THE INELASTIC ANALYSIS OF REINFORCED
AND PRESTRESSED CONCRETE BEAMS

by

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This work was sponsored by the Pennsylvania Department of Transportation; U. S. Department of Transportation, Federal Highway Administration; and the National Science Foundation.

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LEHIGH UNIVERSITY
Office of Research
Bethlehem, Pennsylvania

November, 1972

Fritz Engineering Laboratory Report No. 378B.1
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ABSTRACT

This report describes an approach for the analysis of beams using a tangent stiffness finite element incremental method applied to a layered element. Each layer has its own modified Ramberg-Osgood type stress strain curve and may have inelastic, cracking and crushing non-linearities. Comparisons with laboratory tests of two reinforced and eleven prestressed concrete beams are presented. The agreement between analytic and laboratory load deflection curves is quite good. The method described uses a plane sections assumption to reduce the computational effort. Considerations of flexural shear are also presented.
1. INTRODUCTION

1.1 Purpose and Scope

The purpose of the research being reported was twofold:

1. To develop analytical techniques applicable to the overload analysis of reinforced and prestressed concrete beams so as to adequately describe their load direction behavior, stress distribution, and response to material non-linearities such as yielding, cracking and crushing.

2. To extend the developed techniques, used in objective one, to the analysis of bridge decks composed of reinforced concrete decks and prestressed concrete I-beams.

This report covers the portion of the investigation pertaining to objective one. The ultimate objective of the investigation is the determination of the overloading behavior of beam-slab type highway bridges utilizing prestressed concrete I-beams.

The basic model under consideration is a simply supported, essentially prismatic beam subjected to loading in a plane of symmetry. The formulation is general enough to allow for a wide range of materials and boundary conditions, but does not allow for the inclusion of local or lateral-torsional buckling of the beams.

1.2 Historical Development of Elastic Plastic

Finite Element Method

The finite element method is a recent extension of matrix
analysis techniques to problems of stress analysis. It employs the following steps:

1. The region to be considered (in this context, a beam) is divided into subregions called finite elements.

2. A suitable description of the displacement field is made. A polynomial description is usually assumed.

3. Generalized stresses are related to generalized strains by a suitable stiffness matrix. This stiffness matrix reflects material properties.

Since the material properties used in stiffness matrix are stress dependent, solutions to problems with material non-linearities usually require the employment of an iterative scheme and an incremental loading path.

The application of the finite element method to problems involving material non-linearity has progressed along two different paths, the initial stiffness method and the tangent stiffness method. These two paths are described below. Concepts from both approaches have been used in the research reported herein. The following discussion of the widely used techniques is presented to provide a better appreciation of the problem.

1.2.1 The Initial Stiffness Method

The initial stiffness method utilizes the original stiffness matrix of the system throughout the analysis. This matrix need be inverted only once in the entire process.
Solution of a problem involves a series of linear analyses which requires the representation of previous load history as a state of accumulated stress and strain. This can be written in equation form as:

\[ \{ \sigma \} = [\Gamma] \{ F \} + [G] \{ \varepsilon_1 \} \]  

where  

- \( \{ \sigma \} \) = Stress vector  
- \([\Gamma]\) = Stress matrix  
- \(\{F\}\) = Force vector  
- \([G]\) = A transformation matrix  
- \(\{\varepsilon_1\}\) = A state of initial strain

The initial strains are the plastic strains at the current load level. The obvious difficulty with Eq. 1 is finding \(\{\varepsilon_1\}\) for the current step. This drawback can be overcome by assuming that the inelastic strains of the previous load cycle can be used to approximate the current inelastic strains. Equation 1 may then be rewritten as:

\[ \{ \sigma^{(K)} \} = [\Gamma] \{ \varepsilon^{(K)} \} + [G] \{ \varepsilon_1^{(K-1)} \} \]

There are several ways of incorporating the strain from the previous cycle. Two common methods are the constant stress method and the constant strain method.

1.2.1.1 **Constant Stress Method**

The \(K\) cycle of loading is started with the current
applied loads \{p^{(K)}\} and the initial strains from the previous cycle \{\varepsilon_{1}^{(K-1)}\}. \{\sigma^{(K)}\} is found by using Eq. 2. \{\varepsilon_{1}^{(K)}\} for use with (K+1)th cycle is obtained by using a stress-strain curve to find \{\varepsilon_{1}^{(K)}\} corresponding to \{\sigma_{1}^{(K)}\}. This process is shown in Fig. 1-A. Similar sketches and more detailed descriptions are found at Ref. 4. Experience has shown that the constant stress method has a tendency to diverge at a problem dependent step size and is therefore an undesirable approach .

1.2.1.2 Constant Strain Method

In this approach \{\sigma^{(K)}\} is again found from Eq. 2. \{\varepsilon_{1}^{(K)}\} is found using a stress-strain curve by locating a point whose coordinates are \sigma^{(K)} and \sigma^{(K)}/E + \varepsilon_{1}^{(K-1)}. The strain coordinate defines a total strain. A new estimate of \sigma^{(K)} is found using the total strain. This process is shown in Fig. 1-B. Experience has shown the constant strain method to be numerically stable but less accurate than the constant stress method .

This discussion of the initial stiffness method serves only as an introduction. The concept used in this research is that non-linearities may be mathematically imposed by some set of fictitious forces or displacements (stresses or strains).

1.2.2 The Tangent Stiffness Method

In the tangent stiffness approach the global stiffness matrix is regenerated each time the global equilibrium equations are solved. The stiffness properties of the elements are
continually updated to account for the ongoing stress history. Lansing and Gallagher\textsuperscript{12} state that the tangent stiffness method "appears to be favored by theorists in finite element plasticity analysis. This is presumably the consequence of the consistency of this approach with classical methods of plasticity analysis and because of computational efficiency as well." There are no conceptional difficulties associated with perfect plasticity when using the tangent stiffness approach.

The tangent stiffness method has enjoyed wide application through the use of the elastic-plastic stiffness matrix. The von Mises yield condition and Prandtl-Reuss flow rule are usually assumed to hold. The incremental formulation proceeds as follows:

1. The global equilibrium equations for linear elastic behavior are written as:

\[ \{F\} = [K] \{\delta\} \quad (3) \]

where

\[ [K] = \int_V [B]^T [D] [B] \, dv \]

\([B]\) = Relates element strain to nodal displacements
\([D]\) = Is the elasticity matrix
\([\delta]\) = Is the vector of nodal displacements

2. When an element (or part of one) becomes plastic Eq. 3 must be modified. By using small increments of load Eq. 3 may be replaced by an equation which relates the
increment of stress to the increments of strain. The von Mises yield condition and Prandtl-Reuss flow rule will be used in this introductory discussion. Eq. 4 may be written for plane stress. Their derivation can be found in any elementary plasticity text such as Ref. 14.

\[
d\varepsilon^p_x = \frac{d\tilde{\sigma}}{\partial H} (\sigma_x - \frac{1}{2} \sigma_y)
\]

\[
d\varepsilon^p_y = \frac{d\tilde{\sigma}}{\partial H} (\sigma_y - \frac{1}{2} \sigma_x)
\]

\[
d\gamma^p_{xy} = \frac{d\tilde{\sigma}}{\partial H} (3 \tau_{xy})
\]

\(\tilde{\sigma}\) is defined as the effective stress and is given by Eq. 5. \(d\tilde{\sigma}\) is given by Eq. 6.

\[
\tilde{\sigma} = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{\frac{1}{2}}
\]

\[
d\tilde{\sigma} = \frac{1}{\sigma} \left[ (\sigma_x - \frac{1}{2} \sigma_y) d\sigma_x + (\sigma_y - \frac{1}{2} \sigma_x) d\sigma_y + 3 \tau_{xy} \ d\tau_{xy} \right]
\]

\(H\) is defined as the instantaneous slope of the effective stress-strain curve. Substituting Eq. 6 into Eq. 4 results in a relation between plastic incremental strains and incremental stresses.
\[
\begin{bmatrix}
\frac{d\varepsilon_x}{d\sigma_x} \\
\frac{d\varepsilon_y}{d\sigma_y} \\
\frac{d\gamma_{xy}}{d\tau_{xy}}
\end{bmatrix} = \begin{bmatrix}
(\sigma_x - \frac{1}{2}\sigma_y) \\
(\sigma_y - \frac{1}{2}\sigma_x) \\
3\tau_{xy}
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}\sigma_y \\
\frac{1}{2}\sigma_x \\
\frac{1}{2}\sigma_x
\end{bmatrix} \begin{bmatrix}
\frac{d\sigma_x}{d\sigma_x} \\
\frac{d\sigma_y}{d\sigma_y} \\
\frac{d\tau_{xy}}{d\tau_{xy}}
\end{bmatrix}
\]

Define total strains as \( \varepsilon_e + \varepsilon^P \) and using the elasticity matrix, the elastic-plastic stress-strain relation can be defined as follows:

\[
\begin{bmatrix}
\frac{d\varepsilon_x}{d\sigma_x} \\
\frac{d\varepsilon_y}{d\sigma_y} \\
\frac{d\gamma_{xy}}{d\tau_{xy}}
\end{bmatrix} = \left[ D^{-1} + [D^P]^{-1} \right] \begin{bmatrix}
\frac{d\sigma_x}{d\sigma_x} \\
\frac{d\sigma_y}{d\sigma_y} \\
\frac{d\tau_{xy}}{d\tau_{xy}}
\end{bmatrix}
\]

\([D^P]^{-1}\) is given by Eq. 7. Equation 8 can be inverted and substituted into Eq. 3 to find the increments of nodal displacements corresponding to increments of applied loads.

An iterative process is required because the change in stress field during the current load step alters the material properties. Thus the stiffness matrix is a function of the unknown stress level. If this alteration in material properties is not included, a systematic error will be introduced. This process
is repeated until a convergence criteria is met for each load increment.

The process described above is usually employed in continuum analyses. This type of analysis has been employed by several investigators.

Ngo and Scordelis\textsuperscript{15}, for example, used a continuum approach with triangular elements to model reinforced concrete beams. A pre-existing crack pattern was assumed. A load system was applied and the finite element method was used to find the resulting stress and displacement fields. There was no consideration of successive cracking or yielding or incrementally increasing the loading. Bond was included by finite spring elements with an assumed linear bond stress-bond slip relation.

Nilson\textsuperscript{16} also used triangular elements and a continuum approach. Saenz's concrete stress-strain curve (to be discussed later in detail in Section 2.2) was used to find Young's moduli in two principal directions in an effort to account for the orthotropic nature of biaxially loaded concrete. Springs were used again to model bond action. The bond stress-bond slip relation was assumed to be a cubic parabola. It was noted, however, that the correct relations were not known. Cracking was accounted for by disconnecting the nodes where a cracking stress was reached, and reloading the modified member from an unloaded state.

Reference 22 is one of the most recent papers on the finite element analysis of reinforced concrete beams. Prestressed
concrete beams are not included. The authors employ a continuum approach using the von Mises yield condition and the Prandtl-Reuss flow rule together with simplified stress-strain curves. They also use an initial stiffness approach. They chose this technique to reduce the computational effort. They also assumed perfect bond because of the incomplete state of knowledge about the bond stress-bond slip relationship.

1.3 A Simplified Model

The reported research uses a method especially suited to the analysis of beams of those proportions usually found in bridges. The Bernoulli beam theory, which assume that a plane section before bending remains a plane section after bending, was used instead of a continuum elasticity approach. A relatively small number of elements along the longitudinal axis of the beam are broken into layers. The plane sections assumption is used to relate the strains in the layer to the displacements at the nodes. If a sufficient number of layers is used each layer may be assumed to be in a state of uniaxial tension or compression with the centroid of the layer assumed to be representative of the layer. This has the effect of reducing the plasticity equations to a simple substitution of the instantaneous slope of the stress-strain curve into the conventional elasticity matrix. These assumptions would become tenuous if high shearing stresses were present as in the case of an interior support of a continuous
beam. This consideration would have to be included if this simplified model were to be extended to the study of continuous beams.

The effect of this simplified layered model on the economy of solution via the tangent stiffness approach is apparent from the following example. If 10 elements each having 15 layers is used with a plane section type analysis (as explained in Section 2.1) there are 11 nodes each having 3 degrees of freedom. This results in 33 simultaneous equations. If on the other hand, a continuum approach (as presented in Chapter 2, Ref. 28) utilizing 300 elements with 2 degrees of freedom per node was used, there would be 352 simultaneous equations. Realizing that the solution time increases at the number of equations raised to approximately the 2.5 power and that incremental-iterative approach typically requires 200 to 300 solutions, it is apparent that the savings in computational effort is enormous and allows for a fine tolerance on solution accuracy. The number of elements used in this example was chosen to provide the same area subdivision as 10 elements of 15 layers or a total of 150 layers. In this case there would be two triangles corresponding to each rectangular layer.

Those non-linear behaviors associated with tensile cracking and compressive crushing are included by applying fictitious forces to the surrounding structure to maintain equilibrium and redistribute the accumulated stresses. It is this portion of the research being reported which utilizes the basic concepts of the initial stiffness approach.

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2. THEORETICAL ANALYSIS

2.1 Finite Element Formulation

Consistent with the finite element method\textsuperscript{9,21,28}, the structure to be studied is first subdivided into elements. In this case elements along the longitudinal axis and layers in the cross-section are used. Figure 2 shows the type of elemental idealization and Fig. 3 the type of layering employed for most of the examples included here. Reasonable care should be taken to place the elements and layers in the points of most interest and/or highest strain gradient. This is definitely more important in non-linear than in linear analysis.

A displacement function or functions is then chosen to represent the displacements within the element. In the current context two displacement functions were used.

\[ U = \alpha_1 + \alpha_2 X \]  \hspace{1cm} (9)

\[ W = \alpha_3 + \alpha_4 X + \alpha_5 X^2 + \alpha_6 X^3 \]  \hspace{1cm} (10)

\( U \) is the axial displacement and \( W \) is the transverse displacement. The \( \alpha \)'s are constants to be determined. By using the deflection and slope at both ends of the beam element the four constants in \( W \) can be found. Furthermore, since the bending displacement function is unique and contains the possibility of constant strains this shape function guarantees convergence for bending.
The constants in Eqs. 9 and 10 are evaluated by using the nodal displacements at both ends of the element.

\[
\{\Delta e\} = [C] \{\alpha\}
\]  \hspace{1cm} (11)

Figure 4 shows a beam element, coordinates and positive sign conventions. The vector \(\{\alpha\}\) is evaluated by matrix inversion. Inversion of matrix \([C]\), carried out by hand, resulted in Eq. 13.

\[
[C^{-1}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1/t & 0 & 0 & 1/t & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & -3/t^2 & 2/t & 0 & 3/t^3 & 1/t \\
0 & 2/t^3 & -1/t^3 & 0 & -2/t^3 & -1/t^3
\end{bmatrix}
\]  \hspace{1cm} (13)

The generalized stresses are the normal force and bending moment at the plane of reference defined by \(Z = 0.0\) in Fig. 4. The generalized strains are the axial strain and curvatures at
the plane of reference. The generalized stresses and strains are related by an elasticity matrix, [\text{D}]. These relations are expressed by Eqs. 14, 15, and 16.

\[
\{\sigma\} = \begin{bmatrix} N \\ M \end{bmatrix}
\] (14)

\[
\{\varepsilon\} = \begin{bmatrix} \partial U/\partial x \\ -\partial^2 W/\partial x^2 \end{bmatrix}
\] (15)

\[
\{\sigma\} = [\text{D}] \{\varepsilon\}
\] (16)

The strains can also be related to the coefficients \((\alpha_1, \ldots, \alpha_6)\) by substituting Eqs. 9 and 10 into Eq. 15. This relation is given by matrix \([\text{Q}]\), the elements of which will be defined later by Eq. 30. Further, the strains could be related to the nodal displacements by substituting Eq. 13 into Eq. 11 and solving for \((\alpha_1, \ldots, \alpha_6)\). These operations would lead to Eq. 17

\[
\{\varepsilon\} = [\text{Q}] \{\alpha\} = [\text{Q}] \text{[}c^{-1}\text{]} \{\delta\}
\] (17)

The global stiffness matrix could then be derived by equating internal and external virtual work. The standard forms are given by Eq. 18

\[
[K] = \int_V [B^T] [\text{D}] [B] \, dV = [c^{-1}]^T \int_0^L [\text{Q}]^T [\text{D}] [\text{Q}] \, dx \text{[}c^{-1}\text{]}
\] (18)

The layering technique is employed by supposing that each element is composed of layers such that the element stiffness properties are summations of layer stiffness properties.
Each layer may have its own area, position coordinates \( X \) and \( Z \), material properties such as stress-strain law, tensile and compressive strengths, modulus of elasticity and stress and strain fields. As mentioned in the introduction, continuity between layers is maintained by the assumption of plane sections (Section 1.3). This assumption provides two additional benefits:

1. The strain state in each layer can be found from the displacements of the node points at each end of the element. This materially reduces the number of unknowns as discussed in detail in the introduction (Section 1.3).

2. The layers composing each element provide a bookkeeping technique for accounting for the spread of cracking, yielding or crushing.

The assumption of plane sections enables the problem to be handled by the usual equations of mechanics instead of the theory of elasticity. This is a sacrifice of some accuracy and geometric generality for far greater computational efficiency. Using the plane sections assumption and referring to Fig. 5. The state of strain in a layer can be defined as

\[
\varepsilon_x = \frac{\partial U}{\partial X} = \frac{\partial U}{\partial X} - Z \frac{\partial W}{\partial X^3}
\]

\[
U_z = U - Z \frac{\partial W}{\partial X}
\]

(19)
Employing the assumption of uniform stress in a layer stress can easily be related to strain.

\[ \sigma_x = E \varepsilon_x \]  

(20)

\( E \) in Eq. 20 is an instantaneous modulus of elasticity. The generalized forces can now be compared as a summation of layer contributions.

\[ N_j = \sum_{i=1}^{n} \sigma_i A_i \]  

(21)

\[ M_j = \sum_{i=1}^{n} \sigma_i A_i Z_i + \sum_{i=1}^{n} M_i \]  

(22)

\[ N_j = \sum_{i=1}^{n} E_i A_i \left( \frac{\partial U}{\partial x} - Z_i \frac{\partial^2 W}{\partial x^2} \right) \]  

(23)

\[ M_j = \sum_{i=1}^{n} E_i A_i Z_i \frac{\partial U}{\partial x} - \sum_{i=1}^{n} E_i A_i Z_i \frac{\partial^2 W}{\partial x^2} \]  

(24)

\[ M_i = A_i \int \sigma Z dA \]  

(25)

In Eqs. 23, 24 and 25 \( j \) is an element number and \( i \) is a layer number of \( n \) layers. These equations can be put in the usual elasticity matrix form by defining element stiffness properties \( \tilde{A}, \tilde{S} \)
and \( \bar{I} \) which are an equivalent area, statical moment and moment of inertia times an equivalent modulus of elasticity.

\[
\bar{A} = \sum_{i=1}^{n} E_i A_i
\]  

\[\bar{S} = \sum_{i=1}^{n} E_i A_i Z_i\]  

\[
\bar{I} = \sum_{i=1}^{n} E_i Z_i^2 A_i + \sum_{i=1}^{n} E_i I_{oi}
\]

Arranging these terms in matrix form relating generalized stresses to generalized strains results in Eq. 29.

\[
\begin{pmatrix}
N \\
M
\end{pmatrix}
= \begin{pmatrix}
\bar{A} & \bar{S} \\
\bar{S} & \bar{I}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial U}{\partial X} \\
-\frac{\partial W}{\partial X}
\end{pmatrix}
\]

Once the elasticity matrix has been defined; the generation of the stiffness matrix becomes a routine operation. Equation 30 can be developed by using Eqs. 9, 10 and 15.

\[
\begin{pmatrix}
\frac{\partial U}{\partial X} \\
-\frac{\partial W}{\partial X}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -6X
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_6
\end{pmatrix}
\]
Now Eq. 18 can be used to evaluate the elemental global stiffness matrix. This is given as Eq. 31.

\[
[K] = \begin{bmatrix}
\frac{\bar{A}}{l} & 0 & \frac{\bar{S}}{l} & -\frac{\bar{A}}{l} & 0 & -\frac{\bar{S}}{l} \\
0 & \frac{12\bar{I}}{l^3} & -\frac{6\bar{I}}{l^2} & 0 & -\frac{12\bar{I}}{l^3} & -\frac{6\bar{I}}{l^2} \\
\frac{\bar{S}}{l} & -\frac{6\bar{I}}{l^2} & \frac{4\bar{I}}{l} & -\frac{\bar{S}}{l} & \frac{6\bar{I}}{l^2} & \frac{2\bar{I}}{l} \\
-\frac{\bar{A}}{l} & 0 & -\frac{\bar{S}}{l} & \frac{\bar{A}}{l} & 0 & \frac{\bar{S}}{l} \\
0 & -\frac{12\bar{I}}{l^3} & \frac{6\bar{I}}{l^2} & 0 & -\frac{12\bar{I}}{l^3} & \frac{6\bar{I}}{l^2} \\
\frac{\bar{S}}{l} & -\frac{6\bar{I}}{l^2} & \frac{2\bar{I}}{l} & \frac{\bar{S}}{l} & \frac{6\bar{I}}{l^2} & \frac{4\bar{I}}{l} \\
\end{bmatrix}
\tag{31}
\]

The construction of the global stiffness matrix now follows by summation of stiffness properties of beam elements on each side of a node. The process of writing the stiffness matrix, Eq. 31, in a general form and extracting only those terms necessary to form the stiffness of node number n is illustrated below. Element (n-1) is to the left of node n and element (n) is to the right.
For any member:

\[
\begin{align*}
& \{ F_{xi} \} = \begin{bmatrix} K_{11} \\ K_{a1} K_{a2} \\ K_{31} K_{32} K_{33} \\ K_{41} K_{42} K_{43} K_{44} \\ K_{51} K_{52} K_{53} K_{54} K_{55} \\ K_{61} K_{62} K_{63} K_{64} K_{65} K_{66} \end{bmatrix} \quad \text{Symmetric} \\
& \{ F_{yi} \} = \begin{bmatrix} U_i \\ W_i \\ \theta_i \end{bmatrix} \\
& \{ F_{k} \} = \begin{bmatrix} U_k \\ W_k \\ \theta_k \end{bmatrix} \\
\end{align*}
\]

For the left member:

\[
\begin{align*}
& \{ F_{xn} \} = \begin{bmatrix} K_{41} K_{42} K_{43} K_{44} K_{45} K_{46} \\ K_{51} K_{52} K_{53} K_{54} K_{55} K_{56} \\ K_{61} K_{62} K_{63} K_{64} K_{65} K_{66} \end{bmatrix} \\
& \{ F_{zn} \} = \begin{bmatrix} U_{n-1} \\ W_{n-1} \\ \theta_{n-1} \end{bmatrix} \\
& \{ F_{yn} \} = \begin{bmatrix} U_n \\ W_n \\ \theta_n \end{bmatrix} \\
\end{align*}
\]

For the right member:

\[
\begin{align*}
& \{ F_{xn} \} = \begin{bmatrix} K_{11} K_{12} K_{13} K_{14} K_{15} K_{16} \\ K_{21} K_{22} K_{23} K_{24} K_{25} K_{26} \\ K_{31} K_{32} K_{33} K_{34} K_{35} K_{36} \end{bmatrix} \\
& \{ F_{zn} \} = \begin{bmatrix} U_{n+1} \\ W_{n+1} \\ \theta_{n+1} \end{bmatrix} \\
& \{ F_{yn} \} = \begin{bmatrix} U_n \\ W_n \\ \theta_n \end{bmatrix} \\
\end{align*}
\]
Combining Eqs. 33 and 34:

\[
\begin{bmatrix}
F_x \\
F_z \\
F_y
\end{bmatrix} =
\begin{bmatrix}
K_{41} & K_{42} & K_{43} & (K_{44} + K_{11}) & (K_{45} + K_{12}) \\
K_{51} & K_{52} & K_{53} & (K_{54} + K_{21}) & (K_{55} + K_{22}) \\
K_{61} & K_{62} & K_{63} & (K_{64} + K_{31}) & (K_{65} + K_{32})
\end{bmatrix}
\begin{bmatrix}
U_{n-1} \\
W_{n-1} \\
\theta_{n-1} \\
U_n \\
W_n \\
\theta_n \\
U_{n+1} \\
W_{n+1} \\
\theta_{n+1}
\end{bmatrix}
\]

(35)

Repeating these steps for each node point populates the global stiffness matrix so that the increments of displacement corresponding to an increment of load can be found. This is shown in Eq. 36.

\[ \begin{bmatrix}
F 
\end{bmatrix} = [K] \begin{bmatrix}
\delta 
\end{bmatrix} \]

(36)

Given the incremental displacement vector \( \delta \), Eq. 17 can be used
to find the strain at the centroid of each layer. This strain will be considered representative of the whole layer. Using Eqs. 13, 17 and 30, Eq. 37 was derived.

\[
[B] = \begin{bmatrix}
-\frac{1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\
0 & \frac{6}{l^3} - \frac{12X}{l^3} & -\frac{4}{l} + \frac{6X}{l^3} & 0 & -\frac{6}{l^3} + \frac{12X}{l^3} & -\frac{2}{l} + \frac{6X}{l^3}
\end{bmatrix}
\]

(Eq. 37)

Evaluating matrix [B] at X = \(\frac{l}{2}\) it is possible to define the generalized strains by Eqs. 38 and 39.

\[
\frac{\partial U}{\partial X} = \frac{1}{l} \left(-U_i + U_k\right)
\]

(Eq. 38)

\[
\frac{\partial^3 W}{\partial X^3} = \frac{1}{l} \left(-\theta_i + \theta_k\right)
\]

(Eq. 39)

The engineering strain and the stress can then be computed for the ith layer of the tth element as:

\[
\varepsilon_{xt,i} = \frac{1}{l} \left[U_k - U_i - z_l \theta_i + z_l \theta_k\right]
\]

(Eq. 40)

\[
\sigma_{l,i} = E_{l,i} \varepsilon_{xt,i}
\]

(Eq. 41)

Once the entire stress field is known a convergence check is performed on the increment of the displacement field. Each incremental displacement component is checked against the
corresponding component of the last trial. If all are within a relative tolerance of the last trial the iteration is stopped and the stress and displacement fields are incremented to include the new contributions from this load step. Each layer is then checked for tensile cracking or compressive crushing. These points will be discussed later but it will be stated now that the computer program which performs this analysis makes use of one or more stress-strain curve for each layer to account for inelastic behavior, cracking and crushing. Stress-strain curves are discussed in detail in Section 2.2.

If no cracking or crushing has taken place, another load increment is added and the process is repeated with the generation of a new stiffness matrix which reflects the current state of stress. If cracking or crushing has started or is propagating, a special process to be discussed in Section 2.3 is employed to account for these types of non-linear behavior.

If convergence of the current load step has not been attained the incremental stresses are temporarily added to the total stresses to find new elastic moduli using the layer stress strain laws. A new stiffness matrix is generated and new incremental displacements are computed and compared with the last set to check convergence. This process is repeated until either convergence is attained in a limited number of trials or the maximum number of trials is reached at which time the load increment is reduced by 15% and the whole process is repeated. There is also
an overall trial counter to terminate execution if a large number of load reductions has been tried and convergence is still not attained. Experience with this process applied to materials which have relatively sharp knees in their stress-strain curve has shown that the load reduction process can reduce the load to such an extent that literally hundreds of additional load steps would be required to reach ultimate load. There is, therefore, a load increasing process which increases the load 10% if convergence of the next load step occurs in three trials or less. The amount to reduce or increase the load and the cutoff number of trials were arbitrarily arrived at by observing their effect on several test runs. The fact remains that a load reduction process was needed to assure convergence and a load increasing process was an economic necessity.

2.2 Stress-Strain Curves

The material stress-strain curve is the physical basis of the method used in this research. It is felt that this method employs the most realistic stress-strain curve of any known approach. It will be seen that the method discussed is general enough to accept the following types of stress-strain curves:

1. Elastic-Brittle
2. Elastic-Plastic (not just elastic-perfectly plastic)
3. Elastic-Plastic with linear strain hardening
4. Elastic-Plastic with tensile cracking
5. Elastic-Plastic with tensile cracking and compressive crushing.

The structural stiffness matrix has been shown to be a sum of elemental stiffness matrices which were in turn a summation of layer contributions. The layer stiffness contributions were seen to depend on the instantaneous modulus of elasticity which is the slope of a stress-strain curve at some total stress (or strain).

The Ramberg-Osgood Law has been chosen to provide generality in the shape of the stress-strain curve while maintaining a continuous mathematical expression \(^{17}\). As usually written, the Ramberg-Osgood Curve is given by Eq. 42

\[
\epsilon = \frac{\sigma}{E} + \left(\frac{3\sigma_1}{7E}\right) \left(\frac{\sigma}{\sigma_1}\right)^n
\]  

(42)

This is actually a specialization of the more general form given as Eq. 43.

\[
\epsilon = \frac{\sigma}{E} + \left(\frac{1-m}{m}\right) \left(\frac{\sigma_1}{E}\right) \left(\frac{\sigma}{\sigma_1}\right)^n
\]  

(43)

\(\sigma\) = Stress at some load

\(E\) = Initial modulus of elasticity

\(\sigma_1\) = Secant yield strength equal to the ordinate of intersection of the \(\sigma-\epsilon\) curve and a line of slope \((m) \cdot (E)\)

\(n\) = A constant

\(m\) = A constant defining a line of slope \((m) \cdot (E)\) on a plot of stress and strain
Ramberg and Osgood derived the constants m and n by consideration of log-log plots of strain deviation curves for various materials given by Aitchison and Miller in NACA T.N. 840. Strain deviation was obtained by plotting stress vs. the difference between measured strain and strain from Hooke's Law.

\[ d = \varepsilon - \frac{\sigma}{E} = K \left( \frac{\sigma}{E} \right)^n \]  \hspace{1cm} (44)

\[ K = \left( \frac{1-m}{m} \right) \left( \frac{\sigma_1}{E} \right)^{1-n} \]  \hspace{1cm} (45)

A log-log plot of Eq. 44 should have an intercept at K and a slope of m. From inspections of several such plots it was decided that m should be less than 0.9. Ramberg and Osgood then decided to choose m so as to make \( \sigma_1 \) approximately the yield stress given by the 0.2% offset method. Using test data again, a value of m = .709 was found and rounded off to \( m = 0.7 \).

The constant N is found by using two points on the stress-strain curve to define two strains, two m's and two stresses. Using both sets of data to find K, which is a constant for any stress-strain curve, results in an equation relating \( \sigma_1 \), \( \sigma_2 \), \( m_1 \), \( m_2 \) and n as follows.

\[ K = \left( \frac{1}{m_1} - 1 \right) \left( \frac{\sigma_1}{E} \right)^{1-n} = \left( \frac{1}{m_2} - 1 \right) \left( \frac{\sigma_2}{E} \right)^{1-n} \]
Fig. 6 from NACA Technical Note 902 shows some of the variety in stress-strain curves which can be obtained using the Ramberg-Osgood Law by varying n for a given m.

The application of the Ramberg-Osgood Law to metals is reasonably straightforward. If some means could be developed to find m and n, it could be used to approximate almost any monotonically increasing stress-strain curve. Many specific stress-strain curves have been advanced for concrete. Desayi and Krishnan suggested the equation below:

\[
\frac{1}{\sigma_2} = \frac{1}{m_2} - 1
\]

\[
\log \left( \frac{m_2}{m_1} \cdot \frac{1-m_1}{1-m_2} \right) = 1 + \frac{\log \left( \frac{\sigma_1}{\sigma_2} \right)}{\log \left( \frac{\sigma_1}{\sigma_2} \right)}
\]

\[
f = \frac{E \varepsilon}{1 + \left( \frac{\varepsilon}{\varepsilon_0} \right)^2}
\]

\(f\) = Stress at any strain \(\varepsilon\)

\(\varepsilon_0\) = Strain at the maximum stress, \(f_0\)

\(E\) = A constant such that \(E = \frac{2f_0}{\varepsilon_0}\), i.e. an initial tangent modulus
Saenz\textsuperscript{19} suggested that Desayi and Krishnan's equation was not general enough and suggested the more complicated form shown below because it allowed for a variable ratio of secant to initial modulus.

\[
f = \frac{E_s}{1 + (R + R_E - 2) \frac{\varepsilon}{\varepsilon_0} - (2R - 1) \left( \frac{\varepsilon}{\varepsilon_0} \right)^{a} + R \left( \frac{\varepsilon}{\varepsilon_0} \right)^{3}}
\]

- $E = \text{Initial tangent modulus}$
- $R_E = E / E_s$
- $E_s = \text{Secant modulus through peak of stress-strain curve}$
- $R_f = f_0 / f_f$
- $f_0 = \text{Maximum stress}$
- $f_f = \text{Stress at maximum strain}$
- $\varepsilon_f = \text{Maximum strain}$
- $\varepsilon_0 = \text{Strain at maximum stress}$
- $R_\varepsilon = \varepsilon_f / \varepsilon_0$

\[
R = \frac{R_E (R_f - 1)}{(R_\varepsilon - 1)^{a}} - \frac{1}{R_\varepsilon}
\]

Both of these equations will be compared to the stress-strain curve about to be presented using data reported by Desayi-Krishnan\textsuperscript{5}. 

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The application of the Ramberg-Osgood Law would, however, result in a material independent computer program which would obviously be more versatile. Such a program could handle combinations of materials with the same ease as a homogeneous beam by combining the layering concept with individual stress-strain curves used for each layer.

Consideration will now be given to the approximation of the concrete compressive stress-strain curve by a Ramberg-Osgood Law.

Figure 7 from Ref. 25 shows generally accepted smoothed stress-strain curves for concrete in compression as measured on the compressive side of flexural test. The following characteristics of these will be noted:

1. All curves start as straight lines.
2. All curves reach a peak strength at a strain of approximately 0.002 in/in.
3. All curves, especially those for structural strength concrete have a downward sloping leg.

The approach taken here was to try to find a technique for consistently arriving at an acceptable approximation of these curves given only $f'_c$ and Young's modulus and using the properties above. A preliminary attempt to use a process analogues to that of Ramberg and Osgood as previously described led to results which were difficult to generalize. Typically, the "constants" varied
greatly for different concrete strengths. The following approach has led to reasonably acceptable stress-strain curves and very good agreement between predicted and experimental ultimate strengths.

1. Assume $\sigma_1 = f'_c$. This is the only required assumption to use the analytic stress-strain curve for concrete.

2. Compute the value of Young's modulus from any acceptable equation using $f'_c$ or the results of laboratory tests, if available.

3. Assume that the stress-strain curve must pass through the point $(0.002, f'_c)$. This leads to the following equation for the coefficient $m$.

$$m = \frac{f'_c}{0.002 E} \quad (46)$$

4. Assume the Ramberg-Osgood curve stops at a strain of 0.002 in/in.

5. Assume a horizontal straight line from a strain of 0.002 to a strain given in Table I below as $\epsilon_1$. This value is a variable in the program. The suggested values in Table I were scaled from Fig. 7.

6. Assume a straight line sloping downward from $\epsilon_1$ to a stress of zero. Suggested values for this slope, $E_{down}$, again from Fig. 7 are also found in Table I. The use of $E_{down}$ to compensate for compressive crushing will be explained in Section 2.3. It is noted now that $E_{down}$ is
not a stiffness property and is not used in regenerating the stiffness matrix.

Table I

<table>
<thead>
<tr>
<th>$f'_c$</th>
<th>$F_{down}$</th>
<th>$\varepsilon_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5600</td>
<td>3000</td>
<td>0.0022</td>
</tr>
<tr>
<td>4750</td>
<td>1800</td>
<td>0.0022</td>
</tr>
<tr>
<td>3900</td>
<td>1250</td>
<td>0.0023</td>
</tr>
<tr>
<td>&lt;3000</td>
<td>700</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

7. From trial and error comparisons a value of $n = 9$ has been found to give consistently reasonable results for all strengths tried.

The results of this method of approximating the concrete compressive stress-strain curve are shown in Fig. 8. As can be seen the approximate curves are quite close to the scaled-up versions of Fig. 7 which are shown in dotted lines.

Figure 39 shows a comparison of the proposed method of computing a compressive stress-strain curve for concrete with the equations proposed by Desayi and Krishnan and Saenz. The experimental data are also by Desayi and Krishnan. As was shown before, a great deal of information is needed to use Saenz's equation. Details applicable to the curves shown in Fig. 39 are found in Ref. 19. Figure 39 shows excellent agreement between Saenz curve and the proposed method on the ascending portion of the stress-strain
curve. The descending portion needs more explanation. The experimental data and Saenz's equation are for cylinders whereas the proposed method uses a slope on the downward leg of the curve. Based on flexural tests as shown in Fig. 7. Hognestad, Hanson and McHenry\(^8\) have published comparative flexural and cylinder compressive stress-strain curves which indicate that the slope on the downward leg appears greater for the flexural tests. Thus the difference in the descending portions of the curves shown in Fig. 39 are to be expected. In the current context the flexural behavior is preferred.

Clearly the approximation used here is adequate for analytical use - in fact it represents a considerable refinement over previously used concrete compressive stress-strain curve such as Valliappan and Doolan's bilinear curve\(^{23}\). The downward sloping portion of the stress-strain curve used here is instrumental in creating an unstable in-plane condition. If the curve does not slope down, an artificial termination is sometimes used which is based on exceeding some ultimate compressive strain. It is believed that the technique employed in this research is more realistic. The use of a bilinear stress-strain curve sometimes also requires an "adjustment" of compressive stress to make the stress volume at ultimate load more comparable to a Whitney-like stress block. This is also unnecessary with the proposed curve.

Continued research could result in improved Ramberg-Osgood representations of the concrete stress-strain curve because
an infinity of such curves are possible. It would be difficult, however, to justify significant refinement for use with a material as variable as concrete.

Another illustration will be used to emphasize the power of this stress-strain curve when used with layered elements to model a concrete beam. Consider limit design of continuous beams. One of the major drawbacks to the application of limit design to concrete structures has been the need to provide adequate rotation capacity in the section to allow redistribution of moment and the formation of subsequent hinges. An analogy could be drawn with spiral versus tied columns. Spiral columns exhibit much greater deformation capacity than tied columns because the spiral reinforcement contains the concrete core and induces a multi-axial state of stress. The result is an apparent ductility. In beams it has been proposed to provide the containment through closely spaced stirrups. The result of test reported by Corley\(^3\) show maximum compressive strains of 5, 10, and even 20 times the commonly used ultimate strain of 0.003. Corley proposed Eq. 47 for the maximum compressive strain. It is recognized that there are other equations, but it is the concept which is important here, not the currently favored equation.

\[
\epsilon_u = 0.003 + 0.02 \frac{b}{z} + \left( \frac{P''_{f} V}{20} \right)^{a} \tag{47}
\]
b = Width of the compressive flange of a beam.

z = Distance along the span from section of maximum moment to an adjacent section of zero moment.

p'' = Ratio of volume of binding steel (one stirrup plus compression steel between stirrups) to volume of concrete bound.

fy = Yield point of stirrup steel.

Tests of concrete compressive specimens with tied longitudinal bars have been reported by Roy and Sozen. Figure 9 represents the type of curves reported for specimens with deformed steel bars. It is seen that for medium and close tie spacing a straight line downward leg of the stress-strain curve as used here could be an adequate idealization to a strain of at least 0.03 in/in and that a bilinear downward leg would suffice for all curves presented in Ref. 18. Future research will probably lead to an expression for the downward slope involving some of the same parameters as seen in Eq. 47 and the cylinder strength. At that time the method proposed here and the computer program based on it should be able to predict the entire load-deflection behavior of indeterminant beams complete with zones of cracking and crushing around each hinge. More detailed consideration of shear will have to be included but there is nothing conceptionally prohibitive about this. In fact it is possible that the techniques employed here could be used to set guidelines on the amount of confinement
steel required to produce a given ultimate load capacity for some member.

The shape of the tensile stress-strain curve has been found to be quite important for the prediction of concrete beam behavior - especially for prestressed concrete beams. The exact shape of the curve would appear to be far less important than the recognition of a surprisingly long downward sloping leg. Researchers and practicing engineers have characteristically neglected the tensile properties of concrete other than strength for many reasons. Some of these reasons are listed below.

1. Reinforced concrete is assumed to be cracked so the design process ignores any remaining tensile stress region.
2. Prestressed concrete is not supposed to reach a cracking stress under design load.
3. Concrete tensile strength is small compared to its compressive strength.
4. Concrete is assumed brittle in tension.
5. Tensile properties do not significantly affect the ultimate strength because the volume of concrete still in tension at the critical section and the resulting force are so small as to be negligible.

This research, while agreeing with all of the previous comments except No. 4, would indicate that the tensile properties are quite
important in defining the shape of the load deflection curve. Furthermore, the effect of the tensile properties would appear more significant in prestressed than in reinforced concrete beams. Previous studies of this type have been concentrated on reinforced concrete beams so that the effect of not including this feature would be minimal.

The need to include the downward portion of the tensile stress-strain curve is shown in Fig. 10. This figure shows the experimental load deflection curves for two virtually identical prestressed concrete solid box beams from the test series reported by Walther and Werner. The physical data pertaining to these beams (A-9 and A-10) are given in Table IV. Also shown on the same figure are the analytic load deflection curves obtained by using five different tensile stress-strain curves. These stress-strain curves are drawn to the same scale in Fig. 11 for comparative purposes. It is seen that the results are divided into two easily recognized groups. Curves A, B and C give a reasonable approximation of the non-linear behavior of the beam during cracking whereas curves D and E miss the zone formed by the two tests by a wide margin. The following points deserve mention:

1. Because of the similarity in physical parameters the analytic load deflection curves of beams A-9 and A-10 are quite similar. Therefore, the data for the analytic solution runs necessary to plot Fig. 10 were generated only for beam A-9.
2. Curves D and E show a virtually instantaneous growth of cracked zones extending up about a quarter of the beam's depth. Subsequent cracking occurs at a slower rate. Curves A, B and C show a gradual increase in crack depth with increased load. This is in good agreement with the photographs taken of the actual beams and as shown in Figs. 12 and 13 for two prestressed box beams.

3. The shape of the tensile stress-strain curve has no perceptable effect before initial cracking and virtually no effect on ultimate moment. This is as expected.

4. There is a definite indication that the tensile unloading must be gradual as in curves A, B and C, rather than almost instantaneous as in curves D and E.

Two questions demand immediate answers. They are:

1. Can any evidence be given that the tensile stress-strain curve actually has a gradual downward slope?

2. What guidelines can be given on the choice of a shape and downward slope?

Testing of concrete in direct tension has historically resulted in a brittle type of failure. In the recent past it was thought that concrete had virtually no ductility in tension. During the past two decades increased research into the area of micro-cracking of concrete has lead to tensile testing using special
testing machines which are much stiffer than ordinary machines. Figure 14 represents the curves found in Ref. 6 which show a great variety in shape, peak strains and ultimate strengths. But this figure does show a general shape and a surprisingly long downward leg. Therefore, it can be concluded that the downward leg does exist. There were no corresponding compressive tests reported.

An investigation of tensile behavior and its relation to compressive behavior, Young's modulus and compressive strength is needed.

One of the curves in Ref. 6 had a water-cement ratio of 0.45. The concrete used in the prestressed concrete box beams in Ref. 23 had a water-cement ratio of 0.496. Curve A of Fig. 11 was constructed as an idealization of the experimental stress-strain curve. The downward slope in curve A was chosen as 800 ksi. This compares with approximately 400 ksi to 600 ksi found in Ref. 6.

The maximum tensile stress was chosen as 450 psi (plus about 25 psi dead load tensile stress). This number was chosen because the direct tensile strength of concrete is on the order of 450 to 550 psi. This is higher than any strength reported by Evans and Marathe for a specimen age of about 28 days. Because of the large variation in reported test results and lack of corresponding compression tests the following analogy was tried analytically.

1. Curve A constructed as mentioned above and the results compared to curve B.

2. Curve B, which is the proposed analytic tensile
stress-strain curve, is constructed by using two straight lines. The first line has a slope equal to the compres-
sive modulus of elasticity and stops at a tensile stress of $7.5\sqrt{f'_c}$. This tensile stress is adjusted for the dead
load tensile stress and will be recognized as the ac-
cepted lower estimate of the modulus of rupture for con-
crete. Some engineers might prefer to use another mea-
sure of tensile strength or set a maximum value such
as 500 or 600 psi. This is a matter of judgment on the
part of the analyst. The second line slopes downward from
the end of the first line at a slope of 800 ksi. This line extends to a tensile stress of zero.

It was supposed that if the results using curve B proved
a close approximation to those using curve A then curve B could be used instead. Curve B is easier to construct for all concretes
and requires no additional knowledge save the assumption for the
downward slope. Curve A requires additional Ramberg-Osgood para-
meters which cannot be defined for various concretes at this time. Figure 10 shows the results of using both curves. It can be seen
that curve B appears to be an adequate substitution for a curve shaped like curve A. Figure 15 shows the results of using other
values for the downward slope. The following additional points are mentioned:

1. Figures 21, 22, 24-29, and 31-34 show the results of
applying this method and curve B to 2 reinforced beams, 4 prestressed box beams and 7 prestressed I-beams. The results are encouraging, but more research into stress-strain curves would be quite valuable.

2. The computer program has been left general enough to exert a curve like curve A. Thus, if future research leads to better stress-strain curves, no change will be required. Curve B is seen to be a degenerate form of curve A.

3. A lower limit to the load deflection curve is provided by curves like D and E. These curves are constructed by using one straight line whose slope is the compressive modulus of elasticity from zero to the modulus of rupture stress (see previous discussion). A second straight line runs from the end of the first back to zero on a downward slope which is much larger than the compressive modulus. For curves D and E a slope of 20,000 ksi was used. The resulting load deflection curve is quite good at both ends but fairly conservative in the region of fastest cracking. This is shown in Fig. 10.

The downward slope of the tensile stress strain curve will be referred to as $E_{down}$. $E_{down}$ will not be used in stiffness calculations but will be used to account for the release
of energy caused by cracking. This will be explained in Section 2.3.

Application of the Ramberg-Osgood Law to reinforcing and prestressing steels is virtually exactly what it was intended for and deserves no more comment. The use of the Ramberg-Osgood Law and the layered element to study the bending behavior of steel beams would, however, serve to further illustrate the generality of the method.

Figure 16 shows the results of four analytic investigations of a fixed ended steel I shape. The exact shape used does not correspond to any rolled shape. Figure 16 shows the effect of increasing the parameter "n" in the Ramberg-Osgood Law on the load deflection curve. The effect on the stress-strain curve is to make the knee sharper. This effect is carried over to the load deflection curve whose shape approaches that predicted by the simple plastic theory as the stress-strain curve approaches elastic-perfectly plastic. Using the shape factor for the hypothetical section, the simple plastic theory would predict a ratio $P/P_0 = 2.07$. Adjusting this value for the position of the section actually used for measurement of stresses results in a ratio of 2.28. This is shown by the horizontal line in Fig. 16. Better discretization could make this as close to 2.07 as desired. The value of 2.28 compares quite well with the $n = 100$ and $n = 300$ curves in Figure 16. The curves with $n = 30$ and $n = 50$ are not as good. This is as expected. The cost of solution of these
examples increased as the value of \( n \) increased. Thus it would appear that the Ramberg-Osgood Law combined with the layered elements would allow as close an approximation to the simple plastic theory for steel beams as economically desirable for any section which is symmetric about the plane of loading.

2.3 A Technique for Cracking and Crushing Analysis

When the iterative procedure used to find the incremental displacements and stresses corresponding to a given load step has converged to an acceptable tolerance, the accumulated stresses and displacements are tentatively incremented. A pre-scanning process is then used to check if any layer has a total tension of compression which exceeds given allowable stresses by more than some tolerance. If so, the program returns to the original iterative procedure and reduces the load by 50% for this step. The original problem is resolved to convergence, field quantities are again tentatively incremented and the results pre-scanned again. This process is repeated until all stresses which exceed the tensile or compressive allowable stresses exceed them by less than their respective tolerances. The pre-scanning technique is used to prevent large overstressing of the material for any load step.

As mentioned in Section 2.1, if no stresses exceed the compressive or tensile limit, another load step is taken. If scanning reveals that \( \sigma + d\sigma \) is greater than \( \Gamma_t \) for any layer
then the layer is said to have cracked and steps are taken to adjust its stiffness and redistribute the stresses in that layer. The alteration to stiffness is simply setting that layer's modulus of elasticity equal to zero. Such a layer would then contribute no stiffness to an element and accept no additional increments of stress.

The redistribution of stresses is accomplished by using the downward leg of the tensile stress-strain curve and the basic concept of the initial stiffness method as mentioned in Section 1.2.1. The amount of strain beyond that corresponding to cracking, or the incremental strain, which ever is appropriate, is multiplied by $E_{downt}$ to produce a stress-like quantity called a fictitious stress. This fictitious stress is applied to the layer which has cracked until the sum of the increments of fictitious stress and the accumulated tensile stress are zero. The redistribution to the rest of the beam is accomplished by using the layer area to convert stress to an eccentric force and creating a fictitious load vector with axial force and corresponding moment terms.

During the same scanning operation a test is also made to see if a given layer exceeds a crushing criteria. This crushing criteria for a layer is the attainment of the maximum compressive stress or a strain greater than 0.002. If it is ascertained that crushing has occurred, the first stage is to set Young's modulus equal to zero. If the strain is less than the
value of $\varepsilon_1$ given in Table I, no unloading or redistribution is considered. If the strain exceeds $\varepsilon_1$, the excess strain is converted to fictitious stresses and hence fictitious loads analogously to the tensile cracking analysis.

Once all layers have been scanned the fictitious load vector is used to compute an auxiliary stress and displacement increment. At this time there are two stress and two displacement increments. One corresponds to the actual load step and the other corresponds to cracking and crushing. Essentially the same iterative process is used to find convergence for the auxiliary displacement increment as that used for the actual load step. Once convergence has been obtained, the layers are rescanned to check if the redistribution of cracking and/or crushing stresses has caused any more layers to reach a cracking or crushing criteria. If any layers have reached these criteria the process of assembling a fictitious load vector and iterating to convergence is repeated. If no additional layers have reached cracking or crushing there may still be additional fictitious load vector components as a result of the additional strains computed from the increments of displacements. Therefore, the entire process is repeated until the fictitious load forces are smaller than some tolerance. At that time the cracking-crushing analysis is terminated and the accumulated stress and displacement fields are incremented by both sets of incremental stresses and displacements.

It is this process of crushing generating more crushing,
which is possible using the type of stress-strain curve used here, that enables the beams to reach an unstable in-plane state and a natural termination of execution rather than one forced by an artificial strain limit.

2.4 Extension of the Methodology to Prestressed Concrete Beams

The additional steps used with prestressed concrete beams follow from the physical actions involved in prestressing. An initial stress field is read in for each layer. This provides the initial steel tension. For applications involving prestressed concrete the initial stress input for the concrete layers is zero but other applications could require each layer to have some initial stresses. An eccentric prestressing force is applied using the nodal force vector. It is advisable to compensate this prestressing force for the elastic loss which will occur when it is applied. While the prestress stress could be found for the centroid of each layer by hand calculation and read in it is easier to let the computer do the arithmetic by using nodal forces.

It should be apparent that the object of applying the nodal forces used in prestressing is to produce the same thrust and moment diagrams in the reference plane as would be generated by replacing the prestressing elements at each point along the beam by an eccentric force at that location. This concept is important in generalizing the process for cases other than straight strands or for conditions other than prestressed concrete.
Consider a simply supported prestressed concrete beam pretensioned with a draped strand such that the end eccentricity was \( e_1 \) and the eccentricity at a distance \( L_3 \) from an end was \( e_3 \) and the strand was straight line segments in between. \( e_1 \) and \( e_3 \) are measured from the reference plane. The prestressing forces would then be modeled as follows:

1. An axial force, \( P \), is applied at each end of the beam.
2. End moments are applied to each end of the beam equal to \( (P) (e_1) \).
3. A concentrated load is applied to each drape point such that \( P (e_3 - e_1) = \frac{-P}{L_3} \).

In No. 3 \( P \) is the concentrated load, and \( L_3 \) is the distance from the end of the beam to drape point. If due consideration is given to algebraic sign this system of forces will be equivalent to draped strand prestressing. An alternate technique which used nodal moments at each node to account for the change in eccentricity was also tried and found to be inadequate because of the less accurate representation of the moment diagram produced by prestressing force.

When draped strands are used the inclined strand should be simulated by a series of horizontal line segments to approximate its contribution to the global stiffness matrix.

The beam deflects under the influence of the nodal force and moment used to apply the prestressing force. This
prestress camber may or may not be desired to be part of the displace ment vector output. Both options are provided and the choice is dictated by the physical situation. The prestress camber must, however, be included when displacements are converted to total strains to test against strain based behavior criteria.

The conversion to prestressed concrete beams showed the importance of the tensile stress-strain curve. The flexural cracking of prestressed concrete beams causes a much more pronounced change in the slope of the load deflection curve than it does for reinforced concrete. This is probably because of the relative amounts of steel found in both. The use of the downward leg of the tensile stress-strain curve to improve the analytical load deflection curve was discussed in Sections 2.2 and 2.3.

2.5 An Approximation for Flexural Shear

An approximate method for computing the flexural shear stress has been developed by considering equilibrium of a layer of an element as shown in Fig. 17. Some bending stresses $\sigma$ and $\sigma + d\sigma$ are shown. Since this research has assumed that each layer is in a state of uniform axial stress given at its centroid, the bending stresses could be replaced by the uniform stresses as shown by the dashed lines. If $\sigma_L$ is a uniform stress on the left side of the layer and $\sigma_R$ is the uniform stress on the right, then according to Fig. 17 the following equilibrium equation can be written.
\[ \sigma_L A + \tau bL - \sigma_R A = 0 \]  

where  

- \( A \) = Layer area  
- \( b \) = Layer width  
- \( L \) = Layer length  

Two approaches to finding \( \sigma_L \) and \( \sigma_R \) were considered.  

1. Compute additional stress fields at the ends of the elements and use them in Eq. 48 to find an average shear, \( \tau \), for the layer. Uniform axial stress in a layer implies uniform shear in a layer.  

2. Use an averaging technique to find the shear for a layer using the known centroidal stress fields.  

If strains were computed at the ends of a layer using the nodal displacements at the ends of the parent element and Eq. 17 they would be grossly in error. A considerable improvement can be made by finding the generalized axial and curvature strain on each side of a node point and taking a weighted average of them. This is similar to the concept used in applying Eq. 17 to the mid-point of a layer. This strain averaging technique thus makes the end strains for each layer dependent on three nodal rotations and three nodal axial displacements.  

Consider a simply supported doubly reinforced concrete beam of rectangular cross-section subjected to third point loading. This beam is discussed in detail in Section 3.1. Its
cross-sectional layering and elemental discretization are shown in Figs. 3-B and 2-B. The following strain components were found on both sides of node No. 3.

\[
\begin{align*}
\left( \frac{\partial U}{\partial x} \right)_L &= -0.286 \times 10^{-4} \\
\left( \frac{\partial^2 W}{\partial x^2} \right)_L &= 0.0480 \times 10^{-4}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{\partial U}{\partial x} \right)_R &= -0.447 \times 10^{-4} \\
\left( \frac{\partial^2 W}{\partial x^2} \right)_R &= 0.0665 \times 10^{-4}
\end{align*}
\]

Averaging these strains yields:

\[
\begin{align*}
\left( \frac{\partial U}{\partial x} \right)_{\text{Ave}} &= -0.367 \times 10^{-4} \\
\left( \frac{\partial^2 W}{\partial x^2} \right)_{\text{Ave}} &= 0.0573 \times 10^{-4}
\end{align*}
\]

The corresponding stress in the top layer is:

\[
\begin{align*}
\left( \sigma_x \right)_L &= -0.0673 \text{ ksi} \\
\left( \sigma_x \right)_R &= -0.1064 \text{ ksi} \\
\left( \sigma_x \right)_{\text{Ave}} &= -0.0875 \text{ ksi}
\end{align*}
\]

The average stress agrees quite well with the average of the computed stresses in the first layer of the second and third elements which is - 0.0869 ksi.

While this method of computing shear stress is obviously good enough for interior nodes such as node 3 used in the example above, it does have some drawbacks for the end nodes in that three nodes are required for each shear stress. This method also requires computation of additional stress fields. It was therefore decided to try the second approach.
Consider a beam whose elements are \( j, j+1, j+2, \) etc. The left and right node point of element \( j+1 \) are \( i+1 \) and \( i+2 \) respectively. A two pass operation will then be used to find the layer shears from the known layer stresses.

1. Compute \( Q_{i+1} \) using \( \sigma_j \) and \( \sigma_{j+1} \) and assume this to be the shear at the node point.

2. Compute \( \tau_{\text{first}} = Q_{\text{first}} \)

Compute \( \tau_{j+1} = \frac{1}{2} (Q_{j+1} + Q_{j+2}) \)

Compute \( \tau_{\text{last}} = Q_{\text{last}} \)

Where these shears are assumed to be acting at the centroid of the layer. Putting this in equation form for the \( i^{\text{th}} \) node, \( j^{\text{th}} \) element, \( k^{\text{th}} \) layer results in Eqs. 49 and 50.

\[
Q_{i,k} = \sum_{n=1}^{k} \sigma_{j,n} A_{j,n} + \sum_{n=1}^{k} \sigma_{j+1,n} A_{j+1,n} \frac{1}{2} \left( b_j L_j + b_{j+1} L_{j+1} \right) \quad i = j \quad (49)
\]

\[
\tau_{j,k} = \frac{1}{2} (Q_{j,k} + Q_{j+1,k}) \quad (50)
\]

\[
\tau_{1,k} = Q_{1,k}
\]

\[
\tau_{\text{last},k} = Q_{\text{last},k}
\]

While this process might seem questionable, it actually gives very good numerical results. Consider two examples.
Example I

Consider again the doubly reinforced concrete beam previously referred to. Table II shows a comparison of shears computed using \( VQ/It \) and shears computed using the averaging technique for a total shear of one kip. It is noted that the corresponding load produced no non-linear behavior. This data is compared for the line of centroids of the layers in an end element.

Table II

<table>
<thead>
<tr>
<th>Layer</th>
<th>( VQ/It )</th>
<th>Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0055</td>
<td>0.0054</td>
</tr>
<tr>
<td>2</td>
<td>0.0100</td>
<td>0.0100</td>
</tr>
<tr>
<td>3</td>
<td>0.0890</td>
<td>0.0887</td>
</tr>
<tr>
<td>4</td>
<td>0.0155</td>
<td>0.0154</td>
</tr>
<tr>
<td>5</td>
<td>0.0182</td>
<td>0.0181</td>
</tr>
<tr>
<td>6</td>
<td>0.0200</td>
<td>0.0199</td>
</tr>
<tr>
<td>7</td>
<td>0.0209</td>
<td>0.0208</td>
</tr>
<tr>
<td>8</td>
<td>0.0208</td>
<td>0.0207</td>
</tr>
<tr>
<td>9</td>
<td>0.0198</td>
<td>0.0198</td>
</tr>
<tr>
<td>10</td>
<td>0.0180</td>
<td>0.0179</td>
</tr>
<tr>
<td>11</td>
<td>0.1163</td>
<td>0.1157</td>
</tr>
<tr>
<td>12</td>
<td>0.0127</td>
<td>0.0126</td>
</tr>
<tr>
<td>13</td>
<td>0.0090</td>
<td>0.0089</td>
</tr>
<tr>
<td>14</td>
<td>0.0350</td>
<td>0.0346</td>
</tr>
<tr>
<td>15</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>
The results are obviously satisfactory. The distribution of maximum shear along the beam is shown in Fig. 18. The discrepancy which appears near the third point load is a result of the averaging technique. The length over which this discrepancy occurs can be made arbitrarily small by reducing the length of the element on each side of the concentrated load. It is therefore dependent on discretization. The steel reinforcing rods used as layers 3, 11 and 14 were given a shear thickness equal to the sum of their diameters.

Example II

This example uses the same beam carrying a uniformly distributed load. The two loading cases are different enough to confirm the adequacy of the stress averaging technique. The results of computing the shear stress at the centroid of each layer for the end of the beam, the middle of the first element and 1/4 point are shown in Table III in ksi.
Table III

<table>
<thead>
<tr>
<th>Layer</th>
<th>VQ/It</th>
<th>Computer</th>
<th>VQ/It</th>
<th>Computer</th>
<th>VQ/It</th>
<th>Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0107</td>
<td>0.0094</td>
<td>0.0100</td>
<td>0.0094</td>
<td>0.0053</td>
<td>0.0053</td>
</tr>
<tr>
<td>2</td>
<td>0.0196</td>
<td>0.0172</td>
<td>0.0183</td>
<td>0.0172</td>
<td>0.0098</td>
<td>0.0097</td>
</tr>
<tr>
<td>3</td>
<td>0.1762</td>
<td>0.1534</td>
<td>0.1642</td>
<td>0.1534</td>
<td>0.0881</td>
<td>0.0865</td>
</tr>
<tr>
<td>4</td>
<td>0.0305</td>
<td>0.0267</td>
<td>0.0284</td>
<td>0.0267</td>
<td>0.0152</td>
<td>0.0150</td>
</tr>
<tr>
<td>5</td>
<td>0.0358</td>
<td>0.0314</td>
<td>0.0334</td>
<td>0.0314</td>
<td>0.0179</td>
<td>0.0177</td>
</tr>
<tr>
<td>6</td>
<td>0.0394</td>
<td>0.0344</td>
<td>0.0367</td>
<td>0.0344</td>
<td>0.0197</td>
<td>0.0194</td>
</tr>
<tr>
<td>7</td>
<td>0.0412</td>
<td>0.0359</td>
<td>0.0386</td>
<td>0.0359</td>
<td>0.0206</td>
<td>0.0202</td>
</tr>
<tr>
<td>8</td>
<td>0.0410</td>
<td>0.0358</td>
<td>0.0382</td>
<td>0.0358</td>
<td>0.0205</td>
<td>0.0202</td>
</tr>
<tr>
<td>9</td>
<td>0.0392</td>
<td>0.0342</td>
<td>0.0365</td>
<td>0.0342</td>
<td>0.0196</td>
<td>0.0193</td>
</tr>
<tr>
<td>10</td>
<td>0.0360</td>
<td>0.0309</td>
<td>0.0336</td>
<td>0.0309</td>
<td>0.0180</td>
<td>0.0174</td>
</tr>
<tr>
<td>11</td>
<td>0.2303</td>
<td>0.2002</td>
<td>0.2146</td>
<td>0.2002</td>
<td>0.1151</td>
<td>0.1128</td>
</tr>
<tr>
<td>12</td>
<td>0.0251</td>
<td>0.0219</td>
<td>0.0234</td>
<td>0.0219</td>
<td>0.0126</td>
<td>0.0123</td>
</tr>
<tr>
<td>13</td>
<td>0.0176</td>
<td>0.0155</td>
<td>0.0164</td>
<td>0.0155</td>
<td>0.0086</td>
<td>0.0087</td>
</tr>
<tr>
<td>14</td>
<td>0.0693</td>
<td>0.0599</td>
<td>0.0646</td>
<td>0.0599</td>
<td>0.0347</td>
<td>0.0337</td>
</tr>
<tr>
<td>15</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>

The distribution of maximum shear stress along the length of the beam is shown in Fig. 19. The effect of discretization is apparent. Figure 20 shows the percentage of computed versus VQ/It shear along the beam at the middle of each element. The broken line on Fig. 20 should extend to 200%. While this 200%
figure seems exceedingly high, reference to Fig. 20 shows that
the actual stress is quite small.

These two examples indicate that the averaging techni-
que adequately represents the shear stress at the centroid of each
layer. The principal stresses and directions are computed using
the normal and shear stresses for each layer.

The stress averaging technique contains the assumption
that each layer (except draped strands in a prestressed concrete
beam) is prismatic. This means that the area properties are con-
stant along the beam. If this assumption is violated, the shear
stresses become more approximate in proportion to the degree of
violation. This is also true for draped prestressing strands and
the concrete layers immediately adjacent to them. Special con-
sideration is given to the draped strand in applying Eq. 49, but
error is to be expected adjacent to the draped strand. Prelimi-
nary results have shown that for this case the stress averaging
technique continues to yield good results near the draped strand.

This overall approach to beam problems being reported
should give good deflection and bending stress results for those
types of nonprismatic beams which are typically analyzed with
classical beam theories such as cover plated steel beams and some
haunched beams. In these cases the nonprismatic beam would be
treated as a series of prismatic elements. This process would
obviously require some judgment and experience on the part of the
user.
2.6 Additional Considerations

Several points will now be emphasized regarding the analytical aspects discussed in this chapter and certain implications will be presented.

The procedure for developing analytic concrete stress-strain curves to be used with this analysis procedure requires the assumption of only one material property. That property is the cylinder strength, \( f'_c \). Young's modulus and tensile strength can then be calculated from the cylinder strength using any acceptable equation or conversion tables. All other material properties can then be defined using information in Section 2.2.

Comparisons with laboratory tests will be presented in Chapter 3 in which experimental values for Young's modulus were known. These experimental values were used because it was felt that they were more correct than the corresponding values which would be obtained from empirical formulas. The tensile strength of concrete must also be estimated from the compressive strength. For reinforcing steel, Young's modulus and the yield point must be assumed.

The state of prestress in a prestressed concrete beam must be established by the analyst for each individual case. This state of prestress may be modeled by supplying an initial stress field for each layer of each element or by supplying an initial stress field for only some of the layers (typically the steel layers) and also supplying a nodal force vector. Variations of
prestress due to losses must be modeled by the analyst. Those variations in prestress force which accompany the application of a short term loading are accounted for because perfect bond is assumed between the concrete and the steel. As mentioned in Section 2.4, when the prestressing is modeled by using a combination of an initial stress field and the nodal load vector, the steel stress must be adjusted for the change in stress which will occur when the nodal load vector is applied. This, and other details of input information will be covered in a User's Manual which is to be released as "User's Manual for Program BEAM", (by J. M. Kulicki and C. N. Kostem, Fritz Engineering Laboratory Report No. 378B.2 expected February 1973).

The method being presented is based on the assumption that a flexural failure will be the dominant failure made. For Concrete I-beams of the proportion used as highway bridge beams it is felt that an overloading will produce intolerable flexural damage before shear related cracking or crushing becomes a problem.

As stated in Section 2.5, approximate shear and principal stresses are calculated. These stresses are part of the output of the computer program and can be used by the analyst to judge the state of shear and principal stress in the beam as explained in Section 2.5. These stresses are computed using basically the same fundamental assumptions as are made in the ordinary theory of beams. As a further aid to the analyst, a "flag" appears in the computer output when the principal stresses exceed their respective tolerance.

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The techniques presented in this report will eventually be coupled with similar techniques being developed for reinforced concrete plates to allow the overload analysis of beam-slab bridges using prestressed concrete I-beams. Past research has shown that the torsional stiffness of the I-beams does not drastically affect the lateral distribution of load and that, in fact, a conservative distribution results from neglecting that torsional stiffness. The result is a somewhat higher bending moment in the beams near the load. This conservative estimate of the higher beam bending moments is compatible with an overload analysis. Accordingly, the torsional stiffness of the I-beams will be neglected.
3. CORRELATION WITH TESTS

3.1 Reinforced Concrete Beams

Quantitative comparisons with two under-reinforced concrete beams and qualitative comparisons of an under-reinforced and two over-reinforced concrete beams were made. They are discussed under the subheadings below. All beams had sufficient stirrups to prevent diagonal tension failures. There were no bond failures in the test results.

3.1.1 Under-Reinforced Singly Reinforced Concrete Beam

The cross-sectional layering and elemental discretization are shown in Figs. 3-A and 2-A. Figure 21 shows experimental and calculated load deflection curves for this example. The test beam was a 6 x 12 inch solid rectangular section reinforced with six No. 5 bars with an observed yield strength of 46.8 ksi. The concrete compressive strength was 5 ksi. The test beam was supported with a span of 11 feet and was subjected to third point loading. Figure 21 shows excellent agreement between the experimental and calculated curves. It should be emphasized that all the test comparisons were made after the fact so that adequate input data were usually available. Pretest predictions, on the other hand, would require assumptions for some data and could not be expected to produce as good a representation of the load deflection behavior.
Figure 21 also shows that in this case the analytic solution extends further than the test data. This is misleading. The actual ultimate load was 32.7 kips but no deflection was recorded for that load. It will be shown later that tests to complete destruction usually extend beyond computer generated results.

3.1.2 Under-Reinforced Doubly Reinforced Concrete Beam

The cross-sectional layering and elemental discretization are shown in Figs. 3-B and 2-B. Figure 22 shows experimental and calculated load deflection curves for this case. The same cross-section and test setup as in the singly reinforced example (Section 3.1.1) were used except that the concrete compressive strength was 3.9 ksi and the reinforcement consisted of two layers of two No. 5 bars each as tensile reinforcement and two No. 3 bars as compressive reinforcement. The yield strength of the steel was 54.5 ksi. Figure 22 shows excellent agreement again. As in the singly reinforced test, no deflection corresponding to the test ultimate load was recorded so that the actual upper portion of the load deflection curve is probably closer to the computed curve than Figure 22 would indicate.

This study showed an interesting feature which further demonstrated the generality of the Ramberg-Osgood Law and layering concepts. If a high value of Ramberg-Osgood "N" is used for the steel to simulate the elastic-perfectly plastic stress-strain curve the rounded knee of the analytic load deflection curve is
sharper than the test results would indicate. The agreement with the test is still good, but it can be improved by providing a slightly rounded knee on the steel-stress strain curve. This rounded knee was found experimentally in this example by a strain gage mounted on one reinforcing rod. Guidance in selecting a Ramberg-Osgood "N" was provided by auxiliary plots of elastic-perfectly plastic stress-strain curves for various values of "N".

Figure 40 shows the deflected shape of a half beam for various states of loading. The deflected shape at 12.3 percent of ultimate load corresponds to the formation of the first cracked zones. After cracking has occurred, the deflection continues to grow almost uniformly to about 80% ultimate load. The deflection then more than doubles as the load is increased from about 75% of ultimate to ultimate. This action is also shown in Fig. 41 which shows the midspan deflection versus percent of ultimate load. The initial cracking phase occurs between 12.3% and about 20% of ultimate load. Rapid increases in deflection start at about 80% of ultimate and becomes quite dramatic at over 90% of the ultimate load. This increasingly rapid growth of deflection is accompanied by more cracking and by reinforcement non-linearity.

Figure 42 shows the stress in the lower tensile reinforcement and the compressive reinforcement versus percent of ultimate load. Before first cracking the stress in the compressive reinforcement is greater than in the tensile reinforcement. Figure 3-B shows that the neutral axis of the uncracked section
is below the middle of the section so the larger compressive steel stresses are exactly as would be expected. During the first period of cracking the tensile reinforcement becomes more highly stressed and continues at a higher stress rate until it yields. This is also exactly what would be expected. The response of the tensile steel appears almost linear between 75% and 100% of ultimate load. This observation, taken alone, might seem to indicate that the steel does not yield. Referring back to Fig. 41 it can been seen that there are great increases in deflection during this load range and these would indicate correspondingly large increases in strains. Thus the almost linear response in Fig. 42 does not necessarily imply a linear stress-strain relation. The computer printout of stresses in this load range shows that the tensile reinforcement starts to yield at about 90% of the ultimate load. The effect of this yielding on deflection is seen in Fig. 41. During this same 90% to 100% ultimate load range the stress in the compressive reinforcement increases rapidly as large strain increases occur. In this example the compressive reinforcement did not yield before the beam reached its ultimate load.

3.1.3 Qualitative Curves of One Under-Reinforced and Two Over-Reinforced Beams

Figure 23 shows the effect of varying the amount of reinforcement in a simply supported singly reinforced concrete beam.
The section used here is a hypothetical 10" x 10" solid rectangle of 3 ksi concrete reinforced with 36 ksi steel. The steel area was 2\(\frac{x}{2}\), 4 and 5 square inches for curves A, B and C respectively resulting in \(q = 0.3, 0.48\) and 0.6 for the beams. "q" is the steel percentage times the ratio of \(f_y\) and \(f'\) as in ACI 318-71. Curve "A" is a balanced beam condition. It can be seen that curves B and C show typical over-reinforced behavior while curve A shows typical under-reinforced (or balanced) behavior.

Figure 23 also shows a horizontal line running through each curve. This line is at a load level ratio corresponding to an adjusted value of the ultimate load ratio which would be predicted by ultimate strength analysis techniques. The adjustment was made by multiplying the theoretical ultimate load by 1.068. This number is the average test ultimate load divided by theory ultimate load ratios for the twenty-two tests reported in P.C.A. Bulletin D-49, Table A-1\(^{13}\) which had concrete strengths between 2590 and 3550 psi. This comparison is offered in lieu of laboratory tests.

If desired, a further comparison could be made with the behavior demonstrated by curves 5-.304 and 5-.492 in Fig. 5 of P.C.A. Bulletin D-7\(^{11}\). The same behavior will be noted. It would seem that the method used here would adequately predict over-reinforced beam behavior as well as it predicts under-reinforced beam behavior. There is no conceptual reason why it should not - the use of individual layer stress-strain curves guarantees enough flexibility to handle a wide variety of problems.
3.2 Prestressed Concrete Beams

3.2.1 Solid Box Beams

The prestressed concrete box beams in this study were tested by Walther and Warner\textsuperscript{33}. The cross-sectional layering and elemental discretization are shown in Figs. 3-C and 2-C. A solid rectangular cross-section 8" by 18" was prestressed with six 7/16" diameter seven wire strands using two layers of 3 strands each. There was 4 inches of cover on the lower set and 6 inches on the upper set of strand. A stress-strain curve for the seven wire strand was included in the report. All beams were 15 feet long and were pretensioned at five days. Characteristics of the beams are summarized in Table IV.

<table>
<thead>
<tr>
<th>Beam</th>
<th>Age At Test In Days</th>
<th>( F_i )</th>
<th>( F_o )</th>
<th>( F )</th>
<th>( f'_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A7</td>
<td>38</td>
<td>96.33</td>
<td>92.87</td>
<td>87.26</td>
<td>6.140</td>
</tr>
<tr>
<td>A8</td>
<td>28</td>
<td>96.33</td>
<td>92.47</td>
<td>85.73</td>
<td>6.260</td>
</tr>
<tr>
<td>A9</td>
<td>32</td>
<td>102.15</td>
<td>98.11</td>
<td>92.53</td>
<td>6.320</td>
</tr>
<tr>
<td>A10</td>
<td>33</td>
<td>102.15</td>
<td>97.92</td>
<td>93.85</td>
<td>6.320</td>
</tr>
</tbody>
</table>

\( F_i \) = Total force in the prestressing steel just prior to transfer of the force.

\( F_o \) = Total force in the prestressing steel at the beam midspan just after transfer.
F = Total force in the prestressing steel at beam midspan just prior to testing.

\( f'_c \) = Cylinder strength on day of testing - average of 6 cylinders.

The beams were subjected to third point loading while supported to give a 9' - 0" span. The dead load of the 3' - 0" overhangs offset the dead load tensile stress in the pure moment section. Figures 24 to 27 show the load deflection behavior of each beam. Each test curve gives reasonable agreement with its corresponding analytic curve. The test data were taken in 5 kip intervals.

Table V shows a comparison of the test and calculated ultimate loads applied to each third point.

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Test</th>
<th>Calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>A7</td>
<td>49.9</td>
<td>49.0</td>
</tr>
<tr>
<td>A8</td>
<td>50.2</td>
<td>48.9</td>
</tr>
<tr>
<td>A9</td>
<td>49.8</td>
<td>48.7</td>
</tr>
<tr>
<td>A10</td>
<td>49.9</td>
<td>49.0</td>
</tr>
</tbody>
</table>

Figures 28 and 29 conclusively show the extent of agreement between analytic and experimental results. Test beams A-7 and A-8, and A-9 and A-10 were cast as identical pairs. The initial prestressing forces were identical for each pair. Figure 28
shows test beams A-7 and A-8 plotted together on the same figure with a composite analytic curve. The analytic curves for each identical pair are so close that only one curve was drawn in each case. Figure 29 shows the same information for the identical pair of A-9 and A-10. In both cases it can be seen that the analytic data fits on or between the test curves for most of the load deflection history. Furthermore, by inspecting the properties in Table IV it would be expected that the pair A-7 and A-8 would show more difference in their behavior than the pair A-9 and A-10 because of the relative age differences. The opposite phenomenon is evident in Figs. 28 and 29. It would seem that the variability in test data is on the same order as any errors in the assumptions made in developing the processes used here.

Figures 12 and 13 show the crack growth rate found during the actual test compared to the "crack zones" predicted by the computer program for the specified analytic loads shown in parenthesis. Once again the comparisons are encouraging.

3.2.2 I-Beams

The prestressed concrete I-beams used in this study were tested by Hanson and Hulsbos. The cross-sectional layering and elemental idealization are shown in Figs. 3-D and 2-D. The test setup and cross-sectional data are given in Fig. 30. Six 7/16" diameter seven wire prestressing strands were used as prestressing elements in each beam. A stress-strain curve for the strand was
included in the report. Table VI shows the prestressing data and Table VII shows the properties of concrete used. This data is taken from the report by Hanson and Hulsbos. The beams were simply supported with a clear span of 15'-0". Two concentrated loads were applied to the beam at positions which varied for groups of tests. The position of the loads is shown on the inserts of Figs. 31 through 34.

**Table VI - PRESTRESS DATA**

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Initial Prestress Force (kips)</th>
<th>Elastic</th>
<th>Inelastic</th>
<th>Total (kips)</th>
<th>Prestress Force At Test (kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-5</td>
<td>113.9</td>
<td>8.6</td>
<td>11.9</td>
<td>20.5</td>
<td>90.6</td>
</tr>
<tr>
<td>E-7</td>
<td>114.9</td>
<td>8.1</td>
<td>11.8</td>
<td>19.9</td>
<td>92.0</td>
</tr>
<tr>
<td>E-8</td>
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<td>8.1</td>
<td>11.8</td>
<td>19.9</td>
<td>92.0</td>
</tr>
<tr>
<td>E-9</td>
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<td>8.1</td>
<td>12.7</td>
<td>20.8</td>
<td>91.0</td>
</tr>
<tr>
<td>E-12</td>
<td>113.7</td>
<td>8.5</td>
<td>12.3</td>
<td>20.8</td>
<td>90.0</td>
</tr>
<tr>
<td>E-17</td>
<td>113.3</td>
<td>8.4</td>
<td>10.2</td>
<td>18.6</td>
<td>92.4</td>
</tr>
<tr>
<td>E-18</td>
<td>113.3</td>
<td>8.5</td>
<td>9.9</td>
<td>18.4</td>
<td>92.6</td>
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</table>
### Table VII - PROPERTIES OF CONCRETE

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Age (Days)</th>
<th>$f'_c$ (psi)</th>
<th>$E_c^1$ (ksi)</th>
<th>Age (Days)</th>
<th>$f'_c$ (psi)</th>
<th>$E_c^1$ (ksi)</th>
<th>$E_c^2$ (ksi)</th>
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</thead>
<tbody>
<tr>
<td>E-5</td>
<td>7</td>
<td>5530</td>
<td>3100</td>
<td>60</td>
<td>6610</td>
<td>3800</td>
<td>4600</td>
</tr>
<tr>
<td>E-7</td>
<td>7</td>
<td>5900</td>
<td>3800</td>
<td>62</td>
<td>7230</td>
<td>4100</td>
<td>4700</td>
</tr>
<tr>
<td>E-8</td>
<td>7</td>
<td>5680</td>
<td>3400</td>
<td>70</td>
<td>6970</td>
<td>4400</td>
<td>4700</td>
</tr>
<tr>
<td>E-9</td>
<td>7</td>
<td>5630</td>
<td>3500</td>
<td>74</td>
<td>7140</td>
<td>4200</td>
<td>4700</td>
</tr>
<tr>
<td>E-12</td>
<td>7</td>
<td>5590</td>
<td>3300</td>
<td>68</td>
<td>7020</td>
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<td>4700</td>
</tr>
<tr>
<td>E-17</td>
<td>7</td>
<td>5400</td>
<td>3300</td>
<td>57</td>
<td>6580</td>
<td>3800</td>
<td>4300</td>
</tr>
<tr>
<td>E-18</td>
<td>7</td>
<td>5520</td>
<td>3200</td>
<td>52</td>
<td>6640</td>
<td>3600</td>
<td>4500</td>
</tr>
</tbody>
</table>

$E_c^1$ is determined from cylinder tests, $E_c^2$ is determined from load deflection curve of test beam.

These specimens had an overhang of only 1' - 3" on each end. This was not enough to offset the dead load tensile stress of about 80 psi. The results presented in Figs. 31 through 34 are based on an adjusted tensile strength found by deducting 80 psi from the tensile strength of all layers regardless of their position in the beam. A comparative calculation was performed for beam E-12 by inputting the dead load as part of the
prestressing force nodal load vector. That force vector would not be incremented with the test load. This had the effect of eliminating pure bending and requiring more cycles of cracking-crushing analysis because each layer had a slightly different stress from the combined dead load and live loads. While this is a more realistic situation than having groups of layers with the same stress, the net effect of this extra consideration was less than a 1% change in the load deflection behavior. Execution time, however, was increased considerably. The refined calculations reached an enforced time limit after 151 seconds of central processor time on the CDC 6400 digital computer of the Lehigh University Computing Center. At that stage of the analysis it probably would have required another 30 or 40 seconds to reach completion. These latter figures are, of course, estimates based on experience with the program. The more approximate analysis required only 123 seconds for complete execution. Hence, the refinement would require about 50% more execution time for an increase in accuracy which has no engineering significance. Based on this conclusion it was decided to run all analytic load deflection curves with the adjusted tensile stress instead of including the dead load.

Figures 31 through 34 show very good agreement with the test curves for all beams. Each of the analytic curves shows a pronounced discontinuity which was not evident in the previous examples. This is a result of the cross-sectional layering used
and the approximation for dead load tensile stress just explained which eliminated the moment gradient causing a larger portion of the analytic beam to reach a cracking criteria at a given time under the given loading than was true for the physical beam.

Figure 3-D shows that the fourth layer from the top and bottom contains by far the largest area. Examination of computer output of stresses showed that in each case the discontinuity corresponds to the unloading of the tensile stresses in this layer. This also demonstrates the rather obvious fact that the fewer layers used, the more discontinuous and therefore more approximate the results become.

Table VII shows the two values of Young's modulus recorded for each beam. The question of which value to use for input is valid but somewhat academic. It is valid because different values for the elastic modulus will change the slope of the load deflection curves. It is academic because the problem of predicting the behavior of untested beams would have to rely on an estimate which would be more approximate than either value given in Table VI. Calculations for beams E-5 and E-7 use \( E_1 \) while all others used \( E_\alpha \). Looking at Table VII it can be seen that the difference between \( E_1 \) and \( E_\alpha \) is about 15%. Increasing the deflections in Fig. 31 by 15% and decreasing the deflections in Figs. 32, 33 and 34 by 15% actually has relatively little effect on the overall agreement of the test and calculated curves. In fact it appears that beams E-17 and E-18, Fig. 33, would
benefit by such a reduction in deflection. The gradual non-linear behavior of the beams contributes to mitigating the effect of a small change in elastic modulus.

In some cases the test curves extend beyond the plots shown in Figs. 31 through 34. The ultimate test loads are shown along with the maximum computer generated loads in Table VIII. It can be seen that in some cases the errors are significantly larger than those shown in Table V. Both the extension of the curves past computer output and the larger ultimate load discrepancies are probably explained by the fact that some of these tests were carried to utter destruction. The accompanying very large deflections probably caused the change in the geometry of the prestressing strands to become significant.

Table VIII

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Test</th>
<th>Calculated</th>
</tr>
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<tbody>
<tr>
<td>E-5</td>
<td>42.0</td>
<td>39.2</td>
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<tr>
<td>E-7</td>
<td>41.1</td>
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<tr>
<td>E-8</td>
<td>41.2</td>
<td>39.3</td>
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<tr>
<td>E-9</td>
<td>41.2</td>
<td>38.9</td>
</tr>
<tr>
<td>E-12</td>
<td>41.2</td>
<td>39.0</td>
</tr>
<tr>
<td>E-17</td>
<td>38.0</td>
<td>38.2</td>
</tr>
<tr>
<td>E-18</td>
<td>38.7</td>
<td>38.2</td>
</tr>
</tbody>
</table>

Figure 35 shows the growth of "cracked zones" in one I-beam. These zones are at a stress state which has reached the
cracking criteria. The exact location, number and spacing of the cracks remains undetermined. Broms has suggested, however, that the space between cracks can be approximated as two times the concrete cover of the reinforcement to any edge or to the next piece of reinforcement; and that the average crack width is $2t\varepsilon_s$ where $\varepsilon_s$ is the average strain in the reinforcing and $t$ is the minimum cover. This crack width would presumably be measured at the steel location.

Figure 36 shows the deflected shape of a half beam for various states of loading starting with the prestress camber and continuing to 100% of the calculated ultimate load. The figure corresponds to beam E-7 and shows the catastrophic effect of large overloads. First cracking occurs at 59% of the computed ultimate load. The next 20% of ultimate load more than doubles the deflection. Adding another 16% of load more than doubles the deflection again; in fact the deflection is 8.44 times the deflection at first cracking.

Figure 37 shows the calculated midspan deflection versus the percent of computed ultimate load for beam E-7.

It can be seen that the sudden increase in deflection occurs between 70% and 73% and not at the "first cracking" load of 59.4% of ultimate. This delay is a direct consequence of the unloading leg of the tensile stress-strain curve and the layering used. This delayed behavior is also exemplified in all the load deflection curves for I-beams. Figure 31 shows the load
deflection curve for beam E-7. It is seen that the experimental curve had first cracking at 25 kips and some delay until the non-linearity became significant. This delay was not as long as in the calculated load deflection curve. Part of the difference between the experimental and calculated behavior is the large area in the fourth layer from the top and bottom. This point has already been discussed and is believed to also explain the somewhat longer delay in the calculated results. Similar behavior is also seen in the box beams.

Figure 38 shows the steel stress in the lowest strands versus the percent of ultimate load. The results for the midspan section and a section 45 inches from each end are shown. The midspan curve shows an increase in steel stress corresponding to the growth of cracking shown in Fig. 37. An enlarged plot of the Ramberg-Osgood stress-strain curve for this strand shows that the curve reached a horizontal plateau at 230 ksi. This is also the computed steel stress at the computed ultimate load.

3.3 Example Using a Steel Beam

As discussed in section 2.2 some examples using a stress-strain curve like that of mild steel were also explored for a fixed ended I-shape. The load deflection curves are shown in Fig. 16. A discussion of this figure and suitable comments about the economy of solution are found in section 2.2.
4. CONCLUSIONS AND FUTURE RESEARCH

From this work it can be concluded that the developed analytic tool adequately describes the flexural behavior of beams. Comparisons were made with actual load deflection curves of two reinforced and eleven prestressed concrete beams. As previously presented, these beams contain a wide range of concrete strengths although most are between 5000 and 7000 psi. This is of course the type of concrete most commonly found in the prestressed concrete bridge beams for which the research was intended. In all cases the agreement between the analytic and test results were quite good. The conclusion from this research is that the methods used here for modeling the beam material, accounting for cracking and crushing, and iterating to satisfactory convergence regardless of the stress-strain curve produce acceptable results. Both the load deflection curve and the spread of cracked and plastified regions are adequately reproduced.

There remains, however, large areas of application yet to be researched. They fall into the following classes:

A. Areas within the framework of the current research program and considered as refinements to the basic process.

1. Continued comparison with laboratory tests.

2. Research into the effect of mesh size in layering and elemental discretization.
3. Research into the effect of reducing accuracy requirements in displacement convergence and stress tolerances on the speed and accuracy of solution.

B. The following research areas, while not covered within the scope of the current research program, are feasible as an extension of the reported method.

1. Research into applications as a possible design tool for continuous concrete beams utilizing limit design. It is conceivable that a slightly modified version of the computer program written for this research could be used along with trial contained concrete stress-strain curves to find the necessary stirrup spacing to meet rotational requirements needed to reach a given limit load.

2. Research into applications with beam columns.

3. Research into applications with beams, beam columns and initially imperfect columns having other types of initial stress states such as residual stresses in metal beams. The residual stress could be varied across the cross-section and along the length.

4. More research into the shape of the tensile stress-strain curve and its relation, if any, to the compressive stress-strain curve would be useful.
5. **ACKNOWLEDGMENTS**

This study was conducted in the Department of Civil Engineering and Fritz Engineering Laboratory, under the auspices of the Lehigh University Office of Research, as a part of a research investigation sponsored by the Pennsylvania Department of Transportation, Federal Highway Administration and the National Science Foundation.

The basic research planning and administrative coordination in this investigation were in cooperation with the following individuals representing the Pennsylvania Department of Transportation: Mr. B. F. Kotalik, Bridge Engineer; Mr. H. P. Koretzky, and Mr. Hans Streibel, all from the Bridge Engineering Division; and Messrs. Leo D. Sandvig, Director; Wade L. Gramling, Research Engineer; Kenneth L. Heilman, Research Coordinator; all from the Bureau of Materials, Testing and Research.

The authors would like to thank Mr. William S. Peterson for his encouragement and constructive criticism during the conduct of the reported investigation. The authors would also like to acknowledge the support of Dr. D. A. VanHorn, Chairman, Department of Civil Engineering, Lehigh University. The appreciation is also extended to the staff of Lehigh University Computing Center, and Mr. John H. Morrison in particular, for their cooperation.

The manuscript was typed by Mrs. Ruth Grimes, and the figures were prepared by John M. Gera and Mrs. Sharon Balogh.
6. FIGURES
Fig. 1-A The Constant Stress Method

Fig. 1-B The Constant Strain Method
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Fig. 3 Layer Discretization
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Fig. 5 Generalized Displacements

\[ \theta_y = -\frac{\delta w}{\alpha x} \]
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Fig. 8 Analytic and "Actual" Stress-Strain Curves
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Fig. 11 Analytic Tensile Stress-Strain Curves
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Fig. 13 Crack Growth in a Prestressed Solid Box Beam
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Fig. 18 Shear Distribution Along a Beam
Fig. 19 Percentage of Actual Shear Along a Beam

Fig. 20 Shear Distribution Along a Beam
Fig. 21 Load Deflection Curves for a Singly Reinforced Concrete Beam

Analytic \( V_u = 32.7^k \) (calc.)
Test \( V_u = 32.0^k \) (test)
Fig. 22 Load Deflection Curves for a Doubly Reinforced Concrete Beam
Fig. 23 Under-reinforced and Over-reinforced Concrete Beams
Fig. 24 Load Deflection Curves for a Prestressed Concrete Solid Box Beam

\[ V_u = 48.9^k \text{ (calc.)} \]
\[ V_u = 49.9^k \text{ (test)} \]

- ○ = Analytic
- ● = Test
Fig. 25 Load Deflection Curves for a Prestressed Concrete Solid Box Beam

\[ V_u = 49.4 \text{ (calc)} \]
\[ V_u = 50.2 \text{ (test)} \]
Fig. 26 Load Deflection Curves for a Prestressed Concrete Solid Box Beam

- $V_u = 48.7^k$ (calc.)
- $V_u = 49.8^k$ (test)

$V$ (kips)

MID-SPAN DEFLECTION (inches)

- Analytic
- Test
Fig. 27 Load Deflection Curves for a Prestressed Concrete Solid Box Beam
Fig. 28 Comparisons of "Identical" Prestressed Concrete Solid Box Beams
Fig. 29 Comparisons of "Identical" Prestressed Concrete Solid Box Beams
Fig. 30 Properties of Prestressed Concrete I-Beams
Fig. 31 Load Deflection Curves for Prestressed Concrete I-Beams
Fig. 32 Load Deflection Curves for Prestressed Concrete I-Beams
Fig. 33 Load Deflection Curves for Prestressed Concrete I-Beams
Fig. 34 Load Deflection Curves for a Prestressed Concrete I-Beam
Fig. 35 Growth of Analytic Crack Zones
Fig. 35 (Continued)

$V/V_{ULT} = 0.833$

$V/V_{ULT} = 0.932$

$V/V_{ULT} = 1.000$
Fig. 36 Deflected Half-Shapes for a Prestressed Concrete I-Beam

DEFLECTION (inches)
Fig. 37 Mid-Span Deflection Versus % Ultimate Load for a Prestressed Concrete I-Beam
Fig. 38 Steel Stress Versus % Ultimate Load for a Prestressed Concrete I-Beam
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Fig. 41 Mid-Span Deflection Versus % Ultimate Load for a Doubly Reinforced Concrete Beam
Fig. 42 Steel Stress Versus % Ultimate Load for a Doubly Reinforced Concrete Beam
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