Stability of Steel Structures

Inelastic Stability Analysis of Biaxially Loaded Beam-Columns by the Finite Element Methods

by

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Summary

A finite element method for stability analysis of an initially crooked column subjected to biaxially applied end moments and lateral loads is presented. An expression for the total potential energy of a thin-walled beam-column, including the effect of second-order deformation, is first given and is used to derived the required element stiffness matrix. The governing equations assembled from the element stiffness matrices and the force vector are then solved numerically on a computer. In the inelastic range these equations are solved by an iterative scheme for a series of successively increasing loads until certain convergence criteria are satisfied. Any divergence occurring during the iterative calculations is an indication that the applied load has exceeded the maximum load of the column. The method is applied to determine the theoretical load-deformation relationships of some test columns reported in the literature. Very close agreement between the theoretical and the experimental results is found.

1. Introduction

The problem of a beam-column subjected to loads causing biaxial bending is one of the most general problems in the study of strength of structural members and much research effort has been devoted to develop methods for its solution. The problem becomes very complicated, if it is desired to include all the effects of geometrical and material nonlinearities in the solution. For this reason, numerical methods, such as the finite segment method,1,2 the finite difference method,3,4 and the finite element method,5,6 have been employed by previous researchers to obtain solutions. A summary of these studies is contained in Ref. 7. Among these, the finite element method is generally regarded as the most versatile and can be applied to columns with various boundary and loading conditions. In 1979, a
research was initiated at Tong-Ji University in an effort to develop a comprehensive finite element method which can be applied to a variety of structural members including axially loaded columns, beams, and beam-columns. The work has subsequently been continued at Lehigh University with emphasis on inelastic analysis. The method is now fully developed and the associated computer program thoroughly tested. This paper is intended to be a brief introduction to this work.

In developing the governing equations for a deformed beam-column, a fixed coordinate system \(\mathbf{xyz}\) is adopted (Fig. 1a). A typical element of the members and the forces and moments acting at its ends are shown in Fig. 1b (all quantities as shown are positive). The element is subjected to the following forces or moments: (1) axial force, \(P\), (2) bending moments about the \(x\) and \(y\) axes, \(M_x\) and \(M_y\), (3) bi-moment, \(B_w\), (4) end torque \(M_z\), (5) distributed torque, \(m_z\), (6) concentrated lateral loads, \(P_x\) and \(P_y\), and (7) distributed lateral loads, \(q_x\) and \(q_y\). The last three (not shown in Fig. 1b) are applied along the length of the segment. Figure 1c shows the positive directions of the end forces and moments adopted in the finite element formulation.

![Fig. 1 Forces and Moments Acting on Element](image-url)
In addition to oxyz, a local coordinate system, c~n~z, is used to define the orientation of each section along the member. This system is used to evaluate the internal forces and moments existing at the section.

The displacements in the x, y, and z directions are represented, respectively, by u, v, and w, and the angle of twist is θ. The initial crookedness in the x and y directions are u₀ and v₀, respectively, and the initial angle of twist is θ₀. The solution of each problem establishes the relationship between the applied loads and the resulting displacement.

2. Total Potential Energy of a Thin-Walled Beam Column

The required element stiffness matrix can be derived by applying the Principle of Minimum Potential Energy. The total potential energy of a thin-walled element of length L and subjected to the combined loads as described previously is given by:

\[
\Pi = \frac{1}{2} \int \left[ \frac{E A}{x} \left( u - u_o \right)^2 + \frac{E I_x}{x} \left( v - v_o \right)^2 + \frac{E I_w}{w} \left( \theta - \theta_o \right)^2 
+ \frac{G I_y}{y} \left( \theta - \theta_o \right)^2 - P u_x^2 - P v_y^2 - P \theta_w^2 + M \beta x \theta_x^2 + M \beta y \theta_y^2
+ 2P x \theta_y^2 + 2M v_x \theta_x^2 + \left( Q_x \beta_x + Q_y \beta_y + M \beta_y \right) \theta_y + 2Q_x v_x \theta
- 2Q_y u_y - 2Q_y v_x - 2M \theta_x - q_x (x - a_x)^2
- q_y (y - a_y) \theta_y^2 \right] dZ - \frac{P x}{2} (x - b_x) \theta_x^2 - \frac{P y}{2} (y - b_y) \theta_y^2
- \left[ -P w + Q_x u + Q_y v - M u_x^2 - M v_y^2 + M \theta_x - B \theta_y \right] \frac{L}{2} \left( \theta - \theta_o \right) \right] \right) \right)
\]

in which E and G are the Young's modulus and the shear modulus, Iₓ and Iᵧ are the moments of inertia about the
x and y axes, $I_w$ is the warping moment of inertia, and $I_k$ is the St. Venant torsion constant. $\rho_0^2$, $\beta_x$, $\beta_y$, $\beta_\omega$, and $\bar{R}$ are defined as follows:

\[ r_0^2 = \frac{I_x + I_y}{A} + x_\circ^2 + y_\circ^2 \]  
\[ \beta_x = \int \frac{y(x^2 + y^2) \, dA}{I_x} - 2y_\circ \]  
\[ \beta_y = \int \frac{x(x^2 + y^2) \, dA}{I_y} - 2x_\circ \]  
\[ \beta_\omega = \int \frac{\omega(x^2 + y^2) \, dA}{I_\omega} \]

and

\[ \bar{R} = \int \sigma_r (x^2 + y^2) \, dA \]

in which $x_\circ$ and $y_\circ$ are the coordinates of the shear center, and $\sigma_r$ is the residual stress. The distances $a_x$ and $a_y$ define the location where the distributed loads $q_x$ and $q_y$ are supplied. Similarly, $b_x$ and $b_y$ define the location where the concentrated loads $P_x$ and $P_y$ are applied. The quantities $M_x$, $M_y$, $B_\omega$, $M_\omega$, $Q_x$ and $Q_y$ in the integrated are the internal bending moments, the bi-moment, the warping torsional moment and the shearing forces, which are all functions of $z$.

The detailed derivation of Eq. (1) is presented in a separate report.

### 3. Element Stiffness Matrices

In the development of the method of analysis, the beam-column is divided into $n$ elements along its length. Assume that the displacements $u$, $v$ and $\theta$ and the initial crookedness $u_\circ$, $v_\circ$ and $\theta_\circ$ of each element can be approximated by a cubic function

\[ f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \]
in which \(a_0, a_1, a_2\) and \(a_3\) are the coefficients to be determined from the displacements at the ends of the element. Substituting these displacements into the expression for the total potential energy of Eq. (1) and applying the Principle of Minimum Potential Energy with respect to each displacement, the following equation relating the generalized element displacements \(\Delta_e\) to the generalized element forces \(F_e\) is obtained.

\[
[[K_{fe} + K_{ge}]]\{\Delta_e\} = \{F_e\}
\]

(8)
in which \([K_{fe}]\) is the conventional first-order stiffness matrix and \([K_{ge}]\) is the geometric stiffness matrix. They are given by:

\[
[K_{fe}] = \begin{bmatrix}
EI_y[z_{22}] & 0 & 0 \\
0 & EI_x[z_{22}] & 0 \\
0 & 0 & EI_w[z_{22}] + GI_k[z_{11}]
\end{bmatrix}
\]

(9)

and

\[
[K_{ge}] = \begin{bmatrix}
K_{gel1} & 0 & K_{gel3} \\
0 & K_{gel2} & K_{gel3} \\
K_{gel2} & K_{gel3} & K_{gel3}
\end{bmatrix}
\]

(10)

In Eq. (10), the various sub-matrices are defined as follows:

\[
K_{gel1} = -P[Z_{11}]
\]

\[
K_{gel3} = -Py_o[Z_{11}] - [Z_{11}M_y]^T - [Z_{10}Q_y]
\]

\[
K_{gel2} = -P[Z_{11}]
\]

\[
K_{gel2} = Py_o[Z_{11}] - [Z_{11}M_y] + [Z_{10}Q_y]^T
\]

\[
K_{gel3} = Py_o[Z_{11}] + [Z_{11}M_y] + [Z_{10}Q_y]^T
\]

\[
K_{gel3} = (-Pr_o^2 + \ddot{\vec{R}})[Z_{11}] + \dot{\beta}_x[Z_{11}M_y] + \dot{\beta}_y[Z_{11}M_y]
\]

\[
+ \beta_\omega[Z_{11}M_x] + \frac{\dot{x}}{2}[Z_{10}Q_y] + \frac{\dot{y}}{2}[Z_{10}Q_y]^T
\]

\[
+ \frac{\beta_\omega}{2}[Z_{10}Q_x] + \frac{\beta_y}{2}[Z_{10}Q_x]^T + \frac{\beta_\omega}{2}[Z_{10}M_x]
\]

\[
+ \frac{\beta_\omega}{2}[Z_{10}M_x]^T + (a_y - y_o)[Z_{00}Q_y] + [P_{xy}]
\]
The displacement vector in Eq. (8) is
\[
\{\Delta_e\} = [u_m, u'_m, u_{m+1}, u'_{m+1}, v_m, v'_m, v_{m+1}, v'_{m+1}]^T
\tag{12}
\]
and the force vector is
\[
\{F_e\} = q_x \begin{bmatrix} L/2 \\ L^2/12 \\ L/2 \\ -L^2/12 \end{bmatrix} + q_y \begin{bmatrix} M_{ym} \\ M_{ym} \\ Q_x(m+1) \\ M_x(m+1) \end{bmatrix} + EI_y[Z_{22}]\{u_o\}
\tag{13}
\]
\[
\{F_e\} = q_y \begin{bmatrix} L/2 \\ L^2/12 \\ L/2 \\ -L^2/12 \end{bmatrix} + q_y \begin{bmatrix} M_{ym} \\ M_{ym} \\ Q_y(m+1) \\ M_x(m+1) \end{bmatrix} + EI_x[Z_{22}]\{v_o\}
\tag{13}
\]
\[
\{F_e\} = m_z \begin{bmatrix} L/2 \\ L^2/12 \\ L/2 \\ -L^2/12 \end{bmatrix} + m_z \begin{bmatrix} M_{zm} \\ B_{zm} \\ M_z(m+1) \\ B_{zm}(m+1) \end{bmatrix} + EI_\omega[Z_{22}]\{\theta_o\}
\tag{13}
\]

The matrices on the right-hand side Eqs. (9), (11) and (13) are given as follows:
\[
[Z_{22}] = \begin{bmatrix}
\frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\
\frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\
-\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^2} & -\frac{6}{L^2} \\
\frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L}
\end{bmatrix}
\tag{14a}
\]
\[
[Z_{11}] = \begin{bmatrix}
\frac{6}{5L} & \frac{1}{10} & -\frac{6}{5L} & \frac{1}{10} \\
\frac{1}{10} & \frac{2}{15L} & -\frac{1}{10} & -\frac{L}{30} \\
-\frac{6}{5L} & -\frac{1}{10} & \frac{6}{5L} & -\frac{1}{10} \\
\frac{1}{10} & -\frac{L}{30} & -\frac{1}{10} & 2\frac{15L}{5L}
\end{bmatrix}
\] (14b)

\[
[Z_{10}] = \begin{bmatrix}
-\frac{1}{2} & -\frac{L}{10} & -\frac{1}{2} & \frac{L}{10} \\
\frac{L}{10} & 0 & -\frac{L}{10} & \frac{L^2}{60} \\
\frac{1}{2} & \frac{L}{10} & \frac{1}{2} & -\frac{L}{10} \\
-\frac{L}{10} & -\frac{L^2}{60} & \frac{L}{10} & 0
\end{bmatrix}
\] (14c)

\[
[Z_{11M_x}] = [T]^T \int M_x[Z']^T[Z']dZ[T]
\]
\[
[Z_{11M_y}] = [T]^T \int M_y[Z']^T[Z']dZ[T]
\]
\[
[Z_{11B_\omega}] = [T]^T \int B_\omega[Z']^T[Z']dZ[T]
\]
\[
[Z_{10Q_x}] = [T]^T \int Q_x[Z']^T[Z]dZ[T]
\]
\[
[Z_{10Q_y}] = [T]^T \int Q_y[Z']^T[Z]dZ[T]
\]
\[
[Z_{10M_\omega}] = [T]^T \int M_\omega[Z']^T[Z]dZ[T]
\]
\[
[Z_{00q_x}] = [T]^T \int q_x[Z]^T[Z]dZ[T]
\]
\[
[Z_{00q_y}] = [T]^T \int q_y[Z]^T[Z]dZ[T]
\]

\[
:Z = [1, z, z^2, z^3]^T
\] (14e)

\[
:Z' = [0, 1, 2z, 3z^2]^T
\] (14f)
\[
[p_{x,y}] = \begin{bmatrix}
-P_{xm}(b_{xm} - x_o) & 0 & 0 & 0 \\
-P_{ym}(b_{ym} - y_o) & 0 & 0 & 0 \\
0 & -P_{x(m+1)}(b_{x(m+1)} - x_o) & 0 & 0 \\
0 & 0 & +P_{y(m+1)}(b_{y(m+1)} - y_o) & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]

For the case of uniformly distributed \( q_x \) and \( q_y \),

\[
[Z_{11} M_x] = \frac{1}{2}(M_{xm} - M_{x(m+1)})[Z_{11}] + \frac{1}{2}(-Q_{ym} + Q_{y(m+1)})[Z_{11}L] - \frac{1}{2}Q_{y(m+1)L}[Z_{11}] - \frac{1}{2}q_y L^2[Z_{11}] + q_y L[Z_{11}Z] - q_y[Z_{11}Z^2]
\]

(15)

\[
[Z_{11} M_y] = \frac{1}{2}(M_{ym} - M_{y(m+1)})[Z_{11}] + \frac{1}{2}(-Q_{xm} + Q_{x(m+1)})[Z_{11}L] - \frac{1}{2}Q_{x(m+1)L}[Z_{11}] - \frac{1}{2}q_x L^2[Z_{11}] + q_x L[Z_{11}Z] - q_x[Z_{11}Z^2]
\]

\[
[Z_{10} Q_x] = \frac{1}{2}(-Q_{xm} + Q_{x(m+1)})[Z_{10}] + q_x L[Z_{10}] - 2q_x[Z_{10}Z]
\]

\[
[Z_{10} Q_y] = \frac{1}{2}(-Q_{ym} + Q_{y(m+1)})[Z_{10}] + q_y L[Z_{10}] - 2q_y[Z_{10}Z]
\]

\[
[Z_{00} q_x] = q_x[Z_{00}]
\]

\[
[Z_{00} q_y] = q_y[Z_{00}]
\]

in which,

\[
[Z_{11}Z] = \begin{bmatrix}
\frac{3}{5} & \frac{L}{10} & -\frac{3}{5} & 0 \\
\frac{L}{10} & \frac{L^2}{30} & -\frac{L}{10} & -\frac{L^2}{60} \\
-\frac{3}{5} & -\frac{L}{10} & \frac{3}{5} & 0 \\
0 & -\frac{L^2}{60} & 0 & \frac{L^2}{10}
\end{bmatrix}
\]

(15a)
\[
\begin{bmatrix}
\frac{12}{35} & \frac{1}{14} & -\frac{12}{35} & -\frac{1}{35}^2 \\
\frac{1}{14}^2 & \frac{2}{105} & -\frac{1}{14}^2 & -\frac{1}{77}^3 \\
-\frac{12}{35} & -\frac{1}{14}^2 & \frac{12}{35} & \frac{1}{35}^2 \\
-\frac{1}{35}^2 & -\frac{1}{70} & \frac{1}{35} & \frac{3}{35}^3 \\
\end{bmatrix}
\]  \hspace{1cm} (15b)

\[
\begin{bmatrix}
\frac{13}{70} & \frac{3}{70} & -\frac{11}{35} & \frac{2}{35}^2 \\
\frac{1}{105} & -\frac{1}{210} & -\frac{31}{420} & \frac{1}{84}^3 \\
\frac{13}{70} & \frac{3}{70} & \frac{11}{35} & -\frac{2}{35}^2 \\
-\frac{11}{420} & -\frac{1}{210} & \frac{23}{210} & -\frac{1}{210}^3 \\
\end{bmatrix}
\]  \hspace{1cm} (15c)

\[
\begin{bmatrix}
\frac{13}{35} & \frac{11}{210} & \frac{9}{70} & -\frac{13}{420}^2 \\
\frac{11}{210} & \frac{1}{105} & \frac{13}{420} & -\frac{1}{140}^3 \\
\frac{9}{70} & \frac{13}{420} & \frac{13}{35} & -\frac{11}{210}^2 \\
-\frac{13}{420} & -\frac{1}{140} & \frac{11}{210} & \frac{1}{105}^3 \\
\end{bmatrix}
\]  \hspace{1cm} (15d)
4. Governing Equation and Its Solution

The element stiffness matrices are assembled to form the global stiffness matrices \([K_f]\) and \([K_g]\) for use in the governing equation of a structural member.

\[
[K_f] \{\Delta\} - \{\Delta_o\} + [K_g] \{\Delta\} = \{W\}
\]  

(16)
in which \(\{\Delta\}\) is the member displacement vector and \(\{W\}\) the load vector. The specified boundary conditions are already incorporated in Eq.(16). Within the elastic range, \([K_f]\) is constant, but \([K_g]\) is a function of the applied load. Eq.(16) can be readily solved once the load level is specified. In the inelastic range, however, direct solution of Eq.(16) is very difficult, because \([K_f]\) is no longer constant. Denoting \([K_{fi}]\) and \([K_{gi}]\) as the stiffness matrices of the member when it is subjected to the ith specified load \(\{W_i\}\), Eq. (16) becomes

\[
[K_{fi}] \{\Delta_i\} - \{\Delta_o\} + [K_{gi}] \{\Delta_i\} = \{W_i\}
\]  

(17)
in which \(\{\Delta_i\}\) is the displacements caused by the load. \([K_{fi}]\), which depends on the extent of yielding in the cross sections, cannot be properly evaluated without knowing the magnitude of the forces acting at the sections. But these forces are usually not known until Eq. (17) is solved with the correct \([K_{fi}]\). The evaluation of \([K_{fi}]\) is, therefore, a complicated task.

In this study, an iterative approach attempting to satisfy equilibrium requirements between the external and internal forces is used to obtain solutions to Eq. (17). Recognizing that \([K_{gi}]\) is not affected by cross-sectional yielding (see Eq.(10)) and if it is assumed that \([K_{fi}]\) remains constant and is equal to \([K_f]\), the solution of Eq.(17) will give a load \(\{W_i\}^*\), which is higher than the specified load \(\{W_i\}\) for the same \(\{\Delta_i\}\). This load satisfies the equation

\[
[K_f] \{\Delta_i\} - \{\Delta_o\} + [K_{gi}] \{\Delta_i\} = \{W_i\}^*
\]  

(18)
A comparison of Eqs. (17) and (18) leads to the following expression for \(\{W_i\}^*\)

\[
\{W_i\}^* = \{W_i\} + [[K_f] - [K_{fi}] \{\Delta_i\} - \{\Delta_o\}]
\]  

(19)
In the above equation, \([K_{fi}]\) is unknown, but
\([K_{fi}][\{\Delta_i\} - \{\Delta_0\}]\) may be regarded as a set of nodal forces corresponding to the true internal forces which can be determined by integrating the stresses over the cross sections. On the other hand, \([K_f][\{\Delta_i\} - \{\Delta_0\}]\) may be considered as a set of forces which can be determined by considering the external equilibrium of the member. These forces can, therefore, be treated as external forces. The second term on the right-hand side of Eq. (18) represents the difference between the external and internal forces. Because of the dependence of \([K_{fi}]\) and \(\{W_i\}\) on \(\{\Delta_i\} - \{\Delta_0\}\), the solution of Eq. (19) requires iterative calculations. The modified Newton-Raphson method is adopted for this purpose as illustrated in Fig. 2.

![Diagram](image)
In each cycle of iteration the difference between the external and internal forces is checked and the member is brought back to equilibrium by adding an extra set of nodal forces \( \{W_{in}\} \) (at the end of \( n \)th iteration). For the load vector \( \{W_i\} \), the final displacement vector \( \{\Delta_i\} \) must satisfy the condition that the difference between the external and internal forces is less than certain allowable tolerance (epsilon or \( \text{EPS} \) as indicated in Fig. 2). The final \( \{\Delta_i\} \) is given by
\[
\{\Delta_i\} = \{\Delta_{i1}\} + \{d_{i1}\} + \{d_{i2}\} + \ldots + \{d_{in}\} \quad (20)
\]
which is produced by the fictitious load
\[
\{W_{in}^*\} = \{W_i\} + \{W_{i1}\} + \{W_{i2}\} + \ldots + \{W_{in}\} \quad (21)
\]
A systematic procedure has been developed to check for balance between the external and internal forces and to determine the required load increment \( \{W_{in}\} \). Figure 3 shows the forces to be added to the nodal points and the manner in which the unbalanced torque is distributed. The details of the procedure can be found in Ref. 8.
The load increment \( w_{in} \) becomes smaller and smaller as the calculation continues and the displacement tend to converge to a definite value at the end of the iteration process. Any divergence occurring during the calculation is an indication that the load has exceeded the ultimate load of the member. The last load for which convergence can be obtained have been found to be good estimate of the true ultimate.

5. Numerical Results

The method of analysis has been applied to develop predictions for the load-displacement relationships of a series of biaxially loaded test columns. Table 1 is a summary of the dimensions of these columns, the measured end eccentricities, and the experimental and predicted ultimate loads. The theoretical predictions are based on the measured initial crookedness, residual stresses, and average yield stress of the web and flange. In developing the predictions, it has been found that good accuracy can be achieved using only two elements. Solutions using four element give essentially the same results. A comparison of the loads listed in the last two columns of the table indicates that the theory provides good predictions of the ultimate strength of the test specimens. Further evidence of close correlation between the theory and the test is given in Figs. 4 and 5. The theoretical P-u, P-v, and P-\( \theta \) curves of Specimen No. 7 are compared with the test results in Fig. 4. The theory predicts very well the load-deformation behavior of this initially crooked column. Figure 5 shows the load vs. strain relationships of at four locations where strain gage readings were taken during test. Good correlation can also be observed.
Table 1: Comparison of Experimental and Theoretical Ultimate Loads of H Columns

<table>
<thead>
<tr>
<th>Spec. No.</th>
<th>Spec. Size (mm)</th>
<th>e_x (mm)</th>
<th>e_y (mm)</th>
<th>P_u (ton)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>130 130 11 7 3050</td>
<td>-19.1</td>
<td>-66.5</td>
<td>28.4</td>
</tr>
<tr>
<td>4</td>
<td>164 155 11 8 2440</td>
<td>-42.4</td>
<td>-74.9</td>
<td>39.1</td>
</tr>
<tr>
<td>5</td>
<td>127 127 13 10 2440</td>
<td>-59.9</td>
<td>-82.3</td>
<td>22.5</td>
</tr>
<tr>
<td>7</td>
<td>160 127 12 8 2440</td>
<td>-23.4</td>
<td>-72.9</td>
<td>34.7</td>
</tr>
<tr>
<td>12</td>
<td>128 102 11 7 3050</td>
<td>-20.6</td>
<td>-71.6</td>
<td>23.1</td>
</tr>
<tr>
<td>13</td>
<td>105 102 9 8 3050</td>
<td>-12.7</td>
<td>-67.8</td>
<td>20.9</td>
</tr>
<tr>
<td>16</td>
<td>128 129 9 7 3050</td>
<td>-3.6</td>
<td>-2.8</td>
<td>42.5</td>
</tr>
</tbody>
</table>

* The value above the line is the eccentricity at the upper end of specimen and that below the line is the eccentricity at the lower end.

Fig. 4 Load-Deformation Relationships of H Column
This method has also been used to analyze single angle columns loaded through end gusset plates. The eccentricity of the applied load causes biaxial bending in such columns. Another factor to be considered is that the restraints at the ends of the member do not usually act in the principal-axis directions of the cross section. Also, the stiffness of the restraints may change because of yielding. The method presented in this paper, however, can be easily adapted to such situations without difficulty. To illustrate this, numerical calculations have been performed on a series of angle columns tested at Washington University. The columns were simply supported in the direction perpendicular to the gusset plate and fixed along the plate. Table 2 gives the ultimate loads from the theoretical analysis and from the tests. Remarkably good agreement may be observed. The details of this study can be found in a forthcoming publication.
Table 2: Comparison of Experimental and Theoretical Ultimate Loads of Angle Columns.

<table>
<thead>
<tr>
<th>Spec. Size (mm)</th>
<th>sx (mm)</th>
<th>sy (mm)</th>
<th>( L/r \min )</th>
<th>( E )</th>
<th>Fx/Fy Test</th>
<th>P/Py Theory</th>
<th>Load Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>L 51x51x6</td>
<td>10.4</td>
<td>-20.4</td>
<td>2.69</td>
<td>0.15</td>
<td>0.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L 51x51x6</td>
<td>10.4</td>
<td>-20.4</td>
<td>1.87</td>
<td>0.24</td>
<td>0.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L 51x51x6</td>
<td>10.4</td>
<td>-20.4</td>
<td>1.18</td>
<td>0.35</td>
<td>0.37</td>
<td></td>
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6. Conclusions

A general finite element method for stability analysis of biaxially load thin-walled beam-columns has been presented and applied to develop theoretical predictions of the response of some H and angle columns, which have been tested in previous investigations. The method is a versatile one and can be used to solve a variety of stability problems of structural members.

The required element stiffness matrix has been developed by applying the Principle of Minimum Potential Energy. The total potential energy expression of Eq. (1) is believed to be the most general ever presented. It is valid for any type of loading and boundary conditions. The effects of initial crookedness, initial twist, and residual stress have also been included in the derivation.

The results presented in the paper indicate that (1) The finite element method is an effective method for solving elastic and inelastic stability problems. In the elastic range, direct solution of the governing equation can be readily achieved with minimum computational effort. In the inelastic range, however, an iterative procedure based on the modified Newton-Raphson method has been adopted.
(2) The method developed provides good predictions of the load-deformation relationships and the ultimate strength of biaxially loaded columns.

(3) The use of cubic functions to represent the displacements $u$, $v$ and $\theta$ appears to be quite satisfactory for both elastic and inelastic stability analysis.

(4) Structural members with different boundary conditions can be analyzed conveniently with the same formulation.

The method has also been applied successfully to develop solutions for the post-buckling strength of inelastic beams, columns, and beam-columns by incorporating a very small load eccentricity or initial crookedness. Wider application of the method to other types of stability problems involving both material and geometrical nonlinearities is anticipated.

7. References


