{\partial X^2} + \sum_{i=1}^{n} M_i \]  

(2.16)

\[ M_i = \int A_i \sigma Z dA \]  

(2.17)
In Eqs. 2.15, 2.16 and 2.17 \( j \) is an element number and \( i \) is a layer number of \( n \) layers. These equations can be put in the usual elasticity matrix form by defining element stiffness properties \( \bar{A} \), \( \bar{S} \) and \( \bar{I} \) which are the equivalent area, statical moment and moment of inertia times an instantaneous modulus of elasticity.

\[
\bar{A} = \sum_{i=1}^{n} E_i A_i \tag{2.18}
\]

\[
\bar{S} = \sum_{i=1}^{n} E_i A_i Z_i \tag{2.19}
\]

\[
\bar{I} = \sum_{i=1}^{n} E_i Z_i^2 A_i + \sum_{i=1}^{n} E_i I_{oi} \tag{2.20}
\]

Relating generalized stresses to generalized strains in matrix form results in Eq. 2.21

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} =
\begin{bmatrix}
\bar{A} & \bar{S} \\
\bar{S} & \bar{I}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial U}{\partial x} \\
-\frac{\partial^2 W}{\partial x^2}
\end{bmatrix} \tag{2.21}
\]

Once the elasticity matrix has been defined, the generation of the stiffness matrix becomes a routine operation (Ref. 75). Equation 2.22 can be developed by using Eqs. 2.1, 2.2 and 2.7
Now Eq. 2.10 can be used to evaluate the elemental global stiffness matrix. This is given as Eq. 2.23.

\[
\left[ \frac{\partial U}{\partial x} \right] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \left\{ \begin{array}{c} \alpha_1 \\ \alpha_6 \end{array} \right\}
\]

\[
\begin{array}{cccccc}
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x^2} \\
\end{array}
\]

\[
[K] = \begin{bmatrix}
12\overline{I} & -6\overline{I} & 0 & -12\overline{I} & 6\overline{I} \\
-6\overline{I} & 4\overline{I} & -6\overline{I} & 2\overline{I} & 0 \\
0 & -12\overline{I} & 6\overline{I} & 0 & 12\overline{I} \\
-6\overline{I} & 2\overline{I} & -6\overline{I} & 4\overline{I} & 0 \\
\end{bmatrix}
\]

The construction of the global stiffness matrix now follows by summation of stiffness properties of beam elements on each side.
of a node. The process of writing the stiffness matrix, Eq. 2.23 in a general form and extracting only those terms necessary to form the stiffness of node number \( n \) is illustrated in detail in Ref. 40. If element \((n-1)\) is to the left of node \( n \) and element \((n)\) is to the right the following equation results.

\[
\begin{align*}
\begin{bmatrix}
F_x \\
F_z \\
F_y
\end{bmatrix} &=
\begin{bmatrix}
K_{41} & K_{42} & K_{43} & (K_{44} + K_{11}) & (K_{45} + K_{12}) \\
K_{51} & K_{52} & K_{53} & (K_{54} + K_{21}) & (K_{55} + K_{22}) \\
K_{61} & K_{62} & K_{63} & (K_{64} + K_{31}) & (K_{65} + K_{32})
\end{bmatrix}
\begin{bmatrix}
U_{n-1} \\
W_{n-1} \\
\theta_{n-1} \\
U_n \\
W_n \\
\theta_n \\
U_{n+1} \\
W_{n+1} \\
\theta_{n+1}
\end{bmatrix}
\end{align*}
\]

\( (2.24) \)

The terms \( K_{ij} \) in Eq. 2.24 are the elements of the stiffness matrix given by Eq. 2.23. Repeating these steps for each node point...
populates the global stiffness matrix so that the increments of displacement corresponding to an increment of load can be found. This is shown in Eq. 2.25.

\[
\{f\} = [K] \{\delta\} \tag{2.25}
\]

Given the incremental displacement vector \(\{\delta\}\), Eq. 2.9 can be used to find the strain at the centroid of each layer. This strain will be considered representative of the whole layer. Eq. 2.26 was derived using Eqs. 2.5, 2.9 and 2.22.

\[
[B] = \begin{bmatrix}
-\frac{1}{t} & 0 & 0 & \frac{1}{t} & 0 & 0 \\
0 & \frac{6}{t^2} - \frac{12X}{t^3} & -\frac{6}{t} & \frac{6X}{t^2} & 0 & -\frac{6}{t^2} + \frac{12X}{t^3} & -\frac{2}{t} + \frac{6X}{t^2}
\end{bmatrix} \tag{2.26}
\]

It is possible to define the generalized strains by evaluating matrix \([B]\) at \(X = \frac{t}{2}\). This results in Eqs. 2.27 and 2.28.

\[
\frac{3U}{\partial X} = \frac{1}{t} (-U_i + U_k) \tag{2.27}
\]

\[
-\frac{3W}{\partial X} = \frac{1}{t} (-\theta_i + \theta_k) \tag{2.28}
\]

The engineering strain and the stress can then be computed for the \(l\)\(^{th}\) layer of the \(i\)\(^{th}\) element as:

\[
\epsilon_{xl,i} = \frac{1}{t} \left[ U_k - U_i - z_l \theta_i + z_l \theta_k \right] \tag{2.29}
\]

\[
\sigma_{xl,i} = E_{xl,i} \epsilon_{xl,i} \tag{2.30}
\]
Once the entire stress field is known a convergence check is performed on the increment of the displacement field. Each incremental displacement component is checked against the corresponding component of the last trial. If all are within a relative tolerance of the last trial the iteration is stopped and the stress and displacement fields are incremented to include the new contributions from this load step. Each layer is then checked for tensile cracking or compressive crushing. The computer program which performs this analysis makes use of one or more stress-strain curves for each layer to account for inelastic behavior, cracking and crushing. Stress-strain curves are discussed in detail in Section 2.3 and cracking and crushing are discussed in Section 2.4.

If no cracking or crushing has taken place, another load increment is added and the process is repeated with the generation of a new stiffness matrix which reflects the current state of stress. If cracking or crushing has started or is propagating, a special process discussed in Section 2.4 is employed to account for these types of nonlinear behavior.

If convergence of the current load step has not been attained the incremental stresses are temporarily added to the total stresses to find new elastic moduli using the layer stress-strain laws. A new stiffness matrix is generated and new incremental displacements are computed and compared with the last set to check convergence. This process is repeated until either
convergence is attained in a limited number of trials or the maximum number of trials is reached at which time the load increment is reduced by 15% and the whole process is repeated. There is also an overall trial counter to terminate execution if a large number of load reductions has been tried and convergence is still not attained. Experience with this process applied to materials which have relatively sharp knees in their stress-strain curve has shown that the load reduction process can reduce the load to such an extent that literally hundreds of additional load steps would be required to reach ultimate load. There is, therefore, a load increasing process which increases the load 10% if convergence of the next load step occurs in three trials or less. The amount to reduce or increase the load and the cutoff number of trials were arbitrarily arrived at by observing their effect on several test runs. The fact remains that a load reduction process was needed to assure convergence and a load increasing process was an economic necessity.

2.3 Stress-Strain Curves

The material stress-strain curve is the physical basis of the method used in this research. It is felt that this method uses one of the most realistic stress-strain curves yet employed. It will be seen that the method discussed is general enough to accept the following types of stress-strain curves:
1. Elastic-Brittle
2. Elastic-Plastic, not just elastic-perfectly plastic
3. Elastic-Plastic with linear strain hardening
4. Elastic-Plastic with tensile cracking
5. Elastic-Plastic with tensile cracking and compressive crushing.

The structural stiffness matrix has been shown to be a sum of elemental stiffness matrices which were in turn a summation of layer contributions. The layer stiffness contributions were seen to depend on the instantaneous modulus of elasticity which is the slope of a stress-strain curve at some total stress (or strain).

2.3.1 The Ramberg-Osgood Law

The Ramberg-Osgood Law has been chosen to provide generality in the shape of the stress-strain curve while maintaining a continuous mathematical expression (Ref. 56). As usually written, the Ramberg-Osgood Curve is given by Eq. 2.31.

\[ \varepsilon = \frac{\sigma}{E} + \left( \frac{3\sigma_1}{7E} \right) \left( \frac{\sigma}{\sigma_1} \right)^n \]  \hspace{1cm} (2.31)

This is actually a specialization of the more general form given as Eq. 2.32.

\[ \varepsilon = \frac{\sigma}{E} + \left( \frac{1-m}{m} \right) \left( \frac{\sigma_1}{E} \right) \left( \frac{\sigma}{\sigma_1} \right)^n. \]  \hspace{1cm} (2.32)
Where:  
\[ \sigma = \text{Stress at some load} \]
\[ E = \text{Initial modulus of elasticity} \]
\[ \sigma_1 = \text{Secant yield strength equal to the ordinate of intersection of the } \sigma-\epsilon \text{ curve and a line of slope } (m) \cdot (E) \]
\[ n = \text{A dimensionless constant} \]
\[ m = \text{A dimensionless constant defining a line of slope } (m) \cdot (E) \text{ on a plot of stress and strain} \]

Ramberg and Osgood derived the constants \( m \) and \( n \) by consideration of log-log plots of strain deviation curves for various materials given by Aitchison and Miller (Ref. 4). Strain deviation was obtained by plotting stress vs. the difference between measured strain and strain from Hooke's Law.

\[ d = \epsilon - \frac{\sigma}{E} = K \left( \frac{\sigma}{E} \right)^n \]  
\[ (2.33) \]

\[ K = \left( \frac{1-m}{m} \right) \left( \frac{\sigma_1}{E} \right)^{1-n} \]  
\[ (2.34) \]

A log-log plot of Eq. 2.33 should have an intercept at \( K \) and a slope of \( m \). From inspections of several such plots it was decided that \( m \) should be less than 0.9. Ramberg and Osgood then decided to choose \( m \) so as to make \( \sigma_1 \) approximately the yield stress given
by the 0.2% offset method. Using test data again, a value of
m = .709 was found and rounded off to m = 0.7.

The constant n is found by using two points on the
stress-strain curve to define two strains, two m's and two
stresses. Using both sets of data to find K, which is a constant
for any stress-strain curve, results in an equation relating \( \sigma_1 \),
\( \sigma_2 \), \( m_1 \), \( m_2 \) and n as follows.

\[
K = \left( \frac{1}{m_1} - 1 \right) \left( \frac{\sigma_1}{E} \right)^{1-n} = \left( \frac{1}{m_2} - 1 \right) \left( \frac{\sigma_2}{E} \right)^{1-n}
\]

\[
\left( \frac{\sigma_1}{\sigma_2} \right)^{1-n} = \frac{1}{m_2} - 1 \quad \frac{1}{m_1} - 1
\]

\[
n = 1 + \frac{\log \left( \frac{m_2}{m_1} \frac{1-m_1}{1-m_2} \right)}{\log \left( \frac{\sigma_1}{\sigma_2} \right)}
\]

Fig. 6 from Ref. 56 shows some of the variety in stress-strain
curves which can be obtained using the Ramberg-Osgood Law by varying n for a given m.

Application of the Ramberg-Osgood Law to reinforcing and
prestressing steels is virtually exactly what it was intended for
and deserves no more comment. The use of the Ramberg-Osgood Law
and the layered beam element to study the behavior of steel beams will be illustrated in Section 3.3.

2.3.2 Ramberg-Osgood Law Applied to Concrete in Uniaxial Compression

The application of the Ramberg-Osgood Law to uniaxial stress-strain curves for concrete would result in a material independent computer program which would obviously be more versatile and would conform to the constraints listed in Section 1.1. Such a program could handle combinations of materials with the same ease as a homogeneous beam by combining the layering concept with individual stress-strain curves used for each layer.

Consideration will now be given to the approximation of the concrete compressive stress-strain curve by the Ramberg-Osgood Law. The basic problem is that of defining the parameters m and n.

Figure 7 from Ref. 70 shows generally accepted smoothed stress-strain curves for concrete in compression as measured on the compressive side of flexural tests. The following characteristics of these curves will be noted:

1. All curves start as straight lines.

2. All curves reach a peak strength at a strain of approximately 0.002 in/in.

3. All curves, especially those for structural strength concrete have a downward sloping leg.
The approach taken here was to try to find a technique for consistently arriving at an acceptable approximation of these curves given only the cylinder strength, $f'_c$, and Young's modulus and using the properties above. A preliminary attempt to use a process analogous to that of Ramberg and Osgood as previously described led to results which were difficult to generalize. Typically, the "constants" varied greatly for different concrete strengths. The following approach has led to reasonably acceptable stress-strain curves and very good agreement between predicted and experimental ultimate strengths.

1. Assume $\sigma_1 = f'_c$. This is the only required assumption to use the analytic stress-strain curve for concrete.

2. Compute the value of Young's modulus from any acceptable equation using $f'_c$ or the results of laboratory tests, if available.

3. Assume that the stress-strain curve must pass through the point $(\varepsilon, f'_c)$. This leads to the following equation for the coefficient $m$.

$$m = \frac{f'_c}{\varepsilon E}$$

(2.35)

$\varepsilon$ would typically be 0.002 in/in for normal weight concrete.

4. Assume the Ramberg-Osgood curve stops at a strain of $\varepsilon$ in/in.
5. Assume a horizontal straight line from a strain of \( \varepsilon \) to a strain given in Table I as \( \varepsilon_1 \). This value is a variable in the program. The suggested values in Table I were scaled from Fig. 7.

6. Assume a straight line sloping downward from \( \varepsilon_1 \) to a stress of zero. Suggested values for this slope, "\( E_{\text{down}} \)", again from Fig. 7 are also found in Table I. The use of "\( E_{\text{down}} \)" to compensate for compressive crushing will be explained in Section 2.4. It is noted now that "\( E_{\text{down}} \)" is not a stiffness property and is not used in regenerating the stiffness matrix.

Table I - CONCRETE COMPRESSIVE PROPERTIES

<table>
<thead>
<tr>
<th>( f'_c ) ksi</th>
<th>( E_{\text{down}} ) ksi</th>
<th>( \varepsilon_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.600</td>
<td>3000</td>
<td>0.0022</td>
</tr>
<tr>
<td>4.750</td>
<td>1800</td>
<td>0.0022</td>
</tr>
<tr>
<td>3.900</td>
<td>1250</td>
<td>0.0023</td>
</tr>
<tr>
<td>&lt;3.000</td>
<td>700</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

7. From trial and error comparisons a value of \( n = 9 \) has been found to give consistently reasonable results for all strengths tried.

Figure 8 shows a comparison of the proposed curve and those shown in Fig. 7. Good agreement is noted.
2.3.3 Comparison With Other Compression Stress-Strain Curves for Concrete

Many stress-strain curves have been advanced specifically for concrete in uniaxial compression. Liu (Ref. 47) has tabulated fifteen different curves. Three such curves will be compared to the proposed curve.

Desayi and Krishnan (Ref. 20) suggested the curve below:

\[
\sigma = \frac{E \varepsilon}{1 + \left( \frac{\varepsilon}{\varepsilon_c} \right)^2}
\]

Where:  
\( \sigma \) = Stress at any \( \varepsilon \)  
\( \varepsilon_c \) = Strain at the maximum stress, \( f'_c \)  
\( E \) = A constant such that \( E = \frac{2f'_c}{\varepsilon_c} \), i.e. an initial tangent modulus.

Saenz (Ref. 58) suggested that Desayi and Krishnan's equation was not general enough and suggested the more complicated form shown below because it allowed for a variable ratio of secant to initial modulus.

\[
\sigma = \frac{E \varepsilon}{1 + (R + R_E - 2) \frac{\varepsilon}{\varepsilon_c} - (2R - 1) \left( \frac{\varepsilon}{\varepsilon_c} \right)^2 + R \left( \frac{\varepsilon}{\varepsilon_c} \right)^3}
\]

Where:  
\( E \) = Initial tangent modulus  
\( R_E = \frac{E}{E_s} \)
\[ E_s = \text{Secant modulus through peak of stress-strain curve} \]
\[ R_f = \frac{f'_c}{\sigma_f} \]
\[ f'_c = \text{Maximum stress} \]
\[ \sigma_f = \text{Stress at maximum strain} \]
\[ \varepsilon_f = \text{Maximum strain} \]
\[ \varepsilon = \text{Strain at maximum stress} \]
\[ R_e = \frac{\varepsilon_f}{\varepsilon} \]
\[ R = \frac{R_e (R_f - 1)}{(R_e - 1) R_c} - \frac{1}{R_c} \]

Figure 9 shows a comparison of the proposed method of computing a compressive stress-strain curve for concrete with the equations proposed by Desayi and Krishnan and Saenz. The experimental data are also by Desayi and Krishnan. As was shown before, a great deal of information is needed to use Saenz's equation. Details applicable to the curves shown in Fig. 9 are found in Ref. 58. Figure 9 shows excellent agreement between the Saenz curve and the proposed method on the ascending portion of the stress-strain curve. The descending portion needs more explanation. The experimental data and Saenz's equation are for cylinders whereas the proposed method uses a slope on the downward leg of the curve based on flexural tests as shown in Fig. 7. Hognestad, Hanson and McHenry (Ref. 32) have published comparative
flexural and cylinder compressive stress-strain curves which indicate that the slope on the downward leg appears greater for the flexural tests. Thus the difference in the descending portions of the curves shown in Fig. 9 are to be expected. In the current context the flexural behavior is preferred.

This new stress-strain curve is intrinsically quite similar to Hognestad's stress-strain curve developed for eccentrically loaded columns (Ref. 31). The Ramberg-Osgood curve is somewhat different from Ritter's parabola used by Hognestad and the definition of a peak strain and a slope for the downward leg are also different but the overall shape is quite similar. Some of the downward legs shown in Fig. 8 are steeper than Hognestad's empirical slope and more closely agree with those found in Ref. 32.

The Hognestad curve would use a reduced value of compressive stress equal to 85% of the 6" x 12" cylinder strength to account for the effect of size, shape and casting differences between the column specimens and cylinders. The effect of water gain in the top portions of vertically cast and cured specimens was considered a leading cause of the apparent reduced strength.

Hognestad, Hanson and McHenry (Ref. 32) noted that tests by the U. S. Bureau of Reclamation of reinforced concrete beams with small pressure cells embedded showed measured stresses equal to the corresponding cylinder strength. Breen (Ref. 5) has reported eccentric column tests on horizontally cast specimens which seemed to indicate that Hognestad's equation with 85% f'_c and with
100% $f'_c$ both yielded good results for certain ranges of applied load. Specific results were presented for two cases. In the first example the axial load was kept small enough to produce an initial tension failure and the moment was varied. Experimental curvatures agreed quite well with analytic curvatures computed using both 85% and 100% $f'_c$ as both analytic moment-thrust-curvature curves were quite close in this case.

In the second example the axial load was large enough to cause an initial compression failure. The analytic moment-thrust-curvature curves were significantly different in this case. The experimental values agreed well with the 100% $f'_c$ analytic results up to about 75% of the ultimate moment of the axially loaded section. Above 75% the experimental curvatures fell between the analytic curves.

While cylinder strengths have been used in this research on beam behavior it should be apparent that the proposed curve could also be used with the reduced concrete strength if so desired. This was in fact done in the beam-column study summarized in Appendix B.

2.3.4 Additional Comments on the Concrete Compression Stress-Strain Curve

Clearly the approximation used here is adequate for analytical use - in fact it represents a considerable refinement over many of the compressive stress-stain curves used in the previous
work reviewed in Section 1.3. The downward sloping portion of the stress-strain curve is used here to more accurately describe a failure caused by compressive crushing of the concrete. If the curve does not slope down, an artificial termination is sometimes used which is based on exceeding some ultimate compressive strain. It is believed that the technique employed in this research is more realistic. The use of a bilinear stress-strain curve sometimes also requires an "adjustment" of compressive stress to make the stress volume at ultimate load more comparable to a Whitney-like stress block. This is also unnecessary with the proposed curve.

Continued research could result in improved Ramberg-Osgood representations of the concrete stress-strain curve because of the infinite number of possible choices for m and n. It would be difficult, however, to justify significant refinement for use with a material as variable as concrete.

2.3.5 Concrete Tensile Stress-Strain Curve

The shape of the tensile stress-strain curve has been found to be quite important for predicting the load-deflection behavior of concrete beams - especially for prestressed concrete beams. The exact shape of the curve would appear to be far less important than the recognition of a surprisingly long downward sloping leg. Researchers and practicing engineers have characteristically neglected the tensile properties of concrete other than strength for many reasons. Some of these reasons are listed below.
1. Reinforced concrete is assumed to be cracked so the design process ignores any remaining tensile stress region.

2. Prestressed concrete is not supposed to reach a cracking stress under design load.

3. Concrete tensile strength is small compared to its compressive strength.

4. Concrete is assumed brittle in tension.

5. Tensile properties do not significantly affect the ultimate strength because the volume of concrete still in tension at the critical section and the resulting force are so small as to be negligible.

This research, while agreeing with all of the previous comments except No. 4, would indicate that the tensile properties are quite important in defining the shape of the load-deflection curve. Furthermore, the effect of the tensile properties would appear more significant in prestressed than in reinforced concrete beams. Previous studies of this type have concentrated on reinforced concrete beams so that the effect of not including this feature would be minimal.

The need to include the downward portion of the tensile stress-strain curve is shown in Fig. 10. This figure shows the experimental load-deflection curves for two virtually identical prestressed concrete rectangular beams from the test series.
reported by Walther and Warner (Ref. 64). The physical data pertaining to these beams (A-9 and A-10) are given in Refs. 40 and 64. Also shown on the same figure are the analytic load-deflection curves obtained by using five different tensile stress-strain curves. These stress-strain curves are drawn to the same scale in Fig. 11 for comparative purposes. It is seen that the results are divided into two easily recognized groups. Curves A, B and C give a reasonable approximation of the nonlinear behavior of the beam during cracking whereas curves D and E miss the zone formed by the two tests by a substantial margin. The following points deserve mention:

1. Because of the similarity in physical parameters the analytic load-deflection curves of beams A-9 and A-10 are quite similar. Therefore, the data for the analytic solution runs necessary to plot Fig. 10 were generated only for beam A-9.

2. Curves D and E show a virtually instantaneous growth of cracked zones extending up about a quarter of the beam's depth. Subsequent cracking occurs at a slower rate. Curves A, B and C show a gradual increase in crack depth with increased load. This is in good agreement with the photographs taken of the actual beams and as shown in Figs. 12 and 13 for two prestressed rectangular beams.
3. The shape of the tensile stress-strain curve has no perceptible effect before initial cracking and virtually no effect on ultimate moment. This is as expected.

4. There is a definite indication that the tensile unloading must be gradual as in curves A, B and C, rather than almost instantaneous as in curves D and E.

Testing of concrete in direct tension has historically resulted in a brittle type of failure. In the recent past it was thought that concrete had virtually no ductility in tension. During the past two decades increased research into the area of microcracking of concrete has led to tensile testing using special testing machines which are much stiffer than ordinary machines (Refs. 22,37). Figure 14 represents the curves found in Ref. 22 which show a great variety in shape, peak strains and ultimate strengths. But this figure does show a general shape and a surprisingly long downward leg. Therefore, it can be concluded that the downward leg does exist. There were no corresponding compression tests reported. An investigation of tensile behavior and its relation to compressive behavior, Young's modulus and compressive strength is needed.

One of the curves in Ref. 22 had a water-cement ratio of 0.45. The concrete used in the prestressed concrete rectangular beams in Ref. 64 had a water-cement ratio of 0.496. Curve A of Fig. 11 was constructed as an idealization of the experimental
stress-strain curve. The downward slope in curve A was chosen as 800 ksi. This compares with approximately 400 ksi to 600 ksi found in Ref. 22. The maximum tensile stress was chosen as 450 psi (plus about 25 psi dead load tensile stress). This number was chosen because the direct tensile strength of concrete is on the order of 450 to 550 psi. This is higher than any strength reported by Evans and Marathe for a specimen age of about 28 days (Ref. 22). Because of the large variation in reported test results and lack of corresponding compression tests the following analogy was tried analytically.

1. Curve A was constructed as mentioned above and the results compared to results using curve B.

2. Curve B, which is the proposed analytic tensile stress-strain curve, is constructed by using two straight lines. The first line has a slope equal to the compressive modulus of elasticity and stops at a tensile stress of $7.5\sqrt{f_c'}$. This tensile stress is adjusted for the dead load tensile stress and will be recognized as the accepted lower estimate of the modulus of rupture for concrete. Some engineers might prefer to use another measure of tensile strength or set a maximum value such as 500 or 600 psi. This is a matter of judgment on the part of the analyst. The second line slopes downward from the end of the first line at a slope of 800 ksi. This line extends to a tensile stress of zero.
It was supposed that if the results using curve B proved a close approximation to those using curve A then curve B could be used instead. Curve B is easier to construct for all concretes and requires no additional knowledge save the assumption for the downward slope. Curve A requires additional Ramberg-Osgood parameters which cannot be defined for various concretes at this time. Figure 10 shows the results of using both curves. It can be seen that curve B appears to be an adequate substitution for a curve shaped like curve A. Figure 15 shows the results of using other values for the downward slope. The following additional points are mentioned:

1. Figures 20, 21, 26, 27 and 30 show the results of applying this method and curve B to 2 reinforced beams, 2 prestressed rectangular beams and 2 prestressed I-beams. Ref. 40 contains similar figures for 2 more rectangular and 5 more I-shape prestressed concrete beams. The results are encouraging, but more research into tensile stress-strain curves would be quite valuable.

2. The computer program has been left general enough to accept a curve like curve A. Thus, if future research leads to better stress-strain curves, no change will be required. Curve B is seen to be a degenerate form of curve A.
3. A lower limit to the load-deflection curve is provided by curves like D and E. These curves are constructed by using one straight line whose slope is the compressive modulus of elasticity from zero to the modulus of rupture stress (see previous discussion). A second straight line runs from the end of the first back to zero on a downward slope which is much larger than the compressive modulus. For curves D and E a slope of 20,000 ksi was used. The resulting load-deflection curve is quite good at both ends but fairly conservative in the region of fastest cracking. This is shown in Fig. 10.

The downward slope of the tensile stress-strain curve will be referred to as \( E_{\text{down}} \). \( E_{\text{down}} \) will not be used in stiffness calculations but will be used to account for the release of energy caused by cracking. This will be explained in Section 2.4.

The analytic tensile stress-strain curve has been left general enough to use the "tension-stiffening" type of tensile stress-strain curve developed and used by Cranston and Chatterji, (Ref. 18) in their frame studies.

2.4 Cracking and Crushing Analysis

When the iterative procedure used to find the incremental displacements and stresses corresponding to a given load
step has converged to an acceptable tolerance, the accumulated stresses and displacements are tentatively incremented. A prescanning process is then used to check if any layer has a total tension or compression which exceeds given allowable stresses by more than some tolerance. If so, the program returns to the original iterative procedure and reduces the load by 50% for this step. The original problem is resolved to convergence, field quantities are again tentatively incremented and the results prescanned again. This process is repeated until all stresses which exceed the tensile or compressive allowable stresses exceed them by less than their respective tolerances. The prescanning technique is used to prevent large overstressing of the material for any load step.

As mentioned in Section 2.2, if no stresses exceed the compressive or tensile limit, another load step is taken. If scanning reveals that \( \sigma + d\sigma \) is greater than \( F_t \) for any layer than the layer is said to have cracked and steps are taken to adjust its stiffness and redistribute the stresses in that layer. The alteration to stiffness is simply setting that layer's modulus of elasticity equal to zero. Such a layer would then contribute no stiffness to an element and accept no additional increments of stress.

The redistribution of stresses is accomplished by using the downward leg of the tensile stress-strain curve and the basic concept of the initial stiffness method as mentioned in Section 1.2.1. The amount of strain beyond that corresponding to cracking, or the incremental strain, whichever is appropriate, is multiplied
by \( E_{\text{down}} \) to produce a stress-like quantity called a fictitious stress. This is shown schematically in Fig. 16. This fictitious stress is applied to the layer which has cracked until the sum of the increments of fictitious stress and the accumulated tensile stress are zero. The redistribution to the rest of the beam is accomplished by using the layer area to convert stress to an eccentric force and creating a fictitious load vector with axial force and corresponding moment terms. This is shown in Fig. 17 in which element \( i \) is unloaded by the fictitious stress while the rest of the beam is being held in equilibrium.

During the same scanning operation a test is also made to see if a given layer exceeds a crushing criteria. This crushing criteria for a layer is the attainment of the maximum compressive stress or a strain greater than \( \varepsilon \). If it is ascertained that crushing has occurred, the first step is to set Young's modulus equal to zero. If the strain is less than the value of \( \varepsilon_1 \) given in Table I, no unloading or redistribution is considered. If the strain exceeds \( \varepsilon_1 \), the excess strain is converted to fictitious stresses and hence fictitious loads analogously to the tensile cracking analysis.

Once all layers have been scanned the fictitious load vector is used to compute an auxiliary stress and displacement increment. At this time there are two stress and two displacement increments. One corresponds to the actual load step and the other corresponds to cracking and crushing. Essentially the same
iterative process is used to find convergence for the auxiliary displacement increment as that used for the actual load step. Once convergence has been obtained, the layers are rescanned to check if the redistribution of cracking and/or crushing stresses has caused any more layers to reach a cracking or crushing criteria. If any layers have reached these criteria the process of assembling a fictitious load vector and iterating to convergence is repeated. If no additional layers have reached cracking or crushing there may still be additional fictitious load vector components as a result of the additional strains computed from the increments of displacements. Therefore, the entire process is repeated until the fictitious load forces are smaller than some tolerance. At that time the cracking-crushing analysis is terminated and the accumulated stress and displacement fields are incremented by both sets of incremental stresses and displacements.

It is this process of crushing generating more crushing, which is possible using the type of stress-strain curve used here, that enables the beams to reach a failure caused by crushing of the concrete and a natural termination of execution rather than one forced by an artificial strain limit.

2.5 Application to Prestressed Concrete Beams

The additional steps used in the analytic modeling of prestressed concrete beams follow from the physical actions involved in prestressing. An initial stress field is read in for
each layer. This can be used to account for the initial steel tension. For applications involving prestressed concrete the initial stress input for the concrete layers is zero but other applications such as accommodating residual stresses in metal beams could require each layer to have some initial stress. The prestressing force is applied as an axial force and a moment about the reference plane in the nodal force vector. It is advisable to compensate this prestressing force for the elastic loss which will occur when it is applied. One approximate technique for doing this is illustrated in Ref. 41. While the prestress stress could be found for the centroid of each layer by hand calculation and read in, it is easier to let the computer do the arithmetic by using nodal forces.

It should be apparent that the object of applying the nodal forces used in prestressing is to produce the same thrust and moment diagrams in the reference plane as would be generated by replacing the prestressing elements at each point along the beam by an eccentric force at that location. This concept is important in generalizing the process for cases other than straight strands or for considerations other than prestressed concrete.

Consider a simply supported prestressed concrete beam pretensioned with a draped strand such that the end eccentricity was $e_1$ and the eccentricity at a distance $L_2$ from an end was $e_2$ and the strand was straight line segments in between. $e_1$ and $e_2$
are measured from the reference plane as shown in Fig. 18. The prestressing forces would then be modeled as follows:

1. An axial force, $F_x$, is applied at each end of the beam.

2. End moments are applied to each end of the beam equal to 
   $$(F_x) \cdot (e_1).$$

3. A concentrated load is applied to each drape point such that
   $$F_x (e_2 - e_1) = F_z/L_2.$$

   In No. 3 above $F_z$ is the concentrated load to be applied, and $L_2$ is the distance from the end of the beam to drape point. If due consideration is given to algebraic sign this system of forces will be equivalent to draped strand prestressing.

   When draped strands are used the inclined strand should be simulated by a series of horizontal line segments to approximate its contribution to the global stiffness matrix.

   The beam deflects under the influence of the nodal forces and moments used to apply the prestressing force. This prestress camber may or may not be desired to be part of the displacement vector output. Both options are provided and the choice is dictated by the physical situation. The prestress camber must, however, be included when displacements are converted to total strains to test against strain based behavior criteria.

   The conversion to prestressed concrete beams showed the importance of the tensile stress-strain curve. The flexural cracking of prestressed concrete beams causes a much more pronounced
change in the slope of the load-deflection curve than it does for reinforced concrete. This is probably because of the relative amounts of steel found in each. The use of the downward leg of the tensile stress-strain curve to improve the analytical load-deflection curve was discussed in Sections 2.3 and 2.4.

2.6 Flexural Shear Analysis

The flexural shear stresses can be approximated by considering the equilibrium of each layer of each element. This research has assumed that each layer is in a state of uniform axial stress given at its centroid for the purpose of including material properties. Using the same assumption here the bending stresses could be replaced by the uniform stresses as shown by the dashed lines in Fig. 19. If $\sigma_L$ is a uniform stress on the left side of the layer and $\sigma_R$ is the uniform stress on the right, then according to Fig. 19 the following equilibrium equation can be written.

$$\sigma_L A + \tau b l - \sigma_RA = 0$$

(2.36)

Where:

- $A$ = Layer area
- $b$ = Layer width
- $l$ = Element length

Two approaches to finding $\sigma_L$ and $\sigma_R$ were considered.

1. Compute additional stress fields at the ends of the elements and use them in Eq. 2.36 to find an average shear,
τ, for the layer. Uniform axial stress in a layer implies uniform shear in a layer.

2. Use an averaging technique to find the shear for a layer using the known centroidal stress fields.

If strains were computed at the ends of a layer using the nodal displacements at the ends of the parent element and Eq. 2.9, a significant error could result. This error occurs because the strains are most accurately represented at the centerline of an element. A considerable improvement can be made by finding the generalized axial and curvature strain on each side of a node point and taking a weighted average. This is similar to the concept used in applying Eq. 2.9 to the midpoint of a layer. This strain averaging technique thus makes the end strains for each layer dependent on three nodal rotations and three nodal axial displacements.

A detailed numerical example of this technique and additional discussion can be found in Ref. 40. The second technique was adopted in this research and is explained below.

Consider a beam whose elements are j, j+1, j+2, etc. The left and right node point of element j+1 are i+1 and i+2 respectively. A two pass operation will then be used to find the layer shears from the known layer stresses:

1. Compute $Q_{i+1}$ using $\sigma_j$ and $\sigma_{j+1}$ and assume this to be the shear at the node point,
2. Compute \( \tau_{\text{first}} = Q_{\text{first}} \) for the first element.

\[
\text{Compute } \tau_{j+1} = \frac{1}{2} \left( Q_{j+1} + Q_{j+2} \right) \text{ for a general element,}
\]

\[
\text{Compute } \tau_{\text{last}} = Q_{\text{last}} \text{ for the last element,}
\]

where these shears are assumed to be acting at the centroid of the layer. Putting this in equation form for the \( i^{th} \) node, \( j^{th} \) element, \( k^{th} \) layer results in Eqs. 2.37 and 2.38.

\[
Q_{i,k} = \frac{\sum_{n=1}^{k} \sigma_{j,n} A_{j,n} + \sum_{n=1}^{k} \sigma_{j+1,n} A_{j+1,n}}{\frac{1}{2} (b_j L_j + b_{j+1} L_{j+1})}
\tag{2.37}
\]

\[
\tau_{j,k} = \frac{1}{2} (Q_{j,k} + Q_{j+1,k})
\tag{2.38}
\]

\[
\tau_{\text{first},k} = Q_{\text{first},k}
\]

\[
\tau_{\text{last},k} = Q_{\text{last},k}
\]

While this process might seem quite approximate, it actually gives very good numerical results. Two detailed examples demonstrating this have been included in Ref. 40.

The stress averaging technique contains the assumption that each layer (except draped strands in a prestressed concrete beam) is prismatic. This means that the area properties are constant along the beam. If this assumption is violated, the shear stresses become more approximate in proportion to the degree of
violation. This is also true for draped prestressing strands and the concrete layers immediately adjacent to them. Special consideration is given to the draped strand in applying Eq. 2.37, but error is to be expected adjacent to the draped strand. Preliminary results have shown that for this case the stress averaging technique continues to yield good results near the draped strand.

This overall approach to beam problems being reported should give good deflection and bending stress results even for those types of nonprismatic beams which are typically analyzed with classical beam theories such as cover plated steel beams and some haunched beams. In these cases the nonprismatic beam would be treated as a series of prismatic elements. This process would obviously require some judgment and experience on the part of the analyst.

2.7 Flow Chart and Related Items

A logical flow chart of the operations described in this chapter is presented in Appendix C. Reference 41 is a detailed user's manual describing the computer program written to perform these operations. It also contains a list of the input, a source listing of the program, a discussion of the output and five illustrative example problems.

The computer work used to generate this study was performed on the CDC 6400 installation at the Lehigh University Computing Center. The SCOPE 3.3 operating system (Ref. 15) and
the FORTRAN (Ref. 13) and FORTRAN EXTENDED (Ref. 14) compilers were used.
3. CORRELATION WITH TESTS

This chapter contains comparisons of analytic and experimental load-deflection behavior for several reinforced and prestressed concrete beams and one steel beam.

3.1 Reinforced Concrete Beams

Quantitative comparisons with two under-reinforced concrete beams and qualitative comparisons of an under-reinforced and two over-reinforced concrete beams were made. They are discussed under the subheadings below. All beams had sufficient stirrups to prevent diagonal tension failures. There were no bond failures in the test results. Deviations between analytic and experimental results presented as percentages in this chapter have been computed on the assumption that the experimental ultimate load is the true ultimate load of the beam.

3.1.1 Under-Reinforced Singly Reinforced Concrete Beam

The cross-sectional layering and elemental discretization are shown in Figs. 3-A and 2-A (Ref. 76). Figure 20 shows experimental and calculated load-deflection curves for this example. The test beam was a 6 x 12 inch solid rectangular section reinforced with six No. 5 bars with an observed yield strength of 46.8 ksi. The concrete compressive strength was 5 ksi. The test beam was supported with a span of 11 feet and was subjected to
third point loading. Figure 20 shows excellent agreement between the experimental and calculated curves.

Figure 20 also shows that in this case the analytic solution extends further than the test data. This can be misleading. The actual ultimate load was 32.7 kips but no deflection was recorded for that load. The calculated ultimate load was 32.0 kips which is about 2% low. It will be shown later that tests to complete destruction usually extend beyond computer generated results.

3.1.2 Under-Reinforced Doubly Reinforced Concrete Beam

The cross-sectional layering and elemental discretization are shown in Figs. 3-B and 2-B (Ref. 77). Figure 21 shows experimental and calculated load-deflection curves for this case. The same cross-section and test setup as in the singly reinforced example (Section 3.1.1) were used except that the concrete compressive strength was 3.9 ksi and the reinforcement consisted of two layers of two No. 5 bars each as tensile reinforcement and two No. 3 bars as compressive reinforcement. The yield strength of the steel was 54.5 ksi. Figure 21 shows excellent agreement again. At the last point plotted the test load was 26.0 kips compared to an analytic load of 26.5 kips. As in the singly reinforced test, no deflection corresponding to the test ultimate load of 27.0 kips was recorded so that the actual upper portion of the load deflection curve is probably closer to the computed curve than Fig. 21 would indicate.
Figure 22 shows the deflected shape of a half beam for various states of loading. The deflected shape at 12.3 percent of the analytic ultimate load corresponds to the formation of the first cracked zones. After cracking has occurred, the deflection continues to grow almost uniformly to about 80% ultimate load. The deflection then more than doubles as the load is increased from about 75% of ultimate to ultimate. This action is also shown in Fig. 23 which shows the midspan deflection versus percent of ultimate load. The initial cracking phase occurs between 12.3% and about 20% of ultimate load. Rapid increases in deflection start at about 80% of ultimate and becomes quite dramatic at over 90% of the ultimate load. This increasingly rapid growth of deflection is accompanied by more cracking and by reinforcement nonlinearity.

Figure 24 shows the stress in the lower tensile reinforcement and the compressive reinforcement versus percent of ultimate load. Before first cracking the stress in the compressive reinforcement is greater than in the tensile reinforcement. Figure 3-B shows that the neutral axis of the uncracked section is below the middle of the section so the larger compressive steel stresses are exactly as would be expected. During the first period of cracking the tensile reinforcement becomes more highly stressed and continues at a higher stress rate until it yields. The response of the tensile steel appears almost linear between 75% and 100% of ultimate load. This observation, taken alone,
might seem to indicate that the steel does not yield. Referring back to Fig. 23 it can be seen that there are great increases in deflection during this load range and these would indicate correspondingly large increases in strains. Thus the almost linear response in Fig. 24 does not necessarily imply a linear stress-strain relation. The computer printout of stresses in this load range shows that the tensile reinforcement starts to yield at about 90% of the ultimate load. The effect of this yielding on deflection is seen in Fig. 23. During this same 90% to 100% ultimate load range the stress in the compressive reinforcement increases rapidly as large strain increases occur. In this example the compressive reinforcement did not yield before the beam reached its ultimate load.

3.1.3 Qualitative Curves of One Under-Reinforced and Two Over-Reinforced Beams

Figure 25 shows the effect of varying the amount of reinforcement in a simply supported singly reinforced concrete beam. The section used here is a hypothetical 10" x 10" solid rectangle of 3 ksi concrete reinforced with 36 ksi steel. The steel area was 2\(\frac{1}{2}\), 4 and 5 square inches for curves A, B and C respectively resulting in \(w \approx 0.3, 0.48\) and 0.6 for the beams. \(w\) is the steel percentage times the ratio of \(\sigma_y\) and \(f'_c\) as in ACI 318-71 (Ref. 3). Curve "A" is a balanced condition. It can be seen that curves B
and C show typical over-reinforced behavior while curve A shows
typical under-reinforced (or balanced) behavior.

Figure 25 also shows a horizontal line running through
each curve. This line is at a load level ratio corresponding to
an adjusted value of the ultimate load ratio which would be pre­
dicted by ultimate strength analysis techniques. The adjustment
was made by multiplying the theoretical ultimate load by 1.068.
This number is the average test ultimate load divided by theory
ultimate load ratios for the twenty-two tests reported in P.C.A.
Bulletin D-49, Table A-1, (Ref. 50) which had concrete strengths
between 2590 and 3550 psi. This comparison is offered in lieu
of laboratory tests.

If desired, a further comparison on the effect of steel
percentage could be made with the behavior demonstrated by curves
5 -.304 and 5 -.492 in Fig. 5 of P.C.A. Bulletin D-7 (Ref. 38).
The same behavior will be noted. It would seem that the method
used here would adequately predict over-reinforced beam behavior
as well as it predicts under-reinforced beam behavior. There is
no conceptual reason why it should not; the use of individual
layer stress-strain curves guarantees enough flexibility to handle
a wide variety of problems.
3.2 Prestressed Concrete Beams

3.2.1 Solid Rectangular Beams

The prestressed concrete rectangular beams in this study were tested by Walther and Warner (Ref. 64). The cross-sectional layering and elemental discretization are shown in Figs. 3-C and 2-C. A solid rectangular cross-section 8" by 18" was prestressed with six 7/16" diameter seven wire strands using two layers of 3 strands each. There was 4 inches of cover on the lower set and 6 inches on the upper set of strand. A stress-strain curve for the seven wire strand was included in the report. All beams were 15 feet long and were pretensioned at five days. Analytic and experimental load-deflection curves for four beams from this series are compared in Ref. 40. While tabular data for all four beams will be presented here to show the range of physical tests compared with, only two examples will be discussed in detail, A-7 and A-8. Characteristics of the beams are summarized in Table II.

Table II - PRESTRESS DATA

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Age At Test In Days</th>
<th>$F_i$ (kips)</th>
<th>$F_o$ (kips)</th>
<th>$F_c$ (kips)</th>
<th>$f'_c$ (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A7</td>
<td>38</td>
<td>96.33</td>
<td>92.87</td>
<td>87.26</td>
<td>6.140</td>
</tr>
<tr>
<td>A8</td>
<td>28</td>
<td>96.33</td>
<td>92.47</td>
<td>85.73</td>
<td>6.260</td>
</tr>
<tr>
<td>A9</td>
<td>32</td>
<td>102.15</td>
<td>98.11</td>
<td>92.53</td>
<td>6.320</td>
</tr>
<tr>
<td>A10</td>
<td>33</td>
<td>102.15</td>
<td>97.92</td>
<td>93.85</td>
<td>6.320</td>
</tr>
</tbody>
</table>
\[ F_i \] = Total force in the prestressing steel just prior to transfer of the force.

\[ F_o \] = Total force in the prestressing steel at the beam midspan just after transfer.

\[ F \] = Total force in the prestressing steel at beam midspan just prior to testing.

\[ f'_c \] = Cylinder strength on day of testing - average of 6 cylinders.

The beams were subjected to third point loading while supported to give a 9' - 0" span. The dead load of the 3' - 0" overhangs offset the dead load tensile stress in the pure moment section. Figures 26 and 27 show the load-deflection behavior of beams A-7 and A-8. Both test curves give reasonable agreement with its corresponding analytic curve. The test data were taken in 5. kip intervals.

Figure 28 conclusively shows the extent of agreement between analytic and experimental results. Test beams A-7 and A-8, were cast as an identical pair. The initial prestressing forces were identical for the pair. Figure 28 shows test beams A-7 and A-8 plotted together on the same figure with a composite analytic curve. The analytic curves for the identical pair are so close that only one curve was drawn. It can be seen that the analytic data fits on or between the test curves for most of the load-deflection history.
Table III shows a comparison of the test and calculated ultimate load applied to each third point and the corresponding deviations.

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Test (kips)</th>
<th>Calculated (kips)</th>
<th>Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>A7</td>
<td>49.9</td>
<td>49.0</td>
<td>1.8</td>
</tr>
<tr>
<td>A8</td>
<td>50.2</td>
<td>48.9</td>
<td>2.6</td>
</tr>
<tr>
<td>A9</td>
<td>49.8</td>
<td>48.7</td>
<td>2.2</td>
</tr>
<tr>
<td>A10</td>
<td>49.9</td>
<td>49.0</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Figures 12 and 13 show the crack growth rate found during the actual test compared to the "crack zones" predicted by the computer program for the specified analytic loads shown in parenthesis. The growth of these cracked zones are important in predicting the behavior of single beams. It is expected that they will be even more significant in the overload analysis of bridge superstructures. They will be a convenient and easily recognizable device for limiting the extent of permissible damage to bridge beams from an overload vehicle.

3.2.2 I-Beams

The prestressed concrete I-beams used in this study were tested by Hanson and Hulsbos (Ref. 28). The cross-sectional
layering and elemental idealization are shown in Figs. 3-D and 2-D. The test setup and cross-sectional data are given in Fig. 29. Six 7/16" diameter seven wire prestressing strands were used as prestressing elements in each beam. A stress-strain curve for the strand was included in the report. Table IV shows the prestressing data used and Table V shows the properties of the concrete used. This data is taken from the report by Hanson and Hulsbos.

The beams were simply supported with a clear span of 15'-0". Two concentrated loads were applied to the beam at positions which varied for groups of tests. The position of the loads is shown on the inset of Fig. 30. Analytic and experimental load-deflection curves for seven beams from this series are compared in Ref. 40. As in Section 3.2.1 two examples, E-5 and E-7, will be discussed in detail.

### Table IV - PRESTRESS DATA

<table>
<thead>
<tr>
<th>Beam</th>
<th>Initial Prestress Force (kips)</th>
<th>Percent Losses</th>
<th>Prestress Force At Test (kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Elastic</td>
<td>Inelastic</td>
</tr>
<tr>
<td>E-5</td>
<td>113.9</td>
<td>8.6</td>
<td>11.9</td>
</tr>
<tr>
<td>E-7</td>
<td>114.9</td>
<td>8.1</td>
<td>11.8</td>
</tr>
<tr>
<td>E-8</td>
<td>114.9</td>
<td>8.1</td>
<td>11.8</td>
</tr>
<tr>
<td>E-9</td>
<td>114.9</td>
<td>8.1</td>
<td>12.7</td>
</tr>
<tr>
<td>E-12</td>
<td>113.7</td>
<td>8.5</td>
<td>12.3</td>
</tr>
<tr>
<td>E-17</td>
<td>113.3</td>
<td>8.4</td>
<td>10.2</td>
</tr>
<tr>
<td>E-18</td>
<td>113.3</td>
<td>8.5</td>
<td>9.9</td>
</tr>
</tbody>
</table>
**Table V - PROPERTIES OF CONCRETE**

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Age (Days)</th>
<th>$f'_c$ (psi)</th>
<th>$E_c^1$ (ksi)</th>
<th>Age (Days)</th>
<th>$f'_c$ (psi)</th>
<th>$E_c^1$ (ksi)</th>
<th>$E_c^2$ (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-5</td>
<td>7</td>
<td>5530</td>
<td>3100</td>
<td>60</td>
<td>6610</td>
<td>3800</td>
<td>4600</td>
</tr>
<tr>
<td>E-7</td>
<td>7</td>
<td>5900</td>
<td>3800</td>
<td>62</td>
<td>7230</td>
<td>4100</td>
<td>4700</td>
</tr>
<tr>
<td>E-8</td>
<td>7</td>
<td>5680</td>
<td>3400</td>
<td>70</td>
<td>6970</td>
<td>4400</td>
<td>4700</td>
</tr>
<tr>
<td>E-9</td>
<td>7</td>
<td>5630</td>
<td>3500</td>
<td>74</td>
<td>7140</td>
<td>4200</td>
<td>4700</td>
</tr>
<tr>
<td>E-12</td>
<td>7</td>
<td>5590</td>
<td>3300</td>
<td>68</td>
<td>7020</td>
<td>3900</td>
<td>4700</td>
</tr>
<tr>
<td>E-17</td>
<td>7</td>
<td>5400</td>
<td>3300</td>
<td>57</td>
<td>6580</td>
<td>3800</td>
<td>4300</td>
</tr>
<tr>
<td>E-18</td>
<td>7</td>
<td>5520</td>
<td>3200</td>
<td>52</td>
<td>6640</td>
<td>3600</td>
<td>4500</td>
</tr>
</tbody>
</table>

$E_c^1$ is determined from cylinder tests, $E_c^2$ is determined from the load-deflection curve of the test beams.

These specimens had an overhang of only 1'-3" on each end. This was not enough to offset the dead load tensile stress of about 80 psi. The results presented in Fig. 30 are based on an adjusted tensile strength found by deducting 80 psi from the tensile strength of all layers regardless of their position in the beam. A comparative calculation was performed for beam E-12, shown in Ref. 40, by inputting the dead load as part of the prestressing force nodal load vector. That force vector would not be incremented with the test load. This had the effect of eliminating pure bending and requiring more cycles of cracking-crushing.
analysis because each layer had a slightly different stress from the combined dead load and live load. While this is a more realistic situation than having groups of layers with the same stress, the net effect of this extra consideration was less than a 1% change in the load deflection behavior. Execution time, however, was increased considerably. The refined calculations reached an enforced time limit after 151 seconds of central processor time on the CDC 6400 digital computer of the Lehigh University Computing Center. At that stage of the analysis it probably would have required another 30 or 40 seconds to reach completion. These latter figures are, of course, estimates based on experience with the program. The more approximate analysis required only 123 seconds for complete execution. Hence, the refinement would require about 50% more execution time for an increase in accuracy which has no engineering significance. Based on this conclusion it was decided to run all analytic load-deflection curves with the adjusted tensile strength instead of including the dead load.

Figure 30 shows very good agreement with the test curves for both beams. Each of the analytic curves shows a pronounced discontinuity which was not evident in the previous examples. This is a result of the cross-sectional layering used and the approximation for dead load tensile stress just explained which eliminated the moment gradient causing a larger portion of the analytic beam to reach a cracking criteria at a given time under the given loading than was true for the physical beam. Figure 3-D
shows that the fourth layer from the top and bottom contains by far the largest area. Examination of computer output of stresses showed that in each case the discontinuity corresponds to the unloading of the tensile stresses in this layer.

Table V shows the two values of Young's modulus recorded for each beam. The question of which value to use for input is valid but somewhat academic. It is valid because different values for the elastic modulus will change the slope of the load-deflection curves. It is academic because the problem of predicting the behavior of untested beams would have to rely on an estimate which would be more approximate than either value given in Table V. The additional load-deflection curves presented in Ref. 40 show that good results were obtained using both values of Young's modulus.

In some cases the test curves extend beyond the plots shown in Fig. 30. The ultimate test loads are shown along with the maximum computer generated loads in Table VI. It can be seen that in some cases the errors are somewhat larger than those shown in Table III. Both the extension of the curves past computer output and the larger ultimate load discrepancies are probably explained by the fact that some of these tests were carried to utter destruction. The accompanying very large deflections probably caused the change in the geometry of the prestressing strands to become significant.
Table VI - ULTIMATE LOADS

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Test (kips)</th>
<th>Calculated (kips)</th>
<th>Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-5</td>
<td>42.0</td>
<td>39.2</td>
<td>6.7</td>
</tr>
<tr>
<td>E-7</td>
<td>41.1</td>
<td>39.8</td>
<td>3.2</td>
</tr>
<tr>
<td>E-8</td>
<td>41.2</td>
<td>39.3</td>
<td>4.6</td>
</tr>
<tr>
<td>E-9</td>
<td>41.2</td>
<td>38.9</td>
<td>5.6</td>
</tr>
<tr>
<td>E-12</td>
<td>41.2</td>
<td>39.0</td>
<td>3.3</td>
</tr>
<tr>
<td>E-17</td>
<td>38.0</td>
<td>38.2</td>
<td>0.5</td>
</tr>
<tr>
<td>E-18</td>
<td>38.7</td>
<td>38.2</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Figure 31 shows the growth of "cracked zones" in one I-beam. These zones are at a stress state which has reached the cracking criteria. The exact location, number and spacing of the cracks remains undetermined. Broms (Ref. 6) has suggested, however, that the space between cracks can be approximated as two times the concrete cover of the reinforcement to any edge or to the next piece of reinforcement, and that the average crack width is $2t_s$ where $\varepsilon_s$ is the average strain in the reinforcing and $t$ is the minimum cover. This crack width would presumably be measured at the steel location.

Figure 32 shows the deflected shape of a half beam for various states of loading starting with the prestress camber and continuing to 100% of the calculated ultimate load. The figure corresponds to beam E-7 and shows the catastrophic effect of large
overloads. First cracking occurs at 59% of the computed ultimate load. The next 20% of ultimate load more than doubles the deflection. Adding another 16% of load more than doubles the deflection again; in fact the deflection is 8.44 times the deflection at first cracking.

Figure 33 shows the calculated midspan deflection versus the percent of computed ultimate load for beam E-7. It can be seen that the sudden increase in deflection occurs between 70% and 73% and not at the "first cracking" load of 59.4% of ultimate. This delay is a direct consequence of the unloading leg of the tensile stress-strain curve and the layering used. This delayed behavior is also exemplified in all the load-deflection curves for I-beams. Figure 30 shows the load-deflection curve for beam E-7. It is seen that the experimental curve had first cracking at 25.0 kips and had some delay until the nonlinearity became significant. This delay was not as long as in the calculated load-deflection curve. Part of the difference between the experimental and calculated behavior is the large area in the fourth layer from the top and bottom. This point has been discussed and is believed to also explain the somewhat longer delay in the calculated results. Similar behavior is also seen in the box beams.

Figure 34 shows the steel stress in the lowest strands versus the percent of ultimate load. The results for the midspan section and a section 45 inches from each end are shown. The midspan curve shows an increase in steel stress corresponding to the
growth of cracking shown in Fig. 33. An enlarged plot of the Ramberg-Osgood stress-strain curve for this strand shows that the curve reached a horizontal plateau at 230 ksi. This is also the computed steel stress at the analytic ultimate load.

Figure 35 is a non-dimensional plot of moment versus curvature in the pure moment region of this prestressed concrete beam. While the techniques being reported were not developed to produce moment-curvature diagrams for a given cross-section it is evident that it could perform this function for the same wide range of beam problems being presented here. Moment-thrust-curvature diagrams could also be developed by using the prestressing nodal load vector.

Figure 36 shows the distribution of curvature along the beam for various percentages of the analytic ultimate load. The influence of nonlinear behavior on the growth and spread of curvature can be traced.

3.2.3 Comparison with a Laboratory Test of A Uniformly Loaded Prestressed Concrete I-Beam

Figure 37 shows the analytic and experimental load-deflection curves for a uniformly loaded simply supported prestressed concrete I-beam. This beam, from another test series by Hanson and Hulsbos (Ref. 29) was tested using a fire hose filled with water and loaded by four hydraulic jacks bearing on four wide flange beam segments to simulate a uniform load. The cross-section
was the same as shown in Fig. 29 and the span was 17' - 6". Good
correlation is again noted between the experimental and analytic
load-deflection curves. The following additional data from the
test are given for comparison with the previous examples: The
cylinder strength of the concrete was 6900. psi at an age of 60
days when the beam was tested, Young's modulus was 3700. ksi and
the pretest prestressing force was 89.3 kips.

3.3 Examples Using Steel Beams

As mentioned in Section 2.3 some examples using a stress-
strain curve like that of mild steel were also studied for fixed
ended I-shapes. These examples are discussed below.

3.3.1 The Effect of the Ramberg-Osgood Stress-Strain Curve

Figure 38 shows the results of four analytic investiga-
tions of a fixed ended hypothetical I-shape in which the value of
the Ramberg-Osgood parameter n was taken as 30, 50, 100 and 300.
The corresponding stress-strain curves are shown in Fig. 39 along
with a comparison curve with n = 500.

The effect on the stress-strain curve of increasing the
value of n is to make the knee sharper. This effect is carried
over to the load-deflection curve whose shape approaches that pre-
dicted by the simple plastic theory as the stress-strain curve ap-
proaches elastic-perfectly plastic. Using the shape factor for
the hypothetical section, the simple plastic theory would predict
a ratio \( \frac{P}{P_0} = 2.07 \). Adjusting this value for the position of the section actually used for measurement of stresses results in a ratio of 2.28. This is shown by the horizontal line in Fig. 38. Better discretization could make this as close to 2.07 as desired. The value of 2.28 compares quite well with the \( n = 100 \) and \( n = 300 \) curves in Fig. 38. The curves with \( n = 30 \) and \( n = 50 \) are not as good. This is as expected. The cost of solution of these examples increased as the value of \( n \) increased. It would appear that the Ramberg-Osgood Law combined with the layered elements would allow as close an approximation to the simple plastic theory for steel beams as economically desirable for any section which is symmetric about the plane of loading.

From these studies it appeared that a savings in computational effort could be effected by defining a new yield stress for mild steel Ramberg-Osgood curves. Referring to Fig. 39 it can be seen that if \( \sigma_1 \) in Eq. 2.31 is taken as \( \sigma_y \) a Ramberg-Osgood \( n \) of about 300 is required to produce an almost horizontal post yielding plateau. The curve for \( n = 50 \) would produce about a 5\% higher "yield stress" at a strain/\( \sigma_1 \) value of 0.0002 but it would have reached a reasonably horizontal plateau by that time. Based on these observations the following process would appear to produce a more economical stress-strain curve which would still yield adequate results for many purposes.
1. Using auxiliary stress-strain curves such as Fig. 39 select an adequate value of n.

2. Using that value of n and an appropriate value of strain such as a value midway between yielding and strain hardening find the corresponding value of \( \sigma \).

3. Now use Eq. 2.31 or Eq. 2.32 to find \( \sigma_1 \).

This process would scale a given curve down so that it passed through the chosen strain at stress very close to \( \sigma_y \). There would be some error involved because of the more rounded knee, especially at first yielding, but this error would be decreased as continued straining occurred.

3.3.2 Comparison with a Laboratory Test of a Steel Wide Flange Beam

A comparison of analytic versus experimental behavior of a steel wide flange shape was also conducted. A "fixed ended" 8 x 40 beam 14 feet long under third point loading was selected from the test series reported by Knudsen, Yang, Johnston and Beedle (Ref. 39). The properties of the section are given in the table below taken from Ref. 39.
Young's Modulus: $E = 29.6 \times 10^6$ psi
Lower Yield Point: $\sigma_y = 37,760.$ psi
Strain Hardening Modulus: $c = 630.$ psi
Flange Width: $b = 8.06$ in.
Flange Thickness: $t = 0.552$ in.
Depth: $D = 8.32$ in.
Web Thickness: $d = 0.370$ in.
Area: $A = 11.66$ in.$^2$

Two types of stress-strain curves were used:

1. Elastic-plastic
2. Elastic-plastic-linear strain hardening

In each case a value of 300. was used for the Ramberg-Osgood parameter $n$. Strain hardening was assumed to start at a strain of 0.017 in/in which was scaled from figures in Ref. 39. Each of these stress-strain curves was used with and without an assumed residual stress pattern found in Ref. 24 for a total of four load-deflection curves. It was assumed that the maximum compressive residual stress was 30% of the yield stress. The residual stress pattern is shown in Fig. 40. The equations needed to compute the given values are also presented in Ref. 24.

The elemental discretization and layering used in this example are shown in Fig. 41. It can be seen that in this case the layering has been performed parallel to both axes of the
cross-section rather than parallel to only one axis as shown in earlier examples. This two directional layering will be used to assign different residual stress values to the initial stress field previously discussed in Section 2.5. This use of layering resulted in the approximate residual stress pattern shown dashed in Fig. 40. It also resulted in a relatively crude discretization for accommodating the gradual plastification of the section when residual stresses were not considered. If primary interest in this research had been metal beams with residual stresses more layers would have been used.

The individual layers could also have been assigned separate stress-strain curves to try to account for the change in strain at the onset of strain hardening caused by the residual stresses. This was not actually done and any attempt to do so would have been an approximation.

The four load-deflection curves resulting from the combination of stress-strain curves with and without residual stresses is shown in Fig. 42. Also shown is the experimental load-deflection curve and the results obtained by numerical integration of the distribution of curvature along the beam. This numerical integration scheme is said to be theoretically exact (Ref. 39) but its application involves a trial and error numerical scheme so that some error is to be expected. The numerical integration scheme also included strain hardening but did not include residual stresses. It can be seen that while there were only seven
numerical integration points given they agree quite well with the strain hardening results presented here. It can also be seen that the experimental and analytic results differ significantly between deflections of about 0.3 inches to about 0.9 inches. This difference reaches about \( \frac{7}{\circ} \) at a displacement of about 0.4 inches but is less over the rest of the range. There are several reasons for this discrepancy:

1. The "fixed end" of the beam was framed into a supporting connection which was not perfectly rigid. During this test series several support conditions were tried and this particular specimen had the most end rigidity.

2. The residual stress pattern assumed is only approximately representative of wide flange beams. The welding required at the "fixed ends" would change the residual stress pattern drastically.

3. As previously mentioned, the layering used was relatively crude, although experience would indicate that this would be a minor source of error.

Knudsen et al. made several references to the residual stresses in the "as delivered" beams and indicated that this was a large source of error in comparisons with their calculations. Reference to Fig. 42 shows that the compensation offered by the assumed residual stress pattern is reasonable as plastification
reaches the pure moment section of the beam. It also suggests that a higher level of residual stresses than that assumed is indicated. Plastification of the "fixed ends", however, shows relatively little effect of the assumed residual stresses indicating that the welding in that area and the lack of total fixity are large factors in the apparent discrepancy.

The simple plastic theory would predict a collapse load of 107 kps for this beam. The elastic-plastic stress-strain curve with n = 300. yielded the following results without residual stresses.

<table>
<thead>
<tr>
<th>Load (kips)</th>
<th>Deflection (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>105.0</td>
<td>1.00</td>
</tr>
<tr>
<td>106.0</td>
<td>1.20</td>
</tr>
<tr>
<td>107.0</td>
<td>1.40</td>
</tr>
<tr>
<td>107.8</td>
<td>1.60</td>
</tr>
<tr>
<td>108.5</td>
<td>1.80</td>
</tr>
<tr>
<td>109.0</td>
<td>2.40</td>
</tr>
</tbody>
</table>
4. PARAMETRIC STUDY

4.1 Introduction and Scope

This chapter describes a parametric study conducted using the computer program developed as part of this research (Ref. 41). The object of this study is threefold:

1. To investigate the sensitivity of the analysis technique to such variables as elemental discretization.
2. To investigate the behavior of beams when one or more characteristic parameters are altered.
3. To provide guidance to potential users about the sensitivity of the analysis technique to normal engineering estimates of material properties.

All investigations were carried out using the prestressed concrete I-beam E-5 discussed in Section 3.2.2 and shown in Fig. 29, except as noted. In most cases the two concentrated loads shown in Fig. 29 were applied with the distance "a" equal to four feet. In some cases a uniform load was applied. The values of applied load in the figures to be presented are given as a "load ratio". A load ratio is defined as the value of one of the concentrated loads, \( V \), shown in Fig. 29 divided by 20., or the value of the uniform load divided by its starting value of 2.4 kips per foot. All deflections and positions given in the figures are in inches. The uniformly loaded beam will be 17 ft. 6 in. long.
The parameters investigated in this study are listed below:

1. The effect of the variation of the iteration tolerance using values of 1%, 5%, 10% and 20%.

2. The effect of varying the yield strength of the strand from 225. to 265. ksi.

3. The effect of draped strand, as opposed to straight strand.

4. The effect of the variation of the Ramberg-Osgood parameter $m$ for the values of 0.52, 0.72 and 0.92.

5. The effect of the variation of the Ramberg-Osgood parameter $n$ using the values of 7.0, 9.0 and 11.0.

6. The effect of varying the compressive strength of the concrete ±600. psi from the base value of 6,610. psi.

7. The effect of varying Young's modulus ±600. ksi from the base value of 4600. ksi.

8. The effect of varying the compressive strength ±600. psi and using the procedure discussed in Section 2.3 to compute the other stress-strain curve parameters.

9. The effect of the variation of the tolerance on the tensile strength for the values of 1%, 10% and 20%.
10. The effect of varying the tensile strength of the concrete ±100. psi from the base value of 530. psi.

11. The effect of using 2, 4 and 8 elements of equal length with a small element at the centerline.

12. The effect of using 2, 4 and 8 elements of equal length with a small element at a support.


14. The effect of using 12 concrete layers with 1, 2 and 3 steel layers.

15. The effect of no compressive unloading, no tensile unloading, no compressive and no tensile unloading and, finally, including both types of unloading.

16. The effect of varying the rate of compressive unloading from 1000. ksi to 4000. ksi.

The most significant aspects of these studies will be discussed here in detail. Ref. 42 contains more information about these studies.

The laboratory test beams used as comparative standards in this study all exhibited under-reinforced behavior. Needless to say some of the conclusions drawn here would be different for over-reinforced beams. In general, those conclusions dealing with small changes in external load and involving the yielding of the
strand should be regarded as being especially applicable to the under-reinforced case.

As explained in Section 3.2.2 each analytic load-deflection curve for prestressed concrete I-beams loaded as shown in Fig. 30 has a discontinuity. This discontinuity is exaggerated by the plotting scale used to produce the figures for this chapter.

4.2 The Effect of Iteration Tolerance

Figure 43 shows the effect of varying the convergence tolerance on the load-deflection diagram. The four curves shown correspond to 1%, 5%, 10% and 20% relative error tolerance on the displacement field. It can be seen that this wide range of error tolerance has a surprisingly small effect on the load-deflection curve for the centerline of the beam. This can be explained as follows:

1. The incremental displacement vector is initially null for each increment of load. This means that the first iteration of each increment is never accepted as meeting the error tolerance. If the final vector from the preceding trial were used as the comparative standard it is apparent that many trials could be within say a 10% tolerance of this standard on the first iteration of the next load step. The null initial vector requires more computational effort but the results seem to justify it. Experience has shown that perhaps one-third of the load steps...
are solved with only two iterations making it even more apparent that the error tolerance insensitivity is related to the null initial incremental displacement vector.

2. The error tolerance is the maximum allowed for any displacement. This means that most, and possibly all, of the other displacements have less than the maximum error.

3. The error tolerance is a relative, absolute value so that the error could be positive or negative. This would reduce the accumulation of error in some indefinable manner. It would not, however, increase the accumulation of error to a value over the error tolerance.

4. The same displacement component would probably not consistently be the one with the maximum error. This would tend to distribute the error and aid in making the accumulated error significantly smaller than the maximum allowable error.

5. Figure 43 shows the effect on midspan vertical deflection which is the largest displacement component for this beam. It is plausible that the smaller values would be more susceptible to error than the larger ones. It might therefore be possible to plot some other displacement and see a greater effect of the error tolerance.
Table VII compares the results and execution times of the computer executions required to plot Fig. 43.

**Table VII - The Effect of Iteration Tolerance**

<table>
<thead>
<tr>
<th>Tolerance (%)</th>
<th>Ultimate Load Ratio</th>
<th>Execution Time (Seconds)</th>
<th>Ultimate Deflection (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.9613</td>
<td>199.7</td>
<td>2.290</td>
</tr>
<tr>
<td>5</td>
<td>1.9535</td>
<td>220.0</td>
<td>2.219</td>
</tr>
<tr>
<td>10</td>
<td>1.9513</td>
<td>168.1</td>
<td>2.246</td>
</tr>
<tr>
<td>20</td>
<td>1.9583</td>
<td>197.3</td>
<td>2.311</td>
</tr>
</tbody>
</table>

The results in this table show that no clear conclusion can be drawn about execution time as related to error tolerance. This is a result of the five points previously discussed and the additional fact that the load is continuously being altered so that the tensile strength and convergence tolerances are not exceeded within a limited number of iterations per trial load step. While a similar study was not conducted on a steel beam it would seem that in that case a clearer relation between iteration tolerance and execution time would result because cracking would not be included.

4.3 The Effect of Draped Strand

Figure 44 shows the effect of draping one strand of the I-beam whose cross-section is shown in Fig. 29. Only the single
strand was draped because it was felt that draping the other strand groups would violate practical conditions of cover. Certainly from a mathematic viewpoint the strand could even lie outside the beam but the desirability of practical analytic examples is apparent.

The center of gravity of the strand pattern at the end of the beam is 14.18 inches from the top of the beam. By deflecting the single strand an additional 7 inches at points four feet from each end an eccentricity of 15.33 inches for the center seven feet of the beam is obtained.

Figure 44 shows quantitatively those changes in load-deflection behavior which would be qualitatively deduced. The increase in compressive prestress applied to the bottom of the beam increases the cracking load and ultimate load. There is also a reduction in ultimate deflection. This results because the single strand reaches a higher stress by virtue of its lower position in the beam. A larger compressive stress block is then required for equilibrium causing a lowered neutral axis which, in turn, results in a higher concrete strain for a given deflection. The net result is that the concrete crushes at a smaller vertical deflection.

4.4 **The Effect of Varying Compressive Stress-Strain Parameters**

Figures 45 through 52 are arranged in pairs. The first figure will show portions of three compressive stress-strain
curves for concrete. Each figure stops at the horizontal plateau which was discussed in Section 2.3. The second figure in each pair will show the resulting load-deflection curves obtained by using the corresponding three stress-strain curves. These demonstrate the sensitivity of the load-deflection curves to the parameters \( m, f'c, n \) and \( E \). They will therefore show the degree of variation which could be expected from using estimated concrete properties as input for the analysis. In preparing each curve a 1% iteration tolerance was used and, except as noted, all parameters except the one under investigation were held constant. This means, for instance, that even though \( E' \) or \( f'_c \) were changed, \( m \) would not be changed as would normally be done using Eq. 2.35.

4.4.1 The Effect of \( m \)

Figure 45 shows that as \( m \) increases the strain at which \( f'_c \) is reached decreases. It is also seen that as \( m \) increases the upper one-third of the stress-strain curve would indicate a stiffer material. These observations are seen in Fig. 46. For a given load the load-deflection curves have smaller displacements as \( m \) increases thus indicating a stiffer material. This observation and those to follow apply, of course, to those areas of the load-deflection curve for which the change in \( m \) produces a noticeable effect. It will also be seen that as \( m \) increases the ultimate deflection increases. This is a result of what will be called the stiffer material concept. A stiffer
material is one which has a steeper stress-strain curve thus implying that there is more area under that curve for a given strain.

This "stiffer material concept" will be explained in detail here because it will be used to explain subsequent observations. The stiffer material concept and its influence on load-deflection behavior can be demonstrated as follows:

1. Assume that a displacement field is known.
2. Therefore a strain field can be found.
3. Therefore stresses can be computed.
4. The tensile and compressive stress resultants must be in equilibrium.
5. If the concrete compressive stress-strain curve is steeper and reaches the peak compressive stress at a lower strain a smaller compressive area is required to balance the tensile force. Thus the neutral axis rises.
6. The rise in the neutral axis does three things: (a) it causes a higher steel strain and hence a higher steel stress at a given displacement, (b) it also increases the moment arm of the forces forming the internal couple, and (c) while the steel strain is higher the concrete strain can be lower with a stiffer material and still produce a given compressive resultant. All of these actions contribute to the support of a larger external load at a
given displacement or, conversely, a lower displacement at a given load.

7. At the load required for the less stiff stress-strain curve to reach unloading the stiffer stress-strain curve would have resulted in a lower displacement and a lower strain. Even when a beam made of the stiffer material reaches a deflection corresponding to the ultimate deflection of the less stiff beam the concrete strain is still lower because of the higher neutral axis. Therefore, it takes a larger displacement to reach the unloading portion of the stress-strain curve and hence the ultimate load of the stiffer material beam. The increase in ultimate deflection is much larger than the increase in ultimate load in the example being discussed.

It will also be observed that in the variation of \( m \) the stiffer material produces load-deflection curves which approach their ultimate load at a lower gradient than the less stiff material. This is caused by the fact that the stiffer curves resulting from a higher \( m \) reached their horizontal plateau at a somewhat lower strain than the less stiff curves and hence caused a reduced stiffness when they reached that plateau. It is also noted that this plateau is longer for higher values of \( m \).

For \( m = 0.52 \) Fig. 45 also shows that while the observations above hold for this case too they are exaggerated by the
smaller compressive stress at ultimate load. This explains why the difference in ultimate load shown in Fig. 46 is greater in the pair \( m = 0.52, m = 0.72 \) than in the pair \( m = 0.72, m = 0.92 \).

The change in ultimate load is quite small in this under-reinforced example. This is because internal equilibrium must be maintained as the steel yields. Thus the total ultimate tensile and compressive stress resultants would be essentially the same for all values of \( m \). But as seen in Fig. 45 the moment arm of the internal couple would tend to increase slightly as \( m \) increases and as the ultimate load is reached. This results in a slightly higher ultimate moment as \( m \) increases.

Figure 46 shows that variation of \( m \) from 0.52 to 0.92 has relatively little effect on the shape of the load-deflection curve except, perhaps, at the ultimate load. If the recommendations on defining the compressive stress-strain curve are followed the value of \( m \) will be found from Eq. 2.35 and will not be subject to judgment.

4.4.2 The Effect of Compressive Strength

Figure 47 shows the effect of varying the compressive strength, \( f'_c \), \( \pm 600 \) psi from the base figure of 6.61 ksi. Figure 48 shows the effect on the load-deflection curves. The observations here are very similar to Figs. 45 and 46. As \( f'_c \) increases the ultimate load increases slightly and the ultimate deflection increases. Once again the increase in ultimate deflection is much
greater than the increase in ultimate load. Comparing Figs. 45 and 47 will show that the similarities in observations are a result of the similarities in the stress-strain curves. The explanations employed in the previous discussion are equally valid here and fully explain the observed phenomena—especially the very small increase in ultimate load.

Figure 48 also shows that a variation of almost 10% in the compressive strength of the concrete had much less than a 10% effect on the load deflection curve except perhaps at the ultimate deflection. Beam E-5 was an under-reinforced beam; and over-reinforced beam would probably have shown an increase in ultimate load which was essentially proportional to the increase in $f'_c$.

4.4.3 The Effect of $n$

It was shown in Ref. 42 that varying $n$ from 7.0 to 11.0 made almost no difference in the stress-strain curves and had even less effect on the resulting load-deflection curves. Within practical limits (of $n$) the insensitivity to $n$ is not dependent on the amount of reinforcement.

4.4.4 The Effect of Young's Modulus

Figure 49 shows the effect of varying the modulus of elasticity, $E$, ±600 ksi from the base figure of 4600 ksi or about ±13% variation. Figure 50 shows the effect on the load-deflection curve. Once again a stiffer material is evident.
Varying Young's modulus appeared to have the most effect on the load-deflection curve of any of the stress-strain parameters investigated.

4.4.5 The Effect of Varying the Analytic Compressive Stress-Strain Curve

The comparative studies shown in Figs. 45 through 50 show that the shape of the load-deflection curve is most effected by the estimate of Young's modulus and somewhat effected by $f'_c$ and $m$. While studies of this type are necessary and valuable from a research point of view a more practical study would be to vary $f'_c$ and use the recommendations found in Section 2.3 to find the remaining parameters to define the compressive stress-strain curve. A comparison of the resulting load-deflection curves would be a better indication of the variation an engineer might encounter due to using estimated material properties.

In this study the compressive strength was varied ±600. psi from the base figure of 6.61 ksi. Since the previous studies had shown the importance of Young's modulus it was decided to use Hognestad's equation, given below, instead of the ACI equation.

$$E = 1,800 + 460. f'_c$$

Hognestad's equation gives a wider range of Young's moduli for these concrete strengths than the ACI equation resulting in a more
exaggerated comparison. The stress-strain curve parameters were computed as recommended and are shown in Table VIII.

Table VIII - DATA FOR ANALYTIC STRESS-STRAIN CURVES

<table>
<thead>
<tr>
<th>f'_c (ksi)</th>
<th>E-used (ksi)</th>
<th>m</th>
<th>n</th>
<th>E-ACI (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.01</td>
<td>4560.</td>
<td>0.659</td>
<td>9.</td>
<td>4700.</td>
</tr>
<tr>
<td>6.61</td>
<td>4840.</td>
<td>0.683</td>
<td>9.</td>
<td>4930.</td>
</tr>
<tr>
<td>7.21</td>
<td>5120.</td>
<td>0.704</td>
<td>9.</td>
<td>5150.</td>
</tr>
</tbody>
</table>

Comparing the values given for m in the table above and those used to produce Fig. 45 it can be seen that the effect of m in this study should be much smaller than as indicated in Fig. 46.

The resulting stress-strain curves are shown in Fig. 51. The load-deflection curves are shown in Fig. 52. It can be seen by comparing the ultimate deflections in Figs. 48, 50 and 52 that the effects of estimating the compressive strength and the modulus of elasticity on the ultimate deflection are in some way additive. This is a direct consequence of an increase in compressive strength causing an increase in the modulus of elasticity. Figures 48 and 50 show that an increase in either of these parameters would result in an increase in ultimate deflections, all other parameters being equal.

The intrinsic shape of Fig. 52 is a result of the effect of Young's modulus. A comparison of Figs. 50 and 52 will also
show that the extent of variation shown in Fig. 52 is almost one-half that in Fig. 50. This is roughly the same as the extent of change in Young's modulus corresponding to the two figures.

Figure 52 shows that, with the possible exception of ultimate deflection, the effect of uncertainties inherent in engineering estimates of material properties do not greatly alter the resulting analytic load-deflection behavior of a beam under investigation. This fact is in agreement with observed behavior of test beams which are similar but not truly identical.

4.5 The Effect of Tensile Stress Tolerance

Figure 53 shows the result of varying the tolerance on the tensile cracking strength, FTOL, from 1% to 10% to 20%. The effect is quite small and localized near the region of the load-deflection curve corresponding to the rapid growth of cracking. The variation of the tensile tolerance had the most profound and consistent effect on the execution speed of any parameter tested for a given number of elements and layers. Most parameters made little consistent difference but changing the tensile tolerance from 1% to 20% reduced the execution time approximately 40%. This large change in computational effort is probably due to a smaller number of load reductions required to meet the cracking criterion. Each load reduction causes the original as well as the fictitious loads to be solved again. This represents a considerable effort each time the load reduction is necessary.
4.6 The Effect of Tensile Strength

Figure 54 shows the effect of varying the tensile strength, $f_t$, of the concrete ±100. psi from the base value of 530. psi which was already adjusted for dead load tensile stress as indicated in the discussion in Section 3.2.2. Figure 54 shows that the variation in tensile strength effects that region of the load-deflection curve at and beyond first cracking. The increase in cracking load reflects the increase in tensile strength as would be expected. The effect on the shape of the load-deflection curve diminishes as the ultimate load is approached. There is relatively little effect on the ultimate deflection or ultimate load. The last two observations are consistent with the discussion in Section 2.3.

4.7 The Effect of Elemental Discretization

Figures 55 through 60 will be discussed separately and in sub-groups. The 17 ft. 6 in. beam mentioned in Section 4.1 will be used here. The areas of discussion are:

1. The effect of the total number of elements on the load-deflection behavior.

2. The effect of the total number of elements on the converged solution in the linear elastic region.

3. The effect of the total number of elements on the converged solution in the nonlinear region.
4. The effect of the type elemental discretization, i.e. the position of the small element, on all of the above.

Figure 55 shows the elemental discretization used. The example number used to keep track of the various computer executions will be used in the discussion to distinguish the various examples. As shown, the three figures in Fig. 55 will be considered as "right" for some examples because the centerline of the beam will be on the right side of each sketch. The simple support will be on the left. Symmetry was used in these examples. Some of the test examples were "left", that is to say that the centerline of each sketch in Fig. 55 is now on the left and the support is on the right. The obvious difference is the location of the small element in each case. The various example numbers are shown in Table IX.

<table>
<thead>
<tr>
<th>Example No.</th>
<th>Figure</th>
<th>No. of Elements in Whole Beam</th>
<th>Right or Left</th>
</tr>
</thead>
<tbody>
<tr>
<td>124</td>
<td>55-A</td>
<td>6</td>
<td>R</td>
</tr>
<tr>
<td>125</td>
<td>55-B</td>
<td>10</td>
<td>R</td>
</tr>
<tr>
<td>126</td>
<td>55-C</td>
<td>18</td>
<td>R</td>
</tr>
<tr>
<td>130</td>
<td>55-A</td>
<td>6</td>
<td>L</td>
</tr>
<tr>
<td>131</td>
<td>55-B</td>
<td>10</td>
<td>L</td>
</tr>
<tr>
<td>132</td>
<td>55-C</td>
<td>18</td>
<td>L</td>
</tr>
</tbody>
</table>
Figure 56 shows the load-deflection curves for examples 124, 125, and 126. All three are reasonably close together but it is seen that the load-deflection curves converge to some curve as the number of elements is increased.

Figure 57 shows the load-deflection curves for examples 130, 131, and 132. Convergence with increasing number of elements is again apparent. It is also apparent that the load-deflection curves for examples 130, 131, and 132 are not grouped as closely as examples 124, 125, and 126 as the ultimate load is approached. This, and similar phenomena to be discussed later, is a result of the position of the small element in Fig. 55. In the series 124, 125, 126 this element was near the centerline of a simply supported beam carrying a uniform load. Thus it was quite close to the point of maximum moment. In fact the moment at the center of this element is 99.99% of the centerline moment. For the series 130, 131, 132 the position of the center of the element closest to the centerline reached moments of 93.9%, 98.5%, and 99.6% of the centerline moment respectively. This position at which stress is measured results in increased predicted values of cracking load, as seen in the printed output, and ultimate load. There is also a delay in the initiation and progression of inelastic actions. The computed ultimate loads for examples 130, 131, and 132 have the ratios 1.088, 1.014, and 1.000 compared to the moment ratios from the percentages of centerline moment of 1.061, 1.011, and 1.000. Thus the position of the point where stress is measured (the layer
centroid) accounts for most of the range of ultimate loads. Elemental discretization probably accounts for the rest.

Figure 56 shows that the small element near the centerline produced a much smaller range of ultimate loads and ultimate deflections. The ultimate load ratios were 0.999, 0.997, and 1.000 based on example 126. These values are so close that the approximations inherent in the numerical solution of nonlinear problems precludes any conclusions as to the crudest discretization, example 124, having a "closer" ultimate load than example 125.

Comparison of the elastic range of Figs. 56 and 57 show that the effect of the number of elements is quite small in that range. The effect appears to increase as nonlinear behavior proceeds. These figures show that both types of discretization produce load-deflection curves which converge from above with respect to ultimate load as the number of elements is increased. The ultimate deflections converge from below in the series 124, 125, 126 and from above in the series 130, 131, 132. The reason for the change in direction of convergence is the position of the small element in each series. The previous discussion of the effect of the distance from the point where stress is measured to the point of maximum stress, in this case the centerline node, would indicate that deflection convergence should be from above because these curves are terminated by crushing of the concrete which is dependent on the strain at the centroid of the appropriate element. Thus the effect of increasing the distance to the
point of measure would be to allow more deflection at, in this case, the centerline before reaching the crushing criteria at the point of measure. However, when the small element is close to the point of maximum moment the nonlinearities take place at a more accurate load level and location leaving more of the surrounding length of the beam capable of offering support to the region of high stress. As the distance from the centers of the adjoining elements to the centerline increases these adjoining elements are less subjected to the nonlinearities and more capable of offering support to the system of elements. The effect of the additional stiffness of the adjoining elements is to increase the hinge-like rotation of the small element undergoing a highly nonlinear response. This causes the crushing strain criteria to be met at a lower deflection. Thus the deflection of the beam is reduced and deflection convergence is from below as the number of elements is increased and the small element is kept near the point of maximum moment.

Figures 58 through 60 show these conclusions in different ways than Figs. 56 and 57. Figures 58, 59, and 60 each show the effect of discretization for the same number of elements by comparing load-deflection curves. The need to use a finer element mesh near points of maximum stress is evident. As the number of elements is increased the need for sophisticated discretization is decreased, as would be expected. It is emphasized that the need for good discretization is more important in nonlinear problems.
This fact is evident in Figs. 55 through 60. Reference 42 contains more information about the effects of number and position of elements.

4.8 The Effect of Concrete Layer Discretization

Figure 61 A,B,C show three types of layer discretizations corresponding to examples 133, 134, and 135, respectively. All three examples are prestressed concrete beams loaded as shown in Fig. 29 and have the prestressing strand discretized in the same manner. This test will therefore isolate the effect of increasing the number of layers in general and the number of layers of a cracking-crushing type of material in particular.

Figure 62 shows the resulting load-deflection curves. It can be seen that examples 134 and 135 are quite similar but that example 133 is very different. Actually the curve for example 133 extends beyond what is plotted in this figure. The reason for the tremendous extension of the load-deflection curve lies in the discretization shown in Fig. 61-A. The whole compression flange is modeled as one layer. As the ultimate load is reached the neutral axis lies within this layer. The result is that this layer is reaching a uniform stress of $f'_c$ as the tensile strain approaches infinity. It also means that it is highly unlikely that strain at the centroid of the layer will ever reach the strain at which unloading starts. The discretization shown in Fig. 61-B provides much more accurate results because the two
layers in the flange mean that the stress is being measured much closer to the actual peak stress in the section. Two layers also provide, in this case, for compressive unloading. There was relatively little gained by going from eight layers in Fig. 61-B to twelve layers in Fig. 61-C.

The printed output corresponding to Fig. 62 also shows that as the number of layers is increased the apparent cracking load decreases. This is a result of more layers in the tension flange resulting in the centroidal layer stress being measured closer to the maximum stress in the section.

Figure 63 shows the deflected half-shapes at ultimate loads for examples 134 and 135 and a selected curve for example 133. Again, numerical aspects have to be considered when drawing conclusions from results which are as close together as examples 134 and 135. It appears reasonable to conclude that as the number of layers increases the solutions converge to some deflected shape, especially if the additional layers are added with judgment. It should be apparent that, from a strict consideration of the number of layers, dividing the bottom flange of the beam in Fig. 61-A into ten layers would have almost no effect on the results of example 133 as the ultimate load is approached. It might, however, result in a more realistic speed of release of the strain energy from cracking. This would be minimized by the fact that the cracking extended far above the bottom flange at ultimate load.
In this context the result of better layering of the tensile flange reaches a point of diminishing returns.

The linear range deflected shapes were also plotted. As would be expected from the data shown in Fig. 62 the deflected shapes were virtually identical.

\[4.9\] The Effect of Tensile and Compressive Unloading

Figure 64 shows load-deflection curves indicating the effect of cracking and crushing initiated unloading of layers. Note that there are four curves in Fig. 64. The curve for the E-5 I-beam was used as a standard and will be called example 102.

If cracking unloading is not permitted the curve shown as example 138 results. The ultimate load is higher because the internal resisting couple produced by the tensile stress is still present. The maximum tensile stress is the same as that used in the cracking criteria in example 102 but unloading and redistribution are not performed. The ultimate deflection is decreased because the presence of the tensile stress requires additional compressive stress for equilibrium at the same state of deflection. This causes the neutral axis to be lowered producing a higher concrete strain for a given displacement. This means that compressive unloading starts at a lower displacement and results, eventually, in a failure to converge. Looked at another way, the difference in displacement results from the redistribution of stresses after cracking. The loss of elemental stiffness is the same in both cases.
The effect of not permitting unloading due to crushing is also shown in Fig. 64 and will be called example 139. The maximum compressive stress is the same as the crushing criteria stress in example 102 although it will be recalled (Section 2.3) that compressive unloading is initiated by the attainment of a strain of $\varepsilon_1$. Since there is no unloading a hinge forms in the beam and deflections grow ad infinitum. The curves for examples 139 (and 140) were arbitrarily stopped when drawing these figures so as to produce a scale which also showed the curves for examples 102 and 138 to a reasonable size. Figure 64 shows the importance of the unloading leg of the compressive stress-strain curve; as stated in Section 2.4 it is used to more accurately describe a failure caused by compressive crushing of the concrete. The stress distribution produced in this manner is also more realistic. Example 139 (and 140) have almost rectangular compressive stress blocks at failure. While this might appear reflective of the Whitney stress block there are no $\beta_1$ reduction factor used here so that the stress block volume could be too large. In the under-reinforced example being presented this effect is offset somewhat by a rise in the neutral axis. In an over-reinforced case the effect would be a larger increase in the ultimate load. The increase for example 139 shown in Fig. 64 is a result of the increased moment arm of the internal couple caused by the rise of the neutral axis and by the positive gradient on the post yielding portion of the stress-strain curve for the seven wire strand which
will allow some increase in steel stress to hold the excess compressive force in equilibrium. Including the strain hardening of mild steel reinforcing bars while not including the unloading of the concrete compressive stress-strain curve would also cause an artificial increase in the ultimate moment.

Example 140 has neither cracking or crushing unloading permitted. The increased ultimate load was explained in the discussion of example 138, the increased ultimate deflection was explained in the discussion of example 139.

Figure 65 shows the deflected half-shapes of examples 102 and 139 at the last point for which convergence was attained. The reasons for their relative positions have already been discussed. Deflected half-shapes for examples 138 and 140 are shown for the last point included in Fig. 64. They are included only for reference since the selection of the last point plotted for examples 138 and 140 in Fig. 64 was arbitrary.
5. **APPLICATION OF NONLINEAR PRESTRESSED CONCRETE I-BEAMS TO BRIDGE ANALYSIS**

5.1 **Theoretical Considerations**

As stated in Chapter 1, the ultimate use for the analysis techniques developed in this research is to couple the nonlinear beams with a reinforced concrete deck to perform an overload analysis of beam-slab highway bridges. An investigation is underway (1973) on the development of a refined solution technique for the nonlinear analysis of reinforced concrete slabs (Ref. 55). While this work on slabs is not yet ready to be applied to bridge decks it will be shown in this chapter that the goal of developing a beam analysis technique which will function as part of a nonlinear bridge analysis technique has been achieved. Numerical results will be presented from the inelastic analysis of a bridge composed of prestressed concrete I-beams supporting a deck slab made of either an elastic material or a von Mises, i.e. perfectly plastic, material.

Wegmuller and Kostem (Refs. 66,69) have developed an algorithm and the relevant computer program which performs an elastic-plastic finite element analysis of eccentrically stiffened plates. It is not the intent here to review in any more detail than necessary the work contained in Refs. 66 and 69. Only that information necessary to see how the elastic-plastic-cracking-crushing beam effects this existing algorithm will be provided.
In this earlier work the beam and plate elements were assumed to be made of a material which obeys the von Mises yield criteria and had, at most, a bilinear elastic-perfectly plastic or elastic-linear strain hardening stress-strain curve. Layering was used in both the plate and the beam elements but all layers had the same stress-strain curve. This meant that only a homogeneous beam could be analyzed and that cracking and crushing could not be considered. The plate bending element used was the rectangular, twelve degree of freedom element developed by Adini and Clough (Ref. 1). Each of the four corners of this element has one deflection and two rotational degrees of freedom.

\[
\{\delta_1\} = \begin{bmatrix} W \\
\frac{\partial W}{\partial Y} \\
-\frac{\partial W}{\partial X} \end{bmatrix}
\] (5.1)

Consideration of in-plane action produced two more degrees of freedom at each node.

\[
\{\delta_1\} = \{U\}
\] (5.2)

Thus there are a total of five degrees of freedom at each node. Wegmuller and Kostem (Refs. 66,69) used simple superposition of in-plane and out-of-plane stiffness matrices. Peterson and Kostem (Ref. 55) and Hand, Pecknold and Schnobrich (Ref. 27) have both shown that in-plane and out-of-plane coupling should be considered.

Using the standard approach of the finite element method as shown in Chapter 2, displacement functions are chosen.
The use of superposition or the use of strain coupling leads to one stiffness matrix.

\[ \{F\} = [K] \{\delta\} \]  (5.7)

The beams also contribute to the system stiffness matrix. It can be seen by comparing Eq. 2.4 with Eqs. 5.1 and 5.2 that the nodal displacements used for the beam and plate elements are compatible. Furthermore, comparing the displacement functions given by Eqs. 2.1 and 2.2 and Eqs. 5.3 and 5.4 it can be seen that for any given line defined by a constant Y coordinate the displaced shape of a beam and the corresponding internodal line on the plate
is the same. This means that there are no gaps between the beam and plate elements. Additional study of these equations shows that strain compatibility between beam and plate is maintained. The beam stiffness components are included by adding those terms of the beam stiffness matrix to the corresponding terms of the plate element stiffness matrix. This was done node by node so that a topology matrix was needed to determine if a beam element was present at a given node. A very simplified flow chart of Wegmuller and Kostem's approach is shown in Fig. 66. From the previous discussion it is evident that the following areas of the original eccentrically stiffened plate program have to be modified to permit the use of a concrete beam:

1. Provision for much more extensive beam input.
2. Establishment of new beam stiffness matrix.
3. Provision for a table of materials.
5. Provision for prestressed concrete beams
   a) Include nodal force vector
   b) Include initial stress field
   c) Include special treatment of prestress camber.
7. Provisions for computing and treating the fictitious load vector for unloading.
8. A new initial load scaling technique had to be developed.

Wegmuller and Kostem were able to utilize the von Mises stress correction technique to speed convergence. The use of concrete beams implies that a new iteration scheme will have to be provided. Several schemes were tried and a discussion of two types of iteration schemes is presented in Section 5.3. Load reduction schemes were also needed to assure convergence as the bridge approached failure and to meet stress tolerance requirements.

Some of the modifications listed above were reasonably self-contained units of coding in the program developed for beams and were used virtually intact by making them separate subroutines or overlays.

The problem of having input which was both meaningful to potential users and still be efficient was resolved by allowing for four possible methods of reading in beam properties. These four methods reflect either current design practice or user desirability. They are:

1. Input data for only the first element of one typical beam. This implies that all beams are the same and are prismatic and have straight strand if prestressed.

2. Read in data for all elements of one beam. This implies that all beams are the same but are either non-prismatic, have draped strand, or both. Section 2.6 contains additional comments on non-prismatic beams.
3. Read in data for the first element of all beams. This implies that individual beams may differ but that each beam will be prismatic, have straight strand or both.

4. Read in data for each element of each beam.

Provision for different materials was provided exactly as in the beam program. Arrays of material properties were established and a material code array defines the type of material used in each layer of each element.

The stiffness matrix was assembled exactly as in the three degree of freedom beam and then expanded to the five degree of freedom system to be added to the plate stiffness matrices.

The Ramberg-Osgood stress-strain curve was also handled as it was in the beam program, as explained in Section 2.3. The tensile stress-strain curve was restricted somewhat by providing for only one slope on the downward leg of the stress-strain curve.

Providing for the possibility of prestressed concrete beams was accomplished by utilizing the nodal load vector, stiffness properties, equation solver, iteration scheme and convergence criteria as used for single beams. Single and multiple treatment of beams has also been extended to prestressing to conform to design possibilities. The entire process is performed with the three degree of freedom methods and the results expanded as necessary. Each beam may also have its own boundary conditions for prestressing. Each beam for which data is read in must have its own boundary conditions.
The initial stress field required by prestressing is also provided for each beam. This initial stress field and the possibility of having a nonlinear stress-strain curve complicate the first load scaling considerably. Now it is the difference between the initial stress and the stresses produced by the first load which must be scaled to some percentages of the peak tensile or compressive stress. This can be expressed in equation form as follows:

\[
\text{Ratio} = \text{Max} \left| \frac{\sigma_{i}(I,J) - \sigma_{i}(I,J)}{P \sigma_{0}(k) - \sigma_{i}(I,J)} \right|
\]

(5.8)

Where

- \( \sigma_{i}(I,J) \) = Initial stress in a layer
- \( \sigma(I,J) \) = Current total stress in a layer
- \( P \) = The percentage of the tensile or compressive strength allowed for the first load step
- \( \sigma_{0}(k) \) = Peak tensile or compressive stress as applicable for material "k" with corresponding algebraic sign.
- \( k \) = Material code of layer \((I,J)\).

\( P \) is chosen to keep the scaled up stresses in an essentially linear-elastic range. The applied loads are then scaled up to the inverse of "ratio". The load stresses are then added back into the initial stress field. This procedure eliminates the necessity of loading in small increments from an unloaded state.
The calculation of approximate flexural shear stresses proceeded essentially the same as in the single beam program. The process of computing the shear is essentially that of integration. The value of the limits of this integral are known at both free surfaces of a single beam. When the slab is assumed to be composite with the beam the calculation of shear stresses has to start with the bottom layer and proceed towards the slab.

The fictitious load vector components were calculated as in the single beam program except that they applied to the five degree of freedom system. As will be discussed in Section 5.3, a different iteration technique was used with the modified stiffened plate program. This technique combined the fictitious loads generated by load step "N" with the applied load from load step "N+1". This was done to reduce the computational effort. Further discussion of the iteration scheme and its relation to the fictitious load vector will be deferred to Section 5.3.

5.2 Four Beam Bridge Model Investigations

Figure 67-A shows a quarter piece of a bridge which is symmetric about the X and Y axes and which will carry a symmetric load. This bridge has a reinforced concrete deck 7-1/2" thick supported on four prestressed concrete stringers. This example was chosen for the development phase of the computer program so that the simplest model having all representative elements, double symmetry and distinctly interior and exterior beams could be used.
The basic geometry of this four beam bridge was taken from an in-service box beam structure near White Haven, Pennsylvania (Ref. 26). This provides a geometrically creditable model which, while not necessary, is certainly preferable. Since this research deals with bridges using prestressed concrete I-beams the box beams were replaced with the idealized I-beam shown in Fig. 67-B. The idealized shape has the same area and moment of inertia about the bending axis as the box beams in the actual bridge.

The bridge was a simple span 64'-8" long and 27'-0" wide. Prestressing was assumed to provide an initial compressive stress of 2000. psi at the bottom of the beam, and no tension at the top. A prestressing force of 628. kips centered 7.5" below the centroid of the basic beam section was assumed. The steel layer data was found by assuming that thirty-three 7/16" 250. ksi strand were used to prestress the beam. This provided a steel area of 3.59 square inches at an initial stress of 175. ksi. The beam concrete was assumed to have a cylinder strength of 6.0 ksi and the slab was assumed to have 3.0 ksi concrete. For the first series of examples to be presented only the response to a "live load" will be investigated, i.e. the dead load of the structure has been offset by the relatively low assumed compressive prestress of 2000. psi.

The load-deflection curves in this section will be presented as "force ratio" versus "displacement ratio". A force ratio equal to 1.0 is the normalized load corresponding to the
attainment of a scaled stress resulting from the application of Eq. 5.8. In these examples the result was a stress equal to 90% of the tensile strength in one beam layer. A displacement ratio of 1.0 is the normalized displacement of the point of interest at a force ratio of 1.0

5.2.1 A Uniformly Distributed Load

The first loading case investigated was a uniformly distributed load over the entire structure. The results of this investigation are presented in Figs. 68-72.

Figure 68 shows a series of analytic load-deflection curves for various values of $E_{\text{downt}}$ with the center of the bridge chosen as the point of interest. Also shown are estimated ultimate force ratios.

Figure 69 shows a comparison of the load-deflection behavior of the midspans of both an interior and an exterior beam.

Figure 70 shows the ratio of midspan beam deflections.

Figure 71 shows the effect of varying the iteration tolerance used to check convergence of each load step.

Each of these figures will now be discussed in detail. The results in Figs. 68-71 will be based on an entirely elastic response of the slab. Figure 72 will show the results of a preliminary estimate of including the effect of slab nonlinearity for this load case.
5.2.1.1 *Convergence to an Estimated Ultimate Moment*

Figure 68 shows that a distinctly nonlinear load-deflection response has been obtained. The force ratio value of 1.0 corresponds to a uniform load of 0.441 kips per square foot in this example while the corresponding displacement ratio means that the deflection of the centerline of the bridge, point No. 9 in Fig. 67-A, was 0.917 inches. This loading produced a stress of 0.450 ksi or 90% of the concrete tensile strength (assumed as 500.0 psi) in the most stressed layer of either beam. If the response of the bridge remained linear, the force ratio and displacement ratio would be equal.

It is noted that all the load-deflection curves for non-zero values of $E_{down}$ in Fig. 68 appear to be approaching a similar plateau. The effect of $E_{down}$ appears to diminish with increasing deflection.

The top horizontal line represents an upper estimate of the ultimate load ratio of the cross-section computed by considering that:

1. The whole bridge is one beam.

2. The steel reaches its assumed yield stress of 250.0 ksi.

3. The concrete in the slab reaches an assumed cylinder strength of 3.0 ksi.

4. The compressive and tensile stress resultants can be equated to find the neutral axis using a Whitney stress block.
5. The ultimate moment can be computed by summing moments about the compressive stress resultant.

6. The ultimate moment is adjusted for the longitudinal location of the centroid of beam elements 2 and 4 in Fig. 67-A.

Use of these assumptions results in an ultimate force ratio of 1.737. The lower horizontal line results from using the ultimate strength equations in the AASHO Specifications (Ref. 2) with the single beam idealization. The result was a force ratio of 1.697.

The concrete stress in the longitudinal direction is about 3.0 ksi in the upper plate layers along the transverse centerline at a displacement ratio of about 7.0. Nonlinear behavior would have started before that stress level was reached indicating that the assumed elastic slab behavior is overestimating the stiffness of a structure with an inelastic slab of 3.0 ksi yield strength or ultimate strength.

The stiffness of the "bridge" is 3-1/2% of the original stiffness based on the last line segment of the $E_{\text{down}}$ of 800 ksi curve. Therefore, the elastic stiffness of the slab is quite significant in producing the remaining gradient on the load-deflection curve.
5.2.1.2 The Effect of Varying $E_{\text{down}}$

In the early stages of unloading of cracked zones the effect of increasing the value of $E_{\text{down}}$ is to lower the load-deflection curve. This is the same action noted in Chapter 2 for single beams. However, at higher load levels the behavior may be reversed and a larger value of $E_{\text{down}}$ may result in a higher curve. This is probably because the exterior beam reaches a cracking load after the interior beam has undergone some nonlinear deformation. Thus it appears that for higher values of $E_{\text{down}}$ the unloading of the interior beam is more complete before the exterior beam cracks. This would mean that, for low values of $E_{\text{down}}$, the fictitious forces from the cracking of both beams are being applied simultaneously to a less stiff structure for more of the load-deflection history. This results in larger deflections at a given load. This is reflected in the crossing of some load-deflection curves.

In the range of probable practical interest for overload analysis, say up to a displacement ratio of 2.0 to 3.0 the effect of values of $E_{\text{down}}$ in the range 1000 to 800 ksi is quite small. The instantaneous unloading approximated by $E_{\text{down}}$ equal to 20,000 ksi is significantly different.

The relative positions of two load-deflection curves for two values of $E_{\text{down}}$ is a very complex problem after extensive nonlinearities occur. The size of the basic load step (4.5% of the cracking load was used here), the type of iteration scheme,
the number and location of load reductions, the value or $E_{\text{down}}$ and the geometry of the structure all contribute to determining if and when two load-deflection curves will cross. Thus in Fig. 68 the curves corresponding to $E_{\text{down}}$ of 800. and 1000. ksi cross. The curve for $E_{\text{down}}$ of 20,000. ksi (an essentially brittle material) will cross the $E_{\text{down}}$ of 800. ksi curve between displacement ratios of 8.0 and 10.0. Insufficient data was obtained to determine if it will cross the $E_{\text{down}}$ of 1000. curve. When the non-linear action of the slab is included the relative positions will be even more complex.

Also shown in Fig. 68 is a load-deflection curve for a material which is elastic-plastic in tension as signified by $E_{\text{down}}$ of 0.0. As seen in Section 2.3 this behavior produces a much stiffer load-deflection curve because the deflection contributed by the unloading fictitious load vector has not been included. A somewhat higher ultimate load is also produced because of the extra internal stress in the tensile region which have not been unloaded.

5.2.1.3 Load-Deflection Behavior of An Interior and Exterior Beam

Figure 69 shows the displacement of the interior and exterior beam centerlines versus the load ratio. At first cracking of the interior beam the deflection of the exterior beam was 66.8% of the interior beam's deflection. The load-deflection
curve of the interior beam becomes nonlinear because of its cracked condition. The exterior beam, while still linear if it were a single beam, also has a slightly nonlinear response because of the nonlinear distribution of the increasing load caused by the cracking of the interior beam. When the exterior beam reaches first cracking its deflection is only 47.7% of the interior beam’s deflection. After this both beams are nonlinear in their own rights but the exterior beam continues to take more of the increasing load and will eventually be forced into the same deflected position as the interior beam. At a force ratio of 1.67 the exterior beam has 78.2% of the interior beam’s deflection showing that it has not only returned to its precracked relation to the interior beam but is already approaching the same deflection.

This same information is shown differently in Fig. 70 which shows load ratio versus the ratio of the midspan deflections of the interior and exterior beams. It can be seen that after the interior cracks it deflects faster than the exterior beam until the exterior beam cracks. Then the ratio of their deflections should approach 1.0 as the load ratio increases.

5.2.1.4 The Effect of Iteration Tolerance

Two examples with $E_{down}$ equals 800 were run to determine the effect of iteration tolerance. The first used a 10% iteration tolerance and the second used 2-1/2%. The resulting
displacements for a given load agreed for at least three significant figures until the load ratio exceeded 1.600. At that time different load steps were used which, largely due to the inclusion of the fictitious load vector with the basic load step, resulted in slightly different results. After a load ratio of 1.600 the deviation between the two curves was never more than 1% as shown in Fig. 71. A complete discussion of the effects of iteration tolerance on the behavior of one beam has been given in Chapter 4. There are only three differences in the current context. First, the program used for single beams started each cycle with a null incremental displacement vector. The stiffened plate program has a nonzero first trial vector. This would indicate that the stiffened plate program should be more susceptible to iteration tolerance. The second difference is that the plate elements in this example were entirely elastic so that their longitudinal action as elastic compression flanges reduced the overall nonlinearity of the problem, thus reducing sensitivity to the iteration tolerance.

Further insensitivity is provided by the addition of the fictitious load vector into the next trial instead of solving for a separate displacement increment corresponding to the fictitious load as was done in the single beam program. This means that even with the nonzero initial incremental displacement vector two or more trials are almost always required for a solution. The nonzero initial vector represents an increase in tolerance sensitivity only if one trial proved sufficient for convergence.
The 10% tolerance example was considerably more efficient. A final displacement ratio of 12.02 was attained in 450 central processor seconds using the 10% tolerance compared to a final displacement ratio of 8.87 in 600 seconds using a 2-1/2% tolerance.

5.2.1.5 The Effect of a von Mises Slab

The same example was also run using a slab of a material assumed to obey the von Mises yield condition with a yield stress of 3.0 ksi. A material which obeys the von Mises yield condition is also called a $J_2$ material. The portions of the program necessary to perform an elastic-plastic von Mises ($J_2$) analysis of plates have been reported in Refs. 66 and 69. The refinement in the analysis used in this chapter is made here to make a preliminary assessment of the effect of a nonlinear slab on the response of the bridge. In a sense, the $J_2$ slab is being used to effect a load distribution to the stringers which is more representative of a true bridge than an elastic slab and to alter the longitudinal stiffness of the cross-section.

Figure 72 shows a comparison of the load-deflection behavior of this uniformly loaded bridge with both an elastic and a $J_2$ slab. It can be seen that, in this example, there is no difference in the range plotted. At the last point plotted the maximum effective stress in a slab element was almost equal to the yield stress indicating that at higher displacement ratios some
difference would eventually be noticed. It will be seen that other loadings produce significantly different results.

5.2.2 A Single Concentrated Load at the Bridge Centerline

The load-deflection curves for this example using both the elastic and the $J_2$ slab are shown in Fig. 73. A force ratio of 1.0 corresponds to a force of 79.24 kips at point No. 9 (it should be noted that because of double symmetry the total load is four times as great). The displacement ratio of 1.0 corresponds to a vertical displacement of 1.193 inches at point No. 9.

Considering first the example with the elastic slab, a comparison of Figs. 72 and 73 shows that significantly different responses have been caused by the two loading cases. Figure 72 reflects the spread of cracking along and through both beams as a somewhat overlapping process. The load-deflection curve shows a continuous weakening of the structure. The interior beam cracks first as stated in Section 5.2.1.3 but the cracking of both beams is a continuous and overlapping process.

Figure 73, on the other hand, shows a nonlinear response which indicates that one beam (the interior beam) cracks first resulting in the first major change in the slope of the curve and then a considerable portion of the remaining load-deflection behavior takes place before the exterior beam cracks. The stress fields show this to be exactly the case.
Figure 74 shows this behavior and produces even more insight into the response of the structure. It can be seen that in the linear range the exterior beam deflects only about 21% of the interior beam deflection. These deflections are measured at the beam centerlines. It can also be seen that after an initial reduction in the deflection ratio as the interior beam cracks there is a considerable increase in load ratio during which there is almost no change in deflection ratio. This corresponds to the portion of the load-deflection curve between the widely separated initial cracking of both beams. As the exterior beam cracks the deflection ratio increases to about 50% for the last point plotted.

Also shown in Fig. 73 is a load-deflection curve for the same example with a J₂ slab. It can be seen that the same basic response of one beam cracking long before the second is again evident.

The curve for the J₂ slab is significantly lower (less stiff) than the elastic slab curve. This case of a single concentrated load at the bridge centerline is a very severe loading and emphasizes the effect of the J₂ slab because this loading produces very large transverse bending stresses in plate element No. 4 in Fig. 67. Thus some plate layers reach their yield stress very early in the load-deflection history. This, in turn, causes a reduction in the longitudinal stiffness of the bridge. A single line load on the longitudinal centerline might be an even more severe loading although it would be even less practical than this
single concentrated load. Load-deflection output for this example beyond what is plotted indicates that the $J_2$ curve will also approach the "maximum" and "AASHO" lines but at a significantly increased deflection.

Figure 74 shows the centerline beam deflection ratios for the $J_2$ slab case. It can be seen that in this case the ratio of deflections is reduced more during the period in which only the interior beam is cracked than it was in the elastic slab case. In fact, this ratio approaches 15% for the $J_2$ slab. As in the elastic slab case, the ratio increases again after the second beam cracks.

5.2.3 A Concentrated Load at the Centerline of the Interior Beams

Figure 75 shows the analytic load-deflection curves for a case in which a concentrated load is placed on the centerline of the interior beams. A force ratio of 1.0 corresponds to 87.86 kips at point No. 8 (double symmetry should also be noted again) while a displacement ratio of 1.0 represents a displacement of 0.853 inches at point No. 9. Comparing Figs. 73 and 75 it can be seen that this load case also indicates a relatively complete cracking of the interior beam before significant cracking of the exterior beam. Figure 76 indicates that the relative deflections for the linear history is between those given by Figs. 70 and 74 for the last two load cases. Figure 75 shows that the second
major break in the load-deflection curve occurs somewhat earlier for this case than for the second case.

Figure 75 also shows the result of using a $J_2$ slab with this loading. While the general behavior is similar to that shown in Fig. 73 it can be seen that the effect of the $J_2$ slab occurs much later in the load-deflection history of this case and has a much smaller effect than the $J_2$ slab used in the last case. This difference in the action of the $J_2$ slab is a direct result of the differences in slab bending stresses in the center plate panels. In fact, the difference in behavior caused by the use of the $J_2$ slab compared to the elastic slab was negligible in this case.

Figure 76 shows the centerline beam deflection ratios for this case. The overall response shown lies between the responses in Figs. 70 and 74 for the whole history of loading. The relative deflections start at about 35% for the linear behavior and decrease to about 25% as the first beam cracks and then increases to 50% after the second beam cracks.

Comparison of Figs. 68, 73 and 75 and 70, 74 and 76 show the effect of distributing the load more uniformly across the bridge. The total load indicated by the horizontal lines in Figs. 73 and 75 are the same although the load ratio is different. This is because moving the concentrated load from point No. 9 to point No. 8 in Fig. 67 improved the lateral load distribution enough to allow a 10.9% increase in the insipient cracking load. The effect on slab behavior and hence on longitudinal stiffness is seen in Figs. 68, 73 and 75.
While these examples have shown that the effect of slab nonlinearity is problem dependent it should be emphasized that these slabs did not contain a provision for cracking unloading or for the effect of multiaxial stress states other than the von Mises yield condition. These considerations are currently under study (Ref. 55) and should show that the effects of slab nonlinearity are more significant than demonstrated here.

5.2.4 Lateral Load Distribution

Figures 77 to 81 show the effect of loading and the $J_2$ versus elastic slabs on the lateral distribution of the applied load. While it is expected that the inclusion of a better model for reinforced concrete deck slabs will provide a more realistic estimate of the nonlinear lateral load distribution these figures are informative and reflect the current state of the art.

These figures show the percent of total beam moment versus position along the bridge for one stationary load. The "percent of total beam moment" is essentially a distribution factor. It is the moment carried by one beam as measured about the mid-plane of the plate divided by the total of such moments. These moment percentages were easier to calculate than the conventional distribution factors. Several comparative calculations showed them to be within $\pm 1\%$ of the corresponding distribution factors found by more exact calculations. Wong and Gamble (Ref. 71) have reported similar observations. It is readily apparent that the
reason that these moment percentages agree so closely with the
distribution factors is that the moment in the beam is found in
the reference plane at the midplane of the plate. Thus only the
longitudinal bending moment of the plate about its own middle
plane has been neglected. The sum of these moments along the
width of the bridge is small compared to the sum of the beam mo­
ments. This small discrepancy is further reduced by dividing the
individual beam moments by the sum of beam moments.

In Figs. 77-81 the solid line applies to the center of
elements two and four as shown in Fig. 67 which is representative
of the centerline load distribution. The dashed lines apply to
the centers of elements one and three which are approximately 1/6
the span from the supports. A comparison of the solid and dashed
lines represents the change in lateral load distribution along the
length of the structure as the nonlinear action occurs.

The percentage figures shown on each line are the per­
centage of the ultimate load ratio at the time the line represents.
The lowest percentage corresponds to insipient cracking of the
bottom layer of the center elements of the interior beam. The
middle percentage corresponds to first cracking of the exterior
beam. The highest percentage represents the distribution of load
at the last generated analytic result. A value of 100% corres­
ponds to the estimate of ultimate load of the structure using the
AASHO equations applied to the single beam analogy as explained
in Section 5.2.1.1.
Figures 77, 78 and 80 are for the elastic slab cases while Figs. 77, 79 and 81 are for the J\textsubscript{2} slab cases. For each of these graphs the distribution of moment approaches a uniform distribution as the ultimate load is approached. This is exactly what would be expected and agrees with previous results for a J\textsubscript{2} slab on steel beams (Refs. 66,69).

The concentrated load at bridge center showed the largest distribution to the interior beams in the linear range, followed by concentrated loads at the centerline of the interior beams and finally by the uniform distributed load. This problem dependent linear range distribution to the interior stringers was similar to that shown in the displacement ratio figures but the magnitudes are different. As inelastic action occurs the displacement ratios are decreasing while the moment ratios are increasing until first cracking of the exterior beams. After that the moment ratios and deflection ratios both approach 1.0 as deflections increase. The moment ratio appears to approach unity faster than the deflection ratios for all cases investigated.

Figures 78, 79, 80 and 81 show that the lateral load distribution figures for the end elements involving concentrated load cases become inverted as the nonlinear action occurs. This means that the end exterior beam elements eventually carry more moment than the end interior beam elements. Thus there appears to be a redistribution of load to the ends of the exterior beams in these cases as inelastic action occurs. The redistribution
occurs along both the width and the length of the bridge in some problem dependent fashion. This explains why the inversion occurs only in the concentrated load cases. In these examples the interior beams cracked much earlier in the load-deflection history than the exterior beams allowing for a great deal of redistribution to the exterior beams. Then when the center exterior beam elements cracked the redistributed load was again redistributed to the end external elements. In the uniform load case both the interior and exterior central beam elements cracked at a much closer time in the load-deflection history. More load was redistributed along the length of the bridge than across the width in this case than in the concentrated load case. Thus the total load on the interior end element was greater in this case than the total load on the exterior end element. The question of redistribution of load is obviously dependent on the geometry of the structure and on the type of loading.

The effect of the $J_2$ slab on lateral load distribution is most apparent in Fig. 79 for the case involving the concentrated load at the bridge center. The action of the $J_2$ slab appears to increase the distribution of load to the center interior beam elements in this example. It also appears to increase the redistribution of moment towards the exterior end beam elements as reflected in the dashed line segments. The amount of influence of the $J_2$ slab on lateral load distribution appears to parallel its influence on load-deflection behavior. As expected, there was
no difference in the lateral load distribution in the uniform load case with either the elastic or \( J_2 \) slab.

5.3 A Comparison of Iteration Techniques

The iteration technique used for the analysis of single beams was discussed in Sections 2.2 and 2.4. A logical flow chart for this technique is shown in Fig. 82. This flow chart is extremely simplified and will be used only in the current context. Figure 82 shows that at the end of a load step the effects of applying the fictitious load vector resulting from that load step are accounted for before the next load step is applied. The basic iteration scheme used to analyze the four beam bridge model is shown in Fig. 83. The overall process used in the four beam bridge studies also employed a load reduction scheme similar to that used with the single beams. There was no load increasing scheme as such; the load step was always returned to its original value. The iteration scheme in Fig. 83 represents a considerable reduction in computation effort compared to the one in Fig. 82. This is because the equilibrium equations corresponding to both the applied load increment and the fictitious load vector from the last load step are solved together. However, the second technique is more approximate than the first because it is possible that a large load step could lead to a large fictitious force vector which might then be added to an applied load step which undergoes several load reductions. The physical actions will be somewhat
out of sequence. This subject will be explored in depth through a discussion of two subschemes of the iteration technique used in the four beam bridge studies.

**First Scheme:** If convergence to a given displacement tolerance is not attained in three trials the load step size - not the fictitious load - is reduced by 50%. A maximum of three reductions will be allowed. Convergence to meet the tensile tolerance is accomplished by reducing the load step by 50% for as many reductions as required. This technique appears to contain an endless loop but this did not occur in the trial runs.

**Second Scheme:** This is the scheme actually adopted. This is essentially the same as the first technique except that six trials are allowed for displacement convergence and only three load reductions will be allowed to meet both displacement and tensile stress tolerances. If the tolerances are not met in three load reductions two actions can result.

1. A large number of trials will be allowed to reach the displacement tolerance.

2. There may be temporary tensile overstressing.

Temporary tensile overstressing is a possibility regardless of the iteration scheme because the fictitious load vector representing the unloading from the last cycle of applied load
load may be large enough by itself to raise some stresses above their corresponding allowable tensile stress. The fictitious load vector accounts for most of the possible difference in the results of the two different iteration schemes. Since this load is not subject to the load reduction process its effects will be essentially constant regardless of the extent of reduction in the applied load. This effect is present in a comparison of these two schemes or in a comparison of the iteration techniques used in the programs for the single beam and the four beam bridge analysis.

Assume that the change in incremental deflection from the applied load is directly proportional to the size of the load increment. Considering the small load increments used this is a reasonable assumption for the purpose of discussion. Also assume that:

\[ \begin{align*}
    A &= \text{Total deflection before this load step.} \\
    B &= \text{Total load before this load step.} \\
    C &= \text{Nominal size of the load step.} \\
    D &= \text{Deflection from the fictitious load.} \\
    E &= \text{Deflection from the nominal load step.}
\end{align*} \]

Finally, assume that under the first iteration scheme one 50% load reduction was made and that no reduction was made under the second scheme. The point on the load-deflection curve corresponding to this load step could be defined by the following Cartesian coordinates:
Scheme 1: \((B + 0.5 C), (A + D + 0.5 E)\)

Scheme 2: \((B + C), (A + D + E)\)

Assuming Scheme 2 is the correct solution it can be seen that Scheme 1 will produce a point which does not lie on the line segment joining the points \((B, C)\) and \((B + C), (A + D + E)\). The extent of deviation depends on how often the load reduction process is used differently by the two schemes. This is, in turn, a function of the tensile stress tolerance, deflection convergence tolerance and the rates of unloading used with any problem. Trial comparisons using both schemes to analyze the uniformly loaded four beam bridge produced results which differed by only 1% or 2%. Scheme 2 was somewhat faster than Scheme 1 because it used a smaller total number of load reductions.
6. SUMMARY AND CONCLUSIONS

6.1 A Summary of this Research

Chapter 1 contains the purpose and scope of this research. An analytic model has been developed which accurately describes the flexural load-deflection behavior of beams within the requirements stated in Section 1.1. These requirements were that the technique must be applicable to the overload analysis of beam-slab bridge superstructures using prestressed concrete I-beams and that it also be applicable to the flexural analysis of superstructures with steel I-shaped beams.

Reasons for selecting the finite element method as the basis for this research and a review of the initial and tangent stiffness methods were also presented. Previous finite element research on reinforced concrete beams was reviewed. It was noted that all the methods reviewed used continuum approaches, all but one involved hundreds of simultaneous equations and all would lead to three dimensional analyses if applied to the overload analysis of beam-slab bridge superstructures.

Finally, other approaches suitable for the nonlinear solution of single beam problems were reviewed. It is felt that these other methods were not as suitable for application to bridge overload analysis as the finite element method.

Chapter 2 contained the theoretical development of the chosen method. A simplified layered beam element based on the
plane sections assumption was presented and the economy of solution it provided was demonstrated. When applied to the bridge overload analysis problem this layered element allows a two dimensional algorithm to be used as an approximate solution. Limitations and assumptions were also discussed.

The finite element formulation for beam analysis based on the layered beam element and a concrete compressive stress-strain curve utilizing the Ramberg-Osgood formulation were presented. This curve provided a common base for handling many materials. The proposed concrete compressive stress-strain curve was compared with several other previously reported curves. Similarly, the concrete tensile stress-strain curve has been modeled as a Ramberg-Osgood curve although a degenerate form was used in most of the examples which were presented in Chapters 3 and 4.

Chapter 2 also contained a discussion of techniques developed to handle cracking and crushing of concrete, prestressed concrete beams and approximating the flexural shear stresses in the beam.

Chapter 3 presented correlations between the load-deflection curves computed using this analysis technique and experimentally measured results. Two reinforced concrete beams, four prestressed concrete solid rectangular beams, eight prestressed concrete I-beams and one steel wide flange beam were used as comparative examples. Chapter 3 contained only a comparison with the two reinforced beams, two of the prestressed solid
rectangular beams, three of the prestressed I-beams and the steel beam. The results of all of these comparisons showed that the techniques developed here produced an accurate model of inelastic beam behavior.

A comparative analytic study of under and over-reinforced concrete beam behavior and of the effect of modeling the stress-strain curve for steel were also presented.

Chapter 4 presented a summary of results from a parametric study conducted using the reported analysis technique. The general areas of study were the sensitivity to parameters effecting the internal operations within the computer program, the effect of parameters defining stress-strain curves and the effect of elemental and layering discretizations. An additional study on the effect of draped strands was presented. It is felt that this parametric study will provide valuable background information to potential users of the type of proposed formulation.

Chapter 5 contained the analysis of a four beam bridge using nonlinear prestressed concrete I-beams. Five different types of loading were applied to the structure and its response to these different loadings were compared. It was noted that the load-deflection curves approach the ultimate load predicted by ultimate strength theory applied to a single beam analogy.

The effect of different iteration schemes for the nonlinear analysis of concrete bridges on the analytic load-deflection response was also discussed. The interrelation between
iteration scheme and the unloading of cracked or crushed concrete was also presented. The influence of the nonlinear beams on the distribution of lateral load was also presented in Chapter 5.

The effect of a slab made of an elastic-perfectly plastic material on load-deflection behavior and lateral load distribution was discussed. The influence of slab nonlinearity was seen to be very dependent on the type and position of load applied to the structure.

Two studies relating to extensions of this work were made. The first dealt with torsional considerations and the second with beam-columns. Appendix A contained a discussion of the effects of neglecting the torsional stiffness of the pre-stressed concrete I-beams when they act as part of a bridge superstructure. It was noted that reports describing tests to failure of four full-scale bridges and one-half scale bridge made no mention of noticeable torsional distress.

The compensating effects of including the torsional stiffness of the beams in the four beam bridge model on linear range lateral load distribution and first cracking load were discussed for five load cases. In four of the five cases the effect on the first cracking load was less than 3%.

Load-deflection curves including an upper and lower (i.e. zero) bound on torsional stiffness were compared. Neglecting the torsional stiffness produced conservative but acceptable results in four load cases and virtually no change in the fifth.
An extension of the single beam analysis techniques to inelastic beam-column problems is presented in Appendix B. Good agreement with interaction curves for steel and concrete beam-columns found by the column-deflection curve and column-curvature curve methods was found.

6.2 Conclusions

6.2.1 Conclusions From Modeling Beams

1. The uniaxial compressive stress-strain curve for concrete based on the Ramberg-Osgood law compares well with other curves in the literature and with idealized test results. Using the Ramberg-Osgood stress-strain curve as the basis of material properties makes it possible to handle many materials with the same algorithm.

2. The tensile stress-strain curve for concrete has been developed from a limited amount of test data. The formulation has been left general enough to include the results of future research. Tension stiffening type stress-strain curves can also be handled.

3. A layered beam bending element for concrete flexural members has been developed based on the usual frame analysis type displacements. This element allows a nonlinear analysis to be performed with the beam treated as a line
for the purpose of generating the system stiffness matrix. An extremely efficient analysis for essentially prismatic beams governed by flexural action results. This method is not well suited to shear dominant analysis of beams. An extension into this area seems possible from a material viewpoint but the assumption of plane sections becomes tenuous. An extension to bending about both axes is also possible but if cracking of concrete is included much of the efficiency provided by this method would be lost.

4. The system of fictitious forces, compatible with the displacements used in this analysis, produce an adequate overall response to cracking and crushing of concrete.

5. A method for computing shear stresses based only on equilibrium has been developed for use with the layered beam element. Correlations with laboratory tests have shown that the model explained above gives very good agreement with the observed response of flexural members.

6.2.2 Conclusions From the Parametric Study

1. The method used for single beams is relatively insensitive to the iteration tolerance.

2. Draped strand beams can be analyzed.
3. The analytic load-deflection results for under-reinforced prestressed concrete beams are most affected by the stress-strain parameters $E$ and $f'_c$, somewhat affected by $m$ and virtually unaffected by $n$. Over-reinforced beams would probably show more sensitivity to the concrete stress-strain curve parameters.

4. For under-reinforced beams the normal uncertainties about material properties do not drastically affect the load-deflection curve.

5. The study on discretization has demonstrated that this analysis technique converged in both the elastic and inelastic ranges to both a load-deflection curve and to a deflected shape. Good discretization speeds convergence but is more significant in the inelastic range. Good results can be obtained with relatively few well placed elements.

6.2.3 Conclusions From the Application of the Nonlinear Prestressed Concrete Beam to Bridges

1. The nonlinear prestressed concrete beam analysis technique meets the requirement that it be applicable to the overload analysis of beam-slab I-beam bridges.
2. The load-deflection curves for the four beam bridge model approach the load which results from analyzing the bridge cross-section as a single beam.

3. The distribution of lateral load becomes more uniform as the nonlinear action progresses.

4. The influence of slab nonlinearity is highly problem dependent.
7. FIGURES
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\[ \theta_y = -\frac{\partial w}{\alpha x} \]
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Fig. 19 Free Body Diagram for Shear
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Singly R/C Beam

$P_{\text{CALC}} = 32.7^k$

$P_{\text{TEST}} = 32.0^k$
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Fig. 22 Deflected Half-Shapes for a Doubly Reinforced Concrete Beam
Fig. 23  Midspan Deflection Versus % Ultimate Load for a Doubly Reinforced Concrete Beam
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- The graph shows the load-deflection curves for a prestressed concrete rectangular beam.
- The curves are labeled with data points indicating the load in kips (V) and the mid-span deflection in inches.
- The graph includes a figure showing the dimensions of the beam, with labels indicating the length of the beam segments.
- The analytical load (Vu) is given as 48.9 kips, and the test load (Vu) is given as 49.9 kips.
Fig. 27 Load-Deflection Curves for a Prestressed Concrete Rectangular Beam

- \( V_u = 49.4 \) (calc)
- \( V_u = 50.2 \) (test)

○ = Analytic
● = Test
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\[ V/V_{\text{ULT}} = 0.604 \]

\[ V/V_{\text{ULT}} = 0.606 \]

\[ V/V_{\text{ULT}} = 0.709 \]
Fig. 31 (Continued)

\[ \frac{V}{V_{UL}} = 0.833 \]

\[ \frac{V}{V_{ULT}} = 0.932 \]

\[ \frac{V}{V_{ULT}} = 1.000 \]
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\[ \frac{m}{m_0} = 960 \text{ k-in.} \]

\[ \phi_0 = 0.278 \times 10^{-4} \text{ Rads.} \]
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$m = 0.52$
$m = 0.72$
$m = 0.92$
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- E = 5200 ksi
- E = 4600 ksi
- E = 4000 ksi
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- 6.01 ksi
- 6.61 ksi
- 7.21 ksi
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Fig. 77  Lateral Load Distribution for Uniform Load - \( J_0 \) and Elastic Slabs
Fig. 78 Lateral Load Distribution for Concentrated Load at Bridge Center - Elastic Slab
Fig. 79 Lateral Load Distribution for Concentrated Load at Bridge Center - J₂ Slab
Fig. 80  Lateral Load Distribution for Concentrated Loads at Centerlines of Interior Beams - Elastic Slab
Fig. 81  Lateral Load Distribution for Concentrated Loads at Centerlines of Interior Beams - J₂ Slab
Fig. 82 Single Beam Flow Chart
SET UP $i^{th}$ LOAD VECTOR PLUS THE FICTITIOUS LOAD VECTOR FROM $i-1^{th}$ LOAD STEP

SET UP STRESS DEPENDENT STIFFNESS MATRIX

SOLVE FOR DISPLACEMENTS, STRAINS AND STRESSES

DO DISPLACEMENTS MEET CONVERGENCE CRITERIA?

YES

ACCUMULATE FIELD QUANTITIES

PRINT RESULTS

NO

Fig. 83  Four Beam Bridge Flow Chart
Fig. 84 Load-Deflection Curves - Effect of $K_T$ - Concentrated Load at Bridge Center
Fig. 85 Load-Deflection Curves - Effect of $K_T$ - Concentrated Loads at Centerlines of Interior Beams
Fig. 86 Load-Deflection Curves - Effect of $K_T$ - Uniform Load
Fig. 87 Load-Deflection Curves - Effect of $K_T$ - Partial Interior Uniform Load
Fig. 88 Load-Deflection Curves - Effect of $K_T$ - Partial Exterior Uniform Load
Fig. 89 Distribution of $\phi'$
Fig. 90 Discretization Models for Beam-Columns
Fig. 91 Interaction Curves for 8 x 31 Beam-Columns
Fig. 92 Load-Deflection Curves for 8 x 31, L/r = 140
Fig. 93 Interaction Curves for 8 x 31 Beam-Columns
Fig. 94 Interaction Curves for Reinforced Concrete Beam-Columns
Fig. 95 Load-Deflection Curves for Reinforced Concrete Beam-Columns, L/t = 20
Fig. 96 Interaction Curves for Reinforced Concrete Beam-Columns
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9. NOMENCLATURE

The following Nomenclature was used in Chapters 1 - 5 and is presented in the approximate order found in the text.

\[ \{ \sigma \} = \text{A vector of stresses} \]
\[ \sigma_i = \text{An element of } \{ \sigma \} \]
\[ [\Gamma] = \text{A stress matrix} \]
\[ \{ f \} = \text{A vector of forces} \]
\[ [G] = \text{A transformation matrix} \]
\[ \{ \varepsilon \} = \text{A vector of initial strains} \]
\[ \varepsilon_{i,i} = \text{An element of } \{ \varepsilon \} \]
\[ E_0 = \text{Initial modulus of elasticity} \]
\[ [K] = \text{A stiffness matrix} \]
\[ \{ \delta \} = \text{A vector of nodal displacements} \]
\[ [D] = \text{Elasticity matrix} \]
\[ [B] = \text{A matrix which relates strains to nodal displacements} \]
\[ d\varepsilon^P_x, d\varepsilon^P_y, d\gamma^P_{xy} = \text{Increments of plastic strain} \]
\[ \sigma_z, \sigma_x, \sigma_y = \text{Normal stresses} \]
\[ \tau_{xy}, \tau_{xz}, \tau_{yz} = \text{Shear stresses} \]
\( \bar{\sigma} \) = Effective stress

\( H \) = Slope of effective stress-effective strain curve

\( d\varepsilon^e \) = Elastic strain increment

\( d\varepsilon^p \) = Plastic strain increment

\( \eta \) = \((\sigma_x - \frac{1}{2} \sigma_y)\)

\( \zeta \) = \((\sigma_y - \frac{1}{2} \sigma_x)\)

\( \xi \) = \(3\tau_{xy}\)

\( U_{n-1}, U_n, U_k, U_i, U \) = Axial displacements of a node in the X direction

\( W_n, W_{n-1}, W \) = Transverse displacements of a node in the Z direction

\( X, Y, Z \) = Coordinate axes and positions

\( \{\alpha\} \) = A vector of constants

\( \{\delta^e\} \) = A vector of elemental displacements

\( [C] \) = A matrix relating \( \{\delta^e\} \) and \( \{\alpha\} \)

\(-\partial W/\partial X\) = Rotation about the Y axis

\(l\) = Length of a beam element

\( N_j, N \) = A normal force in reference plane

\( M_j, M \) = A moment in reference plane
\{\varepsilon\} = A vector of generalized strains

\[Q\] = A matrix relating \{\varepsilon\} and \{\alpha\}

\(U_z\) = Axial displacement at a distance \(Z\) from the reference plane

\(\varepsilon_x\) = Engineering strain in \(X\) direction

\(A_1\) = Area of a beam layer

\(M_1\) = Moment about a layer centroidal axis

\(\bar{A}\) = Equivalent area of a beam element

\(\bar{S}\) = Equivalent statical moment of a beam element about the reference axis

\(\bar{I}\) = Equivalent moment of inertia of a beam element about the reference axis

\(F_x, F_y, F_z\) = Elements of vector of forces; \(F_x\) is an axial force, \(F_y\) is a moment and \(F_z\) is a transverse force

\(\theta_{n-1}, \theta_n, \theta_{n+1}\),

\(\theta, \theta_1, \theta_k\) = Rotations about the \(Y\) axis at node points

\(\sigma_2, \sigma_1\) = Yield stresses in Ramberg-Osgood formulation

\(n\) = A Ramberg-Osgood parameter

\(m_1, m_2, m\) = A Ramberg-Osgood parameter

\(d\) = Plastic strain in the Ramberg-Osgood formulation
K = A coefficient in the Ramberg-Osgood formulation

$f'_{c}$ = 6" x 12" cylinder strength of concrete

$\bar{\varepsilon}$ = Strain at stress equal to $f'_{c}$

$E_{s}$ = A secant modulus of elasticity

$R, R_{E}, P_{f}, R_{g}$ = Ratios used in Saenz's concrete stress-strain curve

$E_{down}$ = The rate of compressive unloading

$\varepsilon_{1}$ = The strain at which compressive unloading starts

$E_{down t}$ = The rate of tensile unloading

$e_{1}, e_{2}$ = Eccentricities of prestressing strand

$L_{2}$ = Distance from end of beam to drape point

$\sigma_{R}, \sigma_{L}$ = Stress on right and left side of a layer

$b$ = Width of a layer; width of wide flange beam flange

$\tau_{j,k}$ = Approximate flexural shear stress in a layer

$Q_{1}, Q$ = Approximate shear at a node

$\omega$ = Steel percentage times steel yield stress divided by concrete cylinder strength

$\sigma_{y}$ = Steel yield stress
\( \Gamma_i, \Gamma_o, \Gamma \) = Values of initial, post transfer and test pretensioning force in prestressed concrete beams

\( E_c^1, E_c^2 \) = Measured values of Young's modulus for concrete

\( \varepsilon_s \) = Average strain in reinforcing

\( P_0 \) = An initial value of applied load

\( V, P \) = A value of applied load

\( c \) = Strain hardening modulus of steel

\( D \) = Depth of steel wide flange beam

\( d \) = Web thickness of steel wide flange beam

\( t \) = Flange thickness of steel wide flange beam, or minimum cover on reinforcing steel

\( A \) = Area of beam

\( \text{FTOL} \) = Tolerance on tensile strength

\( f_t \) = Tensile strength of concrete

\( \partial W/\partial Y \) = A rotation about the X axis

\( V \) = The displacement of a node in the Y direction

\( \alpha_1, \alpha_2, \ldots \) = Constants

\( \sigma_i(I,J) \) = Initial stress in a layer

\( \sigma_o \) = An allowable stress

\( \bar{P} \) = A percentage used for initial scaling
k = A material code

\( J_2 \) = The second invariant of the deviator stress tensor; a symbol for the von Mises yield criterion

\( \beta_1 \) = Whitney stress block coefficient

The following additional Nomenclature was used in Appendix A.

\( K_T \) = The St. Venant torsional constant

\( \phi \) = Angle of twist in torsion

\( \phi' \) = Unit angle of twist in torsion

The following additional Nomenclature was used in Appendix B.

\([K_G]\) = Geometric stiffness matrix

\( P \) = An axial load
\( L \) = Length of a beam-column

\( r \) = Radius of gyration about the bending axis

\( t \) = Length of an element

\( P_y \) = Axial yield load of a steel column

\( t \) = Depth of a concrete beam-column

\( Q \) = Lateral load

\( Q_o \) = Ultimate load of a concrete beam-column with no axial load

\( P_o \) = \( f''_c A_c + \sigma_y A_s \)

\( f''_c \) = 0.85 \( f'_c \)

\( A_c \) = Concrete area

\( A_s \) = Steel area

\( Q_p \) = Ultimate load of a steel beam-column with no axial load
10. APPENDICES
A.1 Introduction

As stated in Section 1.1, the application of the single beam analysis techniques to right bridge superstructures using prestressed concrete I-beams has contained the assumption that torsional action involving the beam is much less significant than the flexural action and can be neglected. It is realized that beams in highway bridges will be subjected to shear, bending and torsion. The major emphasis of this study has been on the analysis of bending action. An approximation for flexural shear computations has been presented in Section 2.6. The purpose of this Appendix is to assess the effect of excluding the torsional action.

Tests to destruction of four bridges, using steel I-shaped, prestressed I-shaped and reinforced concrete T-shaped beams, by the University of Tennessee (Ref. 7) indicated that flexural action was the dominant action. Of the four structures tested three were observed to fail by flexure and one by loss of composite action between the beam and slab. Large flexural deflections and rotations were observed. Good agreement between test ultimate loads and calculated ultimate loads was noted. Strain compatibility methods based on stress-strain curves from field specimens were used to compute the ultimate load. This
would indicate that there was very little torsional interaction phenomena involved because such interaction would serve to reduce the flexural capacity of the beams.

Mattock and Kaar (Ref. 49) have tested a half-scale highway bridge continuous over two spans. Once again flexural action appeared to be the dominant cause of failure of the beams; in fact there was no mention of any torsional distress.

While these experimental observations do not serve as proof that torsional stresses do not play a significant role in the overload response of a bridge they do provide evidence that it is not a dominant factor.

Three torsion related areas will now be discussed to provide more insight into the effect of not including torsional considerations in this research:

1. The effect of torsional stiffness on the response of the superstructure as seen in the load-deflection curves.

2. The effect of St. Venant torsional stresses on the load at first cracking.

3. Preliminary considerations on including torsional stresses in future research of this type in light of the current state of the art.
A.2 Torsional Stiffness and Lateral Load Distribution

It is generally agreed that the torsional stiffness of the beams is a factor in the lateral distribution of loads applied to the bridge superstructure (Refs. 49,60,66,67). Work by Wegmuller and Kostem on the lateral distribution of load in the elastic range indicated that a conservative distribution to interior stringers results if torsional stiffness is neglected (Ref. 67). Tamberg noted that, despite the torsional weakness of the AASHO I-beams, live load moments up to 10% larger result if their torsional stiffness is neglected (Ref. 60). The effect of torsional stiffness on lateral load distribution as seen through the load-deflection curves of the four beam bridge model shown in Fig. 67 has been studied. The following five types of loading were investigated including the von Mises slab:

1. A concentrated load at bridge center (Point No. 9).

2. Concentrated loads at centerline of interior beams (Point No. 8, note symmetry).

3. A uniform load over the whole bridge.

4. A patch load covering the central portion of the bridge (element No. 4, note symmetry).

5. A patch load covering part of the edge of the bridge (element No. 2, note symmetry).

As in Section 5.2, doubly symmetry of load and structure was used to reduce computational effort.
Based on observations of the model used in Section 5.2 the following changes were made to produce a more practical condition:

1. The dead load of the bridge was applied to the beams using the prestressing nodal load vector. The beams were assumed to share the load equally.

2. The initial prestress was raised to 3000 psi compression, 0.0 psi tension to more closely approach common design practice. Accordingly, the steel area was increased 50%.

Two extreme cases of St. Venant torsional stiffness were studied:

1. $K_T = 0.0$

2. $K_T$ computed for the flexurally uncracked section using the modified method of el-Darwish and Johnston as shown in Ref. 21.

The second approach for computing $K_T$ does not allow for torsional unloading due to cracking. It therefore represents an upper bound on torsional stiffness. The first approach is obviously a lower bound on torsional stiffness.

Figure 84 shows the resulting load-deflection curves for the case in which one single concentrated load is applied at point No. 9. At first cracking of the interior beam the curve
corresponding to $K_T \neq 0$ has about 5% higher load by virtue of the increased lateral distribution of load. The maximum deviation of the curves reaches 10%. As the deflection continues to increase the two curves converge to less than the original 5% difference.

This load case produced the greatest deviation of any investigated. It also represents an unrealistic loading because there was only one concentrated load applied to the entire bridge deck.

Figure 85 shows the result of applying the concentrated load at point No. 8. Symmetry means that a concentrated load has been applied to the centerline of both interior beams. It can be seen that this more practical loading produced less deviation between load-deflection curves. The maximum deviation reached 6% but was generally less.

Figure 86 shows the result of applying a uniform load to the whole superstructure. It can be seen that there is very little difference between the two load-deflection curves. The maximum deviation reaches only about 3-1/2% and the range over which the deviation occurs is quite small.

Figure 87 shows the results of applying a patch load to element No. 4. Once again the maximum deviation reached about 10% but was slightly less than the case involving the single load at point No. 9.

Figure 88 shows the results of applying a patch load to element No. 2. The results are so close that only one curve was plotted.
From these figures it can be seen that, with the exception of isolated points in Fig. 88, a conservative load-deflection curve has resulted from neglecting the torsional stiffness.

A.3 **Torsional Shear Stresses and the First Cracking Load**

The torsional stresses would, of course, have some effect on the load-deflection curve. It is believed that the secondary torsion stresses combined with the primary bending and prestress stresses would cause a somewhat earlier cracking of some layers. This would have the effect of weakening the analytic model while the inclusion of the additional stiffness would stiffen the model. The effects in this context would tend to offset each other.

This subject was studied further using equations presented in Ref. 21 for the shear stresses in concrete I-beams. The I-beam idealization in Fig. 67 has almost the same $K_T$ value as the AASHO Type III I-beam section tabulated by Eby et al. (Ref. 21). Accordingly, the equations for the AASHO Type III beam were used. The unit angle of twist was calculated using the appropriate rotations at nodes 4, 5, 7 and 8 in Fig. 67 and dividing by 129.33 inches. It was assumed that the diaphragms placed between the beams at the supports provided a torsionally fixed condition. It was also assumed that the load ratio of 1.0 corresponding to a maximum tensile stress of 450 psi in these examples would constitute insipient cracking. The difference between the prestress
All the other cases would have produced reductions in the cracking load of less than 3%. This difference is certainly acceptable, especially when it is recalled that all but one of these five load cases produced an increase in the cracking load by virtue of improved lateral load distribution from including the torsional rigidity.

These calculations have also assumed that the average unit angle of twist, $\theta'$, over the central one-third of the span was an acceptable approximation. It is quite possible that this is significantly conservative. Consider the two types of torsional loadings shown in Fig. 89; the concentrated midspan torque and the uniformly distributed torque. The torsional boundary conditions are the same as previously mentioned in this Appendix. Also shown for both cases is a sketch of the distribution of $\theta'$ as given in Ref. 30. It can be seen that in both of these cases the value of $\theta'$ is zero at midspan and very small over the central one-third of the beam. It seems quite plausible that the point of maximum bending for a load moving across a right bridge and some portion of the beam near it would have smaller values of $\theta'$ than the average values used here. It also seems plausible that the distribution of $\theta'$ in the real structure would lie somewhere in between these two cases. Thus the estimates of shear stresses presented earlier in this Appendix could very well be high over that portion of the beams which will have the highest bending stress and undergo the earliest and most extensive nonlinear behavior.
Summarizing this Appendix so far, it has been noted that reports of tests to destruction of four full scale bridges and one half-scale bridge have not mentioned significant torsional action. It has also been shown that the effect of not including the torsional stiffness of the beams in analyzing the response to five varied loading cases has produced conservative results in four of the cases and virtually no change in the fifth. Finally, it has been shown that, with the exception of the least realistic loading, there would be only a small reduction in cracking load caused by the inclusion of the St. Venant shear stresses. The examples presented do not rigorously prove the assumption that torsional action can be neglected without impairing the application of the single beam techniques to the class of bridges being considered. They do however present a strong and logical argument in support of that assumption. The techniques being used here could employ a nonzero torsional constant but this would include the beneficial effect of better load distribution without including the negative effect of earlier cracking. It is felt that using a zero torsional constant is the best approach at this time.

A.4 Considerations on Including Torsional Effects

In Future Research

There are, of course, large areas of torsion related problems which could be studied as a possible extension of the
reported investigation. This would be especially important if these techniques are eventually applied to structures in which the torsional action is larger. The purpose of this section is to introduce concepts which would be necessary for that future work. Some facets about the current state of knowledge which make including a rigorous torsional analysis virtually impossible at this time will be presented.

The basic problem which should eventually be attacked is that of developing a process to predict the nonlinear load-deflection (torque-twist) curve for prestressed and reinforced concrete beams subjected to torsion. This would involve the following considerations.

1. Development of a shear stress-shear strain (τ-γ) curve for concrete analogous to the uniaxial stress-strain curve.

Hsu has noted that the concrete shear modulus is very difficult to find experimentally (Ref. 34). He has developed post-cracking moduli for torque-twist curves for reinforced concrete sections under pure torsion (Refs. 34,36). This is only a start towards what would be needed for a torsion analysis comparable to the bending analysis presented here.

2. Development of a failure theory.
Assuming that a \( T-Y \) curve were known and that torsional shear, flexural shear and normal stresses were known for some displaced shape, the problem of which failure criteria to use still must be resolved. For entire beams in pure torsion the elastic (soap film analogy) plastic (sand heap analogy), and semi-plastic theories have been developed based on the development of helical cracks. Skewed bending theories have been developed from hi-speed films of actual failures (Ref. 74). For developing a torsional load-deflection curve it would probably be necessary to use a failure criteria similar to Rusch’s criteria as used by Peterson and Kostem for plate bending (Ref. 55). Chen has also developed a failure criteria which may hold promise (Ref. 9).

3. A change in the basic techniques used in this research would have to be made.

Ideally each beam should be analyzed as a three dimensional problem with torsional strains varying across the section as well as along the length. For reinforced concrete beams with connected longitudinal and transverse reinforcement Hsu (Ref. 35) has shown that there is a change in the basic mechanism of equilibrium within the beam after torsional cracking has occurred. Concrete stresses are transferred to the steel cage in a manner which has not yet been mathematically modeled. If items 1 and 2 above were known a three dimensional finite element analysis might hold some promise of solution for a single beam but would be even more
laborious than the two dimensional continuum bending analysis techniques reviewed in Section 1.3. Application to entire bridges would be conceptually feasible but practicably impossible.

Hsu has presented a skewed bending ultimate strength theory for reinforced and prestressed concrete beams without web reinforcement (Ref. 34). This type of beam might be solved approximately by dividing the cross-section into a grid instead of layers. If the shear stress distribution could be found a failure criteria could be employed for each area. In this way an approximation suitable for use with bridge overload analysis might be developed but the problem of finding the shear stress distribution within the cross-section, especially after cracked zones have developed, should not be underestimated.

These comments serve only to introduce the problem and to isolate some salient features. The problem of developing a nonlinear torsional load-deflection curve suitable for use with bridge analysis is a most complex problem indeed, but one which will hopefully be researched in the near future. The development of experimentally verified stress-strain relationships and failure criteria would be a first step in this direction.
APPENDIX B

INELASTIC BEAM-COLUMNS

B.1 Introduction

This Appendix contains a summary of the results from a pilot study on the application of the finite element method to the inelastic analysis of beam-columns. The objective of the study was to extend the formulation described in Chapter 2 and the relevant computer program to perform the analysis of beam-columns. Reference 43 contains a full report on this study.

Consistent with the nature of a pilot study only those changes actually necessary to generate preliminary results were made. The changes in coding were minimal. It was noted during this work that other, more extensive changes to coding and to the iteration techniques employed would enable more efficient and slightly more accurate results to be obtained. Problems associated with iteration schemes and suggested improvements to the existing scheme will be discussed where applicable.

Results of numerical investigations on steel and reinforced concrete beam-columns subjected to concentrated midspan lateral loads were compared with existing analysis techniques via interaction curves.

The load-deflection curve of a given beam-column is also obtained as part of the analysis. This load-deflection curve approaches, but does not extend past, the peak of the curve.
The P-Δ effect caused by the deflection of the beam-column can be included by using the geometric stiffness matrix. It has been shown in Ref. (61) that the geometric stiffness matrix relating the axial force to the bending displacement through the second order strain equations is given by Eq. B.1. \( P \) is the applied axial force assumed positive if it causes tension.

\[
[K_G] = P
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1.2/\ell & -1/10 & 4\ell/30 \\
0 & 0 & 0 & 0 \\
0 & -1.2/\ell & 1/10 & 0 & 1.2/\ell \\
0 & -1/10 & -\ell/30 & 0 & 1/10 & 4\ell/30
\end{bmatrix}
\]  \hspace{1cm} (B.1)

Combining Eqs. 2.23 and B.1 gives the equilibrium equation for the displaced beam-column element.

\[
\{f\} = \left[ [K] + [K_G] \right] \{\delta\} \hspace{1cm} (B.2)
\]

The stiffness matrices of each element can then be assembled to form the global equilibrium equations. After application of the boundary conditions these equations can be solved for each increment of load as described in Chapter 2.

As originally coded the computer program calculated the basic load step load vector from the lateral loads only. As such,
the incremental iterative process would be performed only on the lateral load applied to the beam-column. It would be a relatively simple matter to include a nodal load vector in the incremental load vector and thus allow for the case of proportional loading involving an eccentric (or concentric) axial force and a lateral load. This was not actually done because the comparisons with previous work involved solutions based on the application of a constant axial force first and then the application of an increasing lateral load. One of the advantages of the finite element method is that there is nothing conceptually prohibitive about incrementing all the loads applied to the beam-column.

Another worthwhile change in the iteration procedure would be to allow for two separate iterated load vectors. This would be especially helpful for more extensive work with concrete beam-columns in which the application of a large axial force may, by itself, cause substantial nonlinear behavior to occur before the lateral load is applied. This problem arose in two sets of comparative examples to be presented in Section B.2. Conversion to two incremented load vectors was considered beyond the scope of a pilot study but would be a most desirable addition to the program if an extensive study of beam-columns were to be performed.
B.2 Numerical Results

B.2.1 Steel Wide Flange Beam-Columns

The elemental and layering discretizations are shown in Fig. 90. Previous results from the parametric study using the original computer program reported in Ref. 42 and summarized in Chapter 4 indicate that the number and location of the elements and layers used here are more than adequate for this analysis.

Figure 91 shows a comparison of interaction curves for a concentric axial load reported by Lu and Kamalvand (Ref. 48) using the column deflection curve method, Chen (Ref. 10) using the column curvature method and the results of this study shown by the dashed lines. Lu and Kamalvand used a yield stress of 36 ksi whereas Chen and this study used 34 ksi. Accordingly, the column deflection curve results for $L/r = 80$ had to be adjusted as described in Ref. 48.

$$(L/r)_{36} = (L/r)_{34} \sqrt{36/34}$$

The results for $L/r = 20$ were not adjusted because the change was too small to affect a figure of this scale.

The general agreement appears to be quite good. The three points showing the largest apparent discrepancy compared with the column curvature results are $(L/r = 20., P/P_y = 0.2)$, $(L/r = 20., P/P_y = 0.3)$ and $(L/r = 80., P/P_y = 0.85)$. In each case the stress field output of these examples showed a premature
failure to converge in the next load step beyond what is plotted. In the discussion of iteration procedures in Chapter 2 it was noted that load reduction was used to attain convergence in the original program based on some number of unsuccessful trials. In the case of the three points being discussed here better results would probably be obtained if the load step was reduced when apparent failure to converge developed due to deterioration in the condition of the stiffness matrix. This would still develop at a slightly higher load but such a reduction might allow one or two more load steps to be taken in some cases.

Figure 92 shows load-deflection curves for various $P/P_y$ ratios and an $L/r$ ratio of 140. These figures show the substantial effect that increasing the axial load has on the ability to carry the lateral load. They also show the effect that increasing the axial load has on the stiffness of the beam-column. Increasing the axial load decreases the stiffness. It is noted that without the geometric stiffness matrix all of the curves in Figure 92 would have the same slope in the elastic range. Reference 43 contains similar figures for $L/r = 20$ and 80.

Figure 93 shows interaction curves for the same beam-columns with an eccentric axial load. Comparisons are made only with the column curvature results by Chen (Ref. 10). The agreement between the column curvature and the finite element results are generally even better than those presented for the concentric axial load.
B.2.2 Reinforced Concrete Beam-Columns

The rectangular, doubly reinforced section used by Chen and Chen (Ref. 11) was also used here. The cross-section is shown in Fig. 90. The beam-column is 14 inches deep, 12 inches wide and has equal compressive and tensile steel areas totaling 6.72 square inches. The nominal compressive strength was 3.0 ksi but Chen and Chen used Hognestad's stress-strain curve to define their moment-thrust-curvature relations. This stress-strain curve assumes a 15% reduction in nominal compressive strength as the peak beam-column compressive stress. Accordingly a value of \( \sigma_1 = 2.55 \text{ ksi} \) was chosen to approximate the reduction in the Ramberg-Osgood curve. Chen and Chen neglected the tensile strength of the concrete whereas the method being reported included it. The yield strength of the steel was 45 ksi.

The resulting interaction curve for the concentric axial load case is shown in Fig. 94. It can be seen that the agreement between the results of both analyses is quite good for the curves with \( L/t = 30 \) and \( L/t = 20 \). The agreement with the previously reported results for \( L/t = 10 \) is not as striking but is still within about 5% of the same value of \( Q/Q_0 \) for a given value of \( P/P_0 \). The differences in the stress-strain curves used in both approaches may account for some of the differences.

One set of load-deflection curves is shown in Fig. 95. It will be noted that these curves do not appear as systematic as
those presented for the examples using steel sections. There are
two reasons for this:

1. As shown in Fig. 94 it is possible for the beam-column
to support a larger lateral load for some values of axial
load than it can without the axial load.

2. The effect of cracking is evident in these load-
deflection curves. It appears as a relatively early
change in slope of the load-deflection curve. The amount
of change in slope depends on the extent of the spread of
cracked layers along and into the beam-column.

It will be noticed in Fig. 95 that before cracking be-
comes evident the previously noted decrease in stiffness with in-
creasing axial load is still apparent. Thus the effect of the
gEometric stiffness matrix is also seen in the load-deflection
behavior of concrete beam-columns.

Figure 96 is the interaction diagram for an eccentri-
cally load reinforced concrete beam-column. Good agreement with
the work of Chen and Chen is found. The finite element results do
not extend as far along these interaction curves because of the
limitations in the current iterative procedure. It is felt that
these limitations could be removed. For the higher values of $P/P_0$
for both the eccentric and the concentric case the axial load
alone caused enough nonlinear behavior to result in failure to
converge to the first displacement increment. This first displacement increment had to correspond to the entire axial load because, as explained in Section B.1, there were no provisions to increment the axial load. For the concentric load case it was relatively easy to circumvent this problem by using an initial stress field which satisfied equilibrium and strain compatibility. Strain criteria were adjusted accordingly. No such simple expedient was tried with the eccentric beam-columns because each layer would have to have its own stress-strain curve in order to accommodate the change in strain criteria. It was felt that the results presented in Figs. 91, 93 and 94 were conclusive enough without the added evidence obtained by one or two more points on Fig. 96.

B.3 Conclusions From This Pilot Study and Applications

It can be concluded from this pilot study that the incremental iterative finite element method using a simple layered beam element can provide solutions to inelastic beam-column problems. There is already a large body of information in this area. The method used here is a relatively laborious procedure compared to other existing methods assuming that they have been applied to a given problem. It does, however, have several advantages which might prove useful in future beam-column studies especially if the changes to the iteration scheme mentioned in Sections B.1 and B.2 are made. These advantages are:
1. A wide range of loadings can be handled. There is no intrinsic difference between one concentrated load, several concentrated loads, uniform loads, symmetric loads or unsymmetric loads.

2. Boundary conditions can also be handled easily. There is no change in the formulation required for different boundary conditions.

3. There is no need for an a-priori moment-thrust-curvature curve.

4. There is nothing conceptually prohibitive about changing the order of the loading or using simultaneous (but proportional) axial and lateral loads.

5. The work with prestressed concrete beams reported in this dissertation and in Ref. 40 would indicate that prestressed concrete beam-columns could also be treated. This method might also be used as a check on future extensions of beam-column analysis techniques such as column-deflection curves or column-curvature curves. As such it might provide an independent solution such as seen in Figs. 91, 93, 94 and 96. It is noted that the changes already mentioned should be performed before more extensive beam-column studies are conducted.
APPENDIX C

FLOW CHART

READ IN JOB DESCRIPTORS

READ IN INITIAL STRESS MATRIX IF NEEDED

READ IN ELEMENT LENGTHS

READ IN MATERIAL STRESS-STRAIN CURVE DATA

READ IN ITYPE, ASXLR, AILR, ZCRD, TSHEAR ARRAYS

READ IN BOUNDARY CONDITION CODES

ESTABLISH BOUNDARY CONDITION CODE VECTORS

READ IN MATERIAL TENSILE STRESS-STRAIN DATA IF NEEDED

INITIALIZE LOADS, READ IN LOADS AND POSITIONS - FROM THE LOAD VECTOR

-279-
PRESTRESS IF NECESSARY

IF PRESTRESSED THEN COMPUTE SHEARS AND DRAW PRINTER PLOTS

INITIALIZE COUNTERS

SET UP STIFFNESS MATRIX

IMPOSE BOUNDARY CONDITIONS WITHIN STIFFNESS MATRIX AND LOAD VECTOR

SOLVE BASIC EQUATIONS FOR \( \delta \)

COMPUTE STRAIN INCREMENT

COMPUTE STRESS INCREMENT

TEST FOR CONVERGENCE

NO

COMPUTE NEW TANGENT MODULI

IS CONVERGENCE TAKING TOO LONG?

NO

YES
REDUCE LOAD IF FIRST CYCLE
OR LOAD INCREMENT IF NOT FIRST CYCLE

COMPUTE NEW FORCE RATIO

STORE TRIAL DISPLACEMENTS,
INITIALIZE NEW DISPLACEMENT FIELD

START TENSILE CRACKING

IS THIS LAYER MADE OF A
MATERIAL WHICH DOES NOT
CRACK - OR HAS ALREADY CRACKED
OR CRUSHED?

NO

HAS THE TOTAL STRESS EXCEEDED
TENSILE STRESS BY MORE THAN
A TOLERANCE?

YES

REDUCE LOAD AND FORCE RATIO

STRAIN HARDENING AND CRUSHING CHECK

IN STRAIN HARDENING RANGE?

YES

TYPE OF MATERIAL WHICH DOESN'T CRUSH?

NO

YES

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BELOW COMPRESSIVE CUTOFF STRAIN?

IS DOWNWARD STRESS GREATER THAN $\sigma_y/50$?

SET KODE2 = 1, INITIALIZE FORCE VECTOR

IS THIS A MATERIAL WHICH DOES NOT CRACK?

IN STRAIN HARDENING RANGE?

HAS THIS LAYER ALREADY CRACKED OR CRUSHED AND UNLOADED?

HAS THIS LAYER CRACKED BUT NOT FULLY UNLOADED?

WILL IT CRACK NOW?

COMPUTE DOWNWARD STRAIN. COMPUTE UNLOADING STRESS. COMPUTE FICTICIOUS LOAD VECTOR AND UNLOAD LAYER.

IF FICTICIOUS LOAD VECTOR IS NOT VERY SMALL KODE2 = 2
START COMPRRESSIVE CHECK

IS THIS A MATERIAL WHICH DOES NOT CRUSH?

YES

NO

IN STRAIN HARDENING RANGE?

YES

NO

COMPUTE STRAINS

REACHED CRUSHING STRAIN?

YES

NO

COMPUTE DOWNWARD STRAIN AND UNLOADING STRESS AND FICTICIOUS LOAD VECTOR AND UNLOAD LAYER

IF NEW LOAD VECTOR IS NOT EXCESSIVELY SMALL KODE2 = 2

BEYOND MAXIMUM COMPRESSION STRESS AND WITHOUT LOADING?

YES

SET LAYER STIFFNESS = 0.

NO

REACHED CRACKING?

YES

NO

END

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YES

SET LAYER STIFFNESS = 0.

SET TRIAL COUNTERS

IS KODE2 = 1?

NO

SET UP STIFFNESS MATRIX

ESTABLISH BOUNDARY CONDITIONS

SOLVE EQUATIONS

COMPUTE STRAIN INCREMENTS

COMPUTE STRESS INCREMENTS

TEST FOR CONVERGENCE

NO

COMPUTE NEW TANGENT STIFFNESS

CONVERGENCE TOO SLOW

YES

REDUCE RAMBERG-OSGOOD N'S AS EMERGENCY MEASURE

NO
Initialize Displacement Vector

Increment Stress and Displacement Fields for Cracking and Crushing

Compute new total stresses and displacements from old ones + load step + (crack + crush)

Check for strain hardening - if needed substitute new E's

Draw printer plots

Compute flexural shear

More than 5 cycles to converge?

Yes

Reduce next load increment

No

Less than 3 cycles to convergence?

Yes

Increase next load increment

No

Maximum cycles exceeded?

Yes

Stop
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